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RANGE OF THE HOROCYCLIC RADON TRANSFORM ON TREES

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Introduction.

The *Radon transform* (***RT*** for short), in its original definition by Radon [R], associates to each sufficiently nice function on $\mathbb{R}^2$ its one-dimensional Lebesgue integrals along all affine straight lines. This transform has been widely studied in the last few decades for its highly applicable nature as well as intrinsic interest, both leading to a variety of generalizations.

The other natural ambient spaces of the same dimension are the sphere $\mathbb{S}^2$, its two-to-one quotient $\mathbb{P}^2\mathbb{R}$, and the Poincaré disk $\mathbb{H}^2$, i.e., all the two-dimensional two-point-homogeneous spaces. On these spaces the RT has been studied by Helgason [H] and several others. An instance of application to tomography is described in [BC].

Lines in $\mathbb{R}^2$ have a twofold nature of (maximal) geodesics and of horocycles, so in $\mathbb{H}^2$ their role can be played by two essentially different kinds of one-dimensional submanifolds: geodesics and horocycles, giving rise to two different RTs on $\mathbb{H}^2$.

In recent years discretizations have received considerable attention. Homogeneous trees are widely regarded as discrete counterparts of $\mathbb{H}^2$, as well as objects of thorough study in harmonic analysis in their own right. Exactly like $\mathbb{H}^2$, they feature two distinct kinds of RTs, namely the *geodesic*

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RT (also called *X-ray transform*, since it is reminiscent of the CAT-scan procedure), and the *horocyclic RT*. Several of the standard RT issues in this setting have been investigated over time by various authors: e.g., [BCCP], [A] for injectivity and inversion, [CCP2] for range characterization, and [CC] for function space setting for the geodesic RT; [BP], [BFP], [CCP1] for injectivity and inversion of the horocyclic RT—part of the results therein are rewritten in [CMS] for the *Abel transform*, a multiple of this RT. A related issue, the *Helgason-Fourier transform*, is studied in [CS].

In the present paper, whose results were announced in [CCC], we describe the range of the horocyclic RT R on a homogeneous tree T. We first state the two *Radon conditions*, families of natural explicit relations (one of which had already been observed in [BFPp], [BFP] for radial functions) on functions on the space $\mathcal{H}$ of horocycles of T. We then show that, among compactly supported functions on $\mathcal{H}$, these conditions completely characterize the range of R on finitely supported functions on (the set of vertices of) T. Similar descriptions are valid for the range of R on larger function spaces, although distributions on $\mathcal{H}$ need then to be taken into account. In Theorem 5.5 we show that R is one-to-one on these larger function spaces, thus extending [CCP1, Theorem 3.1]. The boundary Ω of T does not carry any finite Borel measure invariant under the full automorphism group Aut(T) of T, and $\mathcal{H}$ factors (with respect to a reference vertex) as $\Omega \times \mathbb{Z}$. Nevertheless, a natural Aut(T)-invariant measure on $\mathcal{H}$ can be used to express our formulas in an invariant fashion.

In the non-homogeneous case, which we study in a forthcoming paper, analogous results carry over with substantially more complicated expressions of the second Radon condition and, in the non-compact case, of the decay conditions.

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1. Background and notation.

Let T be a *tree*, i.e., an undirected, connected, loop-free graph. Especially with reference to functions, we shall identify T with the set V of its *vertices*, and let $E \subseteq V \times V$ be the set of *edges*. If $(v, w) \in E$ we say that v, w are *neighbors*, and write $v \sim w$ (not an equivalence relation). Given $v, w \in T$, the *path* from v to w is the unique finite sequence of pairwise distinct vertices $v{=}v_0, v_1, \ldots, v_n{=}w$ such that $v_{j-1} \sim v_j$ for all

$j = 1, \ldots, n$. We say that n is the *distance* $d(v, w)$ between v and w, also referred to as the *length* $|w|^v$ of w with respect to v.

We shall assume throughout that T is *homogeneous* of degree $q+1$ for some integer $q \geqslant 2$, that is, each vertex has exactly $q+1$ neighbors; in particular, T is infinite. The group Aut(T) of automorphisms of T (i.e., of distance preserving maps of T onto itself) acts transitively on it. It can be useful to identify T with the free product of $q+1$ copies of $\mathbb{Z}_2$, or (for q odd) of $(q+1)/2$ copies of $\mathbb{Z}$. In either case, two words v, w are neighbors if $v^{-1}w$ is a generator. The left action embeds T as a proper transitive subgroup of Aut(T).

A *ray* starting at $v \in T$ is a one-sided infinite sequence of distinct vertices $v{=}v_0, v_1, \ldots$ such that $v_{j-1} \sim v_j$. The *boundary* of T is the space Ω of *ends*, the classes of rays under the equivalence relation $\simeq$ generated by the unit shift: $\{v_0, v_1, \ldots\} \simeq \{v_1, v_2, \ldots\}$. If we fix $u \in T$, then Ω can be identified with the set of rays beginning at u. Each $\omega \in \Omega$ induces an orientation on the edges of T as follows: an edge (v, w) is positively oriented if there exists a representative ray in ω which starts at v and contains w. For every $\omega \in \Omega$ and $v, w \in T$ define the *horocycle index* $\kappa^v_\omega(w)$ as the number of positively oriented edges minus the number of negatively oriented edges (with respect to ω) in the path from v to w. In particular $\kappa^v_\omega(w) = |w|^v$ if w is in the ray from v to ω, and $\kappa^v_\omega(v) = 0$. It follows from the definitions that the integer $\kappa^u_\omega(v)$ lies in the interval $[-|v|^u, |v|^u]$ at even distance from the endpoints. If w is another vertex, then $\kappa^u_\omega(w) = \kappa^u_\omega(v) + \kappa^v_\omega(w)$.

For $\omega \in \Omega$, $u \in T$, and $n \in \mathbb{Z}$, the set

$$h^u_{\omega,n} = \{w \in T :\, \kappa^u_\omega(w) = n\}$$

is the *horocycle* through ω of index n with respect to u. We use [CCP1] as a general reference for results. The vertices of a horocycle all lie at even distance from each other. Denote by $\mathcal{H}$ the space of horocycles of T. It is immediately seen that the set of horocycles through a fixed $\omega \in \Omega$ does not depend on the choice of the reference vertex u, but indices do. To be precise, if $v \in T$ then

$$h^u_{\omega,n} = h^v_{\omega,n+\kappa^v_\omega(u)}. \tag{1.1}$$

Indeed we have

Remark 1.1.— The map $(\omega, n) \mapsto h^u_{\omega,n}$ is a bijection of $\Omega \times \mathbb{Z}$ onto $\mathcal{H}$ whose dependence on the choice of u shows solely as a shift in the second factor. $\square$

Let Ω_v^u be the set of $\omega \in \Omega$ such that the ray representing ω and beginning at u contains v. There is a compact topology on Ω (independent of u) given by letting $\{\Omega_v^u : v \in T\}$ be a base for the open (and closed) sets. With this topology, Ω is totally disconnected, in fact it is a Cantor set. There is a measure μ^u on Ω given by

$$\mu^u(\Omega_v^u) = \frac{1}{c_{|v|^u}} \qquad \text{where } c_n = \begin{cases} 1 & \text{if } n = 0, \\ (q+1)q^{n-1} & \text{if } n > 0, \end{cases} \tag{1.2}$$

is the number of vertices of distance n from u. The measure μ^u is invariant under the isotropy subgroup at u of Aut(T). The infinitesimal relation

$$d\mu^u(\omega) = q^{\kappa_\omega^v(u)}\, d\mu^v(\omega) \tag{1.3}$$

may be checked on any Ω_w^u containing ω and such that $|w|^u$ is sufficiently large, so that $\Omega_w^v = \Omega_w^u$ and $|w|^v = |w|^u + \kappa_\omega^v(u)$. There is a distance on Ω given by $d^u(\omega, \omega') = \mu^u(\Omega_v^u)$, where Ω_v^u is the smallest basic set containing both ω, ω'.

In view of Remark 1.1, the topology that $\mathcal{H}$ inherits as a product of $\mathbb{Z}$ (with the discrete topology) and Ω is independent of the choice of u. A base is $\{H_{v,n}^u : v \in T,\ n \in \mathbb{Z}\}$, where

$$H_{v,n}^u = \Omega_v^u \times \{n\} = \{h_{\omega,n}^u : \omega \in \Omega_v^u\}.$$

The product measure (used, e.g., in [CCP1]) of the counting measure on $\mathbb{Z}$ and of μ^u on Ω, however, is only invariant under the isotropy subgroup of Aut(T) at u. Instead, if ξ is the measure on $\mathbb{Z}$ given by $\xi(\{n\}) = q^n$, then the product measure $\nu = \mu^u \times \xi$ (cf. [BFP]) does not depend on the choice of u, and is therefore invariant under the whole Aut(T). The corresponding infinitesimal relation is

$$d\nu(h_{\omega,n}^u) = q^n\, d\mu^u(\omega).$$

Indeed, for $v \in T$, the right-hand side equals

$$q^n q^{\kappa_\omega^v(u)}\, d\mu^v(\omega) = d\nu(h_{\omega,n+\kappa_\omega^v(u)}^v)$$

by (1.3), and then (1.1) shows the invariance.

A distance on $\mathcal{H}$ which is invariant under the isotropy subgroup at u is given by $d^u(h_{\omega,n}^u, h_{\omega',n'}^u) = |n - n'| + d^u(\omega, \omega')$. An analogue of (1.1) is then given by

$$H_{u,n}^w = H_{u,n+|u|^v-|u|^w}^v \qquad \text{for } u \text{ not lying on the path from } w \text{ to } v. \tag{1.4}$$

Note that, for u not lying on the path from w to v, we have $\Omega_u^w = \Omega_u^v$.

We shall fix a vertex e as root throughout, although most of the statements below will not depend on its choice. We shall also omit the superscript e in $|v|^e, \Omega_v^e, \mu^e, \kappa_\omega^e(v), h_{\omega,n}^e, H_{u,n}^e$.

For $\omega \in \Omega$ and $n \geqslant 0$, denote by ω_n the n-th vertex of the ray starting at e in the class ω. Similarly, for $v \in T$ and $0 \leqslant n \leqslant |v|$, denote by v_n the n-th vertex of the path from e to v. For $v \in T$ and $n \geqslant |v|$, let

$$D_n(v) = \{u \in T : \ |u| = n, \ u_{|v|} = v\}$$

be the set of *descendants* of v of length n. Then the number of its elements is

$$\#D_n(v) = \frac{c_n}{c_{|v|}}, \tag{1.5}$$

where c_n is given by (1.2). To avoid exceptions to subsequent formulas, set $D_n(v) = \varnothing$ if $n < |v|$. If $v \neq e$, the *parent* of v is $v^- = v_{|v|-1}$. A *family* is a set of the form $D_{|v|+1}(v)$, i.e., the set of all the vertices that share the same parent. The base element $\Omega_v = \{\omega \in \Omega : \ \omega_{|v|} = v\}$ is the boundary of

$$S_v = \{u \in T : \ |u| \geqslant |v|\,, \ u_{|v|} = v\} = \coprod_{n \geqslant |v|} D_n(v).$$

Remark 1.2. — Every horocycle decomposes into a disjoint union as

$$h_{\omega,n} = \begin{cases} \coprod_{k=0}^{\infty} [D_{n+2k}(\omega_{n+k}) \setminus D_{n+2k}(\omega_{n+k+1})] & \text{if } n \geqslant 0, \\ \coprod_{k=0}^{\infty} [D_{-n+2k}(\omega_k) \setminus D_{-n+2k}(\omega_{k+1})] & \text{if } n \leqslant 0, \end{cases}$$

for $\omega \in \Omega$. In particular, $\min_{v \in h_{\omega,n}} |v| = |n|$.

Furthermore, $D_N(\omega_t) \setminus D_N(\omega_{t+1})$ is a disjoint union of families for $N > t+1$, so that any horocycle can be expressed as a disjoint union of families and at most q single vertices. □

For $v \in T$, we may decompose Ω as

$$\Omega = \coprod_{t=0}^{|v|} \Omega_v^{(t)} \qquad \text{where } \Omega_v^{(t)} = \{\omega \in \Omega : \ \kappa_\omega(v) = 2t - |v|\}$$

$$= \begin{cases} \Omega_{v_t} \setminus \Omega_{v_{t+1}} & \text{for } 0 \leqslant t < |v|, \\ \Omega_v & \text{for } t = |v|. \end{cases} \tag{1.6}$$

DEFINITION 1.3. — *The* horocyclic Radon transform R *on* T *is defined as follows: if* $f \in L^1(T)$ *then* $Rf \in L^\infty(\mathcal{H})$ *is given by*

$$Rf(h) = \sum_{v \in h} f(v) \qquad \text{for every } h \in \mathcal{H}.$$

Since f *is an* L^1*-function, it is not necessary to specify the ordering of the sum. We may define* Rf *even when* f *is not in* $L^1(T)$*, by specifying the order of summation:*

$$Rf(h) = \sum_{m=0}^{\infty} \sum_{\substack{|v|=m \\ v \in h}} f(v) \qquad \text{for every } h \in \mathcal{H},$$

as long as the series converges for all $h \in \mathcal{H}$.

In §4 we shall show that the image of R on the space of functions of finite support is the set of functions of compact support on $\mathcal{H}$ that satisfy the Radon conditions, described in §3. We shall see that if $f \in L^1(T)$ then Rf is a continuous function satisfying the Radon conditions. Yet there are continuous functions on $\mathcal{H}$ satisfying these conditions that are of the form Rf for $f \notin L^1(T)$ (cf. Example 3.7). The L^1-condition, however, is the only simply-stated condition on f that ensures that Rf is defined as a function on $\mathcal{H}$. We need to extend the definition of Rf to include generalizations of functions—distributions—on $\mathcal{H}$, and we do so in §2 in terms of the Aut(T)-invariant measure described above.

In §3, in order to extend the result to the case of non-compact support, we introduce the space $\mathcal{A}_s$ (for $0 < s < 1$) of functions on T that satisfy a decay condition depending on s. Likewise, we define the space $\mathcal{B}_s$ of distributions on $\mathcal{H}$ that satisfy a corresponding decay condition and the Radon conditions. Both $\mathcal{A}_s$ and $\mathcal{B}_s$ are increasing one-parameter families, and $\mathcal{A}_s \supset L^1(T)$ for $s \geqslant 1/q$ by Remark 3.6 (the containment is strict by Example 3.7). In §5 we shall show that if $s < 1/\sqrt{q}$ then R is one-to-one on $\mathcal{A}_s$ and the range is precisely $\mathcal{B}_s$. Moreover, we exhibit an example of a function in $\mathcal{A}_s$ whose Radon transform (belonging to $\mathcal{B}_s$ by the above) cannot be evaluated at any horocycle, and thus can be viewed only as a distribution.

2. The distribution-valued Radon transform.

Let $\mathcal{M}^+(\mathcal{H})$ be the set of measurable subsets of $\mathcal{H}$ of positive ν measure.

DEFINITION 2.1.— *A distribution ϕ on $\mathcal{H}$ is a function on $\mathcal{M}^+(\mathcal{H})$ such that the product $\nu\phi$ is a finitely additive complex measure.*

Remark 2.2.— We have chosen to normalize distributions requiring additivity of $\nu\phi$ rather than of ϕ, which is more customary, in order to stress the analogy between forthcoming formulas for functions and those for distributions. □

A distribution ϕ is completely determined by its values on the base elements $H_{u,n}$. Since

$$\nu(H_{v,n}) = \begin{cases} \dfrac{\nu(H_{e,n})}{q+1} & \text{if } |v| = 1, \\ \dfrac{\nu(H_{v^-,n})}{q} & \text{if } |v| > 1, \end{cases}$$

such values must satisfy the averaging property

$$\phi(H_{u,n}) = \begin{cases} \dfrac{1}{q+1} \sum_{|v|=1} \phi(H_{v,n}) & \text{if } u = e, \\ \dfrac{1}{q} \sum_{v^-=u} \phi(H_{v,n}) & \text{if } u \neq e, \end{cases}$$

or more generally

(2.1)
$$\phi(H_{u,n}) = \frac{1}{\#D_m(u)} \sum_{v \in D_m(u)} \phi(H_{v,n}) \qquad \text{for every } u \in T,\ m \geqslant |u|,\ n \in \mathbb{Z}.$$

A locally integrable measurable function ϕ on $\mathcal{H}$ induces a distribution $\tilde{\phi}$ on $\mathcal{H}$ by

$$\tilde{\phi}(H) = \frac{1}{\nu(H)} \int_H \phi(h)\, d\nu(h) \qquad \text{for every } H \in \mathcal{M}^+(\mathcal{H}).$$

In particular, on base elements

(2.2)
$$\tilde{\phi}(H^v_{u,n}) = \frac{1}{\mu^v(\Omega^v_u)} \int_{\Omega^v_u} \phi(h^v_{\omega,n})\, d\mu^v(\omega) \qquad \text{for every } v \in T,\ \omega \in \Omega,\ n \in \mathbb{Z}.$$

Conversely, if $\tilde{\phi}$ is induced by a continuous function ϕ, then ϕ can be recovered from $\tilde{\phi}$ by

$$\phi(h_{\omega,n}) = \lim_{k \to \infty} \tilde{\phi}(H_{\omega_k,n}).$$

We shall henceforth drop the tilde and identify a function on $\mathcal{H}$ with the corresponding distribution.

We now wish to define the Radon transform for a wider class of functions f on T than those that are summable on each horocycle, allowing Rf to be a distribution on $\mathcal{H}$ that is not necessarily induced by a function. Let χ_v denote the characteristic function of the singleton $\{v\}$. Note that, if f is any function on T, its *m-truncation*

$$f_{(m)} = \sum_{|v|\leqslant m} f(v)\chi_v$$

has finite support, therefore $Rf_{(m)}$ exists in the pointwise sense and is a locally integrable measurable function on $\mathcal{H}$, in fact it is uniformly continuous (cf. Proposition 3.3 below).

DEFINITION 2.3.— *Let f be a function on T. Its* horocyclic Radon transform *Rf is the distribution given by*

$$Rf(H) = \lim_{m\to\infty} Rf_{(m)}(H) = \sum_{m=0}^{\infty}\sum_{|v|=m} f(v)R\chi_v(H) \textit{ for every } H \in \mathcal{M}^+(\mathcal{H}),$$

provided the limit exists (i.e., the series is convergent).

If Rf is a function then Definition 2.3 reduces to Definition 1.3. Although the length $|v|$ of $v \in T$ refers to the root e, we now prove:

PROPOSITION 2.4.— *Given a function f on T, the existence and the values of Rf are independent of the root e.*

Proof.— Let e' be another root. We shall show that, for $|u|$ large enough and $n \in \mathbb{Z}$ arbitrary, $Rf(H_{u,n})$ is defined with respect to the root e if and only if it is with respect to e', and, if so, the assigned value is the same. This is enough because for any N the set $\{H_{u,n} : |u| \geqslant N,\ n \in \mathbb{Z}\}$ is itself a base for the measurable subsets of $\mathcal{H}$.

Choose u not in the path from e to e'. Then the parent of u with respect to e or to e' is the same, thus S_u and Ω_u are the same regardless of whether the root is e or e'. For each $n \in \mathbb{Z}$ only finitely many $v \notin S_u$ belong to some (hence all) $h \in H_{u,n}$, i.e., satisfy $R\chi_v(H_{u,n}) \neq 0$. Indeed, if u lies on the ray from v to any $\omega \in \Omega_u$ then $\kappa_\omega(v) = |u| - d(u,v)$, so if this horocycle index equals n, then every such v must be at fixed distance $|u| - n$ from u.

If $\ell = |u| - |u|^{e'}$ then $|v|^{e'} = |v| - \ell$ for all $v \in S_u$, and by (1.4) we have $H_{u,n} = H^{e'}_{u,n-\ell}$. Thus

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{|v|=m} f(v)R\chi_v(H_{u,n}) &= \sum_{v\notin S_u} f(v)R\chi_v(H_{u,n})\\ &\quad+\sum_{m=0}^{\infty}\sum_{\substack{|v|=m\\ v\in S_u}} f(v)R\chi_v(H_{u,n})\\ &= \sum_{v\notin S_u} f(v)R\chi_v(H^{e'}_{u,n-\ell})\\ &\quad+\sum_{m=0}^{\infty}\sum_{\substack{|v|^{e'}=m-\ell\\ v\in S_u}} f(v)R\chi_v(H^{e'}_{u,n-\ell})\\ &= \sum_{m'=0}^{\infty}\sum_{|v|^{e'}=m'} f(v)R\chi_v(H^{e'}_{u,n-\ell})\end{aligned}$$

since, by the above, in each sum for $v \notin S_u$ there are only finitely many non-zero terms. □

LEMMA 2.5. — *For $u, v \in T$ and $n \in \mathbb{Z}$ we have*

$$R\chi_v(H_{u,n}) = \begin{cases} \dfrac{q-1}{q}q^{|u|-(n+|v|)/2} & \text{if } v \in S_u,\ u \neq e,\ n-|v| \text{ even,}\\ & \text{and } 2|u|-|v| \leqslant n < |v|,\\ q^{|u|-|v|} & \text{if } v \in S_u,\ u \neq e,\ n = |v|,\\ \dfrac{q-1}{q+1}q^{-(n+|v|)/2} & \text{if } v \neq u = e,\ n-|v| \text{ even, } |n| < |v|,\\ \dfrac{q}{q+1}q^{-(n+|v|)/2} & \text{if } v \neq u = e,\ |n| = |v|,\\ 1 & \text{if } v \notin S_u \text{ or } v = e,\\ & \text{and } n = |u| - d(u,v),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. — The equality

$$R\chi_v(H^v_{u,m}) = \begin{cases} 1 & \text{if } m = 0,\\ 0 & \text{if } m \neq 0,\end{cases} \qquad \text{for every } u, v \in T, \tag{2.3}$$

follows directly from (2.2). If $v \notin S_u$, by (1.4) and (2.3) we have

$$R\chi_v(H_{u,n}) = R\chi_v(H^v_{u,n+d(u,v)-|u|}) = \begin{cases} 1 & \text{if } n = |u| - d(u,v),\\ 0 & \text{otherwise.}\end{cases}$$

Instead, if $v \in S_u$, then we may decompose $H_{u,n} = \coprod_j H_{w_{(j)},n}$, where the vertices $w_{(j)}$ are all the immediate descendants of the vertices on the path from u to v which are not themselves on this path. Since $v \notin S_{w_{(j)}}$ for any j, we can apply the previous case and the additivity of $\nu R\chi_v$ to get

$$R\chi_v(H_{u,n}) = \sum_j \frac{c_{|u|}}{c_{|w_{(j)}|}} R\chi_v(H_{w_{(j)},n}) = \sum_{|w_{(j)}|-d(w_{(j)},v)=n} \frac{c_{|u|}}{c_{|w_{(j)}|}},$$

which yields the result. □

DEFINITION 2.6. — *In the sequel, for a function f on T we shall use the shorthand*

$$f(v,n) = \frac{1}{\#D_n(v)} \sum_{u \in D_n(v)} f(u) \qquad \text{for } v \in T \text{ and } n \geqslant |v|. \tag{2.4}$$

*In particular, $f(v) = f(v,|v|)$. For a non-negative integer N, the N-*radialization *f_N (with respect to e) of a function f on T is given by*

$$f_N(v) = \begin{cases} f(v) & \text{if } |v| \leqslant N, \\ f(v_N, |v|) & \text{if } |v| \geqslant N. \end{cases}$$

If ρ_N is the left-invariant Haar measure on the compact group

$$\mathrm{Aut_N} = \{\mathrm{g} \in \mathrm{Aut(T)} : g(v) = v \text{ whenever } |v| \leqslant N\},$$

*normalized so that $\rho_N(\mathrm{Aut_N}) = 1$, then $f_N = \int_{\mathrm{Aut_N}} (f \circ g)\, d\rho_N(g)$. We say that f is N-*radial *if $f_N = f$, i.e., if $f(v)$ depends only on v_N and $|v|$ whenever $|v| \geqslant N$. A 0-radial function is usually called a* radial *function* [CD].

*The N-*radialization *ϕ_N of a distribution ϕ on $\mathcal{H}$ is given by*

$$\phi_N(H_{u,n}) = \begin{cases} \phi(H_{u,n}) & \text{if } |u| \leqslant N, \\ \phi(H_{u_N,n}) & \text{if } |u| > N, \end{cases} \tag{2.5}$$

and that of a function ϕ on $\mathcal{H}$ by

$$\phi_N(h_{\omega,n}) = \frac{1}{\mu(\Omega_{\omega_N})} \int_{\Omega_{\omega_N}} \phi(h_{\omega',n})\, d\mu(\omega').$$

*We say that a distribution ϕ is N-*radial *if $\phi_N = \phi$, i.e., if $\phi(H_{u,n})$ depends only on u_N and n whenever $|u| \geqslant N$. In this case it is immediate that ϕ is in fact (induced by) a function for which $\phi(h_{\omega,n})$ depends only on ω_N and n—indeed, $\phi(h_{\omega,n}) = \phi(H_{\omega_N,n})$. Conversely, every function with this property induces an N-radial distribution. We have $\phi_N = \int_{\mathrm{Aut_N}} (\phi \circ g)\, d\rho_N(g)$.*

For a non-negative integer M, a function f on T is M-supported *if its support is contained in $\{v \in T : |v| \leqslant M\}$. A distribution (respectively a function) ϕ on $\mathcal{H}$ is* M-supported *if whenever $|n| > M$ we have $\phi(H_{u,n}) = 0$ for all $u \in T$ (respectively $\phi(h_{\omega,n}) = 0$ for all $\omega \in \Omega$).*

A continuous function ϕ on $\mathcal{H}$ is M-supported if and only if its induced distribution is. A function f on T is finitely supported, or a distribution ϕ on $\mathcal{H}$ is compactly supported, if and only if it is M-supported for some M.

Remark 2.7. — Since $\phi_N(H_{u,n}) = \phi(H_{u,n})$ for all $N \geqslant |u|$, we have $\phi = \lim_{N\to\infty} \phi_N$ in the sense of distributions, and, if ϕ is a continuous function, the limit holds uniformly on compact sets. Of course $f = \lim_{N\to\infty} f_N$ pointwise, or equivalently, uniformly on compact sets. □

Remark 2.8. — The characteristic function χ_v of $v \in T$ is $|v|$-radial and $|v|$-supported. It will follow from Proposition 2.9 below that its Radon transform $R\chi_v$ is also. □

PROPOSITION 2.9. — *If a function f on T is such that Rf is defined, and if $N \geqslant 0$, then Rf_N is defined and equals $(Rf)_N$. In particular, if f is N-radial then so is Rf. Similarly, if f is M-supported then so is Rf.*

Proof. — Assume first that f has finite support. Since R is equivariant for the action of Aut(T) on functions on T and distributions on $\mathcal{H}$, we have

$$\begin{aligned}(Rf)_N &= \int_{\mathrm{Aut_N}} (Rf \circ g)\, d\rho_N(g) = \int_{\mathrm{Aut_N}} R(f \circ g)\, d\rho_N(g)\\ &= R\int_{\mathrm{Aut_N}} (f \circ g)\, d\rho_N(g) = Rf_N,\end{aligned}$$

where the third equality holds because $|R(f \circ g)(h)| = |Rf(g(h))| \leqslant \|f\|_1$ for all $h \in \mathcal{H}$. (A non-group-theoretic proof can be achieved by working out the case $f = \chi_v$ using Lemma 2.5.)

If f is an arbitrary function on T, then it is straightforward to verify that m-truncation commutes with N-radialization, i.e., $(f_N)_{(m)} = (f_{(m)})_N$ for every $N, m \geqslant 0$. Since $f_{(m)}$ has finite support we can apply the previous case, and obtain for every $n \in \mathbb{Z}$ and $|u| \geqslant N$ that

$$R(f_N)_{(m)}(H_{u,n}) = R(f_{(m)})_N(H_{u,n}) = (Rf_{(m)})_N(H_{u,n}) = Rf_{(m)}(H_{u_N,n}).$$

The limit for $m \to \infty$ is then $R(f_N)(H_{u,n}) = Rf(H_{u_N,n}) = (Rf)_N(H_{u,n})$.

If f is M-supported then $Rf(h_{\omega,n}) = 0$ for $|n| > M$, because the length of every $v \in h_{\omega,n}$ is at least $|n|$. So Rf is M-supported. □

Example 2.10. — Fix $\omega \in \Omega$, and let f be the function on T given by

$$f(v) = \begin{cases} q^j & \text{if } v = \omega_{2j} \text{ for } j > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The Radon transform Rf is defined as a function on $\mathcal{H}$, because each horocycle intersects the ray $\{\omega_0, \omega_1, \ldots\}$ in at most two vertices. However, Rf is not defined in the distribution sense, since by Lemma 2.5 $R\chi_{\omega_{2j}}(H_{e,0}) = q^{-j}(q-1)/(q+1)$ for every $j \geqslant 1$, whence $Rf(H_{e,0})$ would be the limit for $m \to \infty$ of

$$\sum_{|v| \leqslant 2m} f(v) R\chi_v(H_{e,0}) = \sum_{j=1}^{m} q^j \frac{q-1}{q+1} q^{-j} = \frac{q-1}{q+1} m.$$

In other words, the function Rf is not locally integrable.

3. The Radon and decay conditions.

Definition 3.1. — *For a distribution ϕ on $\mathcal{H}$ the* Radon conditions *are:*

$$(\mathrm{R}_1) \qquad \sum_n \phi(H_{u,n}) \qquad \textit{is independent of } u \in T.$$

$$(\mathrm{R}_2) \qquad \phi(H^v_{v,-n}) = q^n \phi(H^v_{v,n}) \qquad \textit{for every } v \in T \textit{ and } n \in \mathbb{Z}.$$

If both conditions are satisfied we say that ϕ is R-compatible.

Remark 3.2. — Condition (R_1) is root-independent, since it can be equivalently stated restricting it to $|u|$ large enough. Indeed, the condition for a given u follows easily, using (2.1), from the condition for its descendants. If ϕ is a continuous function on $\mathcal{H}$, letting u approach an arbitrary $\omega \in \Omega$ (i.e., $u = \omega_k$ with $k \to \infty$), condition (R_1) yields

$$(\mathrm{R}'_1) \qquad \sum_n \phi(h_{\omega,n}) \qquad \text{is independent of } \omega \in \Omega,$$

and this in turn implies (R_1) by (2.2) with $v = e$.

Condition (R_2) for radial functions appeared in [BFPp, before Theorem 4.5], although in the final version [BFP, Proof of Theorem 4.1] it was stated only for a specific function.

Condition (R_2) is obviously also root-independent. It may be rewritten in terms of the values at the base elements $H_{u,n}$ as follows:

$$q(q+1)\phi(H_{e,-n-|v|}) + \sum_{t=1}^{|v|} q^t [q^2 \phi(H_{v_t,-n+2t-|v|}) - \phi(H_{v_t,-n+2t-|v|-2})]$$

$$= q^n \left(q(q+1)\phi(H_{e,n-|v|}) + \sum_{t=1}^{|v|} q^t [q^2 \phi(H_{v_t,n+2t-|v|}) - \phi(H_{v_t,n+2t-|v|-2})] \right)$$

(R_2') for every $v \in T$ and $n \in \mathbb{Z}$.

This expression is obtained using the additivity of $\nu\phi$, the decomposition (1.6) which yields

$$H^v_{v,n} = \coprod_{t=0}^{|v|} (H_{v_t,n+2t-|v|} \setminus H_{v_{t+1},n+2t-|v|})$$

(where we formally set $H_{v_{|v|+1},m} = \varnothing$), the observation that $H_{v_{t+1},m} \subseteq H_{v_t,m}$ for every $m \in \mathbb{Z}$ and $t = 0, \dots, |v|$, and finally using (1.2). □

PROPOSITION 3.3.— *If $f \in L^1(T)$, then Rf is R-compatible and uniformly continuous on $\mathcal{H}$. The constant value in* (R_1) *is* $\sum_{v \in T} f(v)$.

Proof.— Given $\epsilon > 0$, choose $N \geqslant 0$ such that $\sum_{|v| \geqslant N} |f(v)| < \epsilon/2$. If $\omega, \omega' \in \Omega$ have distance less than $1/c_N$, then they are both in Ω_v for some $v \in T$ of length N. Then the horocycles $h_{\omega,n}, h_{\omega',n}$ differ only in vertices whose lengths are greater than N, thus $|Rf(h_{\omega,n}) - Rf(h_{\omega',n})| < \epsilon$. So Rf is uniformly continuous.

The horocycles through a given $\omega \in \Omega$ partition T, since every $v \in T$ belongs to exactly one such horocycle, namely $h_{\omega,\kappa_\omega(v)}$. Therefore $\sum_{v \in T} f(v) = \sum_n Rf(h_{\omega,n})$, and (R_1') follows.

We shall prove (R_2') for $\phi = Rf$ when $f = \chi_u$ for arbitrary $u \in T$, and then obtain it for general $f \in L^1(T)$ by linearity. Because condition (R_2) (and hence (R_2')) is root-independent, we can choose $e = u$. By (2.3), the only non-vanishing terms are those in which the second subscript is zero. Such terms are there only if $|n|$ is less than or equal to $|v|$ and has its same parity. The verification of (R_2') in this case is straightforward. □

LEMMA 3.4.— *If a distribution ϕ on $\mathcal{H}$ satisfies either of the Radon conditions, then so does its N-radialization ϕ_N for all $N \geqslant 0$.*

Proof.— The statement for (R_1) follows from the definitions. For $n \in \mathbb{Z}$, the equality in (R_2') at v for ϕ_N is the same as for ϕ if $|v| \leqslant N$, while for $|v| \geqslant N$ it is the average of that for ϕ at v' over all $v' \in D_{|v|}(v_N)$. Indeed, for $0 \leqslant t \leqslant N$ and every such v' we then have $v'_t = v_t$, whereas for $N \leqslant t \leqslant |v|$, by (2.1) and (1.5),

$$\phi_N(H_{v_t,m}) = \phi(H_{v_N}, m) = \frac{c_N}{c_{|v|}} \phi(H_{v'_t,m}).$$ □

DEFINITION 3.5. — *For* $0 < s < 1$ *let*

$$\begin{aligned}\mathcal{D}_s &= \left\{\text{numerical sequences } (a_n)_{n\geqslant n_0} : \limsup_{n\to\infty} |a_n|^{1/n} \leqslant s\right\}\\ &= \left\{(a_n)_{n\geqslant n_0} : \text{the radius of convergence of the series } \sum_{n=n_0}^{\infty} a_n x^n \text{ is} \geqslant \frac{1}{s}\right\}.\end{aligned}$$

Let $\mathcal{A}_s$ *be the set of functions* f *on* T *satisfying the* s-decay condition

$$\big(f(v,n)\big)_{n\geqslant|v|} \in \mathcal{D}_s \qquad \text{for every } v \in T$$

(recall (2.4)*). Equivalently, in view of* (1.5),

$$\left(\sum_{u\in D_n(v)} f(u)\right)_{n\geqslant|v|} \in \mathcal{D}_{qs} \qquad \text{for every } v \in T.$$

With an argument similar to that used at the beginning of Remark 3.2 we see that the condition is root-independent, because it can be equivalently stated by restricting to $|v|$ *large. Indeed, the condition for a given* v *follows from that for its immediate descendants.*

Correspondingly, let $\mathcal{B}_s$ *be the set of R-compatible distributions* ϕ *on* $\mathcal{H}$ *satisfying the* s-decay condition

$$\big(\phi(H_{v,n})\big)_{n\geqslant n_0} \in \mathcal{D}_s \qquad \text{for every } v \in T.$$

Remark 3.6. — The one-parameter families $\mathcal{A}_s$ and $\mathcal{B}_s$ are increasing, and $L^1(T) \subseteq \mathcal{A}_{1/q}$, since for $f \in L^1(T)$ not identically zero we have

$$\left|\sum_{u\in D_n(v)} f(u)\right|^{1/n} \leqslant \|f\|_1^{1/n} \to 1.$$

Conversely, however, $L^1(T)$ does not contain $\mathcal{A}_s$ for any s, as shown below in Example 3.7. Yet, it does contain $\{f \in \mathcal{A}_s : f \text{ is } N\text{-radial for some } N\}$ for every $s < 1/q$. Indeed, if f is N-radial and s-decaying, then

$$\|f\|_1 = \sum_{|v|<N} |f(v)| + \sum_{n=N}^{\infty} \#D_n(v) \sum_{|v|=N} |f(v,n)| < \infty,$$

since $\#D_n(v)$ is proportional to q^n. □

Example 3.7.— Let f be a function on T of constant modulus one (thus $f \notin L^1(T)$), and such that $\sum_{v^-=u} f(v) = 0$ for all $u \in T$ (so $f \in \mathcal{A}_s$ for all s). (We can take $f(v)$ to be a q-th root of unity for $|v| > 1$, a $(q+1)$-st root of unity for $|v| = 1$, and 1 for $v = e$.) Although $f \notin L^1(T)$, we now show that Rf is defined and is (induced by) a continuous R-compatible function on $\mathcal{H}$.

If $\mathbb{F}$ is a union of families, then $\sum_{n=0}^{\infty} \sum_{\substack{|u|=n \\ u \in \mathbb{F}}} f(u) = 0$ because for each n there are only finitely many families with elements of length n. By Remark 1.2, for $\omega \in \Omega$ the horocycle $h_{\omega,n}$ is the disjoint union of

$$\begin{cases} \text{countably many families} & \text{if } n < -1, \\ D_1(e) \setminus \{\omega_1\} \text{ and countably many families} & \text{if } n = -1, \\ \{\omega_n\},\ D_{n+2}(\omega_{n+1}) \setminus \{\omega_{n+2}\} \text{ and countably many families} & \text{if } n > -1. \end{cases}$$

Thus, for every $\omega \in \Omega$ we have

$$Rf(h_{\omega,n}) = \begin{cases} 0 & \text{if } n < -1, \\ -f(\omega_1) & \text{if } n = -1, \\ f(\omega_n) - f(\omega_{n+2}) & \text{if } n > -1. \end{cases}$$

Since for all n the value $Rf(h_{\omega,n})$ is constant on Ω_v for $|v| = n+2$, then Rf is a continuous function on $\mathcal{H}$. Furthermore $\sum_n Rf(h_{\omega,n}) = f(e)$ for all $\omega \in \Omega$, so that (R_1') holds for $\phi = Rf$. We see that Rf satisfies (R_2) for $v = e$, because from (2.2)

$$Rf(H_{e,n}) = \int_\Omega Rf(h_{\omega,n})\, d\mu(\omega) = \begin{cases} f(e) & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases}$$

since $\int_\Omega f(\omega_k)\, d\mu(\omega)$ vanishes for $k \neq 0$, because it is a sum of f over a union of families. For v arbitrary we can either verify (R_2') directly, or else we can observe that, except for the vertices in the path from e to v and their immediate descendants, families are the same with respect to e as with respect to v. Thus f may be written as the sum of a function f_1 with finite support (hence in $L^1(T)$), and a function f_2 with vanishing sums over all families with respect to the root v. Both Rf_1 and Rf_2 exist and satisfy (R_2) by Proposition 3.3 and by the case $v = e$. □

PROPOSITION 3.8.— *If $s < 1/\sqrt{q}$ then R is defined on all of $\mathcal{A}_s$.*

Proof.— Let $f \in \mathcal{A}_s$, $u \in T$ and $n \in \mathbb{Z}$. In the proof of Proposition 2.4 we observed that $v \notin S_u$ only for finitely many non-zero terms of the series defining $Rf(H_{u,n})$ (cf. Definition 2.3), which we may disregard for convergence purposes. By Lemma 2.5 the tail for $m > m_0 = |n| + 2\,|u|$

of the remaining series is

$$C_u \sum_{\substack{m>m_0 \\ m-n \text{ even}}} \sum_{v\in D_m(u)} f(v)q^{|u|-(m+n)/2}$$
$$= C_u q^{|u|-n/2} \sum_{\substack{m>m_0 \\ m-n \text{ even}}} q^{-m/2} \sum_{v\in D_m(u)} f(v)$$

(where C_u equals $(q-1)/(q+1)$ if $u=e$ and $(q-1)/q$ if $u\neq e$), which converges absolutely for $f\in\mathcal{A}_s$ if $s<1/\sqrt{q}$. □

4. Functions of compact support.

NOTATION 4.1. — *Label $v_N^1,\dots,v_N^{c_N}$ all vertices of T of a given length $N\geqslant 0$. If $v=v_N^k$ is one of them, partition the set $\{1,\dots,c_N\}$ as the disjoint union for $t=0,\dots,N$ of*

$$J_v^t=\{j=1,\dots,c_N:\Omega_{v_N^j}\subseteq\Omega_v^{(t)}\}=\{j: d(v_N^j,v_s) \text{ is minimal at } s=t\}.$$

In particular $J_v^N=\{k\}$.

If ϕ is a distribution on $\mathcal{H}$, set

$$a_n^j=\phi(H_{v_N^j,n})\qquad \text{for } j=1,\dots,c_N \text{ and } n\in\mathbb{Z}.$$

If ϕ is N-radial its values can be completely recovered from the knowledge of a_n^j, using (2.1) and (2.5).

LEMMA 4.2. — *Condition* (R_2) *(or* (R_2')*) can be rewritten as*

$$\sum_{t=0}^{|v|} q^{2t}\sum_{j\in J_v^t} a^j_{-n+2t-|v|} = q^n\sum_{t=0}^{|v|} q^{2t}\sum_{j\in J_v^t} a^j_{n+2t-|v|}$$

(R_2'') *for all $n\in\mathbb{Z}$ and $v\in T$, and with $N=|v|$.*

Proof. — For $t=0,\dots,N$ we have $\Omega_{v_t}=\coprod_{\substack{s=t,\dots,N\\ j\in J_v^s}}\Omega_{v_N^j}$. Observing that

$$D_N(v_t)=\{v_N^j: j\in J_v^s,\ t\leqslant s\leqslant N\}$$

and applying (2.1) and (1.5) we obtain

$$\phi(H_{v_t,m})=\frac{c_t}{c_N}\sum_{\substack{s=t,\dots,N\\ j\in J_v^s}} a_m^j.$$

The left-hand side of (R_2') then equals $q(q+1)$ times

$$\sum_{t=0}^{N} q^{2t} \sum_{\substack{s=t,\dots,N \\ j\in J_v^s}} a^j_{-n+2t-N} - q^{-2}\sum_{t=1}^{N} q^{2t} \sum_{\substack{s=t,\dots,N \\ j\in J_v^s}} a^j_{-n+2t-N-2}.$$

A shift in the index t in the double sum on the right yields the left-hand side of (R_2''). The right-hand side is handled similarly. □

For N, M non-negative integers let $E_{N,M}$ be the space of N-radial M-supported R-compatible functions on $\mathcal{H}$. Clearly

$$(4.1) \qquad E_{N,M} \subseteq E_{N',M'} \qquad \text{if } N \leqslant N' \text{ and } M \leqslant M'.$$

LEMMA 4.3. — *If $N \geqslant M \geqslant 0$, all N-radial M-supported R-compatible functions on $\mathcal{H}$ are in fact M-radial, i.e., $E_{N,M} = E_{M,M}$.*

Proof. — It will suffice to prove that $E_{M+1,M} \subseteq E_{M,M}$ for all $M \geqslant 0$. The result will follow from (4.1) and the induction step

$$E_{N+1,M} \subseteq E_{N+1,M} \cap E_{N+1,N} \subseteq E_{N+1,M} \cap E_{N,N} \subseteq E_{N,M}.$$

If ϕ is a 0-supported distribution on $\mathcal{H}$ and $n \neq 0$ then $\phi(H_{u,n}) = 0$ for every $u \in T$. If ϕ also satisfies (R_1'), then $\phi(H_{u,0})$ is independent of $u \in T$, so that ϕ is 0-radial. In particular $E_{1,0} \subseteq E_{0,0}$.

Suppose $M > 0$ and $\phi \in E_{M+1,M}$. We need to show that $a_n^j = a_n^{j'}$ whenever $v_{M+1}^j, v_{M+1}^{j'}$ have the same parent. If $v = v_{M+1}^k$ for a given $k = 1, \dots, c_{M+1}$, this amounts to showing that $a_m^j = a_m^k$ for j in J_v^M, the set of indices of vertices different from v but having its same parent. Since a_m^j vanishes for $|m| > M$, we need to prove this only for $|m| \leqslant M$.

To start a two-step induction, we prove the cases $m = -M, -M+1$. With $N = M+1$, applying (R_2'') for $n = -2M-1, -2M$, respectively, gives

$$\sum_{j\in J_v^0} a_M^j = q a_{-M}^k,$$

$$\sum_{j\in J_v^0} a_{M-1}^j = q^2 a_{-M+1}^k.$$

Observe that $J_v^0 = \{j : \Omega_{v_{M+1}^j} \cap \Omega_{v_1} = \varnothing\}$ depends only on v^- (in fact only on v_1). Thus so do a_{-M}^k, a_{-M+1}^k.

Next let $-M+1 < m \leqslant M$, and consider (R_2'') for $n = -M-1+m$. For $t < M$, since J_v^t depends only on v^-, then so does every corresponding term on either side of the equality. On the left-hand side, the term $t = M+1$ vanishes, because $2t - m > M$. The same is true for $t = M$ provided $m < M$. On the right-hand side, the summands for $t = M$ are a^j_{m-2}, all of which depend only on v^- by induction. Therefore, the unique summand of the remaining term $t = M+1$, namely a^k_m, depends only on v^-.

If $m = M$ we have shown that the difference $qa^k_M - \sum_{j\in J^M_v} a^j_M$ depends only on v^- for every k. Assume the labeling is such that $v^1_N, \dots, v^q_N$ share the same parent. Then for $k \leqslant q$ and $v = v^k_N$ we have $J^M_v = \{1, \dots, \hat{k}, \dots, q\}$. The difference $qa^k_M - \sum_{\substack{j=1,\dots,q\\ j\neq k}} a^j_M$ is therefore independent of $k = 1, \dots, q$, and the resulting $q-1$ equations clearly imply that a^k_M is also independent of k. □

LEMMA 4.4. — *If $M \geqslant 0$, in order for an R-compatible distribution ϕ on $\mathcal{H}$ to be M-supported, it is sufficient that $\phi(H_{u,n}) = 0$ for every $u \in T$ and $n > M$.*

Proof. — Assume that $\phi(H_{u,n}) = 0$ for every $u \in T$ and $n > M$. We shall prove by induction on $|v|$ that $\phi(H_{v,n}) = 0$ whenever $n < -M$. Replacing n by $|v| - n$ in (R_2') we obtain

$$q(q+1)\phi(H_{e,n-2|v|}) + \sum_{t=1}^{|v|} q^t[q^2\phi(H_{v_t,n+2t-2|v|}) - \phi(H_{v_t,n+2t-2|v|-2})] = 0 \qquad \text{if } n < -M.$$

If $v = e$ this reduces to $\phi(H_{e,n}) = 0$, which starts the induction. If $v \neq e$ then, by the induction hypothesis, all summands vanish except possibly those for which $t = |v|$, so that $\phi(H_{v,n-2}) = q^2\phi(H_{v,n})$ whenever $n < -M$. Thus, for every positive integer p, we have $\phi(H_{v,n-2p}) = q^{2p}\phi(H_{v,n})$, and this must vanish in order for the series in (R_1) to converge. □

PROPOSITION 4.5. — *For every $N, M \geqslant 0$, each N-radial M-supported R-compatible function ϕ on $\mathcal{H}$ is the Radon transform of an N-radial M-supported function on T.*

Proof. — Let ϕ be an N-radial M-supported R-compatible function on $\mathcal{H}$. We shall find an M-supported f such that $Rf = \phi$. Then $Rf_N = \phi_N = \phi$, and ϕ_N has the required properties. (Actually $f_N = f$, because R is one-to-one on $L_1(T)$ by [CCP1, Theorem 3.1].)

By (4.1) and Lemma 4.3 we only need to prove the case $N = M$. We shall show that $E_{N,N}$ is linearly generated by $E_{N,N-1}$ and $\{R\chi_v : |v| = N\}$

for all $N \geqslant 0$ (set $E_{0,-1} = \{0\}$ to avoid exceptions). But $E_{N,N-1} = E_{N-1,N-1}$ by Lemma 4.3, proving the inductive step.

If $\psi \in E_{N,N}$ we shall see that $\phi = \psi - \sum_{|v|=N} \psi(H_{v,N}) R\chi_v \in E_{N,N-1}$. Indeed, by Remark 2.8 and Proposition 3.3, we have $\phi \in E_{N,N}$, so by Lemma 4.4 it suffices to show that a_N^j vanishes for all $j = 1, \dots, c_N$. Using Lemma 2.5, we see that $R\chi_{v_N^\ell}(H_{v_N^j,N}) = \delta_{\ell,j}$, the Kronecker symbol, for all $\ell, j = 1, \dots, c_N$, and hence

$$a_N^j = \psi(H_{v_N^j,N}) - \sum_{\ell=1}^{c_N} \psi(H_{v_N^\ell,N})\delta_{\ell,j} = 0. \qquad \square$$

THEOREM 4.6. — *The Radon transform is one-to-one on the space of functions of finite support, and its range is the space of R-compatible distributions on $\mathcal{H}$ of compact support. These distributions are actually continuous functions. In fact, with respect to a root e, the image of the set of M-supported functions on T is the set of continuous M-supported R-compatible functions on $\mathcal{H}$.*

Proof. — The injectivity of R on finitely supported functions derives from that on $L^1(T)$ [CCP1], or will follow from that on $\mathcal{A}_s$ in Theorem 5.5 below.

If f is an M-supported function on T, then Rf is a continuous M-supported R-compatible function on $\mathcal{H}$ by Proposition 2.9 and Proposition 3.3.

Conversely, if ϕ is an M-supported R-compatible distribution on $\mathcal{H}$, then for $N \geqslant 0$ its N-radialization ϕ_N is R-compatible by Lemma 3.4, and is also M-supported. If $N \geqslant M$, then $\phi_N = \phi_M$ by Lemma 4.3, and, by Remark 2.7, $\phi = \lim_{N\to\infty} \phi_N = \phi_M \in E_{M,M}$, so by Proposition 4.5 it is in the image of R. $\square$

5. Functions of non-compact support.

Remark 5.1. — The s-decay conditions are compatible with N-radializations in the following sense. A function f on T is s-decaying if and only if also f_N is for every N. This holds since $f_N(v,n) = f(v_N,n)$ whenever $n \geqslant |v| \geqslant N$ (for the "if" part, take $N = |v|$ for a given $v \in T$). The same equivalence holds for distributions on $\mathcal{H}$ and follows from (2.5) in a similar manner. □

Remark 5.2. — An N-radial function f on T is s-decaying if and only if $\big(f(\omega_n)\big)_{n\geqslant 0} \in \mathcal{D}_s$ for every $\omega \in \Omega$, since, by N-radiality, $f(v,n) = f(\omega_n)$ whenever $n \geqslant |v| \geqslant N$ and $\omega \in \Omega_v$ (take $v = \omega_N$ to prove the "only if" part). Similarly, an N-radial function ϕ on $\mathcal{H}$ is s-decaying if and only if $\big(\phi(h_{\omega,n})\big)_{n\geqslant 0} \in \mathcal{D}_s$ for every $\omega \in \Omega$, because, by N-radiality, $\phi(H_{v,n}) = \phi(h_{\omega,n})$ whenever $|v| \geqslant N$ and $\omega \in \Omega_v$. □

Denote by c the sequence $(c_n)_{n\geqslant 0}$.

LEMMA 5.3. — *If $s < 1/\sqrt{q}$, the operator U on $\mathcal{D}_s$ given by $U(a) = b$, where*

$$b_n = a_n - (q-1)\sum_{k>0} a_{n+2k} \qquad \textit{for every } n \geqslant 0,$$

is a bijection of $\mathcal{D}_s$ onto itself, whose inverse is given by

$$a_n = b_n + (q-1)\sum_{k>0} q^{k-1} b_{n+2k} \qquad \textit{for every } n \geqslant 0.$$

Proof. — The space $\mathcal{D}_s$ can be decomposed as $\mathcal{D}_{s^2} \times \mathcal{D}_{s^2}$, splitting each $a \in \mathcal{D}_s$ into a pair (a^e, a^o), where $a^e_n = a_{2n}$ and $a^o_n = a_{2n+1}$ for every $n \geqslant 0$. Correspondingly, $U(a) = b$ splits as $(b^e, b^o) = (V(a^e), V(a^o))$, where V is the operator on $\mathcal{D}_{s^2}$ given by $V(a) = b$ with

$$b_n = a_n - (q-1)\sum_{m>n} a_m \qquad \text{for every } n \geqslant 0.$$

We need to prove that V is a bijection of $\mathcal{D}_{s^2}$ onto itself, and its inverse is given by

$$a_n = b_n + (q-1)\sum_{m>n} q^{m-n-1} b_m \qquad \text{for every } n \geqslant 0.$$

Identify $\mathcal{D}_{s^2}$ with the space $H(s^{-2})$ of analytic functions on $\{z \in \mathbb{C} : |z| < s^{-2}\}$ by $a \mapsto f(z) = \sum_{n=0}^{\infty} a_n z^n$. We need to show that the function

g corresponding to the sequence $V(a) = b$ is in $H(s^{-2})$. We have

$$\begin{aligned} g(z) = \sum_{n=0}^{\infty} b_n z^n &= \sum_{n=0}^{\infty} a_n z^n - (q-1) \sum_{n=0}^{\infty} \Big(\sum_{m>n} a_m \Big) z^n \\ &= f(z) - (q-1) \sum_{m=1}^{\infty} a_m \sum_{n=0}^{m-1} z^n = f(z) - \frac{q-1}{z-1} \sum_{m=1}^{\infty} a_m (z^m - 1) \\ &= f(z) - (q-1) \frac{f(z) - f(1)}{z-1} = \frac{(z-q)f(z) + (q-1)f(1)}{z-1}. \end{aligned}$$

So g is defined wherever f is, including at $z = 1$, where $g(1) = f(1) - (q-1)f'(1)$ (recall that $s^{-2} > 1$). Thus if $f \in H(s^{-2})$ then $g \in H(s^{-2})$. Since $g(q) = f(1)$, we can solve for $f(z)$ and get

$$f(z) = \frac{(z-1)g(z) - (q-1)g(q)}{z-q} = g(z) + (q-1)\frac{g(z) - g(q)}{z-q}.$$

If $g \in H(s^{-2})$, since $s^{-2} > q$ then $f(q) = g(q) + (q-1)g'(q)$ is defined, thus $f \in H(s^{-2})$. The expression for V^{-1} comes from the identities

$$\begin{aligned} \sum_{n=0}^{\infty} a_n z^n = f(z) &= \sum_{n=0}^{\infty} b_n z^n + \frac{q-1}{z-q} \sum_{n=0}^{\infty} b_n (z^n - q^n) \\ &= \sum_{n=0}^{\infty} b_n z^n + (q-1) \sum_{n=1}^{\infty} b_n \sum_{m=0}^{n-1} z^m q^{n-m-1} \\ &= \sum_{n=0}^{\infty} b_n z^n + (q-1) \sum_{n=0}^{\infty} \Big(\sum_{m>n} q^{m-n-1} b_m \Big) z^n. \end{aligned}$$ □

PROPOSITION 5.4.— *For $s < 1/\sqrt{q}$ the image of R on the set of N-radial s-decaying functions on T is the set of N-radial s-decaying R-compatible functions on $\mathcal{H}$.*

Proof.— Let f be an N-radial s-decaying function on T. Then $\phi = Rf$ is defined by Proposition 3.8, and N-radial by Proposition 2.9. For $\omega \in \Omega$, if $n \geqslant N$ we have $h_{\omega,n} \subseteq S_{\omega_N}$, hence $f(v) = f(\omega_{|v|})$ for every $v \in h_{\omega,n}$. Thus, using Remark 1.2 and (1.5),

$$\phi(h_{\omega,n}) = f(\omega_n) + \sum_{k>0} (q^k - q^{k-1}) f(\omega_{n+2k}) \qquad \text{for } n \geqslant N.$$

Therefore for $n \geqslant N$ the sequence $\big(\phi(h_{\omega,n})\big)_{n\geqslant 0}$ equals $U^{-1}\big(f(\omega_n)\big)_{n\geqslant 0}$, where U is the operator of Lemma 5.3. Hence, by Remark 5.2, ϕ is s-decaying. The Radon conditions for ϕ can be verified by using absolute convergence to rearrange the order of summation.

Conversely, let ϕ be an N-radial s-decaying R-compatible function on $\mathcal{H}$. If v is a vertex with $|v| \geqslant N$, then the sequence $a^{(v)} = \big(\phi(H_{v,n})\big)_{n\geqslant 0} \in \mathcal{D}_s$, hence $b^{(v)} = Ua^{(v)} \in \mathcal{D}_s$ by Lemma 5.3. Define a function f' on T by

$$f'(v) = \begin{cases} 0 & \text{if } |v| < N, \\ b^{(v)}_{|v|} & \text{if } |v| \geqslant N, \end{cases}$$

so f' is N-radial and s-decaying. By the above, Rf' is defined and is N-radial, s-decaying, R-compatible. By the preceding discussion, it coincides with ϕ on $\{h_{\omega,n} : \omega \in \Omega,\ n \geqslant N\}$. Since $\phi - Rf'$ is therefore also $(N-1)$-supported by Lemma 4.4, it equals Rg for some N-radial $(N-1)$-supported function g on T, by Proposition 4.5. Then $f = f'+g$ is N-radial, s-decaying, and $Rf = \phi$. □

THEOREM 5.5.— *For $s < 1/\sqrt{q}$ the Radon transform is one-to-one on $\mathcal{A}_s$ and its range is $\mathcal{B}_s$.*

Proof.— Let $f \in \mathcal{A}_s$ be such that $Rf = 0$. Applying the discussion in the proof of Proposition 5.4 to f_0, which is 0-radial and s-decaying by Remark 5.1, we obtain

$$\big(f_0(\omega_n)\big)_{n\geqslant 0} = U\big(Rf_0(h_{\omega,n})\big)_{n\geqslant 0} = 0 \qquad \text{for every } \omega \in \Omega,$$

because $Rf_0 = (Rf)_0 = 0$ by Proposition 2.9. Then $f(e) = f_0(e) = 0$. Since e is arbitrary, f is identically zero and so R is one-to-one.

Now let $f \in \mathcal{A}_s$ be arbitrary. Thus Rf is defined by Proposition 3.8. By Remark 5.1, $f_N \in \mathcal{A}_s$ for all N. By Proposition 2.9 and Proposition 5.4, $(Rf)_N = Rf_N \in \mathcal{B}_s$. Again by Remark 5.1, $Rf \in \mathcal{B}_s$.

Conversely, assume $\phi \in \mathcal{B}_s$. By Remark 5.1, $\phi_N \in \mathcal{B}_s$ for every $N \geqslant 0$, so by Proposition 5.4 there exists an N-radial function $f^N \in \mathcal{A}_s$ such that $Rf^N = \phi_N$. We shall show that the function f on T given by $f(v) = f^{|v|}(v)$ for all $v \in T$ satisfies $f_N = f^N$ for all N. Then $f \in \mathcal{A}_s$ by Remark 5.1. By Proposition 2.9, $(Rf)_N = Rf_N = \phi_N$ for every N, whence $Rf = \phi$ by (2.5).

If $N' \geqslant N$, by Proposition 2.9 and (2.5) $R(f^{N'})_N = (Rf^{N'})_N = (\phi_{N'})_N = \phi_N = Rf^N$, which implies $(f^{N'})_N = f^N$, because R is one-to-one. In particular, $f^{N'} = f^N$ on $\{|v| \leqslant N\}$, so $f_N(v) = f(v) = f^N(v)$ if $|v| \leqslant N$. If $|v| \geqslant N$, instead, we have $f_N(v) = f(v_N, |v|) = f^{|v|}(v_N, |v|) = (f^{|v|})_N(v) = f^N(v)$ by the above. □

We conclude by showing that the use of distributions is necessary.

Example 5.6. — Assume $s < 1/\sqrt{q}$. Let $\lambda_1, \dots, \lambda_q$ be complex numbers of absolute value 1 and such that $\sum_{j=1}^{q} \lambda_j = qs^2$, and set $\lambda_{q+1} = \lambda_1$. Label the vertices of T by induction: those of length 1 are $x_1, \dots, x_{q+1}$, while the immediate descendants of $v \neq e$ are $vx_1, \dots, vx_q$. Thus any v can be written as $x_{i_1} \cdots x_{i_{|v|}}$, with $1 \leqslant i_1 \leqslant q+1$ and $1 \leqslant i_2, \dots, i_{|v|} \leqslant q$. Define $f(v) = s^{-|v|} \prod_{j=1}^{|v|} \lambda_{i_j}$. We have

$$\Big(\sum_{u \in D_n(v)} f(u) \Big)_{n \geqslant |v|} = \big(s^{-n} (qs^2)^{n-|v|} f(v) \big)_{n \geqslant |v|} \in \mathcal{D}_{qs} \qquad \text{for every } v \in T,$$

so $f \in \mathcal{A}_s$, and $Rf \in \mathcal{B}_s$ by Theorem 5.5.

We now show that the distribution Rf cannot be evaluated pointwise, so it is not induced by a function on $\mathcal{H}$. Let $n \geqslant 0$ and $\omega \in \Omega$. By Remark 1.2, for every $k \geqslant 0$ the absolute value of the sum of f over the vertices of length $n + 2k$ in $h_{\omega,n}$ is greater than or equal to the absolute value of the difference of

$$\Big| \sum_{v \in D_{n+2k}(\omega_{n+k})} f(v) \Big| = s^{-n-2k} (qs^2)^k,$$

$$\Big| \sum_{v \in D_{n+2k}(\omega_{n+k+1})} f(v) \Big| = s^{-n-2k} (qs^2)^{k-1},$$

which is $((qs^2)^{-1} - 1) s^{-n} q^k$. Thus the series defining $Rf(h_{\omega,n})$ does not converge. For $n < 0$ a similar argument yields the same conclusion. □

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