ANNALES DE L'INSTITUT FOURIER

XAVIER GÓMEZ-MONT

PAVAO MARDEŠIĆ

The index of a vector field tangent to a hypersurface and the signature of the relative jacobian determinant

Annales de l'institut Fourier, tome 47, nº 5 (1997), p. 1523-1539

<http://www.numdam.org/item?id=AIF_1997__47_5_1523_0>

Ann. Inst. Fourier, Grenoble
47, 5 (1997), 1523–1539

THE INDEX OF A VECTOR FIELD TANGENT TO A HYPERSURFACE AND THE SIGNATURE OF THE RELATIVE JACOBIAN DETERMINANT

by X. GÓMEZ-MONT and P. MARDEŠIĆ

The (Poincaré-Hopf) index of a holomorphic vector field $X = \sum_{k=0}^{n} X^k \frac{\partial}{\partial x_k}$ on $\mathbb{C}^{n+1}$, with an isolated singularity at 0 may be computed as the dimension of the algebra obtained from the ring of germs of holomorphic functions on $\mathbb{C}^{n+1}$ at 0 by dividing by the ideal generated by the coordinate functions X^k of X:

$$\mathbf{B} = \mathbf{B}_X := \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(X^0,\ldots,X^n)}, \qquad \mathrm{Ind}_{\mathbb{C}^{n+1},0}(X) = \dim_{\mathbb{C}} \mathbf{B}_X.$$

The index of a real analytic vector field $X = \sum_{k=0}^{n} X^k \frac{\partial}{\partial x_k}$ on $\mathbb{R}^{n+1}$ with an algebraically isolated singularity at 0 may be computed from the algebra obtained from the ring of germs of real analytic functions on $\mathbb{R}^{n+1}$ at 0 by dividing by the ideal generated by the coordinate functions X^k:

$$\mathbf{B}^{\mathbb{R}} = \mathbf{B}^{\mathbb{R}}_X := \frac{\mathcal{A}_{\mathbb{R}^{n+1},0}}{(X^0,\ldots,X^n)}.$$

The index in this case is the signature of the non-degenerate bilinear form obtained by composition of the product in the algebra $\mathbf{B}^{\mathbb{R}}_X$ with a linear map ℓ:

$$(1) \qquad <\,,\,>\colon \mathbf{B}^{\mathbb{R}}_X \times \mathbf{B}^{\mathbb{R}}_X \longrightarrow \mathbf{B}^{\mathbb{R}}_X \xrightarrow{\ \ell\ } \mathbb{R}$$

Supported by CONACYT-CNRS and CONACYT 3398-E9307.
Key words: Index of a vector field – Singularity of a hypersurface – Singularity of a vector field – Local algebra – Socle – Bilinear form.
Math. classification: 58C27 – 58F14 – 13H10 .

with $\ell(JX) > 0$, where

$$JX := \det[DX] = \det\left[\frac{\partial X^j}{\partial x_k}\right] \in \mathbf{B}_X$$

is the class of the Jacobian determinant of X ([1], [4], [10]). The ideal (JX) generated by the Jacobian determinant plays a fundamental role in these rings. It is 1-dimensional and minimal in the sense that it is contained in all non-zero ideals. The nature of this ideal allows the definition of the non-degenerate pairing (1).

Assume that the above holomorphic vector field X is tangent to a hypersurface V with an isolated singularity at 0, defined as the zero set of the holomorphic function f and that $n \geq 1$. The tangency condition is expressed algebraically as

$$df(X) = \sum_{k=0}^{n} f_k X^k = fh, \quad f_k := \frac{\partial f}{\partial x_k}, \quad h = h_X := \frac{df}{f}(X) \in \mathcal{O}_{\mathbb{C}^{n+1},0}.$$

If one considers this relation in the ring $\mathbf{B}_X$, it says that f and h annihilate each other, and so we have the inclusions

$$0 \subset (f) \subset \mathrm{Ann}_{\mathbf{B}_X}(h) \subset \mathbf{B}_X. \tag{2}$$

These inclusions give rise to the following geometric points:

$$\mathrm{Spec}\,\frac{\mathbf{B}_X}{\mathrm{Ann}_{\mathbf{B}_X}(h)} \subset \mathrm{Spec}\,\frac{\mathbf{B}_X}{(f)} \subset \mathrm{Spec}\,\mathbf{B}_X. \tag{3}$$

The middle point is obtained by imposing on the zero set $\mathrm{Spec}\,\mathbf{B}_X$ of X the condition $f = 0$. One obtains the smaller point by imposing the further conditions contained in $\mathrm{Ann}_{\mathbf{B}_X}(h)$.

The formula in [5] says that, when studying the index of $X|_V$ at 0, one should "measure" for n even with the smaller point $\mathrm{Spec}\,\dfrac{\mathbf{B}_X}{\mathrm{Ann}_{\mathbf{B}_X}(h)}$, and for n odd with the larger point $\mathrm{Spec}\,\dfrac{\mathbf{B}_X}{(f)}$ (see Theorem 5). The difference between the size of these two subpoints of the zero set of X is bounded by $\dim \dfrac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(f, f_0, \ldots, f_n)}$, a number which is independent of X.

The problem we address in this paper is to extend the method of computing the index at a singular point of a hypersurface from the holomorphic category to the real analytic (or C^∞) category. To do this, we introduce the relative Jacobian determinant of X:

$$J_f(X) := \frac{JX}{h} \in \frac{\mathbf{B}_X}{\mathrm{Ann}_{\mathbf{B}_X}(h)}$$

and show in Theorem 3 that it has similar properties as the Jacobian determinant: It generates a 1-dimensional minimal non-zero ideal in $\dfrac{\mathbf{B}_X}{\mathrm{Ann}_{\mathbf{B}_X}(h)}$. We are then able with its help to introduce a non-degenerate pairing in the smaller point of (3), and hence to count "with sign" the size of the smaller point.

Denote by $\mathfrak{X}(V)$ the space of germs of real analytic vector fields X on $(\mathbb{R}^{n+1}, 0)$ tangent to V with an algebraically isolated singularity at 0. Let $V = f^{-1}(0)$, where f is a real analytic function having an algebraically isolated critical point at 0. Define the **signature** function

$$\mathrm{Sgn}_{f,0} : \mathfrak{X}(V) \to \mathbb{Z}$$

by setting $\mathrm{Sgn}_{f,0}(X) = 0$ if $h_X = 0 \in \mathbf{B}_X$ and for $h_X \neq 0 \in \mathbf{B}_X$ define $\mathrm{Sgn}_{f,0}(X)$ as the signature of the bilinear form

$$< \, , \, >_\ell : \frac{\mathbf{B}_X^{\mathbb{R}}}{\mathrm{Ann}_{\mathbf{B}_X^{\mathbb{R}}}(h)} \times \frac{\mathbf{B}_X^{\mathbb{R}}}{\mathrm{Ann}_{\mathbf{B}_X^{\mathbb{R}}}(h)} \xrightarrow{\cdot} \frac{\mathbf{B}_X^{\mathbb{R}}}{\mathrm{Ann}_{\mathbf{B}_X^{\mathbb{R}}}(h)} \xrightarrow{\ell} \mathbb{R}$$

obtained using any linear map ℓ with $\ell(J_f X) > 0$. This definition is independent of the chosen ℓ ([4] p. 26) and it depends only on the restriction of X to V (Theorem 6).

The method we present gives an algebraic procedure to compute the (Poincaré-Hopf) index of real analytic vector fields on singular hypersurfaces in $\mathbb{R}^{n+1}$ ([2], [7]) with n even:

THEOREM 1. — *Let $n \in \mathbb{N}$ be even, $V = f^{-1}(0) \subset \mathbb{R}^{n+1}$ be a real hypersurface with an algebraically isolated singularity at 0. Then the signature function $\mathrm{Sgn}_{f,0}$ satisfies the law of conservation of number:*

$$\mathrm{Sgn}_{f,0}(X) = \mathrm{Sgn}_{f,0}(X_t) + \sum_{\substack{X_p(t)=0 \\ f(p)=0 \\ p\in\mathbb{R}^n-\{0\}}} \mathrm{Ind}_{V,p}(X_t|_V), \tag{4}$$

for X_t tangent to V, close to X and p close to 0. The signature function coincides with the (Poincaré-Hopf) index function $\mathrm{Ind}_{V,V',0}$ of $X|_V$ at 0 after normalizing by adding an integer K that depends only on the smoothing V' used to compute the index:

$$\mathrm{Sgn}_{f,0}(X) = \mathrm{Ind}_{V,V',0}(X) + K, \quad \forall X \in \mathfrak{X}(V).$$

Formula (4) is valid for n odd under the further assumption that the family of algebras $\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}$ is equidimensional (Theorem 6).

Remark. — In the case of vector fields tangent to an *odd* dimensional real hypersurface V a function satisfying the law of conservation (4) has been constructed in [6]. This function is given as the difference between the signatures of two non-degenerate forms. The first term is $\mathrm{Sgn}_{f,0}$ as in the even-dimensional case. The second term is defined in a similar way, but using the algebra

$$\mathbf{A}^{\mathbb{R}} = \frac{\mathcal{A}_{\mathbb{R}^{n+1},0}}{(f_0,\ldots,f_n)}, \qquad f_i = \frac{\partial f}{\partial x_i},$$

instead of $\mathbf{B}^{\mathbb{R}}$ and the **relative Hessian** of f instead of the relative Jacobian determinant of X.

1. The algebra B.

Let X_t be a holomorphic family of holomorphic vector fields on the closed unit ball $\bar{B} \subset \mathbb{C}^{n+1}$ parametrized by a connected and reduced analytic space T and let each vector field X_t have isolated singularities on $\bar{B}$ and be non-vanishing on the boundary of B. Let $0 \in T$ and assume that X_0 vanishes only at 0. Denote the Jacobian determinant of X_t by $JX_t := \det DX_t$. For every fixed value of t consider the finite dimensional semi-local algebra

(1.1)

$$\mathbf{B}_t := H^0\Big(B, \frac{\mathcal{O}_B}{(X_t^0,\ldots,X_t^n)}\Big) = \bigoplus_{\{X_t(p_t)=0\}} \frac{\mathcal{O}_{(B,p_t)}}{(X_t^0,\ldots,X_t^n)} := \bigoplus_{\{X_t(p_t)=0\}} \mathbf{B}_{t,p_t}$$

having the above decomposition as a direct sum of local algebras. This decomposition corresponds to the geometric decomposition of $X_t = 0$ as a finite number of different points, where the dimension of the algebra over a point corresponds to the multiplicity at that point. The Jacobian determinant decomposes as a sum

$$JX_t = \sum_{\{X_t(p_t)=0\}} J_{p_t}X_t \in \mathbf{B}_t \tag{1.2}$$

obtained by considering the class of JX_t in each local ring of (1.1). Each $J_{p_t}X_t$ generates a 1-dimensional ideal in $\mathbf{B}_t$, which is contained in the corresponding local algebra $\mathbf{B}_{t,p_t}$. This ideal is then the socle of the corresponding local algebra ([4]).

Let $\mathcal{Z}$ be the subvariety of $T \times B$ defined by $X_t^0 = \ldots = X_t^n = 0$, $\mathcal{O}_{\mathcal{Z}}$ its structure sheaf, and $\pi : \mathcal{Z} \longrightarrow T$ the projection to the first factor. Then π is a finite map. Hence

$$\mathcal{B} := \pi_*\mathcal{O}_{\mathcal{Z}} \tag{1.3}$$

is a coherent sheaf on T ([8]).

PROPOSITION 2. — *Under the above hypothesis for X_t, we have:*

(i) *$\mathcal{B}$ is a locally free sheaf of rank μ; i.e. the family of algebras $\{\mathbf{B}_t\}$ forms a holomorphic vector bundle of dimension μ, where the algebra structure is holomorphic in t.*

(ii) *If $t_n \to 0 \in T$, $p_{t_n} \in B$ are zeroes of X_{t_n} converging to $0 \in B$ with $\dim_{\mathbb{C}} \mathbf{B}_{t_n,p_{t_n}} = \nu$, then*

$$\lim_{t_n \to 0} J_{p_{t_n}} X_{t_n} = \frac{\nu}{\mu} J X_0, \tag{1.4}$$

where the limit is taken in the vector bundle $\{\mathbf{B}_t\}$.

(iii) *If $L_t : \mathbf{B}_t \longrightarrow \mathbb{C}$ is a holomorphic family of linear maps such that $L_0(JX_0) \neq 0$ then $L_t(J_{p_t} X_t) \neq 0$, for $t \in T'$ near 0. The family of bilinear forms*

$$< \, , \, >_t : \mathbf{B}_t \times \mathbf{B}_t \stackrel{\cdot}{\longrightarrow} \mathbf{B}_t \stackrel{L_t}{\longrightarrow} \mathbb{C}$$

is holomorphic and non-degenerate for $t \in T'$, with (1.1) an orthogonal decomposition.

Proof. — That $\mathcal{B}$ is locally free follows from the continuity of the intersection multiplicity ([9], p. 664, [3], p. 59). This sheaf corresponds to the sections of a holomorphic vector bundle, where the vector space over the point t is $\mathcal{B} \otimes_{\mathcal{O}_T} \mathbb{C}_t \simeq \mathbf{B}_t$ (see (1.1)).

Now that we know that the family of algebras forms a holomorphic vector bundle, with multiplication varying holomorphically, we can consider the associated projective spaces $\operatorname{Proj} \mathbf{B}_t$. The property that the vector space generated by a non-zero element is an ideal is a closed analytic condition. It is not difficult to show that in the semilocal algebras (1.1) the only such elements are the socles of each factor. Over 0 we only have 1 point, which is the socle of $\mathbf{B}_0$. Hence the family of 1-dimensional spaces $(J_{p_n} X_{t_n})$ converges to the 1-dimensional space (JX_0), and hence the only question that remains is what is the value of the constant(s) in the limit. To find this out, it is enough to apply a linear functional which is non-zero on $J_0 X_0$. Explicitely, the residue map ([9] p. 649)

$$\mathrm{Res}_t : \mathbf{B}_t \longrightarrow \mathbb{C}, \quad \mathrm{Res}_t(g) = \left(\frac{1}{2\pi i}\right)^n \int_{\{|X_t^k| = \epsilon_k\}} \frac{g dz_0 \wedge \ldots \wedge dz_n}{X_t^0 \ldots X_t^n},$$

will do this for us. The (partial) Jacobian determinant $J_{p_n} X_{t_n}$ takes the value 0 on zeroes of X_t different from the chosen p_{t_n} and hence

$\operatorname{Res}_{t_n}(J_{p_n}X_{t_n}) = \nu$ is the local intersection multiplicity ([9] p. 663) at p_{t_n}. Similarly $\operatorname{Res}_0(J_0X_0) = \mu$ is the local intersection multiplicity at 0. This proves (1.4) and (ii).

To prove (iii), note that (1.4) implies that, for small fixed t, the components $J_{p_t}X_t$ in the decomposition (1.2) of the Jacobian determinant are almost parallel, but linearly independent (since they belong to different sumands in (1.1)). The second part follows from [4] and is a consequence of the fact that (JX_0) is the 1-dimensional socle in $\mathbf{B}_0$ and that the pairing $< , >_t$ is continuous. □

2. The algebra $\frac{\mathbf{B}}{\operatorname{Ann}_{\mathbf{B}}(h)}$.

Let f be a holomorphic function defined on the unit ball $B \subset \mathbb{C}^{n+1}$ with the property that $V := f^{-1}(0)$ is a variety that has 0 as its unique singular point. Assume also that the previously introduced family of holomorphic vector fields X_t is tangent to the variety V: $X_t(p) \in T_pV$, $p \in V$. That is, there exists a holomorphic family of holomorphic functions h_t on B such that

$$(2.1) \qquad df(X_t) = \sum_{j=0}^{n} f_j X_t^j = h_t f, \qquad h_t := \frac{df}{f}(X_t).$$

Multiplication by f or by h_t gives endomorphisms of the locally free sheaf $\mathcal{B}$ in (1.3), inducing exact sequences

$$(2.2) \qquad 0 \longrightarrow \operatorname{Ann}_{\mathcal{B}}(f) \longrightarrow \mathcal{B} \xrightarrow{\cdot f} \mathcal{B} \longrightarrow \frac{\mathcal{B}}{(f)} \longrightarrow 0,$$

$$(2.3) \qquad 0 \longrightarrow \operatorname{Ann}_{\mathcal{B}}(h_t) \longrightarrow \mathcal{B} \xrightarrow{\cdot h_t} \mathcal{B} \longrightarrow \frac{\mathcal{B}}{(h_t)} \longrightarrow 0.$$

The images of the maps are isomorphic to the sheaf of ideals (f) and (h_t), respectively. Condition (2.1) then implies

$$0 \subset (f) \subset \operatorname{Ann}_{\mathcal{B}}(h_t) \subset \mathcal{B}$$

and so we obtain surjective sheaf morphisms of algebras

$$(2.4) \qquad \mathcal{B} \longrightarrow \frac{\mathcal{B}}{(f)} \longrightarrow \frac{\mathcal{B}}{\operatorname{Ann}_{\mathcal{B}}(h_t)}.$$

Fix $t \in T$. The maps in (2.2) and (2.3) are represented by diagonal blocks in the decomposition (1.1). Hence we obtain the decomposition

$$(2.5) \qquad \frac{\mathbf{B}_t}{\operatorname{Ann}_{\mathbf{B}_t}(h_t)} = \bigoplus_{\{X_t(p_t)=0\}} \frac{\mathbf{B}_{t,p_t}}{\operatorname{Ann}_{\mathbf{B}_{t,p_t}}(h_t)}.$$

We now show that the smaller point Spec $\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}$ has a natural family of non-degenerate bilinear forms. To define it, we will introduce a **relative Jacobian determinant** $J_f X_t$.

THEOREM 3. — *Let f be a holomorphic function on the unit ball $B \subset \mathbb{C}^{n+1}$ with 0 as its only critical point, X_t a holomorphic vector field on $\bar{B}$ tangent to $V := f^{-1}(0)$ with isolated zeroes on $\{p_{t,0} = 0, p_{t,1}, \ldots, p_{t,k}\}$, none of them on the boundary of B. Then:*

(i) The contribution of $p_{t,j}$ to the sum in (2.5) is trivial for points $p_{t,j}$ in $B - V$, non-trivial for points on $V - \{0\}$ and for the point 0 it is non-trivial if and only if $h_t \neq 0 \in \mathbf{B}_{t,0}$.

(ii) Assume that $h_t \neq 0 \in \mathbf{B}_t$, then the element $J_f X_t \in \dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}$ defined as the class of elements $g \in \mathbf{B}_t$ satisfying $gh_t = JX_t \in \mathbf{B}_t$ is well defined. The decomposition (2.5) induces a decomposition

$$J_f X_t := J_{f,0} X_t + \ldots + J_{f,k} X_t. \tag{2.6}$$

If the corresponding factor $\dfrac{\mathbf{B}_{t,p_{t,j}}}{\mathrm{Ann}_{B_{t,p_{t,j}}}(h_t)}$ is non-trivial, then $J_{f,j} X_t$ generates a 1-dimensional ideal in $\dfrac{\mathbf{B}_{t,p_{t,j}}}{\mathrm{Ann}_{B_{t,p_{t,j}}}(h_t)}$ which is the socle and the minimal non-zero ideal. For $p_{t,j} \in V - \{0\}$, the relative Jacobian determinant equals the Jacobian determinant of the restriction $X|_V$ of X to V.

(iii) Let $\ell : \dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)} \to \mathbb{C}$ be any linear mapping such that $\ell(J_{f,j} X_t) \neq 0$, for $j = 0, \ldots, k$, and define the bilinear form

$$< \, , \, >_\ell : \frac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)} \times \frac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)} \xrightarrow{\cdot} \frac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)} \xrightarrow{\ell} \mathbb{C}. \tag{2.7}$$

Then $< \, , \, >_\ell$ is a non-degenerate bilinear form having (2.5) as an orthogonal decomposition.

Proof. — If $p_{t,j} \notin V$ then f is a unit in the local ring and so by (2.1) we have $h_t = 0 \in \mathbf{B}_{t,p_{t,j}}$. Hence $\mathrm{Ann}_{\mathbf{B}_{t,p_{t,j}}}(h_t) = \mathbf{B}_{t,p_{t,j}}$ and there is no contribution to the sum in (2.5) from these points.

We now show that

$$\mathrm{Ann}_{\mathbf{B}_{t,p_{t,j}}}(h) = (f) \subset \mathbf{B}_{t,p_{t,j}} \qquad p_{t,j} \in V - \{0\}. \tag{2.8}$$

Since f is contained in the maximal ideal we will then have that $h_t \neq 0$ (otherwise $\mathrm{Ann}_{\mathbf{B}_{t,p_{t,j}}}(h_t) = \mathbf{B}_{t,p_{t,j}}$). By the implicit function theorem we may choose coordinates $(w_0, \dots, w_n)$ around $p_{t,j}$ so that $f = w_0$, and in these coordinates we have

$$X_t = w_0 W^0 \frac{\partial}{\partial w_0} + \sum_{k=1}^{n} W^k \frac{\partial}{\partial w_k}, h_t = W^0. \tag{2.9}$$

By additivity of the intersection multiplicity ([9], p. 664) it follows

$$\begin{aligned}\dim \mathbf{B}_{t,p_{t,j}} &= \dim \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(w_0 W^0, W^1, \dots, W^n)} = \dim \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(w_0, W^1, \dots, W^n)} \\ &+ \dim \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(W^0, W^1, \dots, W^n)} = \dim \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(w_0, w_0 W^0, W^1, \dots, W^n)} \\ &+ \dim \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(W^0, w_0 W^0, W^1, \dots, W^n)} = \dim \frac{\mathbf{B}_{t,p_{t,j}}}{(f)} + \dim \frac{\mathbf{B}_{t,p_{t,j}}}{(h_t)}.\end{aligned}$$

But this then means that (f) and (h) have complementary dimensions in $\mathbf{B}_{t,p_{t,j}}$. Since on the other hand inclusions (2) hold, it follows $\mathrm{Ann}_{\mathbf{B}_{t,p_{t,j}}}(h_t) = (f) \subset \mathbf{B}_{t,p_{t,j}}$.

For $p_{t,0} = 0$ we have that $h_t = 0 \in \mathbf{B}_{t,0}$ if and only if its annihilator is the full ring. This proves (i).

To prove (ii), note that the decomposition (1.1) decomposes the multiplication into blocks, allowing to reduce the problem to each component. For any j the ideal $(JX_t) \subset \mathbf{B}_{t,p_{t,j}}$ is 1-dimensional and it is the socle and minimal non-zero ideal. If $h_t \neq 0$ in $\mathbf{B}_{t,p_{t,j}}$, then $(JX_t) \subset (h_t)$, so that there exists $g \in \mathbf{B}_{t,p_{t,j}}$ such that

$$g h_t = JX_t \in \mathbf{B}_{t,p_{t,j}}. \tag{2.10}$$

The element $g \in \mathbf{B}_{t,p_{t,j}}$ is not uniquely defined. However, the difference of any two elements satisfying (2.10) belongs to $\mathrm{Ann}_{\mathbf{B}_{t,p_{t,j}}}(h_t)$, showing that the class $J_{f,j}X_t := [g] \in \dfrac{\mathbf{B}_{t,p_{t,j}}}{\mathrm{Ann}_{\mathbf{B}_{t,p_{t,j}}}(h_t)}$ is well defined. If $h_t = 0 \in \mathbf{B}_{t,p_{t,j}}$, then $p_{t,j} = 0$, as shown in claim (i), and we set $J_{f,0}X_t = 0$.

We now prove that $(J_{f,j}X_t)$ is the socle of a non-trivial component $\dfrac{\mathbf{B}_{t,p_{t,j}}}{\mathrm{Ann}_{\mathbf{B}_{t,p_{t,j}}}(h_t)}$. We begin with the case $p_{t,j} \neq 0$. In this case V is smooth at the point and by choosing new coordinates around this point and computing directly JX_t in (2.9) we obtain that it is equal to $J(X_t|_V)h$. So, in this case the relative Jacobian determinant is the Jacobian determinant of the restriction $(X_t|_V)$. Now the assertion follows from the $\mathbb{C}^n$-case ([4]).

Consider now the point $p = 0$, and assume $h_t \neq 0 \in \mathbf{B}_{t,0}$. By definition the socle of a local algebra is the annihilator of the maximal ideal. The maximal ideal in $\dfrac{\mathbf{B}_{t,0}}{\mathrm{Ann}_{\mathbf{B}_{t,0}}(h_t)}$ is generated by the classes $[x_j]$ of the generators x_j, $j = 0, \ldots, n$, of the maximal ideal in $\mathcal{O}_{\mathbb{C}^{n+1},0}$. We prove first that $J_{f,0}X_t$ is in the socle. We have to prove that $[x_j]J_{f,0}X_t = 0 \in \dfrac{\mathbf{B}_{t,0}}{\mathrm{Ann}(h_{X_t})}$, that is that $(x_j J_{f,0}X_t)h_{X_t} = 0 \in \mathbf{B}_{t,0}$. But $x_j(J_{f,0}h_t) = x_j J_0 X_t = 0$, since $J_0 X_t$ is in the socle of $\mathbf{B}_{t,0}$. So $J_{f,0}X_t$ is in the socle of $\dfrac{\mathbf{B}_{t,0}}{\mathrm{Ann}_{\mathbf{B}_{t,0}}(h_t)}$. On the other hand if $0 \neq [g] \in \dfrac{\mathbf{B}_{t,0}}{\mathrm{Ann}_{\mathbf{B}_{t,0}}(h_t)}$ belongs to the socle, then $[g][x_j] = 0$, for $j = 0, \ldots, n$. But this means that $gx_j \in \mathrm{Ann}_{\mathbf{B}_{t,0}}(h_{X_t})$ *i.e.* $gx_j h_t = 0$, $j = 0, \ldots, n$, implying that gh_t is in the socle of $\mathbf{B}_{t,0}$. As the socle $(\mathbf{B}_{t,0}) = (J_0 X_t)$ is 1-dimensional, it follows that $gh_t = \lambda J_0 X_t$, $\lambda \in \mathbb{C}^*$ (since $[g] \neq 0$). Hence $(\lambda^{-1}g)h_t = J_0 X_t$, so that $[\lambda^{-1}g] = J_{f,0}X_t$ by the definition of $J_{f,0}X_t$. So we obtain $[g] = \lambda J_f X_t$. This proves (ii).

To prove (iii), it suffices to prove that given any $0 \neq g \in \dfrac{\mathbf{B}_{t,p_{t,j}}}{\mathrm{Ann}_{\mathbf{B}_{t,p_{t,j}}}(h_t)}$ there exists $e \in \dfrac{\mathbf{B}_{t,p_{t,j}}}{\mathrm{Ann}_{\mathbf{B}_{t,p_{t,j}}}(h_t)}$ such that $< g, e >_\ell \neq 0$. As $(J_{f,j}X_t)$ is the minimal ideal, there exists $e \in \mathbf{B}_{t,p_{t,j}}$, such that $ge = J_{f,j}X_t$. This implies $< g, e >_\ell = \ell(J_{f,j}X_t) \neq 0$, as claimed. □

Since $\mathcal{B}$ is locally free, the maps in (2.2) and (2.3) may be represented by $\mu \times \mu$ matrices with holomorphic functions on T as entries. If we restrict to the subvarieties where the rank of these matrix maps is constant, then the kernel and cokernel sheaves will also be locally free.

We now show that under the equidimensional hypothesis in (2.3), the family of non-degenerate bilinear forms defined on the smaller point Spec $\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h)}$ varies holomorphically with respect to external parameters:

PROPOSITION 4. — *Let f and the family of holomorphic vector fields X_t, $t \in T$, be as above and assume that $h_0 \neq 0 \in \mathbf{B}_0$. Assume further that the finite dimensional algebras $\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}$ form a holomorphic vector bundle (with fixed rank) over T. Then after shrinking T around 0 we have:*

(i) *The elements $J_f X_t \in \dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}$ defined by Proposition 3 form a holomorphic section of the vector bundle.*

(ii) *If $t_n \to 0 \in T$ and $p_{t_n} \in B$ are zeroes of X_{t_n} converging to $0 \in B$*

with $\dim_{\mathbb{C}} \dfrac{\mathbf{B}_{t_n,p_{t_n}}}{\mathrm{Ann}_{\mathbf{B}_{t_n,p_{t_n}}}(h_t)} = \nu$, *then*

$$\lim_{t_n \to 0} J_{f,p_{t_n}} X_{t_n} = \frac{\nu}{\mu} J_f X_0, \tag{2.11}$$

where the limit is taken in the vector bundle $\left\{ \dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)} \right\}$.

(iii) *For any holomorphic family of* $\mathbb{C}$*-linear maps* $\ell_t : \dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)} \to \mathbb{C}$ *such that* $\ell_t(J_f X_t) \neq 0$ *the family of bilinear forms* $\langle e_t, k_t \rangle_t := \ell_t(e_t k_t)$ *is a holomorphic family of non-degenerate bilinear forms.*

Proof. — The Jacobian determinant JX_t is a holomorphic function on $T \times B$, and by restriction to $\mathcal{Z}$ and pushforward to T, $\pi_{1*} JX_t$ gives rise to a holomorphic section over T of the locally free sheaf $\mathcal{B}$ in (1.3). The equidimensional hypothesis implies that from the sequence (2.3) we obtain an injective non-zero morphism

$$0 \longrightarrow \frac{\mathcal{B}}{\mathrm{Ann}_{\mathcal{B}}(h_t)} \xrightarrow{h_t} \mathcal{B} \tag{2.12}$$

of locally free sheaves. The Jacobian determinant is contained in the image of h_t since for each t the components (1.2) of the Jacobian determinant generate the socles of the components in (1.1) and $(h_{t,p_{t,j}}) \neq 0$ at any point which has a non-trivial component in (2.5). Since the families $\mathbf{B}_t$ and $\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}$ are vector bundles, we can solve holomorphically on t, and hence the relative Jacobian $J_f X_t$ is holomorphic in T. This proves (i).

Relation (2.11) follows from the similar relation for the Jacobian determinant (1.4) after composing with the inverse of the inclusion (2.12). Hence we obtain (ii). Claim (iii) is a consequence of (2.11) and of (iii) of Theorem 3. □

The equidimensional hypothesis in Proposition 4 is satisfied in general for n even since the formula in [5] can be stated as:

THEOREM 5. — *Under the above hypothesis on* f *and* X_t*:*

(i) *For* n *even the sheaves in (2.3) are locally free; let* $r :=$ rank $\dfrac{\mathcal{B}}{\mathrm{Ann}_{\mathcal{B}}(h_t)}$.

(ii) *For* n *odd the sheaves in (2.2) are locally free; let* $r :=$ rank $\dfrac{\mathcal{B}}{(f)}$.

The index of $X_0|_V$ at 0 can then be measured by the rank r, after normalizing by adding the constant term $(-1)^n \dim_{\mathbb{C}} \dfrac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(f, f_0, \dots, f_n)}$.

Proof. — The index in [5] satisfies the law of conservation of number, which is equivalent to the locally freeness ([3], p. 58). Claim (i) follows from [5] by using (2.3):

$$\dim \frac{\mathcal{O}_B}{(X_t^0, \dots, X_t^n)} - \dim \frac{\mathcal{O}_B}{(h_t, X_t^0, \dots, X_t^n)} = \dim \frac{\mathbf{B}_t}{\mathrm{Ann}_{B_t}(h_t)}.$$

The rest is direct from the index formula in [5]. □

3. The algebra $\frac{\mathbf{B}^{\mathbb{R}}}{\mathrm{Ann}_{\mathbf{B}^{\mathbb{R}}}(h)}$.

Let f be a real analytic function defining a holomorphic function on the closed unit ball $\bar{B} \subset \mathbb{C}^{n+1}$ with 0 as its only critical point on $\bar{B}$, $V := f^{-1}(0) \subset B$ and $V_{\mathbb{R}} := V \cap \mathbb{R}^{n+1}$. Let X_t be a real analytic family of real analytic vector fields tangent to $V_{\mathbb{R}}$ parametrized by a reduced real analytic space T, $0 \in T$, and let X_t be a holomorphic family of holomorphic vector fields tangent to V extending the above family in a neighbourhood of $0 \in T$, parametrized by the complex reduced analytic space $T_{\mathbb{C}}$. Assume that X_t defines a holomorphic family of vector fields on the closed unit ball $\bar{B} \subset \mathbb{C}^{n+1}$, each having isolated singularities on $\bar{B}$ and non-vanishing on the boundary of B and that X_0 vanishes only at 0. Define as before

$$\mathbf{B}_t := H^0\Big(B, \frac{\mathcal{O}_B}{(X_t^0, \dots, X_t^n)}\Big), \quad df(X_t) = \sum_{k=0}^{n} f_k X_t^k := f h_t.$$

THEOREM 6. — *Under the above hypothesis, assume also that $h_0 \neq 0 \in \mathbf{B}_0$ and that the finite dimensional algebras $\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}$ form a holomorphic vector bundle (with fixed rank μ_1) over $T_{\mathbb{C}}$. Then after shrinking T around 0 we have:*

(i) *The invariant spaces of the complex conjugation map*

$$\bar{\ }: \frac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)} \longrightarrow \frac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)} \tag{3.1}$$

define a real analytic vector bundle $\left[\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}\right]^+_{\mathbb{R}}$ over T of $\mathbb{R}$-algebras of $\mathbb{R}$-dimension μ_1.

(ii) *The decomposition (2.5) of* $\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}$ *induces a direct sum decomposition*

$$\left[\frac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}\right]^+_{\mathbb{R}} = [\oplus_j D_j] \oplus [\oplus_k E_k], \tag{3.2}$$

where each component D_j *corresponds to a real point on* V *and each component* E_k *corresponds to a pair of conjugate points on* $V - V_{\mathbb{R}}$*. For* $p_j \neq 0$ *and any* k *we have*

$$D_j = \frac{\mathcal{A}_{\mathbb{R}^{n+1},p_j}}{(f, X_t^0, \dots, X_t^n)}, \quad E_k = \left[\frac{\mathcal{O}_{\mathbb{C}^{n+1},p_k}}{(f, X_t^0, \dots, X_t^n)} \oplus \frac{\mathcal{O}_{\mathbb{C}^{n+1},\bar{p}_k}}{(f, X_t^0, \dots, X_t^n)}\right]^+_{\mathbb{R}}. \tag{3.3}$$

(iii) *For any real analytic family of* $\mathbb{R}$*-linear maps* $\ell_t : \left[\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}\right]^+_{\mathbb{R}} \to \mathbb{R}$ *such that* $\ell_0(J_f X_0) > 0$ *the family of bilinear forms* $\langle e_t, k_t\rangle_t := \ell_t(e_t k_t)$ *is a real analytic family of non-degenerate bilinear forms.*

(iv) *The restriction of the above bilinear form to a component* D_j *corresponding to a point in* $V - \{0\}$ *coincides with the Eisenbud-Levine bilinear form of the restriction of* X_t *to* V*, and its signature corresponds to the Poincaré-Hopf index of* $X_t|_V$*. The restriction to a component* E_k *corresponding to a pair of conjugate points in* $V - V_{\mathbb{R}}$ *has index 0.*

(v) *The signature* $\mathrm{Sgn}_{f,0}(X_t)$ *satisfies the law of conservation of number (4) for this family.*

(vi) *The signature* $\mathrm{Sgn}_{f,0}(X)$ *depends only on the restriction of* X *to* V*, i.e.* $\mathrm{Sgn}_{f,0}(X) = \mathrm{Sgn}_{f,0}(X + fY)$*, for* $X, X + fY$ *with an algebraically isolated singularity at* 0.

Proof. — The complex conjugation map (3.1) is induced from the map

$$\sum a_I z^I \to \sum \bar{a}_I z^I.$$

Claim (i) is elementary linear algebra. Complex conjugation leaves invariant the components in (2.5) corresponding to real points and it permutes the components associated with $p_k, \bar{p}_k \in \mathbb{C}^{n+1} - \mathbb{R}^{n+1}$. Hence the sum of these two last components is invariant, and the decomposition in (3.2) and (3.3) is obtained from the $(+1)$-eigenspaces of the restriction of the conjugation map to the above mentioned invariant subspaces. Relation (3.3) follows from (2.8). This proves (ii).

Since multiplication is analytic in the vector bundle $\left[\dfrac{\mathbf{B}_t}{\mathrm{Ann}_{\mathbf{B}_t}(h_t)}\right]^+_{\mathbb{R}}$, claim (iii) follows from the continuity of the bilinear forms and Proposition

4. The first part of (iv) follows from (ii) of Theorem 3. The second part follows from [4] p. 27, since E_k is a local ring with residue field $\mathbb{C} = \frac{E_k}{\mathfrak{M}}$. Here $\mathfrak{M}$ is the maximal ideal. Note that the compatibility of orientations is deduced from (2.11), since the factor is a positive number.

Claim (v) now follows easily, since the signature at 0 and at t coincide due to the non degeneracy of the bilinear form and its continuity. For $t \neq 0$ we have an orthogonal decomposition (3.2), and so using (iv) we obtain formula (4), where the non-real points contribute nothing to the sum since the contribution of each pair of conjugate points to the signature is 0.

To prove (vi) we begin with:

LEMMA 7.

(i) *Let X_0 and $X_1 = X_0 + fY$ be two real analytic vector fields tangent to V, having an algebraically isolated singularity at* 0. *Then there exists a real analytic family of real analytic vector fields $\{X_0+fY_s\}$ tangent to V, with an algebraically isolated singularity at* 0 *parametrized by a connected parameter space containing both vector fields.*

(ii) *Let X_0 and X_1 be two real analytic vector fields tangent to the above hypersurface V and having an algebraically isolated singularity at* 0. *Then there exists a real analytic family of real analytic vector fields tangent to V, with an algebraically isolated singularity at* 0 *parametrized by a connected parameter space containing both vector fields.*

Proof. — The proofs of both statements run in parallel lines. We will provide linear families of vector fields connecting X_0 and X_1. We will prove that in order to have a family with algebraically isolated singularities connecting X_0 and X_1 it suffices to avoid a set Land^2 of codimension at least 2.

The first condition defining Land^2 is that "an external zero lands at 0". The second condition is that "this external zero landing at 0 is of multiplicity at least 2". Removing this set of parameter values does not disconnect our (linear) family. Hence, neither does removing the smaller set of those parameter values with an algebraically isolated singularity disconnect the family.

Let B be a small ball around 0. For (i), let $W_j := f\dfrac{\partial}{\partial x_j}$, $j = 0, \ldots, n$, $W_{n+1} := fY$ and $W_{n+2} = fZ$ where Z is chosen so that $X_0 + W_{n+2}$ has a simple singular point $p \in B - V$ (*i.e.* $\det[D(X_0 + W_{n+2})](p) \neq 0$). For this case we have $m = n + 2$. For (ii), let $W_1, \ldots, W_m$ be a finite collection of real analytic vector fields on B tangent to V such that for every point $p \in B - V$ the collection of values $W_1(p), \ldots, W_m(p)$ generates the tangent space T_pB and for $q \in V - \{0\}$ they generate T_qV. Set $W_0 := X_1$. Assume also that at least one of the vector fields W_j is such that $X_0 + W_j$ has a simple singular point $p \in B - V$, and for some $k \neq j$ the vector field $X_0 + W_k$ has a simple singular point $q \in V - \{0\}$. These vector fields can be easily obtained using Cartan's Theorem A [8].

Consider the family of vector fields

$$\Big\{W_{\mathbf{s}} := X_0 + \sum_{j=0}^{m} s_j W_j\Big\}, \qquad \mathbf{s} := (s_0, \ldots, s_m) \in \mathbb{R}^{m+1},$$

as well as the complexified family with $\mathbf{s} \in \mathbb{C}^{m+1}$. Define the variety $\mathbf{W}$ by

$$\mathbf{W} := \{(\mathbf{s}, z) \in \mathbb{C}^{m+1} \times B \ / \ W_{\mathbf{s}}(z) = 0\},$$

and let π_k be the projections to the factors. The fibers of the map π_2 over $B - V$ are $(m-n)$-dimensional affine spaces, since they are defined by $n+1$ independent equations in $m+1$ variables. Let

$$\mathbf{W}_1 := \overline{\mathbf{W} \cap [\mathbb{C}^{m+1} \times (B - V)]} \subset \mathbb{C}^{m+1} \times B.$$

The variety $\mathbf{W}_1$ is irreducible, since it is the closure of an irreducible variety (fibration by affine spaces over $B - V$). It is then an irreducible component of $\mathbf{W}$ of dimension $m + 1 = (m - n) + (n + 1)$. Similarly, let

$$\mathbf{W}_2 := \overline{\mathbf{W} \cap [\mathbb{C}^{m+1} \times (V - \{0\})]}.$$

For case (i), $\mathbf{W}_2$ is empty, since by shrinking B sufficiently $W_{\mathbf{s}}|_V = X|_V$ will have no zeroes in $V - \{0\}$. For case (ii), the fibers over $V - \{0\}$ of the projection map π_2 are $(m+1-n)$-dimensional affine spaces, since thet are defined by n independent equations in $m+1$ variables. Hence $\mathbf{W}_2$ is in case (ii) an irreducible variety of dimension $m + 1 = (m + 1 - n) + n$.

Denote by $\mathbf{W}_0 = \mathbb{C}^{m+1} \times \{0\}$ the zero section. Clearly $\mathbf{W} = \mathbf{W}_0 \cup \mathbf{W}_1 \cup \mathbf{W}_2$ is its decomposition into irreducible components, all of them of dimension $m + 1$. The set

$$\mathrm{Land}^1 := [\mathbf{W}_0 \cap \mathbf{W}_1] \cup [\mathbf{W}_0 \cap \mathbf{W}_2] \subset \mathbf{W}_0 \simeq_{\pi_1} \mathbb{C}^{m+1}$$

has dimension at most m, since it is properly contained in the varieties $\mathbf{W}_k$ of dimension $m + 1$. The set Land^1 viewed as a subvariety of the parameter space $\mathbb{C}^{m+1}$ corresponds to those vector fields for which a zero has "landed", either from $B - V$ or from $V - \{0\}$.

Let $\mathbf{W}_k^2$, for $k=1,2$, be the closure in $\mathbb{C}^{m+1}\times B$ of

$$\mathbf{W}_k^2 := \overline{\widetilde{\mathbf{W}}_k^2} := \overline{\{(\mathbf{s},p)\in \mathbf{W}_k - \mathbf{W}_0\cap \mathbf{W}_k \ / \ \det[DW_{\mathbf{s}}](p)=0\}} \subset \mathbf{W}_k.$$

By the choice of the generating vector fields W_j, $\widetilde{\mathbf{W}}_k^2$ is properly contained in $\mathbf{W}_k$, and so if non-empty, it has dimension m. Hence its intersection with $\mathbf{W}_0$

$$\mathrm{Land}^2 := [\mathbf{W}_k^2\cap \mathbf{W}_0] \subset \mathbf{W}_0 \simeq_{\pi_1} \mathbb{C}^{m+1}$$

will have at most dimension $m-1$, thus showing that Land^2 has codimension at least 2 in $\mathbf{W}_0$. This subvariety corresponds then to those parameter values for which a point of multiplicity at least 2 has "landed". The analytic variety of $\mathbb{C}^{m+1}$ parametrizing vector fields such that the dimension of its zero set has positive dimension at 0 is contained in Land^2 and so has codimension at least 2 for this family parametrized by $\mathbb{C}^{m+1}$.

Recall that an algebraically isolated singularity for a real analytic vector fields is equivalent to having its complexification with an isolated singularity. Hence the set of vector fields parametrized by $\mathbb{R}^{m+1}$ with non algebraically isolated singularities is contained in the subvariety $\mathrm{Land}^2\cap\mathbb{R}^{m+1}$ of codimension at least 2. Hence its complement is connected in $\mathbb{R}^{m+1}$. □

To finish the proof of (vi), note that Theorem 5 implies that one of the 2 points in (2.4) satisfies the law of conservation of number. The difference between the sizes of these points is $\dfrac{\mathrm{Ann}_{B_t}(h_t)}{(f)}$, a number that depends only on the restriction of $X_0+fY_{\mathbf{s}}|_V = X_0|_V$, since it is the dimension of the non-zero homology of the complex

$$0\longleftarrow \mathcal{O}_{V,0}\xleftarrow{W_{\mathbf{s}}|_V}\Omega^1_{V,0}\xleftarrow{W_{\mathbf{s}}|_V}\cdots\xleftarrow{W_{\mathbf{s}}|_V}\Omega^n_{V,0}\xleftarrow{W_{\mathbf{s}}|_V}\Omega^{n+1}_{V,0}\longleftarrow 0$$

obtained by contracting differential forms on V with $W_{\mathbf{s}}|_V$ ([5], Theorem 2.6). Hence for this family, both points in (2.4) satisfy a law of conservation of number, and we may apply to this family (i-v) of Theorem 6. By (i) of Theorem 3, the only points which give a non-trivial contribution to (2.5) are the points on V. But since $W_{\mathbf{s}}|_V = X_0|_V$, there are no additional singular points on $V-\{0\}$. Hence the decomposition in (2.5) has only one component, the one corresponding to 0. This shows that the component at 0 remains equidimensional throughout the deformation. Claim (vi) now follows from (i) of Lemma 7 and the continuity of the non-degenerate bilinear pairing $<\ ,\ >_{\mathbf{s}}$ ((iii) of Theorem 6). □

Proof of Theorem 1. — By Theorem 5, the equidimensional hypothesis needed to use Theorem 6 is always satisfied for n even. Both the index and the signature satisfy a law of conservation of number (4), and so the difference of these functions is locally constant. Since the space of real analytic vector fields with an algebraically isolated singularity at 0 is connected, by (ii) of Lemma 7, this difference is (globally) a constant. □

The reduction of the C^∞ case to the real analytic case, under the assumption that the ideals $(f_0, \ldots, f_n)$ and $(X^0, \ldots, X^n)$ have finite codimension in the ring of germs of C^∞ functions, is carried out in [4].

BIBLIOGRAPHY

[1] V. ARNOLD, S. GUSEIN-ZADE & A. VARCHENKO, Singularities of Differentiable Maps, I, Birkhauser, 1985.

[2] Ch. BONATTI & X. GÓMEZ-MONT, The index of a holomorphic vector field on a singular variety I, Astérisque, 222 (1994), 9–35.

[3] A. DOUADY, Flatness and Privilige, L'Enseignement Mathématique, 14 (1968), 47–74.

[4] D. EISENBUD & H. LEVINE, An algebraic formula for the degree of a C^∞ map germ, Ann. Math., 106 (1977), 19–38.

[5] X. GÓMEZ-MONT, An Algebraic formula for the index of a vector field on a hypersurface with an isolated singularity, preprint.

[6] X. GÓMEZ-MONT, P. MARDEŠIĆ, The index of a vector field tangent to an odd dimensional hypersurface and the signature of the relative Hessian, preprint.

[7] X. GÓMEZ-MONT, J. SEADE & A. VERJOVSKY, The index of a holomorphic flow with an isolated singularity, Math. Ann., 291 (1991), 737–751.

[8] GRAUERT & H. REMMERT, Coherent Analytic Sheaves, Grundlehren 265, Springer Verlag, 1984.

[9] Ph. GRIFFITHS, J. HARRIS, Principles of Algebraic Geometry, J. Wiley, 1978.

[10] KHIMISHIASHVILI, On the local degree of a smooth map, Trudi Tbilisi Math. Inst., (1980), 105–124.

Manuscrit reçu le 4 décembre 1996,
accepté le 7 juillet 1997.

X. GÓMEZ-MONT,
CIMAT
A.P. 402
Guanajuato, 36000 (Mexique).
gmont@fractal.cimat.mx

P. MARDEŠIĆ,
Université de Bourgogne
Laboratoire de Topologie
UMR 5584 du CNRS
B.P. 400
21011 Dijon Cedex (France).
mardesic@satie.u-bourgogne.fr