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SIMPLICITY OF NERETIN'S GROUP OF SPHEROMORPHISMS

by Christophe KAPOUDJIAN

Introduction.

Answering a question of I.M. Gelfand on the existence of analogues of heighest-weight representations of the diffeomorphism group of the circle in the case of p-adic transformation groups, Yu.A. Neretin constructed a group of transformations of the boundary $\partial\mathcal{T}_p$ of the regular tree $\mathcal{T}_p$ (*cf.* [12] and [13]): the group N_p of spheromorphisms (§1). When p is a prime integer, the boundary $\partial\mathcal{T}_p$ is naturally homeomorphic to the projective line on the field of p-adic numbers, and in any case, to a Cantor set.

Roughly speaking, a spheromorphism is a transformation induced in the boundary by a "piecewise" tree automorphism. The spheromorphism group is generated by two groups: on the one hand a Higman-Thompson group (§2), which is countable and almost-acts on the tree, respecting a local orientation of the edges, and on the other hand, the tree automorphism group (§3).

Exploiting simplicity theorems known for the generating two groups, and adapting some arguments of a simplicity theorem of Epstein, we finally prove the simplicity of N_p (the analogue of M.R. Herman's theorem on the simplicity of the orientation-preserving diffeomorphism group of the circle, *cf.* [7]), and of some of its subgroups (§4):

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THEOREM. — *For each integer $p \geq 2$, the spheromorphism group N_p is simple.*

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1. The Neretin group of spheromorphisms.

1.1. Let $\mathcal{T}_n$ be the regular tree whose vertices have valence $n+1$, with $n \geq 2$, and $\partial\mathcal{T}_n$ its boundary, or set of "ends", see e.g. [14] or [6].

We may describe the boundary $\partial\mathcal{T}_n$ as a compact ultrametric space: choose a vertex o of the tree $\mathcal{T}_n$. Each end is defined by a unique chain (i.e. a sequence of consecutive vertices $(o = x_0, x_1, \ldots)$ with $x_{i+2} \neq x_i$) starting from the origin o. The metric on $\partial\mathcal{T}_n$ is defined in the following way: Let ω, $\omega' \in \partial\mathcal{T}_n$ be respectively represented by the chains $(o = x_0, x_1, \ldots)$ and $(o = x'_0, x'_1, \ldots)$.

• If the intersection of the supports of the chains is reduced to $\{o\}$, then declare the distance between ω and ω' to be equal to 1: $d(\omega, \omega') = 1$.

• If $x_i = x'_i$ for $i = 0, \ldots, k$ and $x_{k+1} \neq x'_{k+1}$, then define $d(\omega, \omega') = \dfrac{n}{n+1} n^{-k}$.

It follows that a closed ball of radius $\dfrac{n}{n+1} n^{-k}$ is the set of all points of $\partial\mathcal{T}_n$ represented by chains containing a fixed finite chain $(o = x_0, x_1, \ldots, x_k)$, and that it is an open set. In fact, $\partial\mathcal{T}_n$ endowed with the metric d is a compact ultrametric space, homeomorphic to a Cantor set.

When p is prime, $\mathcal{T}_p$ is the Bruhat-Tits building of the p-adic Lie group $SL_2(\mathbb{Q}_p)$, just as the Poincaré disk D is the symmetric space of the real group $SL_2(\mathbb{R})$. The boundary $\partial\mathcal{T}_p$, which can be identified with $\mathbb{Q}_pP^1$, the projective line on $\mathbb{Q}_p$, may thus be viewed as the p-adic analogue of the circle.

1.2. Let $\partial\mathcal{T}_n$ still denote the boundary of the tree $\mathcal{T}_n$, $n \geq 2$. The group of spheromorphisms N_n can be defined as the group of transformations of $\partial\mathcal{T}_n$ induced by "piecewise" tree automorphisms:

Take a finite subtree of $\mathcal{T}_n$. Its complementary has finitely many connected components $L_1, \ldots, L_k$, called *branches*, all isomorphic to an infinite n-ary complete rooted tree. A subset ∂L of the boundary is naturally associated to each branch L: it consists of all the ends represented by the chains running over this branch. The k disjoint sets ∂L_j, $j = 1, \ldots, k$ cover the boundary. We call $(L_1, \ldots, L_k)$ a *broom*.

Remark. — Each ball for the metric d is of the form ∂L, and each ∂L is a finite union of balls. The family $\{\partial L : L \text{ branch}\}$ is a basis of closed-open sets for the topology defined by d.

Let $(L_1, \ldots, L_k)$ and $(L'_1, \ldots, L'_k)$ be two brooms of $\mathcal{T}_n$, σ a permutation of $\{1, \ldots, k\}$. Let $\phi_j : L_j \to L'_{\sigma(j)}$ be a rooted tree isomorphism, $j = 1, \ldots, k$. These k mappings induce a bijection $\phi = (\partial\phi_j : \partial L_j \to \partial L'_{\sigma(j)})_{j=1,\ldots,k}$ of the boundary. Such a broom appearing in the definition of ϕ is called *ϕ-adapted*, and is obviously not uniquely associated to ϕ. It is clear that the set of all the ϕ's defined by this procedure is a group of homeomorphisms of the boundary.

DEFINITION 1.1 (Spheromorphism group, [13]). — *For each $n \geq 2$, the set of all bijections $\phi = (\partial\phi_j : \partial L_j \to \partial L'_{\sigma(j)})_{j=1,\ldots,k}$ of the boundary $\partial\mathcal{T}_n$ is the spheromorphism group of Neretin, and is denoted N_n.*

Remarks. — 1) In view of this description, the automorphism group $\operatorname{Aut}\mathcal{T}_n$ of the tree embeds as a subgroup of N_n. The image of $\operatorname{Aut}\mathcal{T}_n$ in N_n is the set of spheromorphisms which possess an adapted broom with two branches.

2) When p is a prime integer, $\partial\mathcal{T}_p$ is homeomorphic to $\mathbb{Q}_pP^1$, and N_p contains the group An_p of locally analytic bijections of $\mathbb{Q}_pP^1$ (see [13]).

2. Higman-Thompson groups.

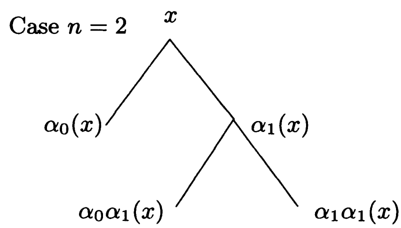
2.1. Definition of Higman-Thompson groups. In 1965, R.J. Thompson, interested in finitely presented groups with non-solvable word problem, introduced a group (denoted $G_{2,1}$ in the following) which happened to be the first known example of finitely generated infinite simple group [11]. Thompson's group was later generalized by G. Higman ([8]). For the description of the Higman-Thompson groups, we refer to [2]. See also [4].

Recall that a finite *n-ary rooted planar tree* is a finite tree T with root x realized in the oriented plane such that

– If T is not reduced to x, the valence of x is equal to n.

– The valence of a vertex $v \neq x$ is equal to 1 or $n+1$: if the valence of v is 1, we call v a *leaf* of the tree; if it is equal to $n+1$, v has n adjacent edges not contained in the geodesic joining the root x to v. We realize them by drawing them down from the vertex v. We order them from the left to the right and label their terminal vertices (opposite to v) $\alpha_0(v), \dots, \alpha_{n-1}(v)$.

The set of leaves of a finite n-ary rooted tree T is called a *basis* and is denoted B_T.



DEFINITION 2.1. — *A simple expansion of a finite n-ary rooted tree T is any finite n-ary rooted tree T' obtained by the following procedure:*

- *Choose a vertex v in the base B_T.*
- *Make an expansion of v by drawing n edges down from it.*

We get a new tree T' whose basis $B_{T'}$ is deduced from B_T by replacing v by $\alpha_0(v), \dots, \alpha_{n-1}(v)$.

An expansion T' of T is a tree obtained from T by making finitely many successive simple expansions. Any two trees T_1 and T_2 always possess a common expansion.

The elements of the Higman-Thompson groups will be represented by "symbols":

DEFINITION 2.2 (symbols). — *Consider a pair (T_1, T_2) of finite n-ary rooted trees with basis having the same cardinality. Let $\sigma : B_{T_1} \to B_{T_2}$ be a bijection from the basis of the first tree to the basis of the second one. We call the triple (T_1, T_2, σ) a symbol.*

A simple expansion of a symbol (T_1, T_2, σ) is any symbol (T_1', T_2', σ') thus obtained:

- *T_1' is a simple expansion of T_1, deduced from T_1 by expanding a vertex $v \in B_{T_1}$.*
- *Then T_2' is the expansion of T_2 realized from the vertex $\sigma(v)$.*

• $\sigma' : B_{T_1'} \to B_{T_2'}$ *is defined by*

$$\sigma'_{|B_{T_1}\setminus\{v\}} = \sigma_{|B_{T_1}\setminus\{v\}},$$
$$\sigma'(\alpha_i(v)) = \alpha_i(\sigma(v))\ , i = 0, \ldots, n-1.$$

An expansion (T_1', T_2', σ') of the symbol (T_1, T_2, σ) is obtained from the latter by making finitely many simple expansions.

Declare now that (T_1, T_2, σ) and (T_1', T_2', σ') are equivalent if they possess a common expansion.

All the necessary vocabulary has been introduced to set the following:

DEFINITION 2.3 (Higman-Thompson groups). — *The set of equivalence classes of symbols $[(T_1, T_2, \sigma)]$ form a set G_n endowed with the following group structure:*

Two elements $[(T_1, T, \sigma)]$ and $[(T', T_2, \sigma')]$ being given, at the price of making expansions of their representing symbols, it may be supposed that $T = T'$. Then $\sigma'\sigma : B_{T_1} \to B_{T_2}$ can be defined, and we set

$$[(T_1, T, \sigma)][(T, T_2, \sigma')] = [(T_1, T_2, \sigma'\sigma)],$$

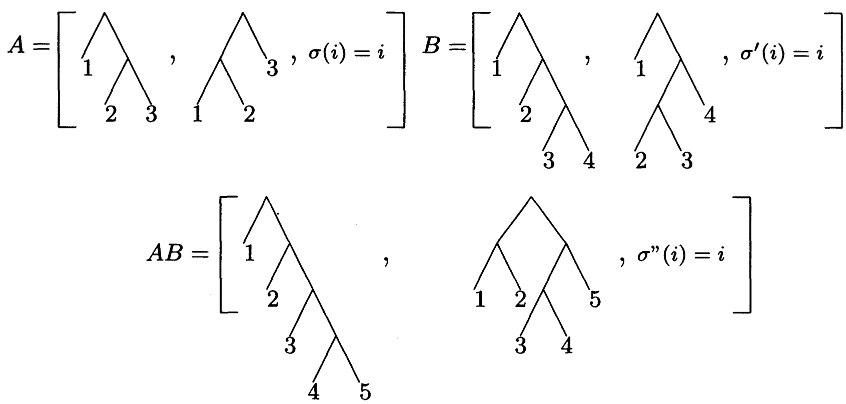
since it is easy to check that this definition is independent of the chosen symbols.

The neutral element is $[(T, T, \sigma = \mathrm{id})]$ represented by any symbol $(T, T, \sigma = \mathrm{id})$.

The inverse of $[(T_1, T_2, \sigma)]$ is $[(T_2, T_1, \sigma^{-1})]$.

The group G_n belongs to the family of Higman-Thompson groups.

Example ($n = 2$).



Recall that the leaves of a tree T (i.e. the vertices in B_T) are always labelled from the left to the right. Let (T, T', σ) be a symbol, and $\sigma : B_T = \{v_1, \ldots, v_k\} \to B_{T'} = \{v'_1, \ldots, v'_k\}$. There exists a unique permutation $\tau \in \mathcal{S}_k$ such that

$$\sigma(v_i) = v'_{\tau(i)} \ \forall i = 1, \ldots, k.$$

Then define $\theta(\sigma) = \epsilon(\tau)$ the signature of τ. An easy calculation shows that if $(\widetilde{T}, \widetilde{T'}, \tilde{\sigma})$ is a simple expansion of the symbol (T, T', σ), then

$$\theta(\tilde{\sigma}) = \theta(\sigma)(-1)^{n-1},$$

so that when n is an odd integer, $\theta(\sigma)$ is independent of the chosen symbol, and we get the group epimorphism

$$\theta : G_n \to \mathbb{Z}/2\mathbb{Z}$$
$$\theta([(T, T', \sigma)]) = \epsilon(\tau).$$

Generalization. Let $r \geq 1$ be a fixed integer. First consider pairs of r-uplets of finite n-ary rooted trees $((T_1, \ldots, T_r), (T'_1, \ldots, T'_r))$, and bijections σ from $B_{T_1} \cup \ldots \cup B_{T_r}$ to $B_{T'_1} \cup \ldots \cup B_{T'_r}$ (We do not ask σ to map B_{T_i} onto $B_{T'_i}$). We always suppose the r-uplet of trees to be ordered from the left (T_1) to the right (T_r). Any triple $((T_1, \ldots, T_r), (T'_1, \ldots, T'_r), \sigma)$ is called an r-symbol. Similarly to the case $r = 1$, we define the group $G_{n,r}$ where the elements are represented by r-symbols. Of course, $G_{n,1} = G_n$.

As in the case $r = 1$, the morphism $\theta : G_{n,r} \to \mathbb{Z}/2\mathbb{Z}$ can be defined provided n is odd. We set $G'_{n,r} = \operatorname{Ker} \theta$. If n is even, we agree that $G'_{n,r} = G_{n,r}$. We are now ready to cite the simplicity theorem:

THEOREM 2.1 ([2]). — *The group $G'_{n,r}$ is the commutator subgroup of $G_{n,r}$, and every non-trivial subgroup normalized by $G'_{n,r}$ contains it. In particular, $G_{n,r}$ is simple if n is even, and if n is odd, $G_{n,r}$ contains a simple group of index 2, namely $G'_{n,r} = [G_{n,r}, G_{n,r}]$.*

2.2. Embedding of $G_{n,1} = G_n$ and $G_{n,2}$ into the Neretin group N_n. The finite n-ary rooted trees we used in the definition of the Higman-Thompson groups may be canonically embedded in a chosen branch L of the regular tree $\mathcal{T}_n$, by simply completing the finite tree to an infinite n-ary rooted tree and then, identifying it to the branch L. Denote by L' the branch opposite to L in $\mathcal{T}_n$ (linked to L by an edge). Each $g \in G_{n,1}$, defined by a symbol (T_1, T_2, σ), induces a spheromorphism $\tilde{g}$ in an obious way: if (v_i^1) (resp. (v_i^2)) are the leaves of T_1 (resp. T_2), denote by L_i^1 (resp. L_i^2) the subbranch of L whose root is v_i^1 (resp. v_i^2). Then $\tilde{g}$ is induced on ∂L by the collection $(L_i^1 \xrightarrow{\cong} L_{\sigma_i}^2)_i$, the isomorphisms respecting the left-to-right order of the edges of the branches. On $\partial L'$, one imposes $\tilde{g}_{|\partial L'} = \mathrm{id}_{|\partial L'}$. The embedding

$$G_{n,1} \hookrightarrow N_n$$

is now obtained.

On the other hand, we need the two branches L and L' like above to realize $G_{n,2}$ in N_n. Each $g \in G_{n,2}$ will induce a spheromorphism by a procedure analogous to the previous one. It will appear in the following that, as far as we are concerned with the Neretin group N_n, $G_{n,2}$ is more relevant than the group $G_{n,1} = G_n$ itself.

3. The group $\mathrm{Aut}\,\mathcal{T}_n$ of automorphisms of the tree $\mathcal{T}_n$, $n \geq 2$.

3.1. Simplicity theorem. In [15], the author gave a theorem of simplicity of a class of groups of automorphisms of a tree:

DEFINITION 3.1. — *Let A be a tree, G be a group of automorphisms of A, C be a (finite or infinite) chain of A, and F the fixator of C in G. For each vertex x of A, let $\pi(x)$ be the nearest vertex from x in C. For each vertex s of C, the set $\pi^{-1}(s)$ (which constitutes a subtree of A) is invariant under the action of F; denote by F_s the group of permutations of this set induced by F. There is a natural homomorphism*

$$(1) \qquad F \longrightarrow \prod_{s \in \mathrm{Vert}(C)} F_s,$$

where Vert(C) *denotes the set of vertices of* C.

We say that the group G *possesses the property* (P) *if the homomorphism* (1) *is an isomorphism for all chains* C *(i.e. the actions of* F *on the sets* $\pi^{-1}(s)$ *are independent from each other).*

For example the group of all automorphisms of A possesses the property (P).

THEOREM 3.1 (J. Tits). — *Let* A *be a tree,* G *be a group of automorphisms of* A*, and* G^+ *be the subgroup generated by the stabilizers of the edges of* A *in* G*. Suppose that* G *possesses the property* (P)*, conserves no proper non-empty subtree of* A *and fixes no end of* A*. Then each subgroup of* G *normalized by* G^+ *and not reduced to the identity contains* G^+*. In particular,* G^+ *is a simple group or is reduced to the identity.*

Example 1. — $A = \mathcal{T}_n$, $n \geq 2$, $G = \mathrm{Aut}\,\mathcal{T}_n$. It happens that $G^+ = \mathrm{Aut}^+\,\mathcal{T}_n$ coincides with the group of type-preserving automorphisms of the tree. So $\mathrm{Aut}^+\,\mathcal{T}_n$ is a simple group, of index 2 in $\mathrm{Aut}\,\mathcal{T}_n$.

Example 2. — Equipped Bruhat-Tits trees.

Let $p \geq 2$ be a prime integer. In [13], the author defines an *equipment* on the tree $\mathcal{T}_p$ as the specification, for each vertex v, of a labelling of its adjacent edges $({l^v}_0, \ldots, {l^v}_{p-1}, {l^v}_\infty)$ by the points of $\mathbb{F}_pP^1$. If v and v' are linked by an edge $l = {l^v}_i = {l^{v'}}_j$, there is no reason that $i = j$.

We denote by $\widetilde{\mathcal{T}_p}$ such an equipped tree, and define the subgroup $\mathrm{Aut}\,\widetilde{\mathcal{T}_p}$ of $\mathrm{Aut}\,\mathcal{T}_p$ as the set of tree automorphisms such that their restrictions to the adjacent edges of a vertex belong to $PSL_2(\mathbb{F}_p)$. Since $\mathrm{Aut}\,\widetilde{\mathcal{T}_p}$ obviously satisfies property (P), conserves no proper non-empty subtree of $\mathcal{T}_p$ and fixes no end, the group $(\mathrm{Aut}\,\widetilde{\mathcal{T}_p})^+$ is simple.

Two equipped trees $\widetilde{\mathcal{T}_p}^1$ and $\widetilde{\mathcal{T}_p}^2$ being given, we use the transitivity of $SL_2(\mathbb{F}_p)$ on $\mathbb{F}_pP^1$ to construct a tree isomorphism $\widetilde{\mathcal{T}_p}^1 \to \widetilde{\mathcal{T}_p}^2$ respecting the equipments. Such an isomorphism conjugates $\mathrm{Aut}\,\widetilde{\mathcal{T}_p}^1$ and $\mathrm{Aut}\,\widetilde{\mathcal{T}_p}^2$.

3.2. A family of subgroups of N_n.

DEFINITION 3.2. — *If* G *is a subgroup of* $\mathrm{Aut}\,\mathcal{T}_n$ *we define*

$$(N_n)_G := < G_{n,2}, G^+ >$$

the subgroup of N_n *generated by* $G_{n,2}$ *and* G^+.

Example 1. — If $G = \mathrm{Aut}\,\mathcal{T}_n$, $(N_n)_G = N_n$. In this case, we can even show:

PROPOSITION 3.1. — *The subgroups* $[G_n, G_n]$ *and* $\mathrm{Aut}^+\,\mathcal{T}_n$ *of the group* N_n, $n \geq 2$, *generate the group* N_n.

Proof. — Let us denote by L the chosen branch of the tree $\mathcal{T}_n$ where we realized the Higman-Thompson group G_n. If L' is the branch opposite to L (i.e., linked with L by an edge), then the boundaries of L and L' partition the whole boundary of the tree: $\partial L \cup \partial L' = \partial \mathcal{T}_n$.

First case. — Suppose that $\phi \in N_n$ possesses a broom $(L_i)_{i=1,\ldots,I}$ such that $\phi_{|\partial L_1} = \mathrm{id}_{|\partial L_1}$. At the price of making an expansion of L_1, one can suppose that L_1 and L' have the same type (i.e. their roots have the same type). Then there exists $k \in \mathrm{Aut}^+\,\mathcal{T}_n$ such that $k(L') = L_1$. So $k^{-1}\phi k_{|\partial L'} = \mathrm{id}_{|\partial L'}$. Let us now consider $k^{-1}\phi k_{|\partial L}$. It may be seen as the composite

$$\partial L \xrightarrow{\tau} \partial L \xrightarrow{\sigma} \partial L$$

with $\tau \in G_n$ and $\sigma \in \mathrm{Aut}^+\,\mathcal{T}_n$, $\sigma_{|L'} = \mathrm{id}_{|L'}$. Then on the whole boundary $\partial\mathcal{T}_n$, $k^{-1}\phi k = \sigma\tau$.

When n is odd, $\mathrm{Aut}^+\,\mathcal{T}_n \cap (G_n \setminus [G_n, G_n]) \neq \emptyset$, so that it can be supposed that $\tau \in [G_n, G_n]$.

Second case: general case. — (a) Suppose there exists L_i in the broom adapted to ϕ such that ∂L_i and $\phi(\partial L_i) = \partial L_i'$ have the same type. Then there exists $k \in \mathrm{Aut}^+\,\mathcal{T}_n$ such that $k\phi(\partial L_i) = \partial L_i$ and $k \circ \phi_{|\partial L_i} = \mathrm{id}_{|\partial L_i}$. The first case enables to conclude.

(b) If not, for all i, the types of ∂L_i and $\phi(\partial L_i)$ are opposite. Then we use an element τ_0 of G_n (it is possible to find it of the form $[\tau_1, \tau_2]$) such that for some branch L_0, $\tau_0(L_0)$ and L_0 have opposite types. At the price of making an expansion of L_1 to make $\phi(\partial L_1)$ and ∂L_0 have the same type, there exists some $k \in \mathrm{Aut}^+\,\mathcal{T}_n$ such that $k\phi(\partial L_1) = \partial L_0$. The types of L_1 and L_0 are still opposite. Then $\tau_0 k\phi(\partial L_1) = \tau_0(\partial L_0) = \partial L_0'$, and the types of L_1 and L_0' coincide. Hence $\tau_0 k\phi$ satisfies the condition of case (a).

It follows that ϕ may be written as a product of elements of G_n and $\mathrm{Aut}^+\,\mathcal{T}_n$.

Example 2. — Now p is a prime integer. Let $\widetilde{\mathcal{T}_p}$ be any equipment on the tree $\mathcal{T}_p$ such that the elements of $G_{p,2}$ are induced by piecewise tree automorphisms of $\mathrm{Aut}\,\widetilde{\mathcal{T}_p}$ (*cf.* §3.1, Example 2).

If $G = \mathrm{Aut}\,\widetilde{\mathcal{T}_p}$, then we claim that $(N_p)_G$ is the group denoted $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ in [13]:

$$\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}) = \{\phi = (\phi_j : L_j \to L'_j)_j,$$
$$\phi_j = \text{ restriction of some element of } \mathrm{Aut}\,\widetilde{\mathcal{T}_p}\}.$$

Indeed, $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ contains G, and because of the condition on the equipment, it contains $G_{p,2}$. So, $< G, G_{p,2} > \subset \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$. On the other hand, every $\phi \in \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ can be written $\phi = \psi \circ \tau$, where $\tau = (L_j \to L'_j)_j$ belongs to $G_{p,2}$, and $\psi = (\psi_j = L'_j \to L'_j)_j$, with ψ_j induced by some element of G, which can be modified to be supported in the branch L'_j. It follows that $\psi_j \in G^+$, and $\psi = \prod_j \psi_j \in G^+$. Thus

$$< G, G_{p,2} > \subset \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}) \subset < G^+, G_{p,2} >,$$

and the inclusions are equalities. Then $(N_p)_{\mathrm{Aut}\,\widetilde{\mathcal{T}_p}} = \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ as claimed.

Remarks. — 1) Any isomorphism of equipped trees $\widetilde{\mathcal{T}_p} \to \widetilde{\mathcal{T}_p}'$ conjugates $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$ and $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}')$.

2) If $p = 2$, the group $PSL_2(\mathbb{F}_2)$ is the full symmetric group $\mathcal{S}_3$, so that $\mathrm{Diff}^+(\widetilde{\mathcal{T}_2}) = N_2$.

4. Simplicity of $(N_p)_G$.

We now give the main theorem of the article, valid for any integer $p \geq 2$:

THEOREM 4.1. — *Let G be a subgroup of* $\mathrm{Aut}\,\mathcal{T}_p$ *such that*

1. *G^+ is simple (e.g. G satisfies the conditions of Theorem 3.1),*
2. *If p is odd, $G^+ \cap (G_{p,2} \setminus [G_{p,2}, G_{p,2}])$ is non-empty,*
3. *G^+ possesses two non-commuting elements supported in a branch of the tree.*

Then the group $(N_p)_G$ is simple.

Condition 2. implies that $(N_p)_G$ is generated by G^+ and $G_{p,2}{}' = [G_{p,2}, G_{p,2}]$, since $G_{p,2}$ is generated by $G_{p,2}{}'$ together with any element in $G_{p,2} \setminus G_{p,2}{}'$.

COROLLARY 4.1. — *For each integer $p \geq 2$, the group N_p of all spheromorphisms is simple.*

For each prime number $p \geq 3$ and for any choice of equipment of the tree $\mathcal{T}_p$, the commutator subgroup $[\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}), \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})]$ is simple, and there is a short exact sequence

$$1 \to [\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}), \mathrm{Diff}^+(\widetilde{\mathcal{T}_p})] \longrightarrow \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}) \xrightarrow{\tilde{\theta}} \mathbb{Z}/2\mathbb{Z} \to 0.$$

In other words, $H_1(\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof of Corollary 4.1. — $G = \mathrm{Aut}\,\mathcal{T}_p$ obviously satisfies all the conditions of the theorem above.

As for the statements about $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p})$, they can be proven by using a particular equipment, since for different equipments the groups are conjugated. So, remembering that $\mathcal{T}_p$ is obtained by gluing by an edge the two branches L and L' appearing in the definition of $G_{p,2}$, define the equipment $\widetilde{\mathcal{T}_p}^0$ in the following way: label the p edges drawn down from a vertex from 0 (on the left) to $p-1$ (on the right), whereas the edge pointing towards the root of the branch (L or L') is labelled ∞. Then setting $G = \mathrm{Aut}\,\widetilde{\mathcal{T}_p}^0$, we have $(N_p)_G = \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0)$ (*cf.* §3.2 Example 2). But condition 2 of Theorem 4.1 fails for such G. We recalled in Section 2 that when p is odd, there is an epimorphism

$$\theta : G_{p,2} \to \mathbb{Z}/2\mathbb{Z}$$

whose kernel is the simple group $[G_{p,2}, G_{p,2}]$. It happens that θ may be extended to the group $\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0)$: if $\phi = (\phi_j : L_j \to L'_{\sigma(j)})_j$, where the indices of the branches label their roots from the left to the right (suppose the branches involved to be subbranches of L or L'), $\tilde{\theta}(\phi)$ will be the signature of σ. Indeed, if we refine some branch L_j into $L_{j_0} \cup L_{j_1} \cup \ldots \cup L_{j_{p-1}}$, then ϕ_j induces

$$\phi_{j_i} : L_{j_i} \to L'_{\sigma(j)_{k_i}} \quad i = 0, 1, \ldots, p-1,$$

with $i \in \mathbb{F}_p \to k_i \in \mathbb{F}_p$ in $B \subset PSL_2(\mathbb{F}_p)$, the stabilizer of ∞. Since B lies in the alternating group $\mathcal{A}_p$ on a set with p elements, the permutation deduced from σ has the same signature as in the case $k_i = i\ \forall i \in \mathbb{F}_p$. But then we saw (*cf.* §2) that, since p is odd, the signature remains unchanged. So

$$\tilde{\theta} : \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

is a well-defined homomorphism.

It is clear that the kernel of $\tilde{\theta}$ is generated by $[G_{p,2}, G_{p,2}]$ and $(\mathrm{Aut}\,\widetilde{\mathcal{T}_p}^0)^+$, and the proof of the theorem will show that this group is simple. Now the kernel contains the commutator subgroup $[\mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0), \mathrm{Diff}^+(\widetilde{\mathcal{T}_p}^0)]$, which is normal and non-trivial, consequently it coincides with the kernel.

Proof of Theorem 4.1. — Let $H \triangleleft (N_p)_G$ be a non-trivial normal subgroup of $(N_p)_G$. Then $H \cap G^+$ is normal in G^+ and $H \cap [G_{p,2}, G_{p,2}]$ is normal in $[G_{p,2}, G_{p,2}]$. Hence either $H \supset G^+$ or $H \cap G^+ = \{\mathrm{id}\}$, and either $H \supset [G_{p,2} G_{p,2}]$ or $H \cap [G_{p,2}, G_{p,2}] = \{\mathrm{id}\}$.

So we will prove that the cases $H \cap G^+ = \{\mathrm{id}\}$ and $H \cap [G_{p,2}, G_{p,2}] = \{\mathrm{id}\}$ do not occur. We will use some arguments of a theorem of Epstein ([5] and [1]):

THEOREM 4.2 (Epstein, 1970). — *Let X be a paracompact Hausdorff topological space, Γ a group of homeomorphisms of X, and $\mathcal{U}$ a basis of open sets for the topology of X. The Epstein axioms for the triple $(X, \Gamma, \mathcal{U})$ are:*

1. *Axiom 1: If $U \in \mathcal{U}$ and $g \in \Gamma$, then $gU \in \mathcal{U}$.*
2. *Axiom 2: Γ acts transitively on $\mathcal{U}$.*
3. *Axiom 3: Let $g \in \Gamma$, $U \in \mathcal{U}$ and $\mathcal{B}$ an open covering of X; then there exists an integer n and $g_1, \ldots, g_n \in \Gamma$ and $V_1, \ldots, V_n \in \mathcal{B}$ such that*

   (i) $g = g_n g_{n-1} \cdots g_1$,

   (ii) $\mathrm{supp}\,(g_i) \subset V_i$,

   (iii) $\mathrm{supp}\,(g_i) \cup (g_{i-1} \cdots g_1 \bar{U}) \neq X,\ 1 \leq i \leq n$.

Suppose the triple $(X, \Gamma, \mathcal{U})$ as above satisfies the Epstein axioms. Then if H is a non-trivial subgroup of Γ that is normalized by $[\Gamma, \Gamma]$, then $[\Gamma, \Gamma] \subset H$. In particular, the group $[\Gamma, \Gamma]$ is simple.

The simplicity of $[\mathrm{Diff}^+(S^1), \mathrm{Diff}^+(S^1)]$ was an easy corollary of this theorem. M.R. Herman finally proved $\mathrm{Diff}^+(S^1)$ was perfect, hence simple ([7]). For more details, we suggest the reader to refer to the very interesting book [1].

In the case of a non-connected topological space and a non trivial group Γ, axiom 3 can never be satisfied (see [5]). Consequently, we will not be able to use the preceding theorem directly to prove the simplicity of $(N_p)_G$. However, setting $X = \partial\mathcal{T}_p$, $\mathcal{U} = \{\partial L : L \text{ branch of } \mathcal{T}_p\}$ and

$\Gamma = (N_p)_G$, it is easy to see that the triple $(\partial\mathcal{T}_p, (N_p)_G, \mathcal{U})$ satisfies axiom 2 and a

"modified axiom 1": If $U \in \mathcal{U}$ and $g \in \Gamma$, then there exists $U' \in \mathcal{U}$, $U' \subset U$, such that $gU' \in \mathcal{U}$.

Then we can show that two lemmas, which are steps in the proof of the Epstein theorem, still hold in our case:

LEMMA 4.1 (from 1.4.2 in [5], or Lemma 2.2.5 in [1]). — *Let $(X, \Gamma, \mathcal{U})$ be a triple satisfying the modified axiom 1 and axiom 2. Let $V_0 \in \mathcal{U}$ and $h \in \Gamma$ with* $\operatorname{supp} h \subset V_0$, *and suppose that $H \triangleleft \Gamma$ is a non-trivial normal subgroup of Γ. Then there exists some $\rho \in H$ such that $\rho_{|V_0} = h_{|V_0}$.*

Proof. — Choose any $\alpha \in H$ with $\alpha \neq \mathrm{id}$, and find $x \in X$ such that $\alpha(x) \neq x$. Choose a small neighborhood $U \in \mathcal{U}$ of x such that $U \cap \alpha^{-1}(U) = \emptyset$. Next, take V, $W \in \mathcal{U}$ such that $V \cap W = \emptyset$, $\bar{V} \cup \bar{W} \subset U$, $x \in V$. Suppose first that $V_0 = V$. By axiom 2, there exists $g \in \Gamma$ with $gW = V$. Define

$$\rho = [\alpha, [g, h]] = \alpha^{-1}[g, h]^{-1}\alpha[g, h].$$

Then $\rho \in \Gamma$ since $H \triangleleft \Gamma$. We can verify that

$$\rho = \begin{cases} h \text{ on } V, \\ g^{-1}h^{-1}g \text{ on } W, \\ \alpha^{-1}h\alpha \text{ on } \alpha^{-1}V, \\ \alpha^{-1}g^{-1}h^{-1}g\alpha \text{ on } \alpha^{-1}W, \\ \text{id elsewhere.} \end{cases}$$

Now if $V_0 \neq V$, choose $k \in \Gamma$ (by axiom 2) such that $k(V) = V_0$. Then $\operatorname{supp} k^{-1}hk = k^{-1}(\operatorname{supp} h) \subset V$, and by the previous case, there exists $\rho \in H$ such that $k^{-1}hk_{|V} = \rho_{|V}$, so that $h_{|V_0} = k\rho k^{-1}_{|V_0}$. Since $k\rho k^{-1} \in H$, the proof is done.

LEMMA 4.2 (variation of 1.4.6 in [5] or Lemma 2.2.7 in [1]). — Γ *still satisfies the modified axiom 1 and axiom 2. Moreover, it is supposed 2-transitive:*

$$\forall(x_1, x_2),\ \forall(y_1, y_2),\ x_1 \neq x_2 \text{ and } y_1 \neq y_2 \Rightarrow \exists\phi \in \Gamma\ \phi(x_i) = y_i,\ i = 1, 2.$$

Let h_1, $h_2 \in \Gamma$ be such that there exists $V_0 \in \mathcal{U}$ with $\operatorname{supp} h_i \subset V_0$, $i = 1, 2$. *Then $[h_1, h_2]$ belongs to H.*

Proof. — Let x be in X. There exist α_1, α_2 in H such that x, ${\alpha_1}^{-1}(x)$ and ${\alpha_2}^{-1}(x)$ are pairwise distinct. Indeed, since $\alpha \neq \mathrm{id} \in H$, there exists some $x \in X$ with $\alpha(x) \neq x$. So, in a neighborhood of x there exists $y \neq x$ such that $\alpha(y) \neq y$. Now one can find $\phi \in \Gamma$ with $\phi(x) = y$ and $\phi^{-1}\alpha\phi(x) \neq \alpha(x)$ (which is equivalent to $\alpha(y) \neq \phi\alpha(x)$). As for the condition $\alpha(y) \neq y$, it is equivalent to $\phi^{-1}\alpha\phi(x) \neq x$. Then one sets $\alpha_1^{-1} = \alpha$, $\alpha_2^{-1} = \phi^{-1}\alpha\phi$. So α_1 and α_2 belong to H, x, ${\alpha_1}^{-1}(x)$ and ${\alpha_2}^{-1}(x)$ are pairwise distinct. Then choose $U \in \mathcal{U}$ a neighborhood of x such that U, ${\alpha_1}^{-1}(U)$ and ${\alpha_2}^{-1}(U)$ are pairwise disjoint. One can also find g_1, g_2 in Γ, and a neighborhood $V \in \mathcal{U}$ of x such that V, $g_1^{-1}(V)$ and $g_2^{-1}(V)$ are pairwise disjoint and included in U. Suppose first that $\operatorname{supp} h_i \subset V$, $i = 1, 2$. Then apply the previous lemma to $(\alpha_i, g_i, h_i, V, W_i = g_i^{-1}V)$, $i = 1, 2$. One gets $\rho_{i|V} = h_{i|V}$. The support of ρ_i is included in $V \cup g_i^{-1}(V) \cup {\alpha_i}^{-1}(V) \cup {\alpha_i}^{-1}g_i^{-1}(V)$. The seven sets involved are disjoint, so that

$$[h_1, h_2] = [\rho_1, \rho_2].$$

To conclude, we may assume $V = V_0$, at the price of making some conjugation.

End of the proof of Theorem 4.1. — Choose $V_0 = \partial L_0$ where L_0 is some branch of the tree, and by condition 3, find two non-commuting elements h_1 and h_2 in G^+ with supports in ∂L_0. Apply Lemma 4.2 to $\Gamma = (N_p)_G$, which is 2-transitive on $\partial\mathcal{T}_p$, since $G_{p,2}$ itself is 2-transitive. Then $[h_1, h_2] \in G^+ \cap H$, so $H \supset G^+$.

Similarly, choose two non-commuting elements h'_1 and h'_2 in $G_p = G_{p,1} \subset G_{p,2}$ (they are supported in a branch), so that $[h'_1, h'_2] \in [G_p, G_p] \cap H \subset [G_{p,2}, G_{p,2}] \cap H$, and $H \supset [G_{p,2}, G_{p,2}]$. Finally, H contains two groups that generate $(N_p)_G$, so $H = (N_p)_G$.

5. Concluding remarks.

The question of the simplicity of the group N_n is a preamble of a series of homological problems. First the result implies $H_1(N_n, \mathbb{Z}) = 0$. As for the second homology group $H_2(N_n, \mathbb{Z})$, though its complete computation could not be achieved (because the group N_n is very huge), we know it is non trivial. Indeed, the group N_n possesses a non-trivial central extension by $\mathbb{Z}/2\mathbb{Z}$, called the "Central Geometric Extension" in [9] and [10], a sort of analogue of the Bott-Virasoro extension of $\mathrm{Diff}^+(S^1)$.

On the other hand, K. Brown proved that the groups G_n are all $\mathbb{Q}$-acyclic, i.e. $H_i(G_n, \mathbb{Q}) = 0$ for all $i > 0$ (*cf.* [3]). By using a description of N_n as the automorphism group of a free object of some appropriate category, it becomes possible to define an N_n-simplicial complex, and to use it to prove the $\mathbb{Q}$-acyclicity of N_n (*cf.* [9] and [10]).

BIBLIOGRAPHY

[1] A. BANYAGA, The Structure of Classical Diffeomorphism Groups, Mathematics and Its Applications, Kluwer Academic Publishers.

[2] K.S. BROWN, Finiteness properties of groups, Proceedings of the Northwestern conference on cohomology of groups (Evanston, I. 11, 1985) J. Pure Appl. Algebra 1-3 (1987), 45–75.

[3] K.S. BROWN, The Geometry of Finitely Presented Infinite Simple Groups, Algorithms and classifications in combinatorial group theory (Berkeley, A, 1989), Math. Sci. Res. Inst. Publ., 23, Springer, New-York (1992), 121–136.

[4] J.W. CANNON, W.J. FLOYD and W.R. PARRY, Introductory notes on Richard Thompson's groups, L'Enseignement Mathématique, 42 (1996), 215–256.

[5] D.B.A. EPSTEIN, The simplicity of certain groups of homeomorphisms, Compos. Math., 22 (1970), 165–173.

[6] A. FIGÀ-TALAMANCA and C. NEBBIA, Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees, London Mathematical Society Lecture Note Series 162, Cambridge University Press.

[7] M.-R. HERMAN, Simplicité du groupe des difféomorphismes de classe C^∞, isotopes à l'identité, du tore de dimension n, C. R. Acad. Sc. Paris, 273 (26 juillet 1971).

[8] G. HIGMAN, Finitely presented infinite simple groups, Notes on pure mathematics, Australian National University, Canberra, 8 (1974).

[9] C. KAPOUDJIAN, Sur des généralisations p-adiques du groupe des difféomorphismes du cercle, Thèse de Doctorat, Université Claude Bernard Lyon1, décembre 1998.

[10] C. KAPOUDJIAN, Homological aspects and a Virasoro type extension for Higman-Thompson and Neretin groups almost-acting on trees, to appear.

[11] R. MCKENZIE and R.J. THOMPSON, An elementary construction of unsolvable word problems in group theory, in "Word problems", Proc. Conf. Irvine 1969 (edited by W.W. Bone, F.B. Cannonito, and R.C. Lyndon), Studies in Logic and the Foundations of Mathematics, 71 (1973), North-Holland, Amsterdam, 457–478.

[12] Yu.A. NERETIN, Unitary representations of the diffeomorphism group of the p-adic projective line, translated from Funktsional'nyi Analiz i Ego Prilozheniya, 18, n° 4 (1984), 92–93.

[13] Yu.A. NERETIN, On combinatorial analogues of the group of diffeomorphisms of the circle, Russian Acad. Sci. Izv. Math., 41, n° 2 (1993).

[14] J.-P. SERRE, Arbres, amalgames, SL_2, Astérisque 46.

[15] J. TITS, Sur le groupe des automorphismes d'un arbre, Essays on topology and related topics, Mémoires dédiés à G. de Rham, Springer-Verlag, Berlin (1970), 188–211.

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