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DEVIL'S STAIRCAISE ROUTE TO CHAOS IN A FORCED RELAXATION OSCILLATOR

by L. ALSEDÀ and A. FALCÓ

1. Introduction and statement of the main result.

In this paper we describe the behaviour of certain sets of solutions of an oscillator of the Van der Pol type with sinusoidal forcing term. The original problem was proposed by Van der Pol [16] in the study of an electrical circuit with a triode valve. Later on, Van der Pol and Van der Mark [17] studied the forced relaxation oscillator in a circuit as the one in figure 1.1. They analyzed the frequency of the circuit as a function of the capacitance C. While increasing C from its initial value they observed that the electrical system takes a period being a multiple of the forcing period and that, for certain parameter values, two different subharmonics may coexist. Furthermore, there are regions where no subharmonics are detected. Plotting the frequency of the circuit against the capacitance they obtained a staircase structure as shown in figure 1.1.

Recently, Kennedy, Krieg and Chua [10] working with a modern version of the Van der Pol and Van der Mark's circuit observed the appearance of secondary staircases. These staircases present a well-known geometric structure called « the Devil's Staircase » (which, roughly speaking, can be defined as the graph of a non-decreasing continuous map with the property that the preimage of any rational number is a closed interval and the preimage of any irrational number is a point). These secondary staircases give the route from the non-chaotic behaviour to the chaotic one in the electrical circuit.

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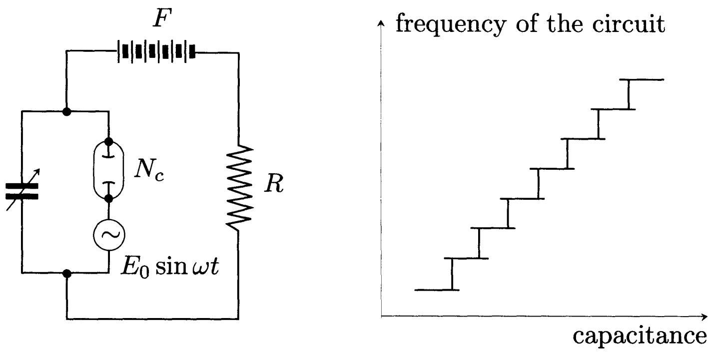

Figure 1.1. The circuit studied by Van der Pol and Van der Mark and the original staircase.

The first mathematical investigation on this model was made by Cartwright and Littlewood [5]. They studied the solutions of the following non-linear differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t} + \nu(x^2 - 1)\frac{\mathrm{d}x}{\mathrm{d}t} + x = \nu\, b(\nu)\, k \cos kt, \tag{1.1}$$

where $\nu \gg 1$ and discovered a family of solutions with chaotic behaviour. Later on Levinson [12] proposed the following version of (1.1)

$$\epsilon \ddot{x} + \Psi_0(x)\dot{x} + \epsilon x = b p_0(t), \tag{1.2}$$

where $\Psi_0 = \operatorname{sgn}(x^2 - 1)$, $p_0(t) = \operatorname{sgn}(\sin(2\pi t/T))$, $\epsilon > 0$ is a small parameter and b varies in some finite interval $[b_1, b_2]$. In this new model the solutions could be analyzed explicitly by piecing together solutions at different linearity intervals.

Afterwards, Levi [11] modified the Levinson's model by replacing the functions $\Psi_0(x)$ and $p_0(t)$ by two differentiable C^0-close functions. Namely, $\Psi(x)$ negative for $|x| < 1$ and positive for $|x| < 1$ and $p(t)$ periodic of period T. In a very complicated process Levi reduced the study of the qualitative behaviour of the solutions of this model to the study of the dynamics of a dissipative diffeomorphism in a region of $\mathbb{R}^2$ that, after identifying the upper boundary with the lower one, can be considered as a dissipative diffeomorphism of an annulus into itself. Moreover this diffeomorphism can be approximated (in some sense) by a circle map.

By using these techniques Levi showed that, for ϵ small enough, the interval $[b_1, b_2]$ can be decomposed into union of alternating closed, proper disjoint intervals A_k and B_k separated by thin gaps g_k and $\hat{g}_k$ as follows :

$$[b_1, b_2] = A_1 \cup g_1 \cup B_1 \cup \hat{g}_1 \cup A_2 \cup g_2 \cup \cdots \cup \hat{g}_{n-1} \cup A_n \cup g_n \cup B_n.$$

When b belongs to one of the intervals A_k a periodic solution of period $(2q-1)T$ appears, where T is the period of the forcing term $p(t)$ and $q = q(k) > 0$ is an integer number. As b increases it crosses one of the small gaps g_k to fall down in one of the intervals B_k. Then, the above periodic solution is preserved and another one of period $(2q+1)T$ is created. Moreover, it is shown that in the intervals B_k the system exhibits chaotic motion. Afterwards, the parameter b crosses another small gap of type $\hat{g}_k$ to arrive to an interval A_{k+1} where only the periodic solution of period $(2q+1)T$ remains and the chaotic motion disappears. Thus, as b moves trough the intervals A_k, g_k, B_k, $\hat{g}_k$ and A_{k+1} one observes a hysteresis phenomenon (frequency demultiplication). However, Levi did not study in detail the evolution of the system as b crosses the intervals g_k and $\hat{g}_k$ but he predicted the existence of orbits of very high period.

The purpose of the present paper is to analyze the bifurcations occurring when the parameter b crosses the gaps of type g_k and $\hat{g}_k$ in the Levi's model of the forced relaxation oscillator. Before stating the main result of this paper we have to introduce some notation and explain Levi's results with more detail.

Levi's model can be conveniently rewritten as

$$\dot{x} = \epsilon^{-1}\big(y - \Phi(x)\big), \quad \dot{y} = \epsilon x + bp(t), \tag{1.3}$$

where $y = \epsilon\dot{x} + \Phi(x)$ is the modified velocity and $\Phi(x) = \int_0^x \Psi(u)\,\mathrm{d}u$.

We shall denote by P_b be the Poincaré map associated to (1.3), defined as $P_b(z) = Z(T, 0, z)$, where $Z(t, t_0, z)$ denotes the solution of the system at time t which starts at z at time t_0. For ϵ small enough and for all $b \in [b_1, b_2]$, the map P_b has the following geometrical properties :

(1) It has exactly two fixed points. One at infinity, and z_0 which is close to the branch of $y = \Phi(x)$ with negative slope.

(2) There exists an annular region $\mathcal{R}$ surrounding z_0 with thickness less than $\sqrt{\epsilon}$ such that any point $z \neq z_0$ enters in $\mathcal{R}$ after sufficiently many iterations of P_b and stays there. In particular $\mathcal{R}$ is P_b-invariant.

(3) The points of $\mathcal{R}$ «circulate clockwise» with respect the point z_0 under the iterates of P_b.

Let $\Pi : \mathbb{R} \times [0,1] \longrightarrow \mathcal{R}$ be the natural projection. That is,

$$\Pi_{\mid [0,2\pi)\times[0,1]} : [0,2\pi) \times [0,1] \longrightarrow \mathcal{R}$$

is a homeomorphism, Π is periodic of period 2π with respect to the first component and $\Pi(x,y)$ moves «clockwise» as x increases. Moreover, Π can be taken in such a way that if $\Pi(x,y) = z$ with $x \in [0,2\pi]$ and $y \in [0,1]$ then x is the «clockwise» angle of the vector $(z - z_0)$ with respect the horizontal line passing through z_0. In what follows, we shall fix a lifting

$$\widetilde{P_b} : \mathbb{R} \times [0,1] \longrightarrow \mathbb{R} \times [0,1]$$

of the map $P_{b\,\mid \mathcal{R}}$. That is, $\widetilde{P_b}$ is a diffeomorphism such that $P_b \circ \Pi = \Pi \circ \widetilde{P_b}$. Let $\pi_1 : \mathbb{R} \times [0,1] \longrightarrow \mathbb{R}$ denote the projection map with respect to the first component. Take $z \in \mathcal{R}$ and $\widetilde{z} \in \Pi^{-1}(z)$. Then, the real number

$$\rho_{\widetilde{P_b}}(z) = \lim_{i\to\infty} \frac{\pi_1(\widetilde{P_b}^i(\widetilde{z})) - \pi_1(\widetilde{z})}{i}$$

will be called the *rotation number of z with respect to $\widetilde{P_b}$* if it exists. We note that this limit is the average angle by which the point z rotates under iteration of the map P_b with respect to the fixed point z_0 (see (3) above). Let $\Omega \subset \mathcal{R}$ be a P_b-invariant set. The *rotation set of Ω with respect to $\widetilde{P_b}$* is defined to be the set of all rotation numbers of all points from Ω with respect to $\widetilde{P_b}$.

The following theorem summarizes Levi's results on the system (1.3) (see [11]).

THEOREM 1.1. — *The interval $[b_1, b_2]$ can be decomposed into union of alternating closed, proper disjoint intervals A_k and B_k separated by gaps g_k and $\hat{g}_k$ as follows :*

$$[b_1,b_2] = A_1 \cup g_1 \cup B_1 \cup \hat{g}_1 \cup A_2 \cup g_2 \cup \cdots \cup \hat{g}_{n-1} \cup A_n \cup g_n \cup B_n.$$

Moreover,

(a) *For b in A_k we have :*

(a$_1$) *P_b has one pair of periodic points of period $(2q-1)$ where $q = q(\epsilon,k) \sim \epsilon^{-1}$ remains constant through the interval A_k, and $q(\epsilon,k+1) = q(\epsilon,k) - 1$, (i.e. the period of the these points decreases as b increases).*

(a$_2$) *One of the two points is a sink and the other a saddle. Moreover, any point which lies off the stable manifold of the saddle (except for the unstable fixed point of P_b) tends to the sink under forward iterations.*

(a_3) *The rotation set of* $\mathcal{R}$ *is* $\{2\pi/(2q-1)\}$.

(b) *For* b *in* B_k, *we have :*

(b_1) *The minimal attractor set of* P_b *is the union of a hyperbolic Cantor set and two pairs of periodic points, one of these pairs has period* $2q+1$ *and the other one has period* $(2q-1)$. *Each of these pairs consists on a sink and a saddle. Moreover, the two saddles belong to the Cantor set.*

(b_2) *The rotation set of* $\mathcal{R}$ *is* $[2\pi/(2q+1), 2\pi/(2q-1)]$.

(c) *There exists* b^* *in* g_k *(respectively in* $\hat{g}_k$*) such that* P_{b^*} *has a nondegenerate homoclinic tangency. Moreover, there exists a small* $\xi > 0$ *and an open subset* B_ξ *in* $B_\xi^* = [b^*, b^* + \xi)$ *(respectively* $B_\xi^* = (b^* - \xi, b^*]$*)such that for* $b \in B_\xi^* \setminus B_\xi$, P_b *is structurally stable. The set* $B_\xi^* \setminus B_\xi$ *consists on infinitely many components, to which there correspond infinitely many different (structurally stable) types of* P_b.

In order to complete statement (c) of Levi's Theorem we study how the Cantor set appearing in the statement (b) and its rotation set associated are formed when b crosses a bifurcation gap g_k or $\hat{g}_k$. This is achieved in the next theorem which is the main result of the paper. We will only state the theorem in the case of the interval g_k. The situation for an interval $\hat{g}_k$ is symmetric. In the rest of the chapter we will use freely the notation introduced above and, in particular, the one from Theorem 1.1.

THEOREM 1.2. — *For each* $b \in g_k$ *the map* P_b *has one pair of periodic points of period* $(2q-1)$; *a sink and a saddle. Moreover, there exist a countable sequence* $\{b_n^k\}_{n=0}^\infty \subset g_k$ *satisfying the following property :*

(a) *For each* b_n^k *the minimal attractor set of* $P_{b_n^k}$ *contains an invariant hyperbolic Cantor set, denoted by* C_n^k, *to which the saddle point belongs.*

(b) *For* $b \geq b_n^k$, *the minimal attractor set of* P_b *contains an invariant hyperbolic Cantor set, denoted by* $C_b^{n,k}$, *which contains the saddle point of* P_b, *such that* $P_{b \mid C_b^{n,k}}$ *is topologically conjugate to* $P_{b_n^k \mid C_n^k}$. *Moreover, if* $b_s^k < b_n^k$, *then* $C_{b_n^k}^{s,k} \subset C_n^k$.

(c) *For each* b_n^k *there exists a rational number* $\alpha_n^k \in [-1,1]$ *such that for* $b \geq b_n^k$ *the* P_b*−rotation set of* $C_b^{n,k}$ *is the closed interval* $[2\pi/(2q+\alpha_n^k), 2\pi/(2q-1)]$. *Moreover* $\{\alpha_n^k\}_{n=0}^\infty = (-1,1] \cap \mathbb{Q}$.

In view of the above two theorems, the bifurcations of P_b when the parameter b crosses g_k from A_k to B_k can be explained in the following way. When b is close to A_k the dynamics of the map P_b is the same as when b lies in A_k (see Theorem 1.1 (a)). This is the situation until b reaches the parameter value b^* from Theorem 1.1 (c). At this point the map P_b has a non-degenerated homoclinic tangency and, in consequence, there exists a wild hyperbolic set by the well known result of Newhouse [15]. Therefore, all parameter values b_n^k considered in Theorem 1.2. must be larger than or equals to b^* and accumulate to $b^{**} \geq b^*$. Then, for $b \geq b^{**}$, the minimal attractor set of P_b contains an invariant hyperbolic Cantor set which is enlarged each time that b crosses one of the parameter values from the sequence $\{b_n^k\}_{n=0}^{\infty}$ (see Theorem 1.2 (a)–(b)). As it will be shown later, the dynamics of the system on each of these Cantor sets can be deduced from a subshift of finite type with a certain transition matrix which can be computed explicitly by using one dimensional techniques (see Corollary 3.5 and Remark 3.6). Finally, when the parameter b is sufficiently close to B_k the dynamics of the map P_b is the same as when b lies to B_k. Moreover, P_b possesses an invariant set, strictly contained in the minimal attractor, with P_b-rotation interval $[2\pi/(2q+1), 2\pi/(2q-1)]$ (see Theorem 1.1 (b) and [11]). The transition of the rotation interval of the system from the point $2\pi/(2q+1)$ into the interval $[2\pi/(2q+1), 2\pi/(2q-1)]$ is also described by the rotation intervals of P_b restricted to the Cantor sets $C_b^{n,k}$ (see Theorem 1.2 (c)).

The paper is organized as follows. In Section 2 we give some more notation and preliminary results. In Section 3 we prove Theorem 1.2 (a)–(b). Lastly, in Section 3, we recall some basic results about rotation numbers for circle maps and we prove Theorem 1.2 (c).

2. Definitions and preliminary results.

We start this section by introducing some of the notions used by Levi to prove Theorem 1.1.

Let $A = \mathbb{S}^1 \times [0,1]$ be the standard annulus. Levi [11] shows that the study of the map P_b can be reduced to the study of the annulus map

$$L_b = L(\cdot, b, \delta, \delta_1) : A \longrightarrow A,$$

depending on three parameters, namely, $b \in [b_1, b_2]$, $0 < \delta \leq \delta'$ and $0 < \delta_1 \leq \delta_1'$, which satisfy the following properties.

Let $\Pi_1 : A \longrightarrow \mathbb{S}^1$ denote the vertical projection on the first component. For each $\sigma \in [0,1]$ we denote by $f_{b,\sigma}(x)$ the circle map $f(x,b,\sigma,\delta,\delta_1) = \Pi_1 \circ L_b(x,\sigma)$ (see Figure 2.1). Then we have :

(L_1) $|f_{b,\sigma} - f_{b,\sigma'}| < \delta$ in the C^0 norm in x, for all σ, $\sigma' \in [0,1]$.

(L_2) There exist $\gamma > 1$, $\vartheta > 32$, $C > 0$ and two intervals $\Delta \subset \Delta_1 \subset \mathbb{S}^1$ whose endpoints depend on b, δ and δ_1 (not on σ) such that $|\Delta_1| < \delta_1$ and for all $\sigma \in [0,1]$ it follows :

(i) $f'_{b,\sigma}(x) > \vartheta\gamma$ for all $x \in \Delta$.

(ii) $-1 + C < f'_{b,\sigma}(x) < -\gamma^{-1}$ for all $x \in \mathbb{S}^1 \setminus \Delta_1$.

(L_3) The oscillation (in x) of $f_{b,\sigma}$ on each of the two components of $\Delta_1 \setminus \Delta$ is less than $\varrho(\delta,\delta_1)$, which is independent on b and $\lim_{\delta,\delta_1\to 0} \varrho(\delta,\delta_1) = 0$.

(L_4) For some $\sigma \in [0,1]$ we have

$$-\frac{\mathrm{d}}{\mathrm{d}b}\Big(f\big(x_i(b), b, \sigma, \delta, \delta_1\big) - x'_i(b)\Big) > \omega(\delta,\delta_1) > 0$$

for $i = 1,2$, where $x_1(b)$, $x_2(b)$, $x'_1(b)$ and $x'_2(b)$, are the endpoints of Δ and Δ_1 respectively (labelled in such a way that $x_1(b)$, $x'_1(b)$ are the endpoints of one of the connected components of $\Delta_1 \setminus \Delta$ and $x_2(b)$, $x'_2(b)$ are the endpoints of the other one), all differentiable in b and $\omega(\delta,\delta_1)$ is independent on b (see Figure 2.1).

(L_5) L_b has a inverse on $L_b(A)$.

(L_6) The map L_b^{-1} in $Q = \Delta \times [0,1]$ maps vertical strips into vertical strips.

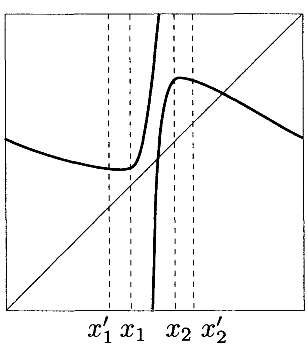

Figure 2.1. The circle map $f_{b\sigma}$.

In view of Levi [11] the relation between L_b and P_b can be described as follows. There exists a homeomorphism h from A to a subset of $\mathcal{R}$ and a positive integer $m = m(\epsilon)$ such that, for each L_b-invariant set $\Omega \subset A$ we have that $\bigcup_{i=0}^{m} P_b^i(h(\Omega))$ also is P_b-invariant. Then, the $\widetilde{P}_b$-rotation number of a point $h(z) \in \mathcal{R}$ with $z \in A$ can be obtained from the L_b-rotation number of z as we shall show next (see Remark 2.2). In a similar way as we did for the map P_b we shall fix a lifting $\widetilde{L}_b$ of L_b to the covering space $\mathbb{R} \times [0,1]$. Then as usual, the $\widetilde{L}_b$-*rotation number* of a point $z \in A$ is defined to be the limit

$$\rho_{\widetilde{L}_b}(z) = \lim_{i \to \infty} \frac{\pi_1(\widetilde{L}_b^i(\widetilde{z})) - \pi_1(\widetilde{z})}{i}$$

if it exists, where $\widetilde{z}$ is a point in $\mathbb{R} \times [0,1]$ projecting to z by the standard projection map (e, id) with $e(x) = \exp(2\pi i x)$. We note that this number can also be computed as $\lim_{i\to\infty} i^{-1}\big(\sum_{j=1}^{i} d_j^b\big)$, where

$$d_j^b = \pi_1\big(\widetilde{L}_b^j(\widetilde{z})\big) - \pi_1\big(\widetilde{L}_b^{j-1}(\widetilde{z})\big).$$

In the sequel we denote $\max\{\delta', \delta_1'\}$ by $\bar{\delta}$. The following lemma is due to Levi [11].

Lemma 2.1. — *There exists a lifting $\widetilde{L}_b$ of L_b such that, if $\bar{\delta}$ is small enough, then for all $\sigma \in [0,1]$ we have*

$$1 + C < \pi_1(\widetilde{z}_1(\sigma)) - \pi_1(\widetilde{z}_2(\sigma)) < 2 - C$$

where $\widetilde{z}_i(\sigma) = \widetilde{L}_b((\widetilde{x}_i(b), \sigma))$ for $i = 1, 2$, $\widetilde{x}_i(b)$ is such that $e(\widetilde{x}_i(b)) = x_i(b)$ for $i = 1, 2$ and $|\widetilde{x}_1(b) - \widetilde{x}_2(b)| < 1$; and the constant $0 < C < 1$ is independent on b, δ and δ_1.

In the sequel we shall assume that the lifting $\widetilde{L}_b$ of L_b we are working with is the one from the statement of Lemma 2.1.

Remark 2.2. — From the above lemma it follows that each map $f_{b,\sigma}$ has degree one and that $d_j^b \in \{0,1\}$ for all $b \in [b_1, b_2]$ and $j \geq 1$. Now set $\tau(t) = 1 - 2t$ for $t \in \{0,1\}$. From Levi [11] it follows that if for some $z \in A$ the $\widetilde{L}_b$-rotation number exists and $\bar{\delta}$ is small enough, then

$$\rho_{\widetilde{P}_b}(h(z)) = \lim_{i\to\infty} 2\pi\Big[2q + \Big(\sum_{j=1}^{i} \tau(d_j^b)/i\Big)\Big]^{-1} = 2\pi\big(2q + 1 - 2\rho_{\widetilde{L}_b}(z)\big)^{-1}. \ \square$$

Next we characterize the intervals A_k, B_k, g_k and $\hat{g}_k$ in terms of the circle maps $f_{b,\sigma}$. From now one we assume that $\bar{\delta}$ is such that Lemma 2.2 holds. For $x, y \in \mathbb{S}^1$ we denote by $[x, y]$ (respectively (x, y), $[x, y)$ and $(x, y]$) the closed (respectively open, open from the right and open from the left) arc from x to y counterclockwise. Such an arc will be called a closed (respectively open, open from the right and open from the left) interval of $\mathbb{S}^1$. If A is a proper interval in $\mathbb{S}^1$ we also will use the notations $\inf A$ and $\sup A$ in the obvious way.

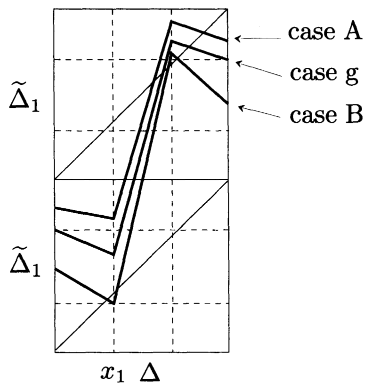

Figure 2.2. The three cases for $f_{b,\sigma}(\widetilde{\Delta}_1)$.

Let $\widetilde{\Delta}_1$ denote the open interval $(x_1'(b) - \varrho(\delta, \delta_1), x_2'(b) + \varrho(\delta, \delta_1))$. Then, one and only one of the following three cases occurs for $f_{b,\sigma}$ (see Figure 2.2) :

Case A. — The set $f_{b,\sigma}^{-1}(\widetilde{\Delta}_1) \cap \Delta$ is an interval such that its endpoints map onto the endpoints of $\widetilde{\Delta}_1$ and its image is $\widetilde{\Delta}_1$.

Case g. — $f_{b,\sigma}(x_i) \in \widetilde{\Delta}_1$, for some $i \in \{1, 2\}$ (i.e. the set $f_{b,\sigma}^{-1}(\widetilde{\Delta}_1) \cap \Delta$ is a union of two disjoint intervals such that the endpoints of one of them map onto the endpoints of $\widetilde{\Delta}_1$ and the image of the other one is strictly contained in $\widetilde{\Delta}_1$).

Case B. — The set $f_{b,\sigma}^{-1}(\widetilde{\Delta}_1) \cap \Delta$ is a union of two disjoint intervals such that the endpoints of both of them map onto the endpoints of $\widetilde{\Delta}_1$ and their images are $\widetilde{\Delta}_1$.

Let A, g and B be the sets of values of $b \in [b_1, b_2]$ for which the corresponding alternative holds. Then, since the endpoints of $f_{b,\sigma}(\Delta)$ move monotonically (clockwise) with respect to the endpoints of $\widetilde{\Delta}_1$

(see (L_4)), the set A (respectively B and g) can be written as $\bigcup_{k\in I_A} A_k$ (respectively $\bigcup_{k\in I_B} B_k$ and $(\bigcup_{k\in I_g} g_k) \cup (\bigcup_{k\in I_{\hat g}} \hat g_k)$), where each of the sets A_k (respectively B_k, g_k and $\hat g_k$) is a connected component of A (respectively of B and g), in such a way that the intervals A_k, B_k, g_k and $\hat g_k$ alternate as stated in Theorem 1.1.

3. Proof of Theorem 1.2 (a)–(b).

To prove Theorem 1.2 (a)–(b) we shall employ the techniques used by Levi in the proof of Theorem 1.1 (a)–(b) to translate the results concerning the circle maps family to the two dimensional setting. Thus, we only will prove in detail the results on the family $f_{b,\sigma}$ which are necessary to prove Theorem 1.2 (a)–(b).

We start by constructing the sequence of parameter values appearing in the statement of the theorem. First we have to fix some notation.

Note that for each $b \in A_k \cup g_k$ there exist $u_{b,\sigma} \in \operatorname{Int}(\Delta)$ depending continuously on b such that $u_{b,\sigma}$ is a unstable fixed point of $f_{b,\sigma}$ (see Case A, Case g and Figure 2.2). Then, for $\sigma \in [0,1]$, we define

$$\alpha_\sigma^k = \sup\{b \in g_k : f_{b,\sigma}(x_1(b)) = u_{b,\sigma}\},$$

$$\beta_\sigma^k = \inf\{b \in g_k : \ f_{b,\sigma}(x_1(b)) = x_1'(b) - \varrho(\delta,\delta_1)\}.$$

In view of (L_4) we see that $\alpha_\sigma^k < \beta_\sigma^k$.

In the sequel we shall denote the closed interval $[x_1(b), u_{b,\sigma}] \subset \Delta$ by Δ_σ^L. We note that for $b \in (\alpha_\sigma^k, \beta_\sigma^k)$ we have that $f_{b,\sigma}^{-1}(\Delta_\sigma^L) \cap \Delta_\sigma^L$ is the union of two closed disjoint intervals $I_{b,\sigma}$ and $J_{b,\sigma}$ such that $x_1(b) \in I_{b,\sigma}$, $u_{b,\sigma} \in J_{b,\sigma}$, $f_{b,\sigma}(J_{b,\sigma}) = \Delta_\sigma^L$ and $f_{b,\sigma}(I_{b,\sigma}) \subset \Delta_\sigma^L$ (see Figure 3.1). Let $A_{b,\sigma}$ be the open interval $\Delta_\sigma^L \setminus (I_{b,\sigma} \cup J_{b,\sigma})$. Observe that $f_{b,\sigma}(\sup A_{b,\sigma}) = x_1(b)$, $f_{b,\sigma}(\inf A_{b,\sigma}) = u_{b,\sigma}$ and $f_{b,\sigma}(A_{b,\sigma}) = \mathbb{S}^1 \setminus \Delta_\sigma^L$.

Let

$$\begin{aligned} W_{b,\sigma} &= \{x \in \Delta_\sigma^L : f_{b,\sigma}^i(x) \in A_{b,\sigma} \text{ for some } i \in \mathbb{Z}^+\} \\ &= \bigcup_{i=0}^{\infty} f_{b,\sigma}^{-i}(A_{b,\sigma}) \cap \Delta_\sigma^L. \end{aligned}$$

LEMMA 3.1. — *For all $\sigma \in [0,1]$ and for all $b \in (\alpha_\sigma^k, \beta_\sigma^k)$ there exists $\{K_i^{b,\sigma}\}_{i=1}^\infty \subset \Delta_\sigma^L$, a countable sequence of open (in Δ_σ^L) disjoint subintervals of Δ_σ^L, such that $W_{b,\sigma} = \bigcup_{i=1}^\infty K_i^{b,\sigma}$*

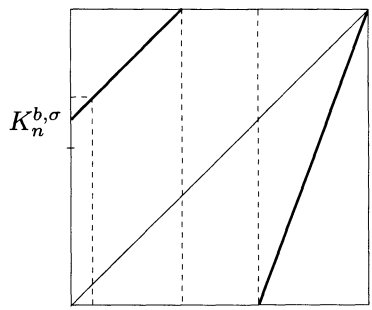

Figure 3.1. The map $f_{b,\sigma}$ in Δ_σ^L.

Proof. — It uses a standard argument. Clearly, $W_{b,\sigma}$ is open in Δ_σ^L. Then, we only have to prove that $W_{b,\sigma}$ is dense in Δ_σ^L. Suppose not. Then $D = \Delta_\sigma^L \setminus \mathrm{Cl}(W_{b,\sigma})$ is a countable union of open intervals (in Δ_σ^L). Number these intervals and let d_i be the length of the i–th one. Each d_i is positive and $\sum_{i=1}^{\infty} d_i \leq 1$. So $\lim_{i\to\infty} d_i = 0$. Hence there is an i_0 such that $d_i \leq d_{i_0}$ for all i. Now, observe that $f_{b,\sigma}(D) \subset D$ and that the image of the i_0–th interval of D by $f_{b,\sigma}$ is a larger interval because $f'_{b,\sigma}|_\Delta > 1$; a contradiction. □

In the sequel we shall assume that the sequence $\{K_i^{b,\sigma}\}_{i=1}^{\infty}$ is labelled in such a way that if $n < m$, then $\sup K_n^{b,\sigma} \leq \inf K_m^{b,\sigma}$. Note that the whole sequence depends on b and σ.

Now, set $K_0^{b,\sigma} = (x_1'(b) - \varrho(\delta, \delta_1), x_1(b))$. From ($L_4$) we have that for each $n \geq 0$ and for all $\sigma \in [0,1]$ there exists $b_{n,k}^\sigma$ such that $f_{b,\sigma}(I_{b,\sigma}) \cap K_n^{b,\sigma} \neq \emptyset$ for all $b \geq b_{n,k}^\sigma$ and $b_{n,k}^\sigma$ is the smallest one having this property.

In view of Lemma 3.1 and the definition of $W_{b,\sigma}$, for $n > 0$ there exists $\ell = \ell(n) \in \mathbb{Z}^+$ such that $f_{b,\sigma}^{\ell}(K_n^{b,\sigma}) = A_{b,\sigma}$. Additionally, we set $\ell(0) = 0$. The following result will be crucial in the proof of Theorem 1.2 (a)–(b).

PROPOSITION 3.2. — *Let $n \geq 0$ and let $b \in (b_{n,k}^\sigma, \beta_\sigma^k)$. Then there exist a set $R_{n,k}^{b,\sigma}$ such that :*

(a) *$R_{n,k}^{b,\sigma}$ is union of $R_1, \ldots, R_{\ell(n)+2}$, a finite sequence of closed disjoint intervals in $\Delta_\sigma^L \setminus A_{b,\sigma}$ whose endpoints are preimages of $x_1(b)$ or $u_{b,\sigma}$ by $f_{b,\sigma}^m$ for some $m \geq 0$.*

(b) *If $f_{b,\sigma}(x_1(b)) \in \mathrm{Int}(K_n^{b,\sigma})$, then the closed $f_{b,\sigma}$-invariant set $\Delta_\sigma^L \setminus W_{b,\sigma}$ is strictly contained in $R_{n,k}^{b,\sigma}$.*

Proof. — If $n = 0$ then the proposition holds trivially by taking $R_1 = I_{b,\sigma}$ and $R_2 = J_{b,\sigma}$. Assume $n > 0$. Clearly, there exists $z \in (x_1(b), \inf A_{b,\sigma})$ such that $f_{b,\sigma}(z) = \sup K_n^{b,\sigma}$ (see Figure 3.1). Observe that for all m such that $0 \leq m < \ell(n)$, $f_{b,\sigma}^m(K_n^{b,\sigma})$ is an open interval (in Δ_σ^L) whose endpoints map onto the endpoints of $f_{b,\sigma}^{m+1}(K_n^{b,\sigma})$. The complement of

$$[x_1(b), z) \cup \left(\bigcup_{i=0}^{\ell(n)} f_{b,\sigma}^i(K_n^{b,\sigma})\right)$$

in Δ_σ^L is union of $\ell(n) + 2$ closed pairwise disjoint intervals. Call them $R_1, \ldots, R_{\ell(n)+2}$. By construction this sequence satisfies (a). Assume now that $f_{b,\sigma}(x_1(b)) \in \operatorname{Int}(K_n^{b,\sigma})$. Then the complement of $R_{n,k}^{b,\sigma}$ in Δ_σ^L is strictly contained in $W_{b,\sigma}$. From this, statement (b) follows. □

Remark 3.3. — Let $\beta_\sigma^k > b > b_{n,k}^\sigma > b_{m,k}^\sigma$. Then, Proposition 3.2 gives us two different sequences of intervals. Namely,

$$R_{n,k}^{b,\sigma} = \bigcup_{i=1}^{\ell(n)+2} R_i \quad \text{and} \quad R_{m,k}^{b,\sigma} = \bigcup_{i=1}^{\ell(m)+2} \widetilde{R}_i.$$

From the construction of the sets $R_{n,k}^{b,\sigma}$ and $R_{m,k}^{b,\sigma}$ (see Figure 3.2) it is not difficult to see that $\ell(n) \geq \ell(m)$ and that there exist $\{k_1, k_2, \ldots, k_{\ell(m)+2}\} \subset \{1, 2, \ldots, \ell(n) + 2\}$ such that $R_i \cap f_{b,\sigma}(R_j) \neq \emptyset$ if and only if $\widetilde{R}_{k_i} \cap f_{b,\sigma}(\widetilde{R}_{k_j}) \neq \emptyset$ for $i, j \in \{1, 2, \ldots, \ell(m) + 2\}$.

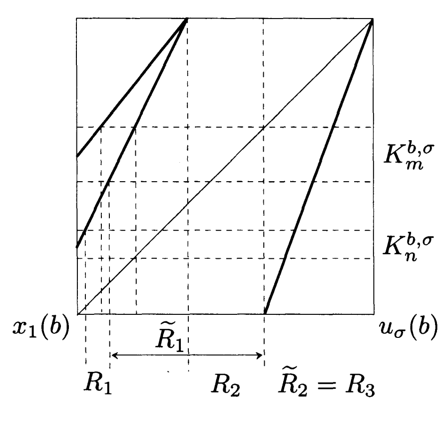

Figure 3.2. The sets $R_{n,k}^{b,\sigma}$ and $R_{m,k}^{b,\sigma}$.

Now we are ready to define the sequence of parameter values appearing in the statement of Theorem 1.2.

In the sequel we shall assume that $\bar{\delta}$ is such that Proposition 3.2 holds.

In view of (L_4), for $\bar{\delta} > 0$ small enough there exists $\eta_\sigma > 0$ such that for all $b \geq b^\sigma_{n,k} + \eta_\sigma$ we have $f_{b,\sigma}(I_{b,\sigma}) \cap K^{b,\sigma}_n \neq \emptyset$ for all $\sigma \in [0,1]$. Then we define b^k_n as $\sup_\sigma b^\sigma_{n,k} + \eta_\sigma$.

Now, the proof of Theorem 1.2 (a)–(b) follows directly from the following results.

PROPOSITION 3.4. — *Let $b \in g_k$ with $b \geq b^k_n$ and let $R^{b,\sigma}_{n,k} = \bigcup_{i=1}^{\ell(n)+2} R_i$. Then for $\bar{\delta} > 0$ small enough there exists a finite sequence $V^b_1, \dots, V^b_{\ell(n)+2}$ of disjoint vertical strips contained in Q such that $V^b_i \cap L_b(V^b_j) \neq \emptyset$ if and only if $R_i \cap f_{b,\sigma}(R_j) \neq \emptyset$.*

Proof. — The implicit function theorem implies that $u_{b,\sigma}$ is a smooth function in σ. First we claim that for a fixed b, $(u_{b,\sigma}, \sigma)$ considered as function of σ is a vertical curve in Q. To prove the claim, fix b and σ. From Case g we know that, if $b \in g_k$, then there is a closed interval $V^1_{b,\sigma} \subset \Delta$ such that $u_{b,\sigma} \in V^1_{b,\sigma}$, $f_{b,\sigma}(V^1_{b,\sigma}) = \Delta$ and the endpoints of $V^1_{b,\sigma}$ map onto the endpoints of Δ. Now we set $V^i_{b,\sigma} = f^{-1}_{b,\sigma}(V^{i-1}_{b,\sigma}) \cap V^1_{b,\sigma}$ for all $i \geq 2$. It is easy to see that $V^i_{b,\sigma} \supset V^{i+1}_{b,\sigma}$ and $u_{b,\sigma} \in V^i_{b,\sigma}$ for all $i \geq 1$. From (L_2) (i) it follows that the limit of the length of $V^i_{b,\sigma}$ as i tends to infinity is zero. Then, $\bigcap_{i=1}^\infty V^k_{b,\sigma} \times \{\sigma\} = (u_{b,\sigma}, \sigma)$. Now, set $V^i = \bigcup_{\sigma \in [0,1]} V^i_{b,\sigma} \times \{\sigma\}$ for all $i \geq 1$. Clearly, V^i is a vertical strip and $V^i \supset V^{i+1}$ for all $i \geq 1$. Moreover, the width of V^i tends to zero as i tends to infinity. Then, by a standard result (see for instance Guckenheimer and Holmes [7]; Lemma 5.2.1) we get that

$$
\begin{aligned}
V^\infty = \bigcap_{i=1}^\infty V^i &= \bigcap_{i=1}^\infty \Big[\bigcup_{\sigma \in [0,1]} V^i_{b,\sigma} \times \{\sigma\} \Big] \\
&= \bigcup_{\sigma \in [0,1]} \Big[\bigcap_{i=1}^\infty V^i_{b,\sigma} \times \{\sigma\} \Big] = \bigcup_{\sigma \in [0,1]} (u_{b,\sigma}, \sigma)
\end{aligned}
$$

is a vertical curve. This ends the proof of the claim.

Our next step will be the construction of the set of vertical strips. Assume that $b \geq b^k_n$. Then Proposition 3.2 holds for all $\sigma \in [0,1]$ and $f'_{b,\sigma} \neq 0$ on each interval R_i. Therefore, from the implicit function theorem we get that the endpoints of R_i are smooth functions in σ.

Let $v_i^b = \bigcup_\sigma(\inf R_i, \sigma)$ and $w_i^b = \bigcup_\sigma(\sup R_i, \sigma)$. Then, by the construction of the sets R_i, we have that v_i^b and w_i^b are pre-images of the vertical curves $(u_{b,\sigma}, \sigma)$ and $(x_1(b), \sigma)$ under L_b (or L_b^m for some $m \geq 0$). Then by (L_6) and by using the same techniques employed by Levi in the proof of Theorem 1.1 (see [11] pp. 76–86) we obtain the vertical character of v_i^b and w_i^b. Let $V_i^b = [v_i^b, w_i^b] \times [0,1]$. By construction we have $V_i^b \cap L_b(V_j^b) \neq \emptyset$ if and only if $R_i \cap f_{b,\sigma}(R_j) \neq \emptyset$. □

Now, for each b_n^k we define the $(\ell(n)+2) \times (\ell(n)+2)$-matrix, $T_n^k = (t_{ij})$ by

$$t_{ij} = \begin{cases} 1 & \text{if } V_i^b \cap L_b(V_j^b) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we denote by Σ_n^k the set of infinite sequences $\mathbf{a} = (a_i)_{i=-\infty}^{\infty}$ such that $a_i \in \{1, 2, \ldots, \ell(n)+2\}$ and $t_{a_i a_{i+1}} = 1$ for all $i \in \mathbb{Z}$. The next corollary follows in the standard way (see Moser [13], p. 76 and Levi [11], p. 78).

Corollary 3.5. — *For $b \geq b_n^k$ there exists an L_b-invariant hyperbolic Cantor set $S_b^{n,k}$, which contains the saddle point of L_b, such that $L_{b\,|\,S_b^{n,k}}$ is topologically conjugate to the standard shift map on Σ_n^k. Moreover, for each $z \in S_b^{n,k}$ there exists a unique $\mathbf{a}(z) \in \Sigma_n^k$ such that $L_b^i(z) \in V_{a_{-i}}^b$ for all $i \in \mathbb{Z}$.*

Remark 3.6. — We note that by Proposition 3.4 we can compute the transition matrix T_n^k by using the one dimensional map $f_{b,\sigma}$ and the construction of the set $R_{n,k}^{b,\sigma}$ given in Proposition 3.2. Moreover, from Remark 3.3 we obtain that if $b > b_n^k > b_m^k$ then there exists an injective map $i : \Sigma_m^k \longrightarrow \Sigma_n^k$ which commutes with the standard shift maps on the spaces Σ_m^k and Σ_n^k (i.e. Σ_m^k is a subsystem of Σ_n^k).

Proof of Theorem 1.2 (a)-(b). — Theorem 1.2 (a) and the first assertion of Theorem 1.2 (b) follow immediately from Corollary 3.5 and the relation between the maps L_b and P_b. In view of Remark 3.3 and the proof of Proposition 3.4 we obtain the second assertion of Theorem 1.2 (b) in a similar way. □

4. Proof of Theorem 1.2 (c).

Prior to start the proof of Theorem 1.2 (c) we have to introduce some notation and state some preliminary results. We start by introducing the notions of rotation number and rotation interval of a circle map of degree one.

Denote by $\mathcal{L}$ the class of all liftings of all continuous maps of the circle into itself of degree one. That is, $\mathcal{L}$ is the class of maps $F : \mathbb{R} \longrightarrow \mathbb{R}$ such that $F(X+1) = F(X)+1$ for each $X \in \mathbb{R}$. Clearly, for $F \in \mathcal{L}$ we have $F(X+k) = F(X)+k$ for all $X \in \mathbb{R}$ and $k \in \mathbb{Z}$.

We shall say that a point $X \in \mathbb{R}$ is *periodic (mod. 1) of period s with rotation number r/s* for a map $F \in \mathcal{L}$ if $F^s(X) - X = r$ and $F^i(X) - X \notin \mathbb{Z}$ for $i = 1, \dots s-1$. A periodic (mod. 1) point of period 1 will be called a *fixed (mod. 1) point.* Clearly, if F is a lifting of f, then X is periodic (mod. 1) for F if and only if x is periodic for f and their periods are equal.

Let $F \in \mathcal{L}$. We define the *rotation number* of $X \in \mathbb{R}$ as $\limsup_{i\to\infty}(F^i(X) - X)/i$ and we denote it by $\rho(X)$ or $\rho_F(X)$. We note that if $X, X' \in e^{-1}(x)$ where $x \in \mathbb{S}^1$, then $\rho_F(X) = \rho_F(X')$ and if X is a periodic (mod. 1) point of F with rotation number r/s, then $\rho_F(X) = r/s$.

For $F \in \mathcal{L}$ we denote by R_F the set $\{\rho_F(X) : X \in \mathbb{R}\}$. From [8] it follows that R_F is a closed interval in $\mathbb{R}$ (perhaps degenerate to a single point). Also, for an F-invariant set $\Lambda \subset \mathbb{R}$ (i.e. $F(\Lambda) \subset \Lambda$) we set $R_F(\Lambda) = \{\rho_F(X) : X \in \Lambda\}$.

We define the maps F_ℓ and F_r by

$$F_\ell(X) = \inf\{F(Y) : Y \geq X\},$$

$$F_r(X) = \sup\{F(Y) : Y \leq X\}.$$

It is well known (see for instance [6]) that F_ℓ and F_r are non-decreasing maps from $\mathcal{L}$. Thus, the numbers

$$a^-(F) = \lim_{i\to\infty} \frac{F_\ell^i(X) - X}{i},$$
$$a^+(F) = \lim_{i\to\infty} \frac{F_r^i(X) - X}{i},$$

are well defined (recall that for any continuous non-decreasing map $G \in \mathcal{L}$ and for all $X \in \mathbb{R}$ the $\lim_{i\to\infty}(G^i(X) - X)/i$ exists and is independent on X). Moreover $R_F = [a^-(F), a^+(F)]$ (see for instance [14] or [3]).

Let $F \in \mathcal{L}$ and $X \in \mathbb{R}$. The set

$$\{Y \in \mathbb{R} : Y = F^i(X) \text{ (mod. 1) for } i = 1, 2, \dots\}$$

will be called the *(mod. 1) orbit* of X by F. We note that if P is a (mod. 1) orbit and $X \in P$, then $X + k \in P$ for all $k \in \mathbb{Z}$. Moreover, for each $Y \in P$

we have $\rho_F(Y) = \rho_F(X)$. If X is a (mod. 1) orbit of F, we denote by P_i the set $P \cap [i, i-1)$ for all $i \in \mathbb{Z}$ (obviously $P_i = i + P_0$ for all $i \in \mathbb{Z}$). The orbit of a periodic (mod. 1) point will be called a *periodic (mod. 1) orbit.* We also note that if it has period s then $\mathrm{Card}(P_i) = s$ for each $i \in \mathbb{Z}$.

Let P be a (mod. 1) orbit of a map $F \in \mathcal{L}$. We say that P is a *twist orbit* if F restricted to P is strictly increasing. It is well known that if a twist periodic orbit has rotation number r/s with $(r, s) = 1$ then it has period s. Also, if a twist orbit P has rotation number a then, for each $X \in P$ we have $\lim_{i\to\infty}(F^i(X) - X)/i = a$.

In the sequel we shall denote the $f_{b,\sigma}$-invariant set $\Delta_\sigma^L \setminus W_{b,\sigma}$ by $\Lambda_{b,\sigma}$.

PROPOSITION 4.1. — *Let $b \geq b_n^k$, $b \in g_k$. Then for each $\sigma \in [0,1]$ there exists an open interval $I_\sigma^{n,k} \subset (\alpha_\sigma^k, \beta_\sigma^k)$ satisfying that for all $c \in I_\sigma^{n,k}$ there exists a homeomorphism $\phi_\sigma : S_b^{n,k} \longrightarrow \Lambda_{c,\sigma}$ (here we use the notation from Corollary 3.5) such that $\phi_\sigma \circ L_b = f_{c,\sigma} \circ \phi_\sigma$.*

Proof. — Recall that for each $\sigma \in [0,1]$ there exists $b_{n,k}^\sigma$ such that $f_{b_{n,k}^\sigma,\sigma}(x_1(b_{n,k}^\sigma)) = \sup(K_n^{b_{n,k}^\sigma,\sigma})$. Then there exists $\kappa_\sigma > 0$ such that for all $c \in I_\sigma^{n,k} = (b_{n,k}^\sigma, b_{n,k}^\sigma + \kappa_\sigma)$ we have that $f_{c,\sigma}(x_1(c)) \in \mathrm{Int}(K_n^{c,\sigma})$. Let $R_{n,k}^{c,\sigma} = \bigcup_{i=1}^{\ell(n)+2} R_i$.

Take $z \in S_b^{k,\ell}$. Then, by Corollary 3.5, there exists unique $\mathbf{a}(z) \in \Sigma_n^k$ such that $L_b^i(z) \in V_{a_{-i}}$ for all $i \geq 0$. We recall that $V_{a_{-i}} = \bigcup_\sigma (R_{a_{-i}} \times \{\sigma\})$ and $L_b(V_{a_{-i}}) \cap V_{a_{-i-1}} \neq \emptyset$ if and only if $f_{b,\sigma}(R_{a_{-i}}) \cap R_{a_{-i-1}} \neq \emptyset$. Now, for $i > 0$ we define the set $R_{a_{-i}\ldots a_0}$ as $R_{a_{-i+1}\ldots a_0} \cap f_{b,\sigma}^{-i}(R_{a_{-i}})$. By Proposition 3.4 we have that $R_{a_{-i}\ldots a_0} \neq \emptyset$ and $R_{a_{-i-1}\ldots a_0} \subset R_{a_{-i}\ldots a_0}$. Moreover, for each $i > 0$, the set $R_{a_{-i}\ldots a_0}$ is a closed interval in Δ_σ^L and the diameter of $R_{a_{-i}\ldots a_0}$ is smaller than or equal to $(\gamma)^{-i}$ because $f'_{b,\sigma\,|\,\Delta} > \gamma > 1$. Therefore, $\bigcap_{i=0}^\infty R_{a_{-i}\ldots a_0}$ contains a unique point, denoted by $x(z,\sigma)$, such that $f_{b,\sigma}^i(x(z,\sigma)) \in R_{a_{-i}}$ for all $i \geq 0$. Hence, $\{x(z,\sigma) : z \in S_b^{k,\ell}\} \subset \Lambda_{c,\sigma}$. Moreover, from Proposition 3.2 (b) it follows that $\{x(z,\sigma) : z \in S_b^{k,\ell}\} = \Lambda_{c,\sigma}$. Lastly, the map $\phi_\sigma(z) = x(z,\sigma)$ is a homeomorphism. □

From the above proposition and its proof we have that the twist periodic orbits of period s and rotation number r/s of the map $f_{c,\sigma}$ in $\Lambda_{c,\sigma}$ for $c \in I_\sigma^{n,k}$ correspond to (r/s)-Birkhoff orbits of the annulus map L_b in $S_b^{n,k}$ (see [9]).

In what follows we shall fix a lifting $F_{b,\sigma}$ of the circle map $f_{b,\sigma}$ by setting $F_{b,\sigma} = \pi_1 \circ \widetilde{L}_b$.

COROLLARY 4.2. — *Let $z \in S_b^{n,k}$. Then for all $c \in I_\sigma^{n,k}$ we have that $\rho_{\widetilde{L}_b}(z) = \rho_{F_{c,\sigma}}(Z)$, if it exists, where $Z \in e^{-1}(\phi_\sigma(z))$.*

In view of the above corollary we see that the computation of the rotation set of $S_b^{n,k}$ reduces to the computation of the rotation set of $F_{b,\sigma \mid e^{-1}(\Lambda_{b,\sigma})}$. Unfortunately, this rotation set is different from the rotation interval of $F_{b,\sigma}$. However, from the family $f_{b,\sigma}$, it is possible to obtain a logistic family of circle maps of degree one such that they still have $\Lambda_{b,\sigma}$ as invariant set and the rotation interval of these maps coincides with the rotation set of $e^{-1}(\Lambda_{b,\sigma})$. This is achieved simply by modifying the maps $f_{b,\sigma}$ in such a way that they loose the differentiability at the endpoints of Δ. To be more precise, we define $h_c = h(\cdot, c, \bar{\delta}) : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ with $c \in [b_1, b_2]$ such that (see Figure 4.1) :

(ALS_1) h_c depends continuously on c.

(ALS_2) The map h_c satisfies (L_2) and (L_4) with $\Delta = \Delta_1$.

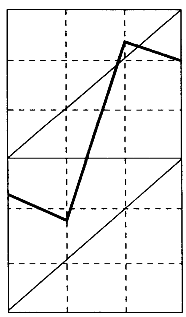

Figure 4.1. The logistic family of circle maps.

This family of maps was used by Alsedà, Llibre and Serra [2] to study the bifurcations of the Levi's circle maps at the level of the set of periods. In [1] the complete bifurcations diagram for the above family has been depicted.

In the rest of this section we shall use, for the family h_c (and their liftings H_c), the notation and definitions introduced in the preceding sections extended in the natural way.

From (ALS_2) it is easy to see that the unique h_c-invariant set strictly contained in Δ is $\Lambda_{c,\sigma}$. Moreover, if $c \in I_\sigma^{n,k}$, then by Proposition 4.1 and

Corollary 4.2 we have that the $\widetilde{L}_b$-rotation set of $S_b^{n,k}$ coincides with the H_c-rotation set of $e^{-1}(\Lambda_{c,\sigma})$. We note that by a change of variables, if necessary, we may assume that $e(0) = x_1(c)$ for each $c \in [b_1, b_2]$. Then we denote by $X_2(c)$ the unique element of $[0,1) \cap e^{-1}(x_2(c))$. Let $\widetilde{\Lambda}_c$ be the union of all (mod. 1) orbits of H_c contained in $e^{-1}(\Delta)$.

The next result states that the H_c-rotation set of $e^{-1}(\Lambda_{c,\sigma})$ coincides with R_{H_c}, which is the property we are looking for. It follows from Theorem B of [14], the proof of Theorem 2 of [6] and the Theorem B of [4].

THEOREM 4.3. — *For the maps $H_c \in \mathcal{L}$ we have :*

(a) *The maps $c \to a^-(H_c)$ and $c \to a^+(H_c)$ are continuous.*

(b) *Let $a \in R_{H_c}$. Then there exists a twist orbit of H_c with rotation number a contained in $e^{-1}(\Delta)$. That is, $R_{H_c} = R_{H_c}(\widetilde{\Lambda}_c)$.*

(c) *If $a^-(H_c) \in \mathbb{R} \setminus \mathbb{Q}$ (respectively $a^+(H_c) \in \mathbb{R} \setminus \mathbb{Q}$) then*

$$\{H_c^n(0) : n \in \mathbb{Z}^+\} \subset e^{-1}(\Delta)$$

and $\lim_{i\to\infty} i^{-1} H_c^i(0) = a^-(H_c)$ (respectively

$$\{H_c^i(X_2(c)) : i \in \mathbb{Z}^+\} \subset e^{-1}(\Delta)$$

and $\lim_{i\to\infty} i^{-1}(H_c^i(X_2(c)) - X_2(c)) = a^+(H_c)$.

The following two lemmas allow us to study the $\widetilde{L}_b$ rotation set of $S_b^{n,k}$. Let $U_{c,\sigma}$ be the unique element of $e^{-1}(u_{c,\sigma}) \cap [0,1)$.

LEMMA 4.4. — *Let $c \in I_\sigma^{n,k}$. Then $a^+(H_c) = 1$, $a^-(H_c) \in \mathbb{Q}$ and the $\widetilde{L}_b$-rotation set of $S_b^{n,k}$ is equal to $[a^-(H_c), 1]$.*

Proof. — Without loss of generality we may assume that $H_c(0) \in [0,1)$. Since $H_{c\,|\,e^{-1}(\mathbb{S}^1 \setminus \Delta)}$ is strictly decreasing we have that $(H_c)_r(X) = H_c(X_2(c))$ for all $X \in [X_2(c), 1]$. By Lemma 2.1 and (ALS$_2$) we see that $(H_c)_r(U_{c,\sigma}) = U_{c,\sigma} + 1$. Therefore, $a^+(H_c) = 1$.

We note that in the proof of Proposition 4.1 the definition of $I_\sigma^{n,k}$, the set $K_n^{c,\sigma}$ and the point $x_1(c)$ depend only on $f_{c,\sigma\,|\,\Delta}$. Hence, in view of the definition of the family h_c and since $c \in I_\sigma^{n,k}$, it follows that $h_c(x_1(c)) \in \mathrm{Int}(K_n^{c,\sigma})$. On the other hand, since $h_c(A_{c,\sigma}) = \mathbb{S}^1 \setminus \Delta_\sigma^L$ there exists $j \geq 0$ such that $H_c^j(0) \in e^{-1}(\mathbb{S}^1 \setminus \Delta_\sigma^L)$. Moreover, for each $X \in e^{-1}(\Delta \setminus \Delta_\sigma^L)$ there exists some $i \geq 0$ such that $H_c^i(X) \in e^{-1}(\mathbb{S}^1 \setminus \Delta)$ because $H_{c\,|\,e^{-1}(\Delta \setminus \Delta_\sigma^L)}$ is strictly increasing and $U_{c,\sigma}$ is a unstable fixed (mod. 1) point of H_c. Therefore, $H_c^j(0) \in e^{-1}(\mathbb{S}^1 \setminus \Delta)$ for some $j \geq 0$. Hence, from Theorem 4.3 (c) we get that $a^-(H_c) \in \mathbb{Q}$.

From the construction made in Section 3 we see that the definition of Δ_σ^L and $A_{c,\sigma}$ depend only on $f_{c,\sigma\,|\,\Delta}$. Thus, $e(\widetilde{\Lambda}_c) \subset \Delta_\sigma^L$. Since $f_{b,\sigma}(A_{c,\sigma}) = \mathbb{S}^1 \setminus \Delta_\sigma^L$, from (ALS2), we have that $e(\widetilde{\Lambda}_c) = \Lambda_{c,\sigma}$. So, from Corollary 4.2 and Theorem 4.3 (b) it follows that the $\widetilde{L}_b$-rotation set of $S_b^{n,k}$ is $[a^-(H_c), 1]$.

LEMMA 4.5. — *For each $a \in [0,1)$ there exists $c \in (\alpha_\sigma^k, \beta_\sigma^k]$ such that $a^-(H_c) = a$. Moreover, for each $c \in (\alpha_\sigma^k, \beta_\sigma^k]$ we have that $a^-(H_c) \in [0,1)$.*

Proof. — From the definitions of α_σ^k and β_σ^k we have that for $c \in (\alpha_\sigma^k, \beta_\sigma^k]$ we may assume, without loss of generality, that $H_c(0)$ is an element of $(U_{c,\sigma} - 1, U_{c,\sigma})$.

We recall that, for $c \in (b_{0,k}^\sigma, \beta_\sigma^k]$ we have $h_c(x_1(b)) \in \mathbb{S}^1 \setminus \Delta_\sigma^L$. Then, $H_c(0) < 0$. Thus, there exists $\widetilde{U}_{c,\sigma} \in [0, U_{c,\sigma}]$ such that $H_c(\widetilde{U}_{c,\sigma}) = \widetilde{U}_{c,\sigma}$. We have that $H_c(1) = H_c(0) + 1 > U_{c,\sigma} \geq \widetilde{U}_{c,\sigma}$. Therefore, by the definition of H_c we have $(H_c)_\ell(\widetilde{U}_{c,\sigma}) = H_c(\widetilde{U}_{c,\sigma})$. So, $a^-(H_c) = 0$.

Let $c = \alpha_\sigma^k$. Then, $H_c(0) = U_{c,\sigma}$. Clearly, $(H_c)_\ell(U_{c,\sigma}) = H_c(U_{c,\sigma}) = U_{c,\sigma} + 1$. Thus, $a^-(H_c) = 1$. Then, in view of Theorem 4.3 (a), the first statement of the lemma follows.

Since $H_c(0) < U_{c,\sigma}$ it is not difficult to see that $(H_c)_\ell(X) < X + 1$ for all $X \in \mathbb{R}$ (recall that $H_c(U_{c,\sigma})) \in \{U_{c,\sigma}, U_{c,\sigma} + 1\}$). Hence $a^-(H_c) < 1$.

Proof of Theorem 1.2 (c). — From Lemma 4.4 and Remark 2.2, we get that the $\widetilde{P}_b$-rotation set of $C_b^{n,k}$ for $c \in I_\sigma^{n,k}$ is the closed interval $[2\pi/(2q + 1 - 2a^-(H_c)), 2\pi/(2q - 1)]$. Then Theorem 1.2 (c) follows from Lemma 4.5. □

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