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BERNSTEIN CLASSES

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1. INTRODUCTION

Let $\Omega \subseteq \mathbb{C}$ be a bounded domain, $K \subseteq \Omega$ - a compact, and let f be analytic in Ω and continuous on $\bar{\Omega}$.

We call the ratio

$$B(f, K, \Omega) = \max_{\bar{\Omega}} |f| / \max_{K} |f|$$

the Bernstein constant of f on (K, Ω).

This definition is motivated by one of the classical Bernstein's inequalities: *let $p(x)$ be a polynomial of degree d. Then*

$$\max_{x \in E_R} |p(x)| \leqslant R^d \max_{[-1,1]} |p(x)|,$$

where $E_R \subseteq \mathbb{C}$ is the ellipse with the focuses at $-1, 1$ and the semiaxes R. In other words, $B(P, [-1,1], E_R) \leqslant R^d$. ([3]).

A kind of an inversion of this result is true: it is well-known, that the number of zeros of f on $K \subseteq \Omega$ is bounded from above by $\gamma \log B(f, K, \Omega)$, where γ depends only on the couple $K \subseteq \Omega$. (See 2.2 below.) This fact has been intensively used in transcendental number theory, in particular in the estimates of the number of zeros of exponential sums. (See [48], where also some survey of a vast literature on the subject is given.)

Bounding the number of zeroes of certain analytic functions is also one of the central questions in a vast field of investigations around the

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second part of the Hilbert's 16-th problem (which asks for the maximal possible number of limit cycles in a system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ on the plane, with P and Q polynomials of degree d. See [1], [11], [13], [21], [22], [27], [28], [30], [32], [36], [37], [40], [56].

Various approaches have been used to produce some partial information on this problem: Khovanskii's theory of fewnomials and Pfaffian functions ([33], [25]), Gabrielov's existence result for the bound on the number of components of semianalytic families ([23], [20]), etc. Recently Bernstein inequalities have been applied to an infinitesimal version of the above problem in [31], [32], [37] (see also [38]).

Besides bounding the number of zeroes, recently some other important applications for the Bernstein-type inequality have been found, in particular, in PDE's ([16], [17], [18], [19]), in Dynamical Systems ([50], [51], [52], [53]) and in Potential Theory ([5], [6], [46]).

The main purpose of the present paper is to show that the Bernstein constant can be a convenient tool in the analysis of much wider classes of functions, then those treated traditionally. We present several natural such classes for which the Bernstein constant can be effectively estimated. Roughly, the following classes are considered:

1. Algebraic functions.
2. Solutions of algebraic differential equations.
3. Functions, whose Taylor coefficients satisfy "Lipschitzian" recurrency relations.
4. Functions, whose Taylor coefficients are polynomials or rational functions in the parameters of the problem.

Notice that functions of classes 1 and 2, as well as most of the functions, naturally appearing in analysis, belong to both the classes 3 and 4.

In this paper we show in detail, how to bound Bernstein constant for functions in classes 1, 2, 3, and describe shortly a situation for the class 4. The complete proofs for the class 4 are given in [21], [7], [8]. An alternative approach to the classes 1 and 2, which sometimes produces sharper results, is given in [7].

The following main results are obtained:

1. A stronger version of the Fefferman-Narasimhan inequality for algebraic functions ([16], [17], [18], [19]) is obtained. In particular, this answers a question, posed in [17].

2. A similar inequality is obtained for much larger classes of functions (classes 2 and 3 above). Notice however, that these results, being quite general, are not very sharp. A much sharper inequality, which is a direct generalization of Fefferman-Narasimhan's one to the solutions of first order algebraic differential equations, is obtained in [7].

3. A stronger version of the result of [5] is obtained.

We would like this paper to also be a kind of a survey of some new approaches to bounding the Bernstein constant, so we discuss without proof some results of [7], [8], [21], [41], [45] and [53], [55].

Since the paper is rather long, let us present now its content in more detail.

We restrict ourselves in this paper almost completely to the special case of $(K, \Omega) = (\mathcal{D}_{R_1}, \mathcal{D}_{R_2})$ - the couple of disks of radii R_1, R_2, respectively, centered at $0 \in \mathbb{C}$.

Part 2 is devoted to general properties of the Bernstein Classes. In 2.1 two types of Bernstein classes, B^1 and B^2, are defined. B^1 consists of functions with the uniformly bounded Bernstein constant. B^2 consists of functions, whose Taylor coefficients, starting from $d+1$, are uniformly bounded through the first d Taylor coefficients. We prove that these two types of classes essentially coincide.

In Section 2.2 we state a bound on the number of zeros of functions in B^1, mentioned above, and then, prove a corresponding bound for functions in B^2 using an equivalence of B^1 and B^2, established in 2.1.

Finally, in Section 2.3 we prove, using some classical results on p-valent functions, that both the properties above are essentially equivalent to the following property of f: for any $c \in \mathbb{C}$, the number of solutions of $f = c$ in the disk is uniformly bounded.

Part 3 is devoted to algebraic functions. Recently Bernstein-type inequalities for algebraic functions found applications in differential equations (see [16]-[19]), in potential theory ([5], [6], [46]) and in dynamical systems ([50] [53]). In particular, Fefferman and Narasimhan ([16], [17], [18], [19]) proved Bernstein-type inequalities for restrictions of real polynomials to real algebraic manifolds. L. Bos, N. Levenberg, P. Milman and B.A. Taylor in [5] gave a characterization of algebraic sets in terms of Bernstein and Markov-type inequalities for restrictions of polynomials to these sets.

The constants in the inequalities in [16], [17], [18], [5] depend on the "measure of nondegeneracy" of the algebraic manifold at the origin (and, of course, on the degrees of the polynomials involved). We prove in Section 3.3 a stronger version of these results: the Bernstein inequality is true for algebraic functions in the maximal disk of their regularity, with the constants, depending on the degree only. (Below we call such an inequality a "structural" one.)

The proof is based on the equivalence of the Bernstein bound and the bound on the number of solutions of $f = c$, combined with the Bezout theorem.

In Section 3.4, the result of 3.3 is extended to algebraic functions in several variables, providing in particular stronger versions of the results from [5], [16], [17], [18].

However, in most of the applications we have in mind, the situation is quite opposite to that of algebraic functions: we want to bound the number of solutions of $f = c$ by bounding the Bernstein constant $B(f)$, and not vice-versa.

Considering algebraic functions as a model example, we discuss in 3.5 some alternative approaches, which allow one to bound the Bernstein constant without knowing the bound on the number of zeros.

In part 4 solutions of analytic differential equations are considered. Usually in this situation one can produce a differential inequality, which helps to compare the values of the function on two sets, one smaller and one bigger. This approach is classical (see [34], [29]), and recently it was further developed in [31], [32], [37]. In contrast to the method applied in [31], where only the number of zeros is bounded for equations of higher orders, we use the equivalence of the classes B^1 and B^2 and thus bound Bernstein constant directly for solutions of equations of any order. We also extend the results of [31], [32], [37], obtained for linear equations to the non-linear ones.

In part 5 a somewhat "dual" situation is considered. We discuss functions, whose Taylor coefficients satisfy "Lipschitzian" recurrency relations. Using the equivalence of the classes B^1 and B^2 also here we obtain explicit bounds on Bernstein constants.

In part 6 we discuss shortly the approach, developed in [21] [22], [7], [8]. We deal here with families $f_\lambda(x) = \sum_{k=0}^{\infty} a_k(\lambda)x^k$ of analytic functions, whose Taylor coefficients $a_k(\lambda)$ are polynomials in λ. Following the pioneering work of Bautin [2], we relate the structure of the ideals, generated by $a_k(\lambda)$ in the ring of polynomials in λ, with the analytic properties of f_λ, in particular, with the property of f_λ to belong to the Bernstein classes B^2 and B^1.

It is important to notice that most of the functions we want to investigate, can be treated by any one of the above approaches. In particular, this is true for algebraic functions, and each of the results of parts 4, 5 and 6 produces a Bernstein inequality for algebraic functions. However, the constants in each of these inequalities depend not only on the degree, but also on the "size" of the data.

We consider a question of obtaining Bernstein-type inequalities with the constants, depending on the degree only, for solutions of algebraic differential equations as a very important one. At present (except for the case of algebraic functions) we have only very partial results in this direction (see [8], [41], [43]).

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In particular, the idea of the first proof of the existence of a Bernstein-type inequality for algebraic functions, with the constant depending only on the degree, was suggested to us by M. Gromov (see [45] and Section 3.5 below). A possibility of "inverting" the bound on the number of zeroes and combining it with the Bezout theorem for algebraic functions (which is used in Section 3.3 below) was proposed to us by P. Milman and S. Yakovenko.

In conclusion, we would like to thank the referee for correcting a mistake in the initial version of the paper, which allowed for a serious improvement of the results and presentation.

2. BERNSTEIN CLASSES - B^1 AND B^2

Let $\mathcal{D}_R$ denote, as above, the closed disk of radius $R > 0$, centered at $0 \in \mathbb{C}$. In this paper we mostly restrict ourselves to the Bernstein classes with respect to a couple of such disks.

2.1. Definition of B^1, B^2 and their equivalence.

DEFINITION 2.1.1. — *Let $R > 0$, $0 < \alpha < 1$ and $K > 0$ be given and let f be holomorphic in a neighbourhood of $\mathcal{D}_R$. We say that f belongs to* *the Bernstein class* $B^1_{R,\alpha,K}$ *if* $\max_{\mathcal{D}_R}|f|/\max_{\mathcal{D}_{\alpha R}}|f| \leqslant K$ *(or, in other words, if* $B(f,\alpha\mathcal{D}_R,\mathcal{D}_R) \leqslant K$*).*

DEFINITION 2.1.2. — *Let a natural* N, $R > 0$ *and* $c > 0$ *be given, and let* $f(z) = \sum_{i=0}^{\infty} a_i x^i$ *be an analytic function in a neighbourhood of* $0 \in \mathbb{C}$. *We say that* f *belongs to the Bernstein class* $B^2_{N,R,c}$, *if for any* $j > N$, $|a_j|R^j \leqslant c \max_{i=0,\cdots,N} |a_i|R^i$.

The following theorem shows that the classes B^1 and B^2, essentially coincide:

THEOREM 2.1.3. — *Let* $f \in B^2_{N,R,c}$. *Then* f *is analytic in an open disk* $\overset{\circ}{\mathcal{D}}_R$, *and for any* $R' < R$ *and* $\alpha < 1$ *and* $K = \frac{1}{\alpha^N}\left(1 + \frac{\alpha(1-\alpha^N)}{1-\alpha} + \frac{c\beta}{1-\beta}\right)$, *where* $\beta = R'/R$, f *belongs to* $B^1_{R',\alpha,K}$.

Conversely, if f belongs to $B^1_{R,\alpha,K}$, then it belongs to $B^2_{N,R,c}$, with $N = [(\log\ K - \log(1-\alpha) + 1)/\log(1/\alpha)]$, $c = 3K/(1-\alpha)^2$. In particular, for $\alpha = \frac{1}{2}$, $N = [\log\ K + 2]$ and $c = 12K$.

Proof. — For $f \in B^2_{N,R,c}$ the convergence of $f(x) = \sum_{i=0}^{\infty} a_i x^i$ inside $\mathcal{D}_R$ is immediate. Let $m = \max_{\mathcal{D}_{\alpha R'}}|f|$. Then by the Cauchy formula, $|a_i| \leqslant m/(\alpha R')^i$ for any i. In particular, for $i = 0,\cdots,N$, $|a_i|R^i \leqslant m/\alpha^i(R'/R)^i \leqslant m/\alpha^N(R'/R)^N$. Hence for any $j \geqslant N+1$,

$$|a_j|R^j \leqslant cm/\alpha^N(R'/R)^N \ .$$

Now we can bound $|f|$ on $\mathcal{D}_{R'}$ as follows:

$$\begin{aligned}
\max_{\mathcal{D}_{R'}}|f| &\leqslant \sum_{i=0}^{N} |a_i|R'^i + \sum_{j=N+1}^{\infty} |a_j|R'^j \\
&\leqslant m\sum_{i=0}^{N}\left(\frac{1}{\alpha R'}\right)^i R'^i + (cm/\alpha^N(R'/R)^N)\sum_{j=N+1}^{\infty}(R'/R)^j \\
&= m\cdot\frac{1}{\alpha^N}\,\frac{1-\alpha^{N+1}}{1-\alpha} + \frac{cm}{\alpha^N}\,\frac{\beta}{1-\beta} \\
&= \frac{m}{\alpha^N}\,(1 + \frac{\alpha(1-\alpha^N)}{1-\alpha} + \frac{c\beta}{1-\beta}), \quad \beta = R'/R\ .
\end{aligned}$$

Remark. — One can show that K above can be chosen also as $\frac{1}{\alpha^N}\left(1+\frac{c\alpha\beta}{1-\alpha\beta}+\frac{c\beta}{1-\beta}\right)$, which in some cases gives a better estimate.

To prove an opposite inclusion, assume that $f \in B^1_{R,\alpha,K}$. We want to prove, that $f \in B^2_{N,R,c}$, with the same R, and with N and c as given in the statement of the theorem.

Let $f = \sum_{i=0}^{\infty} a_i x^i$. Let us fix some natural N. Denote by σ the $\max_{i=0,\cdots,N}\{|a_i|R^i\}$. For $P_N(x) = \sum_{i=0}^{N} a_i x^i$, we have: $\max_{\mathcal{D}_{\alpha R}} |P_n(x)| \leqslant \sigma \sum_{i=0}^{N} (\alpha R/R)^i \leqslant \frac{\sigma}{1-\alpha}$. Denote by a the $\max_{\mathcal{D}_{\alpha R}} |R(x)|$, where $R(x) = \sum_{i=N+1}^{\infty} a_i x^i$. Then $\max_{\mathcal{D}_{\alpha R}} |f| \leqslant a + \frac{\sigma}{1-\alpha}$, and since $f \in B^1_{R,\alpha,K}$, we get $\max_{\mathcal{D}_R} |f| \leqslant K\left(a+\frac{\sigma}{1-\alpha}\right)$.

By the Cauchy inequality, we obtain now

$$(*) \qquad |a_j|R^j \leqslant K\left(a + \frac{\sigma}{1-\alpha}\right), \quad \text{for any } j.$$

This inequality would give us the required bound on $|a_j|R^j$, would we know a. So let us continue as follows:

$$(**) \qquad \begin{aligned} a \leqslant \max_{\mathcal{D}_{\alpha R}} \left| \sum_{j=N+1}^{\infty} a_i x^i \right| &\leqslant K\left(a + \frac{\sigma}{1-\alpha}\right) \sum_{i=N+1}^{\infty} (\alpha R/R)^i \\ &= K\left(a + \frac{\alpha}{1-\alpha}\right) \cdot \frac{\alpha^{N+1}}{1-\alpha}, \qquad \text{or} \\ a\left(1 - \frac{K\alpha^{N+1}}{1-\alpha}\right) &\leqslant \frac{\sigma\alpha^{N+1}}{(1-\alpha)^2}. \end{aligned}$$

Let us fix N in such a way, that $\frac{K\alpha^{N+1}}{1-\alpha} < \frac{1}{2}$; it is enough to take $N = [(\log\ K - \log(1-\alpha) + 1)/\log(1/\alpha)]$, where logarithm is taken with the basis 2. In particular, for $\alpha = 1/2$ we have $N = [\log\ K + 2]$. This is the choice, given in the statement of the theorem.

For this specific N, $(**)$ gives

$$a \leqslant \frac{2\sigma\alpha^{N+1}}{(1-\alpha)^2} \leqslant \frac{2\sigma}{(1-\alpha)^2}.$$

Now we return to $(*)$ and obtain $|a_j|R^j \leqslant K\sigma \frac{3-\alpha}{(1-\alpha)^2} \leqslant \frac{3K\sigma}{(1-\alpha)^2}$ for any j. This completes the proof of Theorem 2.1.3.

The equivalence of the classes B^1 and B^2, established by Theorem 2.1.3, allows one to combine different techniques in investigating functions in Bernstein classes. In particular, if a function satisfies a certain first order homogeneous differential equation, one can usually compare its values on a smaller and a bigger disks, using a corresponding differential inequality. In this case the classes B^1 are appropriate. On the other hand, if the Taylor coefficients of a certain function satisfy "Lipschitzian" recurrency relations, this function can be shown to belong to some of the B^2-classes.

The following simple proposition compares the Taylor polynomial (of an appropriate degree) and the remainder term for functions in the Bernstein classes. Here, clearly, the B^2 classes are appropriate.

PROPOSITION 2.1.4. — *Let $f \in B^2_{N,R,c}$, and let $P(x) = \sum_{i=0}^{N} a_i x^i$ be the N-th order Taylor polynomial of f, and $R(x) = \sum_{j=N+1}^{\infty} a_j x^j$ - the corresponding remainder term. Let $\|P\|_R = \max_{i=0,\cdots,N} |a_i| R^i$. Then for any $R' < R$ and $\beta = R'/R$, $\max_{\mathcal{D}_{R'}} |R(x)| \leqslant \|P\|_R \cdot \dfrac{c\beta^{N+1}}{1-\beta}$.*

Proof. — By the definition of the class B^2, for any $j \geqslant N+1$, $|a_j| R^j \leqslant c\|P\|_R$. Hence for any $x \in \mathcal{D}_{R'}$,

$$|R(x)| \leqslant \|P\|_R \sum_{j=N+1}^{\infty} c(R'/R)^j = \|P\|_R \cdot \frac{c\beta^{N+1}}{1-\beta}.$$

In this paper we usually do not consider more complicated domains, than concentric disks. However, in the next proposition we replace the inner disk by the real segment. This proposition is used in Sections 3.3 - 3.4 below, to obtain real versions of Bernstein inequalities in one and several variables.

Let $R > r > 0$ be given. Consider $Ir \subseteq \mathcal{D}_R$, $Ir = [-r, r]$. Let f be a function, analytic in $\overset{\circ}{\mathcal{D}}_R$ and continuous on $\mathcal{D}_R$. Assume that for any $\alpha < 1$, $f \in B^1_{R,\alpha,K(\alpha)}$, with $K(\alpha) = K_1(1/\alpha)^N$. (This assumption is motivated by the form of the constant K in Theorem 2.1.3. See also Remark 2 below.)

PROPOSITION 2.1.5. — *Let f, R, r, K_1 be as above. Then the Bernstein constant $B(f, I_r, \mathcal{D}_R)$ does not exceed $K_1^2(R/r\kappa)^{2N}$, with $\kappa =$*

$\frac{1}{2}\left(\rho - \frac{1}{\rho}\right), \rho = (R/r + \sqrt{(R/r)^2 - 1})^{1/2}$. *In particular, for* $r = \frac{1}{2} R$, $B(f, I_{\frac{1}{2}R}, \mathcal{D}_R) \leqslant K_1^2 5^{2N}$.

Proof. — It will be convenient to rescale the domain in such a way that $r' = 1$, $R' = R/r$, and to assume that $\max\limits_{I_1} |f| = 1$.

We apply a transformation $x = \psi(w) = \frac{1}{2}\left(w + \frac{1}{w}\right)$ and consider $g(w) = f(\psi(w))$. ψ transfsorms any circle S_ρ of radius ρ in the w-plane into the ellipse E_ρ with the focuses -1, 1 and the semiaxes $\frac{1}{2}\left(\rho + \frac{1}{\rho}\right)$ and $\pm\frac{1}{2}\left(\rho - \frac{1}{\rho}\right)$. In particular, the unit circle S_1 is transformed to the interval I_1.

Define ρ_* by the equation

$$\frac{1}{2}\left(\rho_* + \frac{1}{\rho_*}\right) = R', \quad \text{or} \quad \rho_* = R' + \sqrt{R'^2 - 1}.$$

The ellipse E_{ρ_*} is the maximal one, contained in $\mathcal{D}_{R'}$. Now let $\rho = \rho_*^{1/2}$. Consider the following three concentric circles in the w-plane: S_1, S_ρ and S_{ρ^2}. By assumptions for $g(w) = f(\psi(w))$, $\max\limits_{S_1} |g(w)| = 1$. Denote $\max\limits_{S_\rho} |g(w)|$ by a. Since the ellipse $\psi(S_\rho)$ contains the disk of radius $\kappa = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)$, $|f|/\mathcal{D}_\kappa \leqslant a$. Now for $\alpha_0 = \kappa/R'$, $f \in B^1_{R', \alpha_0, K_1(1/\alpha_0)^N}$ by assumptions. Hence $\max\limits_{\mathcal{D}_{R'}} |f| \leqslant K_1(1/\alpha_0)^N \cdot a$ $(*)$.

Since the ellipse $\psi(\rho^2) = E_{\rho_*}$ is contained in $\mathcal{D}_{R'}$, we conclude that $\max\limits_{S_{\rho^2}} |g(w)| \leqslant K_1(R'/\kappa)^N a$.

Now we apply Hadamard's inequality, which asserts, that the function $\log \max\limits_{S_t} |g(w)|$ is a convex function of $\log t$. Since the maxima of $|g(w)|$ on S_1 and S_ρ are 1 and a, and $\max\limits_{S_{\rho^2}} |g(w)| \leqslant K_1(R'/\kappa)^N a$, we get $2 \log a \leqslant \log a + \log K_1 + N \log(R'/\kappa)$ or $a \leqslant K_1(R'/\kappa)^N$. By $(*)$ above we conclude that $\max\limits_{\mathcal{D}_{R'}} |f|$ (which is the Bernstein constant under question) does not exceed $K_1^2(R'/\kappa)^{2N}$, with $R' = R/r$ and $\kappa = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)$, with $\rho = (R/r + \sqrt{(R/r)^2 - 1})^{\frac{1}{2}}$.

For $R/r = \frac{1}{2}$ we obtain $B(f, I_{R/2}, \mathcal{D}_R) \leqslant K_1^2 5^{2N}$.

Remark 1. — The bound above is not sharp. Using more accurate computations, involving estimates of the Lorent coefficients of $f(\psi(w))$ through the majorizing series $f(u+v)$, one can improve it significantly. In particular, K_1^2 can be replaced by K_1.

Remark 2. — Hadamard inequality, quoted above, produces the following general property of Bernstein constants: if we write $B(f, \mathcal{D}_{\alpha R}, \mathcal{D}_R)$ as α^N (for an appropriate N), then for any $\beta > \alpha$,

$$B(f, \mathcal{D}_{\beta R}, \mathcal{D}_R) \geqslant \beta^N.$$

2.2. Zeros of functions in Bernstein classes.

The most important property of the functions in the Bernstein classes is that they demonstrate "effectively - polynomial-like" behaviour. In other words, for any property of polynomials one can expect that the functions in B^1, B^2 will have this property, with a "correction term", uniformly bounded in terms of the constants of the Bernstein class.

Many examples of such properties can be given. Proposition 2.1.4 above gives, probably, the simplest one: a "relative error" in an approximation of a function $f \in B^2_{N,R,c}$ by its N-th degree Taylor polynomial is effectively bounded through N, R and c. In this section we discuss one of the most important results of this type - the bounds on the number of zeros for functions in Bernstein classes.

These results are mostly well-known and appear in different forms in various parts of the theory of analytic functions. In particular, as it has been mentioned in the introduction, for any $K \subseteq \Omega$, K, $\bar{\Omega}$-compact, the number of zeros in K of any f, analytic in Ω and continuous on $\bar{\Omega}$, is bounded by $\gamma \log B(f, K, \Omega)$, with γ depending only on $K \subseteq \Omega$, (we remind that the Bernstein constant $B(f, K, \Omega)$ is defined as $\max\limits_{\bar{\Omega}} |f| / \max\limits_{K} |f|$). This inequality is closely related to the classical Jensen inequality. The proof of one of its versions can be found in [31].

For the case $\Omega = \mathcal{D}_R$, $K = \mathcal{D}_{\alpha R}$ (to which we mostly restrict ourselves in this paper), the most precise bound we are aware of, belongs to Van der Poorten [48]:

LEMMA 2.2.1 ([48], Lemma 1, $S = R$). — *Let $f \in B^1_{R,\alpha,K}$. Then the number of zeros of f on $\mathcal{D}_{\alpha R}$ does not exceed* $\log K / \log[(1+\alpha^2)/2\alpha]$.

Now we want to prove a similar result for functions in B^2. There are several possible ways to do this. First, one can compare the N-th order Taylor polynomial and the function itself, using Proposition 2.1.4. Alternatively, one can use very similar results, obtained by Hayman [29]. However, both these approaches require introduction of some techniques, which are not related to the rest of the paper. Instead we obtain the required bound, combining the equivalence of B^2 and B^1, given by Theorem 2.1.3, and Lemma 2.2.1. This way is, probably the simplest, technically, but the bounds it produces are far from being sharp.

PROPOSITION 2.2.2. — *Let $f \in B^2_{N,R,c}$. Then for any $R_1 < R$ the number of zeros of f in $\mathcal{D}_{R_1}$ does not exceed* (∗)

$$N \cdot \min_{\alpha} \left(\log\left(\frac{1}{\alpha}\right) + \frac{1}{N} \log\left(1 + \frac{\alpha(1-\alpha^N)}{1-\alpha} + \frac{c\gamma}{1-\gamma}\right)\right) \Big/ \left(\log\left(\frac{1}{\alpha}\right) + \log\left(\frac{1+\alpha^2}{2}\right)\right),$$

where the minimum is taken for α varying between R_1/R and 1 and $\gamma = R_1/\alpha R < 1$.

Proof. — For any α, $R_1/R < \alpha < 1$, let $R' = \dfrac{1}{\alpha} R_1$. Then by Theorem 2.1.3, f belongs to $B^1_{R',\alpha,K}$, with $K = \dfrac{1}{\alpha^N} \left(1 + \dfrac{\alpha(1-\alpha^N)}{1-\alpha} + \dfrac{c\gamma}{1-\gamma}\right)$, where $\gamma = R'/R = R_1/\alpha R$.

By Lemma 2.2.1, the number of zeroes of f on $\mathcal{D}_{R_1} = \mathcal{D}_{\alpha R'}$ is bounded by $\log K/\log[(1+\alpha^2)/2\alpha]$, which gives the expression (∗).

Now substituting to (∗) some special values of R_1 and α, we get the following bounds:

LEMMA 2.2.3. — *Let $f \in B^2_{N,R,c}$. Put $R_1 = \dfrac{1}{4} R$, $R_2 = R/2\max(c,2)$ and $R_3 = R/2^{3N}\max(c,2)$. Then the number of zeroes of f in $\mathcal{D}_{R_1}$, $\mathcal{D}_{R_2}$ and $\mathcal{D}_{R_3}$ does not exceed $5N + \log_{5/4}(2+c)$, $5N+10$ and N, respectively.*

Proof. — For $R_1 = \dfrac{1}{4} R$ put $\alpha = \dfrac{1}{2}$. Then $\gamma = \dfrac{1}{2}$ and we get (remember that the logarithms are with the basis 2):

$$\begin{aligned}\#\{f = 0 \text{ on } \mathcal{D}_{R_1}\} &\leqslant N\left(1 + \frac{1}{N} \log(1+1+c)\right)/\log\frac{5}{4} \\ &= \frac{1}{\log 5/4} N + \log_{5/4}(2+c) \leqslant 5N + \log_{5/4}(2+c).\end{aligned}$$

For $R_2 = R/2\ \max(c,2)$ we still put $\alpha = 1/2$. Hence $\gamma = R_2/\alpha R = \dfrac{1}{2\alpha\ \max(c,2)} = \dfrac{1}{\max(c,2)}$. We get for the number of zeroes of f on $\mathcal{D}_{R_2}$ the bound

$$N\Big(1 + \frac{1}{N}\ \log(2+2)\Big)/\log\ 5/4 \leqslant 5N + 10.$$

Finally, let $R_3 = R/2^{3N}\max(c,2)$. We put $\alpha = 1/2^{3N}$ and hence $\gamma = 1/\max(c,2)$. For the number of zeroes of f on $\mathcal{D}_{R_3}$ we get the bound

$$\begin{aligned} & N\Big(3N + \frac{1}{N}\ \log(1+1+2)\Big)/(3N-1) \\ & = N\Big(1 + \frac{2}{3N^2}\Big)/\Big(1 - \frac{1}{3N}\Big) \leqslant N\Big(1 + \frac{2}{3N^2} + \frac{2}{3N}\Big) \\ & \leqslant N + \frac{2}{3N} + \frac{2}{3} < N + 1 \ \ (\text{assuming } \ N > 2). \end{aligned}$$

But the number of zeroes is an integer, and hence it does not exceed N. Lemma 2.2.3 is proved.

2.3. Local valency and Bernstein classes.

The following classical result can be found in [4], (see also [29]):

THEOREM A. — *Suppose that* $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *is p-valent in* $\mathcal{D}_R$, *i.e. that* $f(z)$ *is regular in* $\mathcal{D}_R$ *and assumes no value more than p times there. Then for* $j > p$

$$(**) \qquad |a_j|R^j \leqslant (A/p)^{2p} j^{2p} \max_{i=1,\cdots,p} |a_i|R^i,$$

with A - an absolute constant.

Clearly the inequality $(**)$ is very similar to the definition of B^2. (One can define a finer scale of classes B^2 using this inequality.) However, chosing any $R' < R$, we get:

COROLLARY 2.3.1. — *Let* f *be p-valent in* $\mathcal{D}_R$. *Then for any* $R' < R$ *the derivative* f' *belongs to* $B^2_{p-1,R',c}$ *with* $c = (A/p)^{2p} \max_{j \geqslant p+1} (R'/R)^{j-p} j^{2p+1} \leqslant (A/p)^{2p} \cdot [(2p+1)/e]^{2p+1}(R/R')^p[1/\ell n(R/R')]^{2p+1}$.

Proof. — By $(**)$,

$$|ja_j|R^{j-1} \leqslant (A/p)^{2p} j^{2p+1} \max_{i=1,\cdots,p} |ia_i|R^{i-1}.$$

Therefore for $R' < R$ we get

$$|ja_j|(R')^{j-1}(R/R')^{j-1} \leqslant (A/p)^{2p}j^{2p+1}(R/R')^{p-1} \cdot \max_{i=1,\cdots,p} |ia_i|(R')^{i-1}, \text{ or}$$

$$|ja_j|(R')^{j-1} \leqslant (R'/R)^{j-p}(A/p)^{2p}j^{2p+1} \cdot \max_{i=1,\cdots,p} |ia_i|(R')^{i-1}.$$

Now an easy computation of the $\max\limits_{j\geqslant p+1}(R'/R)^{j-p}j^{2p+1}$ completes the proof.

Clearly, if the derivative f' belongs to $B^2_{N-1,R,c}$, then $f+\alpha \in B^2_{N,R,c}$ for any constant α. Conversely. if $f + \alpha \in B^2_{N,R,c}$ for any α, then $|a_j|R^j \leqslant c \max\limits_{i=1,\cdots,p} |a_i|R^i$, and hence the derivative f' belongs to B^2_{N-1} on any smaller disk. (This statement can be easily generalized: $f^{(s)} \in B^2$ if and only if $f + P_{s-1} \in B^2$ for any polynomial P_{s-1} of degree $s-1$. Easy examples show that $f \in B^2$ does not imply any conclusion about f'.) Now combining these remarks with Lemma 2.2.3 above we obtain the following result:

THEOREM 2.3.2. — *If f is p-valent on $\mathcal{D}_R$, then f' belongs to $B^2_{p-1,R',c}$ on any smaller disk $\mathcal{D}_{R'}$, with c as given in corollary 2.3.1. If $f' \in B^2_{p-1,R,c}$, then f is p-valent on $\mathcal{D}_{R_3}$, with R_3 as defined in Lemma 2.2.3.*

In the rest of this paper the usual approach will be to prove that a certain function belongs to the Bernstein classes, in order to bound the number of its zeroes. However, for an important class of algebraic functions the number of zeroes is bounded by Bezout theorem, and the equivalence above can be used in an opposite direction. This will be done in the next section.

3. BERNSTEIN TYPE INEQUALITIES FOR ALGEBRAIC FUNCTIONS

A problem of computing Bernstein constant of algebraic functions has recently appeared in several quite different situations.

In [16], [19] this problem is investigated in relation with estimates of a symbol of some pseudodifferential operators. In [5] this problem is connected with some results in Potential Theory and with a characterization of algebraic subsets. In [50], [51] and [52] various forms of Bernstein inequality are used to prove results on a "C^k-reparametrization" of semialgebraic sets, which, in turn allow one to control the complexity growth in iterations of smooth mappings. It was exactly the absence of the Bernstein inequality for algebraic functions, which restricted the results of [52] to one and two-dimensional dynamics only.

Let us consider first of all some special cases and examples of the problem, which will clarify a possible statement of a general result.

3.1. Some examples.

In general, by algebraic functions we shall understand restrictions of polynomials to algebraic subsets in $\mathbb{C}^n$, (see 3.2 and 3.3 below). For one variable an algebraic function $y = \psi(x)$ can be equivalently defined as a (multivalued) function, satisfying an equation $P(x, y) \equiv P(x, \psi(x)) \equiv 0$, with P - a polynomial in x and y. It is convenient to write this equation in a form

$$p_d(x)y^d + p_{d-1}(x)y^{d-1} + \cdots + p_1(x)y + p_0(x) = 0, \tag{3.1}$$

with $p_j(x)$ - polynomials in x of degree m.

For $d = 1$ and $p_1(x) \equiv 1$ we obtain polynomials $y = -p_0(x)$ of degree m. As it was shown below, the classical Bernstein inequalities imply that for any R and $\alpha < 1$, these polynomials belong to $B^1_{R,\alpha,(1/\alpha)^m}$ and to $B^2_{m,R,o}$.

If $p_1(x) \not\equiv 1$, we obtain rational functions $y = -p_0(x)/p_1(x)$ with the numerator and the denominator of the degree at most m.

Let us assume that $0 \in \mathbb{C}$ is not a pole of y and that the nearest pole is at the distance $R > 0$ from the origin. Then for any $R' < R$ and $\alpha < 1$ we can ask for the Bernstein constant of y on $(\mathcal{D}_{\alpha R'}) \subseteq \mathcal{D}_{R'})$.

PROPOSITION 3.1.1. — *$y = p(x)/q(x)$, with* deg p, deg $q \leqslant m$, *and the nearest pole at the distance $R > 0$ from the origin, for any $R' < R$ and $\alpha < 1$ belongs to $B^1_{R',\alpha,K}$ with $K = \left[\dfrac{R+R'}{\alpha(R-R')}\right]^m$.*

Proof. — We can assume that $q(x) = \prod_{i=1}^{m}(x - \xi_i)$, with $|\xi_i| \geqslant R$ for $i = 1, \cdots, m$. Hence for any $i = 1, \cdots, m$, $\max_{\mathcal{D}_{R'}} |x - \xi_i| / \min_{\mathcal{D}_{R'}} |x - \xi_i| \leqslant \dfrac{R+R'}{R-R'}$ and hence $\max_{\mathcal{D}_{R'}} |q(x)| / \min_{\mathcal{D}_{R'}} |q(x)| \leqslant \left(\dfrac{R+R'}{R-R'}\right)^m$. Since for any

$\alpha < 1$, $p(x) \in B^1_{R',\alpha,(1/\alpha)^m}$, we get $y = \dfrac{p(x)}{q(x)} \in B^1_{R',\alpha,K}$ with $K = \left[\dfrac{R+R'}{\alpha(R-R')}\right]^m$.

By equivalence of the Bernstein classes B^1 and B^2 we can now find the parameters of the class B^2, to which $y = \dfrac{p(x)}{q(x)}$ belongs, but a general result in Section 3.3 below gives a sharper bound.

Notice, that the constant in the Bernstein inequality for rational functions, obtained in Proposition 3.1.1, depends only on the degree m and on the relative position of the two disks $\mathcal{D}_{R'}$ and $\mathcal{D}_{\alpha R'}$ in the maximal disk of regularity $\mathcal{D}_R$.

Now let us return to an algebraic function, given by equation (3.1) and rewrite it as

$$y^d + R_{d-1}(x)y^{d-1} + \cdots + R_1(x)y + R_0(x) = 0, \tag{3.2}$$

where $R_i(x) = p_i(x)/p_d(x)$ is a rational function of degree m, $i = d-1, \cdots, 0$. Assume that the closest to the origin pole of $R_i(x)$ is at the distance $R > 0$ from the origin.

Proposition 3.1.2. — *Let $R' < R$ and $\alpha < 1$ be given. If each of the branches $y_j(x)$ of the function $y(x)$ satisfies over $\mathcal{D}_{\alpha R'}$ an inequality $|y_j(x)| \leqslant M$, $j = 1, \cdots, d$ then $|y_j(x)| \leqslant CM$ over $\mathcal{D}_{R'}$, $j = 1, \cdots, d$, with C depending only on the degrees d and m and on R' and α.*

Proof. — By Vieta formula we conclude that $R_i(x)$ are bounded on $\mathcal{D}_{\alpha R'}$ in terms of M and d. Applying Proposition 3.1.2 we get a bound on R_i over $\mathcal{D}_{R'}$ which provides the required bound for the roots y_j.

Notice, that in this proposition y may have branching points inside the disks $\mathcal{D}_{R'}$ or $\mathcal{D}_{\alpha R'}$.

The results of Propositions 3.1.1 and 3.1.2 determine the form of the inequality we would like to have: (below we call such inequalities "structural ones").

a) It must concern one branch of an algebraic function, without assuming any information about the behaviour of other branches.

b) It must be valid in a maximal disk of regularity of this branch.

c) The constant in this inequality must depend only on the degree of the algebraic function considered, and on the relative position of the couple of domains inside the disk of regularity.

We prove such an inequality in Section 3.3 below. Let us now give an example, showing that in the case of one branch the Bernstein constant can blow up, as we approach a ramification point.

Example 3.1.3. — Consider a family of algebraic functions $y_{\lambda,R}(x)$, defined by a quadratic equation

$$y^2 + 2\lambda(x - R)y - \lambda = 0. \tag{3.3}$$

We have $y_{\lambda,R} = \lambda(R - x) \pm \sqrt{\lambda^2(x - R)^2 + \lambda}$. Let us fix $R > 0$ and consider the branch y_1 of $y_{\lambda,R}$, defined by

$$y_1 = \lambda(R - x)(1 - \sqrt{1 + 1/\lambda(x - R)^2}),$$

with the square root taking the value 1 at 1. y_1 has the following properties:

a) It is regular on $\mathcal{D}_R$ for $\lambda > 0$. Indeed the ramification points of y are given by $\lambda^2(x-R)^2+\lambda = 0$, or $x-R = \sqrt{-1/\lambda}$, or $x_{1,2} = R \pm i\sqrt{1/\lambda}$. Notice that these ramification points approach the boundary of $\mathcal{D}_R$ as $\lambda \to \infty$.

b) For any x inside $\mathcal{D}_R$, $y_1(x)$ tends to $1/2(x-R)$ as $\lambda \to \infty$. It follows from $y_1 = \lambda(R - x)(1 - 1 - 1/2\lambda(x - R)^2 + o(1/\lambda))$.

In particular, y_1 remains uniformly bounded by $1/(R - R')$ on any disk $\mathcal{D}_{R'}$ with $R' < R$, but for λ big enough it can take arbitrarily big values as $x \to R$.

Remark 1. — The second branch y_2 of y, given by $y_2 = \lambda(R-x)(1 + \sqrt{1 + 1/\lambda(x - R)^2})$, tends to ∞ for any x in $\mathcal{D}_R$. Otherwise Proposition 3.1.2 would imply a bound on the Bernstein constant on any disk, since the coefficients of (3.3) have no poles.

Remark 2. — The equation (3.3) could be written in a form $\varepsilon y^2 + 2(x - R)y - 1 = 0$, representing y as a family of second degree algebraic functions, degenerating into $y = 1/2(x - R)$ for $\varepsilon = 0$.

Remark 3. — Replacing $x-R$ by $(x-R)^m$ in (3.3), we obtain a family, tending to $1/2(x - R)^m$ and hence the growth of the Bernstein constant on $\mathcal{D}_{R'}$ of order $1/(R - R')^m$. Compare with Lemma 3.1.1 and Theorem 3.3.1 below.

3.2. Results of Fefferman-Narasimhan and Bos-Levenberg-Milman-Taylor.

Recently Bernstein and Markov-type inequalities for algebraic functions have been investigated by C. Fefferman and R. Narasimhan in [16], [17], [18], [19], and by L. Bos, N. Levenberg, P. Milman and B.A. Taylor in [5]. (See also [6], [46], [9] [10].) In this section we state and discuss these results, and in the next one we prove, in particular, their refined versions.

Let $P_1, \cdots, P_r$, $1 \leqslant r \leqslant n-1$, be polynomials on $\mathbb{R}^n$ of degree at most $\mathcal{D}$. We assume that the coefficients of these polynomials have absolute value at most C.

Let an algebraic variety $V \subseteq \mathbb{R}^n$ be defined by $P_1 = P_2 \cdots = P_r = 0$. We assume that $0 \in V$ and that V is regular at $0 \in \mathbb{R}^n$. Moreover, we assume that $\det \left(\frac{\partial P_j}{\partial x_k}(0) \right)_{1 \leqslant j,k \leqslant r}$ is in absolute value not smaller than $c > 0$.

Let $\pi : V \to \mathbb{R}^{n-r}$ be the projection $(x_1, \cdots, x_n) \to (x_{r+1}, \cdots, x_n)$. From the assumption it follows that π has a smooth local inverse $\pi^{-1} : B(0, \delta_*) \to V$, defined on a ball $B(0, \delta_*)$ of a sufficiently small radius δ_*.

The following result is obtained in [17]:

THEOREM B (see [17], Theorem 2). — *Let V, π, n, c, C, $\mathcal{D}$ be as above. There exist constants δ_*, C_*, depending only on n, c, C, $\mathcal{D}$, such that π has a smooth inverse $\pi^{-1} : B(0, \delta_*) \to V$ and such that if P is a polynomial of degree $\leqslant \mathcal{D}$ and $F = P \circ \pi^{-1}$ on $B(0, \delta_*)$, then the following inequality holds: for any $\delta, 0 < 2\delta < \delta_*$,*

$$\sup_{B(0,2\delta)} |F| \leqslant C_* \sup_{B(0,\delta)} |F|. \tag{3.4}$$

The important point here is that C_* does not depend on δ and on a specific choice of the polynomial P. In applications, which are assumed in [16], [17] (see [17], an introduction), the constants C and c (clearly, by a normalization, only a ratio C^r/c can be considered) can be assumed to be known.

However, in applications to the complexity growth in dynamical systems ([50], [52]) no information on C^r/c is available. Fortunately, one can show that in fact (3.4) is true in a maximal ball $B(0, R)$ of regularity of $P \circ \pi^{-1}$, with C_* depending only on n, $\mathcal{D}$ and δ/R.

Remark 1. — Bernstein inequality for P/V is obtained in [17] as a consequence of the main extension theorem ([17], Theorem 1), which allows one to extend P/V in a controllable way to the whole ball (as a rational function). It seems to be an important question, whether such an extension can be controlled by the degrees of the involved polynomials only.

Remark 2. — Clearly δ_* above gives a lower bound for the radius of the maximal ball of regularity of $P \circ \pi^{-1}$ in terms of the degree and C^r/c. The following example shows that for a given C^r/c the actual radius of the maximal ball of regularity can be arbitrarily big.

Let $V \subseteq \mathbb{C}^2$ be given by $P_1(x,y) = y^2 + 2c(1-x)y + c = 0$, $P(x,y) = y$. The ramification points of the algebraic function $y = -c(1-x) \pm \sqrt{c^2(1-x)^2 - c}$ are given by $c^2(1-x)^2 - c = 0$ or $x = 1 \pm \sqrt{1/c}$. Thus the regularity domain of each branch of y grows to infinity as $c \to 0$. Notice that here $C = 1$ and $\dfrac{\partial P_1}{\partial y}(0,0) = c$. Of course, the reason for c not to reflect the distance to the nearest critical point is that two branches of y approach one another as $c \to 0$.

An accurate representation of the distance to the nearest singularity for algebraic functions, and, more generally, for solutions of algebraic differential equations, seems to be very important. Some results in this direction are given in [8].

Remark 3. — The results of [16] and [17] are stated for a real case. However, it is mentioned there, that the extension to a complex case can be easily obtained. Compare also Proposition 2.1.5 above.

Let us turn now to the results of [5]. Let K be a smooth compact m-dimensional submanifold of $\mathbb{R}^n$ without boundary. The following result is obtained in [5]:

THEOREM C ([5], Main Theorem, parts 2, 3).

1. *K is algebraic if and only if it satisfies a local tangential Markov inequality with exponent one:*

$$|\mathcal{D}_T p(x)| \leqslant M(\deg p/\varepsilon) \max_{K \cap B(x,\varepsilon)} |p|, \tag{3.5}$$

for any $x \in K$ and all polynomials p. Here $\mathcal{D}_T$ denotes any tangential derivative along K and M is a constant, depending only on K.

2. *The same is equivalent to the following global tangential Markov inequality with exponent one:*

$$|\mathcal{D}_T p(x)| \leqslant M'(\deg p) \max_K |p|, \tag{3.6}$$

for any polynomial p and any $x \in K$, with M' depending only on K.

The character of dependence of M and M' on K (for K algebraic) is not specified in the proof of this result. What is important, is the linear dependence on the degree of p.

In this setting, as one can easily show, $M(M')$ depends not only on the degree of K but also on its geometry. The following examples demonstrate this fact:

Example 3.2.1. — $K = \{x^2+y^2 = \varepsilon^2\}$, $p = (1/\varepsilon)x$. Then $\max_K |p| = 1$, while the tangential derivative at the points $(0, \pm\varepsilon)$ is $1/\varepsilon$.

Example 3.2.2. — K is the curve $y = \varepsilon^3/(x^2+\varepsilon^2)$, $p = (1/\varepsilon)y$. Then $\max_K |p| = 1$, but the tangential derivative of p on K is at least $\frac{1}{\varepsilon} \mid \frac{dy}{dx} \mid = \frac{1}{\varepsilon} \frac{2\varepsilon^3 x}{(x^2+\varepsilon^2)^2}$, and for $x = \pm\varepsilon$ we get $1/2\varepsilon$.

In the next sections we shall show that in fact K depends only on the degree and on the "distance to complex singularities", which, in examples above are at the points $\pm\varepsilon$ and $\pm i\varepsilon$, respectively.

In [5] it is shown that a tangential Markov inequality of exponent 1 is implied by the Bernstein inequality, with the constant, depending exponentially on the degree of p. Accordingly, in Sections 3.3 - 3.4 we will restrict ourselves to Bernstein inequalities only, but we shall analyze the dependence of the constant on various parameters involved, including the degree of p.

3.3. Structural Bernstein inequality for algebraic function.

By a *structural* Bernstein inequality for a certain class of functions, defined by algebraic data (algebraic functions, solutions of algebraic differential equations, etc.) we understand an inequality bounding the Bernstein constant of the function on a couple of concentric disks in terms of the degree and the relative position of these concentric disks in the maximal concentric disk of regularity only.

As the example of a rational function (Proposition 3.1.1 above) shows, in a sense, this is the best possible inequality one can expect.

Structural Bernstein inequalities exist only for specific families of "algebraically defined" analytic functions. For example, the family $y_\lambda = e^{\lambda x}$ of the solutions of an algebraic differential equation

$$y' = \lambda y$$

possesses no such an inequality. Indeed, for any $R > 0$, $1 > \alpha > 0$, $B(e^{\lambda x}, \mathcal{D}_{\alpha R}, \mathcal{D}_R) = e^{(1-\alpha)\lambda R}$ depends on λ and blows up as λ tends to ∞.

On the other hand, the family of the solutions of

$$y' = \lambda y^2,$$

given by $y_{\lambda, y_0} = \dfrac{y_0}{1 - y_0 \lambda x}$ gives an example of two-parametric family of rational functions of degree 1, satisfying a structural Bernstein inequality.

We consider a problem of characterizing algebraic families with a structural Bernstein inequality as a very important one. In this section we prove that algebraic functions of a given degree form such a family. In [8] some results for solutions of algebraic differential equations are given. However, this problem seems to require a further serious investigation.

Let $y(x)$ be an algebraic function, given by an equation (3.1):

$$p_d(x)y^d + p_{d-1}(x)y^{d-1} + \cdots + p_1(x)y + p_0(x) = 0,$$

with $p_j(x)$ - polynomials in x of degree m. Let $\tilde{y}(x)$ be one of the branches of y and assume that $\tilde{y}$ is regular over $\mathcal{D}_R$. (We can assume that $\mathcal{D}_R$ is a maximal disk of regularity of $\tilde{y}$, so its boundary contains poles or branching points of $\tilde{y}$).

THEOREM 3.3.1. — *For any $R' < R$ $\tilde{y} \in B^2_{m,R',c}$, with $c = C(m) \cdot (R/R')^m [1/\ell n(R/R')]^{2m+1}$, $C(m) = (A/m)^{2m}[(2m+1)/e]^{2m+1}$.*

Proof. — First of all, $\tilde{y}$ is m-valent in $\mathcal{D}_R$. Indeed, for any $\alpha \in \mathbb{C}$, the equation $\tilde{y} = \alpha$ is equivalent to $p_d(x)\alpha^d + \cdots + p_0(x) = 0$, and the last equation is a polynomial one of degree m.

Now by Corollary 2.3.1, for any $R' < R$, the derivative f', and hence f itself, belongs to $B^2_{m,R',c}$, with

$$\begin{aligned} c &= (A/m)^{2m} \cdot [(2m+1)/e]^{2m+1} (R/R')^m [1/\ell n(R/R')]^{2m+1} \\ &= C(m) \cdot (R/R')^m [1/\ell n(R/R')]^{2m+1}. \end{aligned}$$

COROLLARY 3.3.2. — *For any* $R'' < R$, $\alpha < 1$, $\tilde{y} \in B^1_{R'',\alpha,K}$ *with* $K = \dfrac{1}{\alpha^m}\left(1 + \dfrac{\alpha(1-\alpha^m)}{1-\alpha} + \dfrac{c\beta}{1-\beta}\right)$, *where* $\beta = \dfrac{2R''}{R+R''}$ *and* $c = C(m)\gamma^m[1/\ell n\gamma]^{2m+1}$, $\gamma = \dfrac{2R}{R+R''}$.

Proof. — We put $R' = \dfrac{R+R''}{2}$, apply Theorem 3.3.1 to get $\tilde{y} \in B^2_{m,R',c}$ and then apply Theorem 2.1.3 to get $\tilde{y} \in B^1_{R'',\alpha,K}$.

Now let us fix certain $\alpha < 1$ and let $R'' \to R$. Writing $\gamma = 1 + \delta$, with $\delta = \dfrac{R-R''}{R+R''}$, we have $\ell n\ \gamma \sim \delta$ and hence $c \sim C(m)\left(\dfrac{R+R''}{R-R''}\right)^{2m+1}$. Since $\dfrac{\beta}{1-\beta} = \dfrac{2R''}{R+R''} \cdot \dfrac{R+R''}{R-R''} = \dfrac{2R''}{R-R''}$, we obtain

COROLLARY 3.3.3. — *For* R'' *approaching* R, $\tilde{y} \in B^1_{R'',\alpha,K}$ *with* $K \lesssim \dfrac{C(m)}{\alpha^m}\left(\dfrac{R+R''}{R-R''}\right)^{2m+2}$.

Notice that Proposition 3.1.1 and Example 3.1.3 above bound K from below by $\left[\dfrac{R+R''}{\alpha(R-R'')}\right]^m$. Thus an asymptotic behavior of the bound of Corollary 3.3.3 as $\alpha \to 0$, is a correct one. For a fixed α and $R'' \to R$, there is a gap between the powers m and $2m+1$ of $\dfrac{R+R''}{R-R''}$ in the lower and upper bounds.

As well as the asymptotic behaviour for $m \to \infty$ is concerned, the above expressions give $K \sim C^m$, with C depending on R''/R, which is a correct bound, by the example 3.1.3.

3.4. Multidimensional structural Bernstein inequality for algebraic functions.

Let $V \subseteq \mathbb{C}^n$ be an algebraic subset. Assume that $0 \in V$, and that V is regular of dimension ℓ near the origin and that the algebraic mapping $\pi : (V, 0) \to (\mathbb{C}^\ell, 0)$ is given, which is locally invertible. Let us stress that no specific representation of V as a set of zeroes of certain polynomials is considered. The same concerns also the mapping π.

Let $B(0,R) \subseteq \mathbb{C}^\ell$ be the maximal ball of regularity of π^{-1}. For each polynomial P on $\mathbb{C}^n$ of degree N, $f = P \circ \pi^{-1}$ is a regular algebraic function on $B(0,R)$.

THEOREM 3.4.1. — *For any $R' < R$ and $\alpha < 1$,*

$$\begin{aligned}\tilde{B}(f,R',\alpha) &= \max_{B(0,R')}|f|/\max_{B(0,\alpha R')}|f| \\ &\leqslant \frac{1}{\alpha^m}\left(1+\frac{\alpha(1-\alpha^m)}{1-\alpha}+\frac{c\beta}{1-\beta}\right), \quad \text{where } \beta=\frac{2R'}{R+R'},\end{aligned}$$

$m = C(\deg V,\ \deg \pi)\cdot N$, and c is given in Corollary 3.3.2.

Proof. — Let $\max_{B(0,R')}|f|$ be attained at a certain point $z \in B(0,R')$. Let $L = \{x\cdot z,\ x\in\mathbb{C}\}$ be a complex line, passing through the point z and through the origin. The restriction $\tilde{f}(x) = f/L$ is an algebraic function of one variable. Instead of explicitly representing $\tilde{f}$ by an equation of the form (3.1) we notice that by a multidimensional Bezout theorem, the number of solutions of $\tilde{f} = \alpha$, for any $\alpha\in\mathbb{C}$, does not exceed the degree of an algebraic mapping $P\circ\pi^{-1}$, which, in turn has the form $C(\deg V,\ \deg\pi)\cdot N$, where the constant C depends only on the degrees of V and π. Now an application of Corollary 3.3.2 provides the required bound, since $\max_{\mathcal{D}_{R'}}|\tilde{f}| = \max_{B(0,R')}|f|$ by the choice of the line L, and $\max_{B(0,\alpha R')}|f| \geqslant \max_{\mathcal{D}_{\alpha R'}}|\tilde{f}|$.

Remark 1. — Clearly one can replace in the statement of Theorem 3.4.1 the maximal ball of regularity of π^{-1}, $B(0,R)$, by the maximal symmetric starlike domain of regularity $S(0)$. (This domain intersects each complex line, passing through the origin in a disk). The balls $B(0,R')$ and $B(0,\alpha R')$ are replaced by the corresponding concentric starlike domains.

Remark 2. — The discussion of the sharpness of the constants, concluding Section 3.3, is applicable also to the inequality in Theorem 3.4.1. Notice, however, that these constants are given explicitly, which answers, in particular, a question in [17].

Now we pass on to the real version of the above inequality. Assume the V, π, P are now a real algebraic set, a real mapping and a real polynomial on $\mathbb{R}^\ell$, respectively. In fact, we assume that all these objects are given by polynomials with real coefficients, so, their complexifications are naturally defined.

Define $R>0$ as follows: R is a supremum of all $\tilde{R}>0$, such that π^{-1} is regular on a real ball $B^R(0,\tilde{R})$ and that for any $z\in B(0,\tilde{R})$ the restriction $\pi^{-1}/\{\gamma z, \gamma\in\mathbb{C}\}$ is regular on the disk $\{\gamma x, |\gamma|\leqslant 1\}$.

Let P be a real polynomial on $\mathbb{R}^\ell$ of degree N and let $f = P\circ\pi^{-1}$.

THEOREM 3.4.2. — *For any $R' < R$, $1 > \alpha > 0$,*

$$\max_{B^R(0,R')} |f| / \max_{B^R(0,\alpha R')} |f| \leqslant K(\alpha, R'/R, m),$$

where $m = C(\deg V, \deg \pi) \cdot N$, and K is given in Theorem 3.4.1 and Proposition 2.1.5.

Proof. — Let $z \in B^R(0,R')$ be the point, where the $\max_{B^R(0,R')} |f|$ is attained. We restrict f to the complex line $L = \{\gamma z, \gamma \in \mathbb{C}\}$. For $\gamma \in \mathbb{R}$ γz belongs to $\mathbb{R}^\ell$, and we can use Proposition 2.1.5 instead of Corollary 3.3.2.

Remark 1. — In the same way one can compare the maxima of $|f|$ on a *complex* ball $B(0,R')$ and the *real* smaller ball $B^R(0,\alpha R')$. The only problem is that for $z \notin \mathbb{R}^\ell$ the complex line $\{\alpha z, \alpha \in \mathbb{C}\}$, usually intersects $\mathbb{R}^\ell$ only at zero. However, we can first bound $|f|$ on the real ball $B^R(0,R')$ and then to use restrictions of f to the complex lines of the form $(\mathrm{Re} z + \alpha \mathrm{Im}\, z, \alpha \in \mathbb{C})$. Such lines contain real ones, corresponding to $\alpha \in \mathbb{R}$, and Proposition 2.1.5 can be applied.

Remark 2. — According to the results above, the constant in a global Markov inequality on a compact algebraic manifold $K \subseteq \mathbb{R}^n$ of dimension ℓ, depends on the following geometric invariant of K, which may present an independent interest:

Consider the coverings Ω of K by neighbourhoods U_j together with the linear projections $\pi_j : K \to \mathbb{R}^\ell$, such that π_j is regular and invertible on U_j. For each π_j define $R_j > 0$ as in Theorem 3.4.2 above. Finally define $R(\Omega)$ as min R_j and $R(K)$ as the maximum of $R(\Omega)$ over all the coverings as above. $R(K)$ plays the role of the distance to the nearest singularity. In Examples 3.2.1 and 3.2.2 above, $R(K) = \varepsilon$.

3.5. Some alternative approaches to algebraic functions.

The method, used in Sections 3.3 - 3.4 is very specifically tailored for algebraic functions. We use Bezout theorem to bound the valency and then use the classical results relating it to the Bernstein constants.

For wider classes of functions, in particular, those arising in connection to algebraic differential equations, the number of zeroes is usually the quantity under the question.

We consider algebraic function as a (relatively) simple model for the general problems of the above type. Hence it may be important to investigate alternative approaches to bounding the Bernstein constant of algebraic functions, not relying on Bezout bounds.

Several such approaches have been investigated. In this section we present shortly the main results obtained.

Let us start with the structural Bernstein inequality (i.e. the one with the constant depending only on the degree and the relative geometry of the domains in a maximal disk of regularity).

The original proof of the existence of such an inequality for algebraic functions has been obtained in [45]. The idea of this proof was suggested to us by M. Gromov and it goes as follows: assuming that the required bound does not exist, we construct a sequence of algebraic functions of a given degree, regular on $\mathcal{D}_1$, converging to zero on $\mathcal{D}_{\frac{1}{2}}$ and with a growing to infinity Bernstein constant. Using the compactness of the space of *projective* algebraic curves of a given degree, we find a subsequence, converging to a certain projective algebraic curve, which contains $\{y = 0\}$ as a component. Now analyzing the geometry of the converging sequence, we find that it must be uniformly bounded on $\mathcal{D}_1$. (See [45] and [55] for a detailed proof and some generalizations.)

We hope that this compactness argument can be applied also in other situations related to algebraic differential equations.

Structural Bernstein inequality for some special classes of algebraic functions has been obtained in [41], [43], using a certain general inequality between Wronskians of a system of functions. This inequality allows one to analyze also some more general classes of functions, in particular, exponential quasipolynomials (see [43]).

Yet another approach is based on the analysis of the ideals, generated by the Taylor coefficients of the considered algebraic function (these coefficients are polynomials in the parameters of the problem). This approach is developed in [21], [7], [8]. In a somewhat more detailed form it is discussed in Section 6 below.

Now, if we allow the constant in the Bernstein inequality to depend on the "size" of the equation (C/c in Theorem B above), algebraic functions can be considered as a special subclass of either one of the following wide classes of functions:

1. those, satisfying "Lipschitzian" differential equations (it is known, that any algebraic function satisfies a linear homogeneous differential equation with polynomial coefficients. See e.g. [14], [15]).

2. those, whose Taylor coefficients satisfy "Lipschitzian" recurrency relations. (See [7]).

These two classes (which contain also solutions of algebraic differential equations) are considered in Sections 4 and 5 below.

As it was mentioned above, another class of functions, for which the Bernstein constant can be computed, consists of those, whose Taylor coefficients are polynomials in the parameters (with certain restrictions on the growth of the degrees and of the norms). This class is studied in [21], [7] and [8] and some results are shortly discussed in Section 6 below. All the functions, arising in connection with the algebraic differential equations, belong naturally also to this third class.

4. BERNSTEIN CONSTANT FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS

Consider an r-th order ordinary differential equation

$$y^{(r)} = F(x, y, y', \cdots, y^{(r-1)}), \tag{4.1}$$

with F an analytic function, defined on a certain domain U of $\mathbb{C}^{r+1} = \mathbb{C} \times \mathbb{C}^r$, containing the origin.

We assume that $F(x, 0, 0, \cdots, 0) \equiv 0$ for any x.

Let for some $R > 0$, $\rho > 0$, $\mathcal{D}_R \times B(0,\rho) \subseteq U$. Then for a certain $K > 0$, for any $x \in \mathcal{D}_R$ and $w = (w_0, \cdots, w_{r-1}) \in B(0,\rho)$, $|F(x, w_0, \cdots, w_{r-1})| \leqslant K\|w\|$. The norm $\|\ \|$ on $\mathbb{C}^r$ is defined as $\|w\| = \max(|w_0|, \cdots, |w_{r-1}|)$.

THEOREM 4.1. — *Let $\rho_1 = \rho e^{-KR}$. Then for any $w = (w_0, \cdots, w_{r-1}) \in B(0,\rho_1) \subseteq \mathbb{C}^r$, the solution $y_w(x)$ of (4.1), satisfying $y_w^{(i)}(0) = w_i$, $i = 0, \cdots, r-1$, exists and is regular for $x \in \mathcal{D}_R$, and $y_w \in B^2_{r-1,R,c}$, with $c = e^{KR}(r-1)![\max(1, 1/R)]^{r-1}$.*

Proof. — Transform (4.1) into a first order system

$$Z' = \tilde{F}(x, Z), \quad Z \in \mathbb{C}^r, \tag{4.2}$$

by substituting $z_0 = y, \cdots, z_{r-1} = y^{(r-1)}$. By assumptions (and because of the choice of the norm) we have

$$\|\tilde{F}(x, Z)\| \leqslant K\|Z\|, \tag{4.3}$$

for $x \in \mathcal{D}_R$, $Z \in B(0,\rho)$. By the standard differential inequality argument, any solution $Z(x)$ of (4.2) satisfies

$$\|Z(x)\| \leqslant \|Z(0)\| e^{K|x|}, \tag{4.4}$$

until $|x| \leqslant R$ and the solution remains in $B(0,\rho)$. In particular, for $\rho_1 = \rho e^{-KR}$, any solution with $Z(0) \in B(0,\rho_1)$ is defined for $x \in \mathcal{D}_R$, remains in $B(0,\rho)$ and satisfies (4.4).

In particular, it follows, that the function y itself, being the first component of Z, is bounded by $\|Z(0)\|e^{KR}$ for $x \in \mathcal{D}_R$. By the Cauchy formula we get for any $j \geqslant r$

$$|a_j|R^j \leqslant e^{KR}\|Z(0)\| = e^{KR}\max(|a_0|, |a_1|, 2!|a_2|, \cdots, (r-1)!|a_{r-1}|).$$

Hence $|a_j|R^j \leqslant c \max\limits_{i=0,\cdots,r-1} |a_i|R^i$, with $c = e^{KR}(r-1)![\max(1, 1/R)]^{r-1}$.

Remark 1. — The bound, given by Theorem 4.1, is essentially sharp. As far as the index $r-1$ is concerned, the polynomials of degree $r-1$ satisfy $y^{(r)} = 0$, and they belong to B^2_{r-1}, but not to B^2_ℓ for $\ell < r-1$. Notice that for the first order equations their solutions belong to B^2_0, and indeed, the corresponding differential inequality bounds the values of $|y(x)|$ on $\mathcal{D}_R$ through the value $|y(0)|$ only.

Remark 2. — An approach, used in the proof of theorem 4.1 in the case of a scalar differential equation, does not allow for a straightforward generalization to systems. Indeed, for a general system $Z' = \Phi(x, Z)$ each component can be bounded on a bigger disk through the bound on *all the components* of $Z(0)$, but not through its own values.

However, the Bernstein constant of each component of the solution of a system of algebraic differential equations can be effectively bounded by a different method (see [21], [54], [24]).

In a special case of a linear homogeneous differential equation

$$y^{(r)} + p_{r-1}(x)y^{(r-1)} + \cdots + p_1(x)y = 0 \tag{4.5}$$

with the coefficients, analytic in $\Omega \subseteq \mathbb{C}$, for any $R > 0$ with $\mathcal{D}_R \subset \Omega$ the right hand side is defined over $\mathcal{D}_R \times \mathbb{C}^r$, and the Lipschitz constant K does not exceed $K(r) = \max\limits_{x \in \mathcal{D}_R} (|p_1(x)| + \cdots + |p_{r-1}(x)|)$. We obtain hence the following corollary:

Corollary 4.2. — *Solutions of (4.5) for any $R > 0$ with $\mathcal{D}_R \subset \Omega$ belong to $B^2_{r-1,R,c}$, with $c = e^{R\cdot \max_{x\in\mathcal{D}_R}(|p_i|+\cdots|p_{r-1}|)}\cdot(r-1)![\max(1,1/R)]^{r-1}$.*

This bound can be written in a more precise form for p_j - entire functions of the prescribed order of growth. In particular, for p_j - polynomials of degree $\leqslant m$, $K(r) = \max\limits_{x\in\mathcal{D}_R}(|p_1|+\cdots+|p_{r-1}|) \leqslant C(r,m) \max\limits_{j=1,\cdots,r-1} \|p_j\|\cdot R^m$, and we get

Corollary 4.3. — *If the coefficients of (4.5) are polynomials in x of degree m, then any solution of (4.5) for any $R > 0$ belongs to $B^2_{r-1,R,C(R)}$, with $C(R) = C_1 e^{C_2 R^{d+1}}$, with C_1, C_2 depending only on r and m.*

Via the equivalence of the classes B^2 and B^1, and bounds on the number of zeroes of functions in Bernstein classes, given in Section 2, one can produce bounds on zeroes distribution of solutions of linear equations (4.5). These types of bounds have been obtained by different methods in a general theory of distribution of zeroes in analytic differential equations (see [29], [34]).

Consider now the case of a homogeneous polynomial r-th order differential equation

$$y^{(r)} = P(x, y, y', \cdots, y^{(r-1)}), \tag{4.6}$$

with P - a polynomial in x and $(y, y', \cdots, y^{(r-1)})$ of the degrees m and d, respectively. (In other words, $P(x,w) = \sum\limits_{0<|\alpha|\leqslant d} p_\alpha(x) w^\alpha$, where $\alpha = (\alpha_0, \cdots, \alpha_{r-1})$ is a multi-index and $p_\alpha(x) = \sum\limits_{i=0}^{m} a_{\alpha i} x^i$. We define the norm of P as $\|P\| = \sum\limits_{\alpha,i} |a_{\alpha i}|$.)

Then for $x \in \mathcal{D}_R$ and $w \in B(0,\rho)$ we have $|P(x,w)| \leqslant \|P\|(1+R)^m (1+\rho)^{d-1}\|w\|$.

Substituting this value of K into the expression of theorem 4.1 we get the following result:

Corollary 4.4. — *Let for a given $R > 0$ and $\rho > 0$, $\rho_1 = \rho e^{-KR}$, with $K = \|P\|(1+R)^m(1+\rho)^{d-1}$. Then for any $w = (w_0, \cdots, w_{r-1}) \in B(0,\rho_1) \subseteq \mathbb{C}^r$, the solution $y_w(x)$ of (4.6), satisfying $y_w^{(i)}(0) = w_i$, $i = 0, \cdots, r-1$, exists and is regular for $x \in \mathcal{D}_R$, and $y_w \in B^2_{r-1,R,c}$, with $c = e^{KR}(r-1)![\max(1,1/R)]^{r-1}$.*

Also here one can produce a corresponding bound on the number of zeroes of y_w. Notice also that instead of estimating the Lipschitz constant of $P(x,w)$, one can use in the proof of Theorem 4.1 a sharp differential inequality

$$\|Z'\| \leqslant C\|P\|(\|Z\| + \|Z\|^d),$$

and to express the domain of existence and the Bernstein constant in terms of the solution of the equation $y' = C(y+y^d)$. We shall present these results, as well as the bounds on the number of zeroes, separately.

A specific feature of the above bounds is that the Bernstein class of the solution of a homogeneous equation is determined by its order and by the Lipschitz constant of the coefficients. In particular, the Bernstein classes of these coefficients do not influence at least the direct computations above. Easy examples show that this is not the case for nonhomogeneous equations. Indeed, an equation $y' = f(x)$ prescribes the Bernstein class B^2 of y', and hence of y, to be exactly the same as of f. Thus taking $f(x) = x^N$ we get the right hand side bounded by 1 in $\mathcal{D}_1$, while the solutions $y = \frac{1}{N}\, x^{N+1}$ have there an arbitrarily high Bernstein constant as N tends to ∞.

One can prove a general inequality, determining the Bernstein class of the solution of a nonhomogeneous equation through the Lipschitz constant of a homogeneous part and the Bernstein class of the "free term". However here we just mention the following simple fact:

LEMMA 4.5. — *Any solution of a nonhomogeneous polynomial differential equation $y^{(r)} = P(x, y, \cdots, y^{(r-1)})$ satisfies also a homogeneous polynomial equation of order at most $r+m+1$, where m is the degree in x of the free term.*

Proof. — We differentiate the equation $m+1$ times, and notice that any term, containing y or its derivatives, will contain them also after differentiation.

Thus the Bernstein class of any solution of a nonhomogeneous polynomial equation can be determined using Lemma 4.5 and Corollary 4.4.

Returning to algebraic functions, let us mention that any algebraic function satisfies a linear differential equation with rational coefficients (see for example [14], [15]). However, these coefficients may have poles inside the domain of regularity of the function. Hence to reproduce the results of Section 3 some additional considerations are required. As far as the structural Bernstein inequality is concerned (Section 3.3 above) at present we can obtain it by the methods of this section only in some special cases (see [41], [43]).

A detailed analysis of zeroes distribution of solutions of some special linear differential equations (in particular, for the Abelian integrals) can be found in [31], [32], [37]. Some other results, closely related to the results of this section, can be found in [34], [48], [49].

5. LIPSCHITZIAN RECURRENCY RELATIONS ON TAYLOR COEFFICIENTS

In this section we discuss an approach, which is, in a sense, dual to that of Section 4. While in Section 4 a Lipschitzian behaviour of differential equations has produced a bound for the ratio of the maxima of $|y|$ on concentric disks, here we do exactly the same in the "dual" space of the Taylor coefficients of y.

Assume that a sequence ϕ of mappings $\varphi_k : \mathbb{C}^{d+1} \to \mathbb{C}$ is given, $k = d+1,\ d+2, \cdots$. For any $w = (w_0, \cdots, w_d) \in \mathbb{C}^{d+1}$ construct a sequence $a_k(w)$ as follows: $a_i = w_i,\ i = 0, \cdots, d$, and $a_j = \varphi_j(w)$ for $j > d$. We also consider a (formal) power series $f_w(x) = \sum\limits_{k=0}^{\infty} a_k(w)x^k$.

Assume also that each φ_k is a Lipschitzian mapping, satisfying

$$|\varphi_k(w)| \leqslant C^k \|w\| \tag{5.1}$$

for any $k \geqslant d+1$ and any $w \in B(0,\delta)$, with some given $\delta > 0$ and $C > 0$.

Usually recurrency relations are written in a slightly different form: a_k is expressed as the function of $a_{k-1},\ a_{k-2}$ etc. However iterating these expressions one can always express a_k as the function of the first d terms (starting from which the recurrency is valid). Main examples are once more provided by solutions of algebraic differential equations. In [7] it is shown, that the recurrency relations, which arise there, satisfy the assumptions above.

THEOREM 4.1. — *For C and δ as above and for any $w \in B(0,\delta)$, the series $f_w(x) = \sum\limits_{k=0} a_k(w)x^k$ converges on $\mathcal{D}_R$, $R = 1/C$ and belongs to $B^2_{d,R,\alpha}$, with $\alpha = [\max(1,C)]^d$.*

Proof. — By (5.1) we have for any $j > d$

$$|a_j(w)|R^j \leqslant \max(|a_0|, \cdots, |a_d|) \leqslant \alpha \max_{i=0,\cdots,d} (|a_i|R^i), \text{ with}$$

$$\alpha = [\max(1, C)]^d.$$

Theorem 4.1 reduces the problem of determining the Bernstein class of f_w to a study of the recurrency relation, producing its coefficients. We refer to [7] where a detailed study of this question for algebraic functions and for the first order differential equations can be found. Notice, that the recurrency relations of the above type have been studied in Transcendental Number Theory (see [35], [47]).

6. A_0-SERIES

In this section we shortly discuss, following [7] and [21] another wide class of functions, whose Bernstein constant can be explicitly estimated. It consists of power series, whose coefficients are polynomials in a finite number of parameters (with certain restrictions on the growth rate of the degree and of the norm). As it will be clear from the results below, this class suits well for producing Bernstein inequalities, depending on the size of the parameters (as the two classes, considered in Sections 4 and 5 do). However, also here determining subclasses, which allow structural Bernstein inequalities, is not easy. Some results in this direction are given in [8].

Let for $\lambda \in \mathbb{C}^p$, $f_\lambda(x) = \sum_{k=0}^{\infty} a_k(\lambda)x^k$, with $a_k(\lambda)$ - polynomials in λ for any $k = 0, 1, \cdots$. As above, we use the norm of polynomials, equal to the sum of the absolute values of their coefficients.

DEFINITION 6.1 (See [7]). — *$f_\lambda(x)$ is called an A_0-series, if the following two conditions are satisfied:*

1. deg $a_j(\lambda) \leqslant K_1 \cdot j + K_2$, $j = 0, 1, \cdots$.
2. $\|a_j(\lambda)\| \leqslant K_3 \cdot K_4^j$, $j = 0, 1, \cdots$, *for some positive* K_1, K_2, K_3, K_4.

Below we denote by C_i constants, depending only on K_1 to K_4.

A_0-series form a subring of the ring of formal power series in x with coefficients - polynomials in λ. All the basic analytic operations, like substitution to a given analytic function, inversion, etc., transform A_0-series into themselves.

An A_0-series $f_\lambda(x)$ converges for any $\lambda \in \mathbb{C}^p$ in a disk $\mathcal{D}_R$ with $R = C_1/(1 + \|\lambda\|)^{C_2}$. One can show that if a series with polynomial coefficients (of linearly growing degrees) converges uniformly for λ in a certain ball in $\mathbb{C}^p$, it is an A_0-series.

Analytic families, similar to A_0-series, were considered in [26], [30].

Consider in the ring $\mathbb{C}[\lambda]$ of all polynomials in $\lambda = (\lambda_1, \cdots, \lambda_p)$ the ideal $I = \{a_0(\lambda), a_1(\lambda), \cdots\}$, generated by all the coefficients $a_k(\lambda)$ of f_λ, and let $Y = Y(I)$ be the set of zeroes of this ideal. We call I the Bautin ideal of f_λ (following [39], [40]). In his pioneering work [2], Bautin used this ideal to bound (by 3) the number of limit cycles, which can bifurcate from the center in quadratic families of plane vector fields.

By Hilbert's finiteness theorem, I is in fact generated by a finite number of $a_k(\lambda)$. Let d be the minimal number such that $I = \{a_1(\lambda), \cdots, a_d(\lambda)\}$. We call d the Bautin index of f_λ.

The following theorem (which is implied by one of the main results of [21]) shows that the Bautin index plays a role of the "degree" of a family f_λ:

THEOREM D (See [21], Theorem 2.3.7). — *Let $f_\lambda(x)$ be an A_0-series and let d be its Bautin index. Then for any $\lambda \in \mathbb{C}^p$, $f_\lambda \in B^2_{d,R,\alpha}$, with $R = C_3/(1 + \|\lambda\|)^{C_4}$, (where α, C_3 and C_4 do not depend on λ).*

In particular, for $R_1 = R/2^{3d} \max(\alpha, 2)$, the number of zeroes of f_λ in $\mathcal{D}_R$ does not exceed d.

Remark. — In fact, the constants, α, C_3 and C_4 depend, in addition to d and $K_1 - K_4$, on the Gröbner basis of the ideal I and on the transformation matrix from the generators $a_0(\lambda), \cdots, a_d(\lambda)$ to this Gröbner basis.

Theorem D produces an effective bound on the Bernstein class and the number of zeroes of f_λ in any case, where the Bautin ideal and its generators can be computed explicitly. It is shown in [7] that this is the case for algebraic functions and solutions of algebraic differential equations. (Recurrency relations on the Taylor coefficients are used in [7] to produce the generators of the Bautin ideal). Another case, where these computations can be done effectively concerns restrictions of polynomials to solutions of systems of polynomial differential equations. This is done in [54] by combining some recent results of Gabrielov [24] with simple computations of ideals of linear polynomials. In [53] some results for iterations of a fixed A_0-series are given. In the original Bautin's setting (i.e., for the Poincaré mapping of algebraic differential equations) an effective computation of the generators of the Bautin ideal is in general an open problem. (See [11]-[13], [20], [27], [28], [40], [56]).

In conclusion, let us mention that the class of A_0-series is certainly too large to possess much stronger "finiteness" properties than the one, given by Theorem D. In particular, an example of an A_0-series $e^{\lambda x} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} x^k$ shows, that in general A_0-series do not allow the structural Bernstein inequality.

The solutions of algebraic differential equations form a small subclass of A_0-series. Indeed, their coefficients $a_k(\lambda)$ are completely determined by several initial ones, while the coefficients $a_k(\lambda)$ of a general A_0-series can be picked, essentially, at random (with mild restrictions on the degree and the norm). Thus to have stronger finiteness properties and, in particular, the structural Bernstein inequality, one has to restrict significantly the class of A_0-series, considered. Some steps in this direction are taken in [8].

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