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# THE BERGMAN KERNEL OF THE MINIMAL BALL AND APPLICATIONS

by K. OELJEKLAUS, P. PFLUG & E.H. YOUSSFI

## 1. Introduction.

Let $\mathbb{B}_*$ be the domain in $\mathbb{C}^n, n \geq 2$, defined by

$$\mathbb{B}_* := \{z \in \mathbb{C}^n : |z|^2 + |z \bullet z| < 1\},$$

where $z \bullet z := \sum_{j=1}^{n} z_j^2$. This is the ball of radius $\frac{\sqrt{2}}{2}$ with respect to the norm

$$N_*(z) := \sqrt{\frac{|z|^2 + |z \bullet z|}{2}}, \quad z \in \mathbb{C}^n.$$

The norm $N_*$ was introduced by Hahn and Pflug [HP], and was shown to be the smallest norm in $\mathbb{C}^n$ that extends the euclidean norm in $\mathbb{R}^n$ under certain restrictions. The automorphism group of $\mathbb{B}_*$ is compact and its identity component is $\mathrm{Aut}^0_{\mathcal{O}}(\mathbb{B}_*) = S^1 \cdot SO(n, \mathbb{R})$, where the $S^1$-action is diagonal and the $SO(n, \mathbb{R})$-action is the matrix multiplication, see [K] or [OY]. This shows that for $n \geq 3$, the ball $\mathbb{B}_*$ is not biholomorphic to any Reinhardt domain. For $n = 2$, $\mathbb{B}_*$ is linearly biholomorphic to the Reinhardt triangle $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$.

The main purpose of this note is to establish the following

*Key words*: Bergman kernel - Minimal ball - Proper holomorphic mapping.
*Math. classification*: 32H10 - 32H35.

THEOREM. — *The Bergman kernel of $\mathbb{B}_*$ is given by the formula*

$$K_{B_*}(z,w) = \frac{1}{n(n+1)V(B_*)} \frac{\sum_{j=0}^{[\frac{n}{2}]} \binom{n+1}{2j+1} X^{n-1-2j} Y^j \left(2nX - (n-2j)[X^2 - Y]\right)}{(X^2 - Y)^{n+1}},$$

*where*

$$X = 1 - \langle z,w \rangle, \quad \text{and} \quad Y = (z \bullet z)\overline{w \bullet w},$$

*and $V(\mathbb{B}_*)$ is the Lebesgue volume of $\mathbb{B}_*$.*

In particular, when $n = 2$, the Bergman kernel of $\mathbb{B}_*$ is

$$K_{\mathbb{B}_*}(z,w) = \frac{2}{\pi^2} \frac{3(1 - \langle z,w \rangle)^2(1 + \langle z,w \rangle) + (z \bullet z)\overline{w \bullet w}(5 - 3 \langle z,w \rangle)}{\left((1 - \langle z,w \rangle)^2 - (z \bullet z)\overline{w \bullet w}\right)^3}.$$

It should be noted that for $n = 2$ this formula can be obtained from the Bergman kernel of the above mentioned Reinhardt triangle whose Bergman kernel can be found in ([JP], p. 176).

*Remark.* — To the best of our knowledge, the domain $\mathbb{B}_*$ is the first bounded domain in $\mathbb{C}^n$ which is neither Reinhardt nor homogeneous, and for which we have an explicit formula for its Bergman kernel.

## 2. Preparatory results.

Let $G$ be a semi-simple complex Lie group and $K$ a maximal compact subgroup of $G$. Suppose that $G$ acts irreducibly on a finite dimensional complex vector space $E_\Lambda$ via a representation $(\Pi_\Lambda, E_\Lambda)$ with dominant weight $\Lambda$ and dominant vector $v_\Lambda$. Assume further that $E_\Lambda$ is furnished with a $K$-invariant hermitian scalar product $[\cdot,\cdot]$. If $G = KAN$ is the Iwasawa decomposition of $G$, let $\varrho$ denote the sum of the roots associated with the complex decomposition in the Lie algebra $\mathfrak{g}$ of $G$. If $\mathfrak{a}$ is the Lie algebra of $A$, then we have the following orthogonal decomposition with respect to the Cartan-Killing form

$$\mathfrak{a} = a_\Lambda \oplus a_\Lambda^\perp,$$

where $a_\Lambda$ is the annihilator of $\Lambda$. If $H_0$ is the unique vector in $a_\Lambda^\perp$ such that $\Lambda(H_0) = 1$, we set

$$\sigma := 2\varrho(H_0). \tag{2.1}$$

Let $\mathbb{M}^*$ be the intersection of the $G$-orbit of $v_\Lambda$ and the unit ball in $E_\Lambda$.

In his work [Lo], Loeb proved that the Bergman kernel of the manifold $\mathbb{M}^*$ with an invariant form on $\mathbb{M}^*$ is given for $\zeta = \Pi_\Lambda(g_1)v_\Lambda, \eta = \Pi_\Lambda(g_2)v_\Lambda, g_1, g_2 \in G$, by

$$K_{\mathbb{M}^*}(\zeta,\eta) = \sum_{j=0}^{\infty}(2j+\sigma)T_\Lambda(j)[\zeta,\eta]^j, \tag{2.2}$$

where $T_\Lambda(j)$ is the dimension of the representation with dominant weight $j\Lambda$.

Here we consider the special case $G = SO(n+1,\mathbb{C})$ with its natural linear representation on the complex hermitian space $(\mathbb{C}^{n+1}, < \cdot,\cdot >)$, where $\Lambda$ is the dominant weight associated to this representation and $v_\Lambda = \frac{\sqrt{2}}{2}(1,i,0,\cdots,0)$. The intersection of the $G$-orbit of $v_\Lambda$ and the unit ball in $\mathbb{C}^{n+1}$ is $\mathbb{M}^* = \mathbb{M}\setminus\{0\}$, where

$$\mathbb{M} := \{z = (z_1,\cdots,z_{n+1}) \in \mathbb{C}^{n+1} : |z| < 1, z \bullet z = 0\}.$$

Then by formula (2.2), we see that the Bergman kernel of $\mathbb{M}^*$ with respect to an $SO(n+1,\mathbb{C})$-invariant form $\alpha(z)\wedge\overline{\alpha(z)}$ is given (up to a multiplicative constant) by

$$K_{\mathbb{M}^*}(\zeta,\eta) = \sum_{j=0}^{\infty}(2j+\sigma)T_\Lambda(j) < \zeta,\eta >^j, \tag{2.3}$$

for $\zeta,\eta \in \mathbb{M}^*$.

LEMMA 2.1. — *If $\alpha(z)$ is an $SO(n+1,\mathbb{C})$-invariant nonzero $n$-form on $\mathbb{M}^*$ (the invariant $n$-from $\alpha$ is unique up to a constant), then the Bergman kernel of $\mathbb{M}^*$ with respect to the invariant form $\alpha(z)\wedge\overline{\alpha(z)}$ is given (up to a multiplicative constant) by*

$$\begin{aligned} K_{\mathbb{M}^*}(\zeta,\eta) &= \frac{2(n+1) < \zeta,\eta > +2n-2}{(1- < \zeta,\eta >)^{n+1}} \\ &= \frac{4n}{(1- < \zeta,\eta >)^{n+1}} - \frac{2n+2}{(1- < \zeta,\eta >)^{n}}, \end{aligned}$$

*for $\zeta,\eta \in \mathbb{M}^*$.*

*Proof.* — Using the notations above, a calculation involving the Weyl character formula implies that

$$T_\Lambda(j) = \frac{n+2j-1}{n-1}\binom{n-2+j}{j}, \text{ for all positive integers } j.$$

See ([FH], pp. 267-315). In addition, some computing shows that $\sigma = 2n-2$. See ([FH], pp. 399-414). The lemma now follows from (2.3). □

LEMMA 2.2. — *The $n$-form on $(\mathbb{C}\setminus\{0\})^{n+1}$*

$$\widetilde{\alpha}(z) := \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1},$$

*induces by restriction an $SO(n+1,\mathbb{C})$-invariant and holomorphic $n$-form $\alpha$ on $\mathbb{M}^*$.*

*Proof.* — Let $A \in SO(n+1,\mathbb{C})$, $z \in \mathbb{M}^*$ and set $w = Az$. Denote by $A_{jk}$ the $n\times n$ matrix obtained from $A$ by deleting the $j$th row and the $k$th column. Since $A \in SO(n+1,\mathbb{C})$, Cramer's rule gives that

$$a_{jk} = (-1)^{k+j}\det A_{jk}. \tag{2.4}$$

Note also that for $z \in \mathbb{M}^*$ and $z_j \neq 0$, then

$$dz_j = -\sum_{l\neq j} \frac{z_l}{z_j} dz_l \quad \text{on } T_z\mathbb{M}^*, \tag{2.5}$$

where $T_z\mathbb{M}^*$ denotes the tangent space of $\mathbb{M}^*$ at the point $z$. Denote by $A^*\alpha$ the pull-back of $\alpha$. Then

$$\begin{aligned}
(A^*\alpha)(z) &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{n+1} \\
&= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} \sum_{k=1}^{n+1} \det A_{jk} dz_1 \wedge \cdots \wedge \widehat{dz_k} \cdots \wedge dz_{n+1} \\
&= \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{j=1}^{n+1} \frac{(-1)^{k+j}}{w_j} \det A_{jk} dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1}
\end{aligned}$$

$$\text{by (2.4)} \quad = \sum_{k=1}^{n+1}(-1)^{k-1}\sum_{j=1}^{n+1}\frac{a_{jk}}{w_j}dz_1\wedge\cdots\wedge\widehat{dz_k}\wedge\cdots\wedge dz_{n+1}$$

$$\text{by (2.5)} \quad = \sum_{k=1}^{n+1}(-1)^{k}\sum_{j=1}^{n+1}\frac{a_{jk}}{w_j}dz_1\wedge\cdots\wedge dz_{j-1}\wedge\widehat{dz_j}\wedge\Big(\sum_{l\neq j}\frac{z_l}{z_j}dz_l\Big)$$
$$\wedge\, dz_{j+1}\wedge\cdots\widehat{dz_k}\wedge\cdots\wedge dz_{n+1}$$
$$= \sum_{k=1}^{n+1}(-1)^{k-1}\sum_{j=1}^{n+1}(-1)^{j-k}\frac{a_{jk}}{z_jw_j}z_kdz_1\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\wedge dz_{n+1}$$
$$= \sum_{j=1}^{n+1}\left(\frac{(-1)^{j-1}}{w_jz_j}\sum_{k=1}^{n+1}a_{jk}z_k\right)dz_1\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\wedge dz_{n+1}$$
$$= \sum_{j=1}^{n+1}(-1)^{j-1}\frac{1}{z_j}dz_1\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\wedge dz_{n+1} = \alpha(z).$$

That the restriction of $\alpha$ to $\mathbb{M}^*$ is holomorphic can be seen by evaluating the form $\alpha$ on the $n$-fold exterior power of the tangent space. □

## 3. Proper holomorphic mappings from $\mathbb{M}$ into $\mathbb{C}^n$.

Consider the projection $pr : \mathbb{C}^{n+1} \to \mathbb{C}^n$ defined by

$$pr(z_1,\cdots,z_{n+1}) := (z_1,\cdots,z_n).$$

The restriction $F := pr|_{\mathbb{M}}$ of $pr$ to $\mathbb{M}$ gives a proper holomorphic mapping of degree 2 from $\mathbb{M}$ onto $\mathbb{B}_*$. Let $W$ be the branching locus of $F$ and $V$ the image of $W$ under $F$. Denote by $\varphi$ and $\psi$ the two local inverses of $F$ defined for $z = (z_1,\cdots,z_n) \in \mathbb{B}_* \setminus V$ by

$$\varphi(z) = (z, i\sqrt{z\bullet z})$$
$$\psi(z) = (z, -i\sqrt{z\bullet z}).$$

LEMMA 3.1. — *If $\varphi := (\varphi_1,\cdots,\varphi_{n+1})$ and $\psi := (\psi_1,\cdots,\psi_{n+1})$ are the local inverses of $F$ defined on $\mathbb{B}_*\setminus V$, then*

$$\varphi^*(\alpha) = \frac{1+n}{i\sqrt{z\bullet z}}(-1)^n dz_1\wedge\cdots\wedge dz_n \tag{3.1}$$

$$\psi^*(\alpha) = \frac{1+n}{-i\sqrt{z\bullet z}}(-1)^n dz_1\wedge\cdots\wedge dz_n. \tag{3.2}$$

*Proof.* — The pull back of $\alpha$ under $\varphi$ is

$$\begin{aligned}\varphi^*(\alpha) &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{n+1} \\ &= \sum_{j=1}^{n} \frac{(-1)^{j-1}}{w_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge dz_n \wedge dw_{n+1} + \frac{(-1)^n}{w_{n+1}} dz_1 \wedge \cdots \wedge dz_n.\end{aligned}$$

But for $1 \leq j \leq n$

$$\begin{aligned}&\frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge d\varphi_{n+1} \\ &= \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge \Big( -\sum_{k=1}^{n} \frac{z_k}{w_{n+1}} dz_k \Big) \\ &= \frac{(-1)^{j}}{w_{n+1}} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge dz_j \\ &= \frac{(-1)^{j+n-j}}{w_{n+1}} dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \\ &= (-1)^n \frac{dz_1 \wedge \cdots \wedge dz_n}{w_{n+1}}.\end{aligned}$$

Thus

$$\begin{aligned}\varphi^*(\alpha) &= \left( \frac{(-1)^n}{w_{n+1}} + (-1)^n \frac{n}{w_{n+1}} \right) dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1+n}{w_{n+1}} (-1)^n dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1+n}{i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n.\end{aligned}$$

Similarily one has that

$$\psi^*(\alpha) = \frac{1+n}{-i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n.$$

□

If $P_{\mathbb{M}^*}$ denotes the Bergman projection of $\mathbb{M}^*$ with respect to the volume form $\alpha(z) \wedge \overline{\alpha(z)}$, and if $P_{\mathbb{B}_*}$ denotes the Bergman projection of $\mathbb{B}_*$, then we have the following transformation rule

LEMMA 3.2. — *If $\varphi := (\varphi_1, \cdots, \varphi_{n+1})$ and $\psi := (\psi_1, \cdots, \psi_{n+1})$ are the local inverses of $F$ defined locally on $\mathbb{B}_* \setminus V$, then*

$$P_{\mathbb{M}^*}(z_{n+1}(h \circ F))(z) = z_{n+1}((P_{\mathbb{B}_*} h) \circ F)(z)$$

*for all $h \in L^2(\mathbb{B}_*)$, where $V$ is the image of the branching locus of $F$.*

*Proof.* — First observe that the lemma holds for all holomorphic functions $h \in L^2(\mathbb{B}_*)$. Indeed, by virtue of Lemma 3.1 we have that

$$\begin{aligned}
&\int_{\mathbb{M}^*} |z_{n+1}(h\circ F)(z)|^2 \alpha(z)\wedge\overline{\alpha(z)} \\
&\qquad = \int_{\mathbb{M}^*\setminus W} |(z_{n+1}(h\circ F))(z)|^2 \alpha(z)\wedge\overline{\alpha(z)} \\
&\qquad = \int_{\mathbb{B}_*\setminus V} |\varphi_{n+1}(w)h(w)|^2 \varphi^*(\alpha)(w)\wedge\varphi^*(\overline{\alpha})(w) \\
&\qquad\quad + \int_{\mathbb{B}_*\setminus V} |\psi_{n+1}(w)h(w)|^2 \psi^*(\alpha)(w)\wedge\psi^*(\overline{\alpha})(w) \\
&\qquad = 2(n+1)^2 \int_{\mathbb{B}_*\setminus V} |h(w)|^2\, dv(w) < +\infty.
\end{aligned}$$

Thus $z_{n+1}(h\circ F)(z) \in L^2(\mathbb{M}^*, \alpha(z)\wedge\overline{\alpha(z)})$.

Next let $f \in L^2(\mathbb{M}^*, \alpha(z)\wedge\overline{\alpha(z)})$ be a holomorphic function, and let $g$ be an element of the space $C_0^\infty(\mathbb{B}_*\setminus V)$ of all $C^\infty$-function with compact support in $\mathbb{B}_*\setminus V$. Then

$$\begin{aligned}
&\int_{\mathbb{M}^*} f(z)\overline{z_{n+1}\left(\frac{\partial g}{\partial w_j}\circ F\right)(z)}\alpha(z)\wedge\overline{\alpha(z)} \\
&= (n+1)^2\left[\int_{\mathbb{B}_*} \frac{(f\circ\varphi)(w)}{\varphi_{n+1}(w)}\overline{\frac{\partial g}{\partial w_j}(w)}\, dv(w) + \int_{\mathbb{B}_*} \frac{(f\circ\psi)(w)}{\psi_{n+1}(w)}\overline{\frac{\partial g}{\partial w_j}(w)}\, dv(w)\right]
\end{aligned}$$

so that by integration by parts we obtain that

$$P_{\mathbb{M}}\left(z_{n+1}\Big(\frac{\partial g}{\partial w_j}\circ F\Big)\right) = 0, \quad \text{for all } \ j = 1,\cdots,n.$$

Since the space

$$\mathcal{H} := \left\{\frac{\partial g}{\partial w_j} : g \in C_0^\infty(\mathbb{B}_*\setminus V)\right\}$$

is dense in the orthogonal complement in $L^2(\mathbb{B}_*)$ of the subspace $L_h^2(\mathbb{B}_*)$ of all square integrable holomorphic functions on $\mathbb{B}_*$, the lemma follows. □

LEMMA 3.3. — *If $\varphi$ and $\psi$ are as before, then*

$$z_{n+1}K_{\mathbb{B}_*}(F(z), w) = (n+1)^2\left[\frac{K_{\mathbb{M}^*}(z,\varphi(w))}{\overline{\varphi_{n+1}(w)}} + \frac{K_{\mathbb{M}^*}(z,\psi(w))}{\overline{\psi_{n+1}(w)}}\right], \qquad z\in\mathbb{M}^*, w\in\mathbb{B}_*.$$

*Proof.* — Let $w \in \mathbb{B}_*\setminus V$ and let $r > 0$ be chosen so small that $w + r\Delta^n \subset \mathbb{B}_*\setminus V$, where $\Delta$ is the unit disc in $\mathbb{C}$. By Remark 6.1.4 in [JP], there is a $\mathcal{C}^\infty$-function $u : \mathbb{C}^n \to [0, +\infty)$ with compact support in $w + r\Delta^n$ such that

$$P_{\mathbb{B}_*} u = K_{\mathbb{B}_*}(\cdot, w).$$

By virtue of Lemma 3.2 we see that for $z \in \mathbb{M}^*$,

$$\begin{aligned} z_{n+1} K_{\mathbb{B}_*}(F(z), w) &= z_{n+1}(P_{\mathbb{B}_*} u)(F(z)) = P_{\mathbb{M}^*}(z_{n+1}(u \circ F)(z)) \\ &= \int_{\mathbb{M}^*} \zeta_{n+1}(u \circ F)(\zeta) K_{\mathbb{M}^*}(z, \zeta)\, \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \\ &= (n+1)^2 \int_{\mathbb{B}_*} u(\eta) \left[ \frac{K_{\mathbb{M}^*}(z, \varphi(\eta))}{\overline{\varphi_{n+1}(\eta)}} + \frac{K_{\mathbb{M}^*}(z, \psi(\eta))}{\overline{\psi_{n+1}(\eta)}} \right] dv(\eta) \\ &= (n+1)^2 \left( \frac{K_{\mathbb{M}^*}(z, \varphi(w))}{\overline{\varphi_{n+1}(w)}} + \frac{K_{\mathbb{M}^*}(z, \psi(w))}{\overline{\psi_{n+1}(w)}} \right), \end{aligned}$$

and the lemma is proved. □

## 4. Proof of the main result.

*Proof of the theorem.* — For $z, w \in \mathbb{B}_* \setminus V$, set

$$\begin{cases} s := 1- < z, w >, & t := \varphi_{n+1}(z)\overline{\varphi_{n+1}(w)} \\ x :=< z, w > +t \quad \text{and} \quad y :=< z, w > -t. \end{cases}$$

Then using the notations in the main theorem we have $X = s$ and $Y = t^2$. By Lemma 2.1 we see that for some positive constant $C$ we have

$$\begin{aligned} K_{\mathbb{M}^*}(\varphi(z), \varphi(w)) &= C \left( \frac{4n}{(1-x)^{n+1}} - \frac{2n+2}{(1-x)^n} \right) \\ K_{\mathbb{M}^*}(\varphi(z), \psi(w)) &= C \left( \frac{4n}{(1-y)^{n+1}} - \frac{2n+2}{(1-y)^n} \right), \end{aligned}$$

so that by Lemma 3.3 we obtain that

$$\begin{aligned} K_{\mathbb{B}_*}(z, w) &= 4C(n+1)^2 \frac{f(x) - f(y)}{x - y}, \quad \text{where} \\ f(u) &= \frac{2n}{(1-u)^{n+1}} - \frac{n+1}{(1-u)^n}. \end{aligned}$$

On the other hand,

$$\frac{f(x) - f(y)}{x - y} = n \frac{(s+t)^{n+1} - (s-t)^{n+1}}{t(s^2 - t^2)^{n+1}} - \frac{n+1}{2} \frac{(s+t)^n - (s-t)^n}{t(s^2 - t^2)^n},$$

and

$$\begin{aligned}\frac{(s+t)^{n+1}-(s-t)^{n+1}}{t} &= \frac{(s+t)^{n+1}-s^{n+1}}{t}+\frac{(s-t)^{n+1}-s^{n+1}}{-t}\\ &= \sum_{j=1}^{n+1}\binom{n+1}{j}s^{n+1-j}t^{j-1}\\ &\qquad + \sum_{j=1}^{n+1}\binom{n+1}{j}s^{n+1-j}(-t)^{j-1}\\ &= \sum_{j=1}^{n+1}\binom{n+1}{j}s^{n+1-j}[t^{j-1}+(-t)^{j-1}]\\ &= \sum_{k=0}^{n}\binom{n+1}{k+1}s^{n-k}[t^k+(-t)^k]\\ &= 2\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}s^{n-2k}t^{2k}.\end{aligned}$$

Similarily we have that

$$\frac{(s+t)^n-(s-t)^n}{t}=2\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}s^{n-1-2k}t^{2k}.$$

Therefore,

$$\begin{aligned}\frac{f(x)-f(y)}{x-y} &= \frac{2n\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}s^{n-2k}t^{2k}}{(s^2-t^2)^{n+1}}\\ &\qquad -(n+1)\frac{\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}s^{n-1-2k}t^{2k}}{(s^2-t^2)^n}.\end{aligned}$$

But $\binom{n}{2k+1}=\frac{n-2k}{n+1}\binom{n+1}{2k+1}$. Thus

$$\begin{aligned}\frac{f(x)-f(y)}{x-y} &= \frac{2n\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}s^{n-2k}t^{2k}}{(s^2-t^2)^{n+1}}\\ &\qquad - \frac{\sum_{k=0}^{[\frac{n-1}{2}]}(n-2k)\binom{n+1}{2k+1}s^{n-1-2k}t^{2k}(s^2-t^2)}{(s^2-t^2)^{n+1}}\\ &= \frac{\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}s^{n-1-2k}t^{2k}[2ns-(n-2k)(s^2-t^2)]}{(s^2-t^2)^{n+1}}.\end{aligned}$$

It follows that

$$K_{\mathbb{B}_*}(z,w) = 4C(n+1)^2 \frac{\sum_{k=0}^{[\frac{n}{2}]} \binom{n+1}{2k+1} X^{n-1-2k} Y^k [2nX - (n-2k)(X^2-Y)]}{(X^2-Y)^{n+1}},$$

where $X$ and $Y$ are as in the statement of the theorem. To compute the constant we use the formula

$$1 = \int_{\mathbb{B}_*} K_{\mathbb{B}_*}(0,w) dV(w).$$

□

## 5. Applications.

THEOREM 5.1. — *Let $D \subset \mathbf{C}^n$ be a pseudoconvex domain with $C^2$-boundary and let $f : D \to \mathbb{B}_*$ be a proper holomorphic mapping. Then $A_f \subset V(f)$ where*

$$A = A_f := \{z \in D : f(z) \bullet f(z) = 0\}, V(f) := \{z \in D : \det f'(z) = 0\}.$$

*Proof.* — Observe that $A$ is an analytic subset of $D$. Assume that there exists a point $a \in A$ with $\det f'(a) \neq 0$. To get a contradiction it suffices to show that:

$$\text{if } z^\nu \in A, z^\nu \to z^0 \in \partial D, \text{ then } \det f'(z^\nu) \to 0.$$

We choose a ball $B(z^0, s)$ , $0 < \eta < 1$ so that $\eta(n+2) > n+1$, and a defining function $r$ of $D \cap B(z^0, s)$ such that $\widetilde{r} := -(-r)^\eta$ is plurisubharmonic on $D \cap B(z^0, s)$; this can be achieved using a result of Diederich-Fornaess [DF]. Moreover, we may assume that $f(z^\nu) \to w^0 \in \mathbb{H} \cap \partial\mathbb{B}_*$, where $\mathbb{H} := \{\zeta \in \mathbb{C}^n : \zeta \bullet \zeta = 0\}$.

Assuming that $f$ is a mapping with multiplicity $m$, we know by Pinchuk [Pi2] that

$$mK_D(z,z) \geq |\det f'(z)|^2 K_{\mathbb{B}_*}(f(z), f(z)),\ z \in D.$$

It is well known that $K_D(z,z) \leq C_1 \mathrm{dist}\,(z, \partial D)^{-(n+1)}$, $z \in D$. Hence we get

$$|\det f'(z)|^2 \leq C_2 \left(K_{\mathbb{B}_*}(f(z), f(z))\right)^{-1} \mathrm{dist}\,(z, \partial D)^{-(n+1)}.$$

Now we apply the theorem to obtain that

$$|\det f'(z^\nu)|^2 \leq C_3(1-|f(z^\nu)|^2)^{n+2}/\mathrm{dist}\,(z^\nu,\partial D)^{n+1},\ \nu >> 1.$$

Fix $s' < s$ and define on $D$ the following function:

$$v(z) := \begin{cases} \max\{\widetilde{r}(z), |z-z^0|^2 - s'^2\} \text{ if } z \in D \cap \overline{B(z^0,s')}, \\ |z-z^0|^2 - s'^2, \text{ if } z \in D \setminus B(z^0,s'). \end{cases}$$

It is clear that $v$ is plurisubharmonic on $D$ and that $v(z) = \widetilde{r}(z)$ for $z \in D \cap B(z^0,s'')$, $0 < s'' < s'$ sufficiently small.

For $w \in \mathbb{B}_*$ we put $\rho(w) := \max\{v(z) : z \in D, f(z) = w\}$. Obviously, $\rho$ is plurisubharmonic on $\mathbb{B}_*$. In particular, for $\nu >> 1$ we have $\rho(f(z^\nu)) \geq v(z^\nu) = \widetilde{r}(z^\nu)$.

Exploiting that $\mathbb{B}_*$ is balanced and the Hopf-Lemma on $\mathbb{H} \cap \mathbb{B}_*$ leads to the following estimate: $\rho(f(z^\nu)) \leq C_4(|f(z^\nu)|^2 - 1)$, $\nu >> 1$; $C_4 > 0$ independent of $z^\nu$. Therefore

$$|\det f'(z^\nu)|^2 \leq C_5(-r(z^\nu))^{\eta(n+2)}/\mathrm{dist}\,(z^\nu,\partial D)^{n+1} \to 0,\ \text{if } \nu \to \infty,$$

which leads to that contradiction we mentioned at the beginning of the proof. □

COROLLARY 5.2. — *There are no unbranched proper holomorphic mappings from $D$ onto $\mathbb{B}_*$ for any bounded pseudoconvex domain with a $C^2$- boundary; in particular, such a $D$ is never biholomorphically equivalent to $\mathbb{B}_*$.*

Moreover, if $D$ is assumed to be strongly pseudoconvex we get even more:

THEOREM 5.3. — *Let $D \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with $C^2$-boundary. If $f : D \to \mathbb{B}_*$ is a proper holomorphic mapping, then $A = V(f)$.*

*Proof.* — Assume the inclusion $V(f) \subset A$ is not correct. Then, by the maximum principle, there is a sequence $z^\nu \in V(f)$, $z^\nu \to z^0 \in \partial D$ such that $|f(z^\nu) \bullet f(z^\nu)| > C > 0$. Without loss of generality we assume that $f(z^\nu) \to w^0$. Since $|w^0 \bullet w^0| > 0$ we conclude that $w^0$ is a strongly pseudoconvex boundary point of $\mathbb{B}_*$. By Theorem 3 of [Ber] there is a neighborhood $U = U(z^0)$ such that $f$ extends to a continuous mapping on $U \cap \bar{D}$. Then using Theorem 3′ of [Pi1] we obtain that $f \in C^1(\bar{D} \cap U)$. Finally using Theorem 1 of [Pi2] we finally get the contradiction to the fact that $z^0 \in \overline{V(f)}$. □

We recall that a bounded domain $\Omega$ is said to satisfy condition $(Q)$ if the Bergman projection of $\Omega$ maps $C_0^\infty(\Omega)$ into the space $\mathcal{O}(\overline{\Omega})$ of all holomorphic functions on a neighborhood of $\overline{\Omega}$. It was proved recently in [Th] that $\Omega$ satisfies condition $Q$ is if and only if that for every compact subset $L$ of $\Omega$, there is an open neighborhood $U = U(L)$ of $\overline{\Omega}$ such that the Bergman kernel $K_\Omega(z, w)$ of $\Omega$ extends to be holomorphic on $U$ as a function of $z$ for each $w \in L$, and $K_\Omega$ is continuous on $U \times L$.

LEMMA 5.4. — *The ball $\mathbb{B}_*$ satisfies condition $(Q)$.*

*Proof.* — For $z, w \in \mathbb{B}_*$, we have

$$\begin{aligned}\left|(1-<z,w>)^2-(z\bullet z)\overline{(w\bullet w)}\right| &\geq |1-<z,w>|^2-|z\bullet z||w\bullet w|\\ &\geq (1-|z||w|)^2-|z\bullet z||w\bullet w|\\ &\geq \left(1-|z||w|-\sqrt{|z\bullet z|}\sqrt{|w\bullet w|}\right)^2\\ &\geq \left(1-\sqrt{|z|^2+|z\bullet z|}\sqrt{|w|^2+|w\bullet w|}\right)^2\end{aligned}$$

where the last inequality holds because of Cauchy-Schwarz's inequality. Therefore for some positive constant $C$ we have

$$|K_{\mathbb{B}_*}(z,w)| \leq \frac{C}{\left(1-\sqrt{|z|^2+|z\bullet z|}\sqrt{|w|^2+|w\bullet w|}\right)^{2n+4}}, \text{ for all } z, w \in \mathbb{B}_*.$$

This shows that $\mathbb{B}_*$ satisfies condition $(Q)$. □

THEOREM 5.5. — *Let $D \subset \mathbb{C}^n$ be an arbitrary bounded circular domain which contains the origin.*

(1) *If $f : \mathbb{B}_* \to D$ is a proper holomorphic mapping, then $f$ extends holomorphically to a neighborhood of $\overline{\mathbb{B}_*}$.*

(2) *If $D$ is smooth then there is no proper holomorphic mapping from $\mathbb{B}_*$ into $D$.*

*Proof.* — Since, by Lemma 5.4, $\mathbb{B}_*$ satisfies condition $Q$, part (1) of the theorem becomes a consequence of Theorem 2 of [Bel]. To see that part (2) of theorem holds, it is enough to notice that if there is proper holomorphic mapping $f : \mathbb{B}_* \to D$, and if $\varrho$ is a defining function of $D$, then $\varrho \circ f$ is a defining function for $\mathbb{B}_*$, which will imply that $\mathbb{B}_*$ is smooth and thus leads to a contradiction. □

THEOREM 5.6. — *Let $L$ be a compact subset of $\mathbb{B}_*$ and let $\zeta$ be a boundary point of $\mathbb{B}_*$. Then every holomorphic function $f$ in a* *neighborhood of L is the uniform limit of functions in the complex span of the functions*

$$\frac{\partial}{\partial\zeta^\beta}K_{\mathbb{B}_*}(.,\zeta), \quad \beta\in\mathbb{N}_0^n.$$

*Proof.* — Since $\mathbb{B}_*$ is a Runge domain and satisfies condition $(Q)$ the proposition follows from Theorem 2.5 of [Th]. □

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