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REMARKS ON HOLMGREN'S UNIQUENESS THEOREM

by Lars HÖRMANDER

1. Introduction.

Holmgren's uniqueness theorem states that a solution of a linear differential equation with (real) analytic coefficients must vanish in a full neighborhood of a non-characteristic surface if it vanishes on one side. The original proof of Holmgren [5] for the case of two variables was given for n variables by John [9] who also treated C^1 surfaces by a geometric deformation argument. (John considered only classical solutions, but the result was later extended first to distribution solutions and then to hyperfunction solutions.) The method of Holmgren consisted in proving that certain mean values of solutions over non-characteristic surfaces depending analytically on a parameter λ must vanish if they vanish initially. Fritz John observed that they are always analytic functions of λ, and by developing this idea he proved in particular the analyticity of solutions of linear elliptic differential equations with analytic coefficients.

It was realized by the author [6] and Kawai [11] that one can reverse this argument and consider the Holmgren theorem as a combination of a (microlocal) analyticity theorem for solutions of a linear differential equation with analytic coefficients, and a general relation between the support and the analytic singularities of a distribution (hyperfunction). To state the result we must first recall the notion of exterior conormal set of an arbitrary closed set, as given in [7, Def. 8.5.7]:

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Definition 1.1. — *If F is a closed subset of a C^2 manifold X, then the exterior conormal set $N_e(F) \subset T^*(X) \setminus 0$ is defined as the set of all (x^0, ξ^0) such that $x^0 \in F$ and there exists a real valued function $f \in C^2(X)$ with $df(x^0) = \xi^0 \neq 0$ and*

$$f(x) \leqslant f(x^0) \quad \text{when } x \in F \cap U, \tag{1.1}$$

for some open neighborhood U of x^0.

Naturally it suffices to have f defined in U. The symmetric conormal set $N(F)$ is defined by allowing $df(x^0)$ to be $-\xi^0$ also. The classical Holmgren argument can be adapted to a proof that

$$N(\operatorname{supp} u) \subset WF_A(u), \tag{1.2}$$

where $WF_A(u)$ is the analytic wave front set of u. (A definition is given for distributions in [7, Section 8.4] and for hyperfunctions in [7, Section 9.4]; in the second case the original definition was given in Sato-Kawai-Kashiwara [13], where the notation $SS(u)$ is used.) A proof of (1.2) is given in [7, Theorem 8.5.6′] for distributions, and it extends to hyperfunctions as indicated in [7, Section 9.3]. We shall give another in Section 2. The deformation argument of John [9] proves that if $x^0 \in F$ and (1.1) is valid with $f \in C^1$, $df(x^0) = \xi^0 \neq 0$, then $(x^0, \xi^0) \in \overline{N_e(F)}$. Since $WF_A(u)$ is closed it follows that (1.2) remains valid if one allows C^1 functions in the definition of the conormal set.

The Holmgren-John theorem is obtained by combining (1.2) with the non-characteristic regularity theorem which states that

$$Pu = 0 \Longrightarrow WF_A(u) \subset \operatorname{Char} P \tag{1.3}$$

if P is a linear differential operator with analytic coefficients and characteristic set $\operatorname{Char} P$. This gives

$$Pu = 0 \Longrightarrow N(\operatorname{supp} u) \subset \operatorname{Char} P. \tag{1.4}$$

As observed above we may allow C^1 functions in the definition of the conormal set here.

The advantage of the splitting of Holmgren's theorem just outlined is that any improvement of the two component facts (1.2), (1.3) can be combined to give an improved uniqueness theorem. For example, combining (1.2) with results on the propagation of $WF_A(u)$ when u satisfies a partial differential equation gives unique continuation across some characteristic surfaces. This was the original motivation in [6].

In Section 2 we shall review earlier results related to (1.2) and prove some new refinements. In Section 3 we then recall some purely geometric properties of the conormal set, and in Section 4 we state some uniqueness theorems for solutions of analytic differential equations which follow. Section 5 is devoted to a uniqueness theorem for a second order hyperbolic equation with non-analytic coefficients which are translation invariant in time. The result is close to what Holmgren's theorem gives in the analytic case but so far it is an open question if (1.4) holds with no change. In a final Section 6 we have collected miscellaneous related remarks.

The author wants to thank Jan Boman at the University of Stockholm for suggesting the example discussed at the end of Section 2. He has also made various conjectures which would improve the results in Section 2 and in particular settle the problem left open in that example. In addition he raised the question whether Theorem 6.1 is true. It turned out that there was a proof in unpublished notes of lectures given by the author in Stockholm 1974; it is included here in a somewhat simplified form.

2. The analytic wave front set and the support.

The basic inclusion (1.2) is a consequence of the following much stronger result:

THEOREM 2.1 (Kashiwara). — *If u is a hyperfunction in a real analytic manifold and $(x^0, \xi^0) \in N_e(\operatorname{supp} u)$, then*

$$(2.1) \quad (x^0, \xi) \in WF_A(u) \Longrightarrow (x^0, \xi + t\xi^0) \in WF_A(u) \quad \text{for all } t \in \mathbf{R} \text{ with } \xi + t\xi^0 \neq 0.$$

For a proof see [7, Corollary 9.6.8]; another will be given below. Theorem 2.1 implies (1.2), for if x^0 is a boundary point of $\operatorname{supp} u$ we must have $(x^0, \xi) \in WF_A(u)$ for some ξ since u would otherwise be analytic in a neighborhood of x^0. Since $WF_A(u)$ is conic it follows from (2.1) that $(x^0, \xi/|t| + t\xi^0/|t|) \in WF_A(u)$ for large t, and when $t \to \pm\infty$ we obtain $(x^0, \pm\xi^0) \in WF_A(u)$, for this is a closed set.

Thus the fiber $WF_A(u)_{x^0}$ can be identified with a closed conic subset of $T^*_{x^0}/\mathbf{R}\xi^0$. At a corner we may have $(x^0, \xi) \in N_e(\operatorname{supp} u)$ for several vectors ξ. Then Theorem 2.1 yields that $WF_A(u)_{x^0}$ is invariant under translation along the vector space Ξ^0 generated by these covectors, so it can be regarded as a closed cone in $T^*_{x^0}/\Xi^0$.

Theorem 2.1 is only a special case of Kashiwara's theorem. To state it in full generality we need another definition (see [7, Vol. I, pp. 364–365]).

DEFINITION 2.2. — *If $x^0 \in M \subset \mathbf{R}^n$ then the tangent cone $T_{x^0}(M)$ is defined as the set of limits of sequences $t_j(x_j - x^0)$ where $t_j \to +\infty$ and $x_j \in M$.*

It is clear that $T_{x^0}(M)$ is a closed cone, and it is easy to show (see [7]) that it is invariant under a C^1 change of coordinates. Thus $T_{x^0}(M)$ is invariantly defined as a closed cone $\subset T_{x^0}(X)$ if M is a subset of a C^1 manifold X.

THEOREM 2.3 (Kashiwara). — *If u is a hyperfunction in a real analytic manifold X, $x^0 \in X$, and $W_0 = WF_A(u)_{x^0}$ is considered as a subset of the vector space $T^*_{x^0}(X)$ with the origin removed, then*

$$N(W_0) \subset \partial W_0 \times T_{x^0}(\operatorname{supp} u). \tag{2.2}$$

This is [7, Theorem 9.6.6]. Note that if $(x^0, \xi^0) \in N_e(\operatorname{supp} u)$, then

$$T_{x^0}(\operatorname{supp} u) \subset \{t \in T_{x^0}(X), \langle t, \xi^0\rangle \leqslant 0\}.$$

This is also true if C^1 functions are allowed in the definition of N_e. Thus it follows from (2.2) that the component of the symmetric conormal set $N(W_0)$ in $T_{x^0}(X)$ is in the orthogonal space of ξ^0. Hence the standard consequences of Holmgren's theorem are valid as if we were considering the continuation of a solution of a differential equation with constant coefficients for which all characteristic hyperplanes contain a fixed direction. This implies that if an open set does not intersect W_0 then this remains true for the cylinder it generates in the ξ^0 direction, so Theorem 2.1 follows even with N_e defined using C^1 functions. For an even more general version of Kashiwara's theorem we refer to [7, Theorem 9.6.6′].

If the boundary of $\operatorname{supp} u$ is in C^1, the preceding results do not strengthen (1.2). For example, if the support of u is $\{x \in \mathbf{R}^2; x_2 \geqslant |x_1|^{1+a}\}$ for some $a \in (0,1)$ it only follows that $WF_A(u)_0 \supset \{(0,\xi_2); \xi_2 \neq 0\}$. However, we shall see that $WF(u)_0 = T^*_0(\mathbf{R}^2) \setminus \{0\}$ as a very special case of results taking curvature into account. Before stating them we shall make some technical preparations which allow us to analyze $WF_A(u)$ at exposed boundary points using the Fourier-Laplace transformation.

Let u be a hyperfunction with compact support in $\mathbf{R}^n$, that is, an analytic functional carried by a compact subset of $\mathbf{R}^n$, and let

$$H(\xi) = \sup_{x \in \operatorname{supp} u} \langle x, \xi \rangle, \quad \xi \in \mathbf{R}^n,$$

be the supporting function of $\operatorname{supp} u$. Then the Fourier-Laplace transform defined by $\hat{u}(\zeta) = u(e^{-i\langle \cdot, \zeta \rangle})$ has a bound

$$|\hat{u}(\zeta)| \leqslant \exp(C_\varepsilon + \varepsilon|\zeta| + H(\operatorname{Im} \zeta)), \quad \zeta \in \mathbf{C}^n, \tag{2.3}$$

for every $\varepsilon > 0$, so the plurisubharmonic and positively homogeneous indicator function

$$j_{\hat{u}}(\zeta) = \varlimsup_{\theta \to \zeta} \varlimsup_{t \to +\infty} t^{-1} \log |\hat{u}(t\theta)|$$

has the bound

$$j_{\hat{u}}(\zeta) \leqslant H(\operatorname{Im} \zeta), \quad \zeta \in \mathbf{C}^n. \tag{2.4}$$

By the three line theorem

$$h(\eta) = \sup_{\xi \in \mathbf{R}^n} j_{\hat{u}}(\xi + i\eta), \quad \eta \in \mathbf{R}^n,$$

is positively homogeneous and convex, so h is the supporting function of a convex compact set K, and (2.3) holds with H replaced by h. By the analogue of the Paley-Wiener theorem for analytic functionals, due to Pólya and Ehrenpreis, it follows that $\operatorname{supp} u \subset K$. Hence $H \leqslant h \leqslant H$ so $H = h$. From the Phragmén-Lindelöf theorem and the fact that $j_{\hat{u}} \leqslant 0$ in $\mathbf{R}^n$ it follows if $\operatorname{Im} z_0 > 0$ that

$$j_{\hat{u}}(\xi + z\eta) \leqslant \frac{\operatorname{Im} z}{\operatorname{Im} z_0} j_{\hat{u}}(z_0\eta), \quad \operatorname{Im} z \geqslant 0,$$

for $\varlimsup_{t \to +\infty} j_{\hat{u}}(\xi + tz_0\eta)/t \leqslant j_{\hat{u}}(z_0\eta)$ since $j_{\hat{u}}$ is upper semi-continuous. Hence

$$\begin{aligned} H(\operatorname{Im} z\eta) \leqslant \frac{\operatorname{Im} z}{\operatorname{Im} z_0} j_{\hat{u}}(z_0\eta) &\leqslant \frac{\operatorname{Im} z}{\operatorname{Im} z_0} H(\operatorname{Im} z_0\eta) \\ &= \operatorname{Im} z H(\eta) = H(\operatorname{Im} z\eta), \quad \operatorname{Im} z > 0, \end{aligned}$$

which proves that $j_{\hat{u}}(z_0\eta) = \operatorname{Im} z_0 H(\eta) = H(\operatorname{Im} z_0\eta)$ when $\operatorname{Im} z_0 > 0$. By the upper semicontinuity of plurisubharmonic functions it follows that

$j_{\hat{u}}(z_0\eta) \geqslant H(\operatorname{Im} z_0\eta) = 0$ when $\operatorname{Im} z_0 = 0$, and since the opposite inequality holds by (2.4), we conclude that

$$j_{\hat{u}}(\zeta) = H(\operatorname{Im}\zeta), \quad \zeta \in \mathbf{C}\mathbf{R}^n. \tag{2.5}$$

In addition to these basic and well known facts we need a supplement to the Paley-Wiener theorem (see [14, Theorem 2.3.1]):

PROPOSITION 2.4. — *Let u be a hyperfunction in $\mathbf{R}^n$ with compact support, and let H be the supporting function of* $\operatorname{supp} u$. *Let $\xi^0, \eta^0 \in \mathbf{R}^n \setminus \{0\}$. Then*

$$j_{\hat{u}}(\xi^0 + i\eta^0) = H(\eta^0) \tag{2.6}$$

if and only if $(x, \xi^0) \in WF_A(u)$ for some $x \in \operatorname{supp} u$ with $\langle x, \eta^0\rangle = H(\eta^0)$.

Note that if $x^0 \in \operatorname{supp} u$ and $\operatorname{supp} u \subset B$ where B is a ball with $x^0 \in \partial B$, it follows that $(x^0, \pm\eta^0) \in WF_A(u)$ if η^0 is the exterior conormal of B at x^0. In fact, (2.6) follows from (2.5) if $\xi^0 = \pm\eta^0$, and since x^0 is the only point $x \in \operatorname{supp} u$ with $\langle x, \eta^0\rangle = H(\eta^0)$, the assertion follows. By a localization and a change of local coordinates we obtain (1.2). Also (2.1) is an immediate consequence. In fact, if $j_{\hat{u}}(\xi^0 + i\eta^0) = H(\eta^0)$, then

$$v(z) = j_{\hat{u}}(\xi^0 + z\eta^0) - \operatorname{Im} z H(\eta^0)$$

is a subharmonic function of z with $v(z) \leqslant 0$ when $\operatorname{Im} z > 0$, and since $v(i) = 0$ it follows that $v(z) = 0$ when $\operatorname{Im} z > 0$. Hence $j_{\hat{u}}(\xi^0 + t\eta^0 + i\eta^0) = H(\eta^0)$ for every $t \in \mathbf{R}$. If $\{x; x \in \operatorname{supp} u; \langle x, \eta^0\rangle = H(\eta^0)\}$ consists of a single point x^0, it follows by Proposition 2.4 that $(x^0, \xi^0) \in WF_A(u)$ implies $(x^0, \xi^0 + t\eta^0) \in WF_A(u)$ for every $t \in \mathbf{R}$. Localization and a change of local coordinates (and notation) gives (1.2) and also (2.1) in full generality.

Our improvements of (1.2) and (2.1) will depend on an analysis of the condition (2.6). First we prove a lemma:

LEMMA 2.5. — *If φ is a subharmonic function and $\varphi \leqslant 0$ in $\mathbf{C}_+ = \{z \in \mathbf{C}; \operatorname{Im} z > 0\}$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(z + re^{i\theta})\, d\theta \geqslant C(z, r) t\varphi(ti), \quad t \geqslant 1, \tag{2.7}$$

with a constant $C(z, r)$ which is locally bounded when $z \in \mathbf{C}_+$ and $0 < r < \operatorname{Im} z$.

Proof. — By the Riesz representation formula

$$\varphi(z) = \gamma \mathrm{Im}\, z + \int_{\mathbf{R}} \frac{\mathrm{Im}\, z}{|z-x|^2}\, d\sigma(x) + \int_{\mathbf{C}_+} \log \left| \frac{z-\zeta}{\bar{z}-\zeta} \right| d\mu(\zeta), \quad z \in \mathbf{C}_+,$$

where $\gamma \leqslant 0$, $d\sigma \leqslant 0$ and $d\mu \geqslant 0$. Normalizing φ we can assume that $t\varphi(ti) = -1$, that is,

$$-t^2\gamma - \int_{\mathbf{R}} \frac{t^2}{x^2+t^2}\, d\sigma(x) + t \int_{\mathbf{C}_+} \log \left| \frac{ti+\zeta}{ti-\zeta} \right| d\mu(\zeta) = 1,$$

where all terms are non-negative so each of them is $\leqslant 1$. We have for $0 < r < \mathrm{Im}\, z$

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(z + re^{i\theta})\, d\theta = \gamma \mathrm{Im}\, z + \int_{\mathbf{R}} \frac{\mathrm{Im}\, z}{|z-x|^2}\, d\sigma(x) + \int_{\mathbf{C}_+} \log \frac{\max(r, |z-\zeta|)}{|\bar{z}-\zeta|}\, d\mu(\zeta).$$

Here $\gamma \geqslant -1$ when $t \geqslant 1$, and since

$$\frac{\mathrm{Im}\, z}{|z-x|^2} \leqslant \frac{1+2|z|^2}{\mathrm{Im}\, z} \frac{1}{1+x^2} \leqslant \frac{1+2|z|^2}{\mathrm{Im}\, z} \frac{t^2}{x^2+t^2}, \quad t \geqslant 1,$$

we conclude that the second term is $\geqslant -(1+2|z|^2)/\mathrm{Im}\, z$. Since $\mathrm{Im}\, \zeta/(1+|\zeta|^2) = A$ implies

$$|\zeta+i|^2/|\zeta-i|^2 = (|\zeta|^2+1+2\mathrm{Im}\, \zeta|)/(|\zeta|^2+1-2\mathrm{Im}\, \zeta) = (1+2A)/(1-2A),$$

and $(1+2A)/(1-2A) \geqslant e^{4A}$, $0 \leqslant 2A < 1$, by Taylor's formula, we have $2 \log |(\zeta+i)/(\zeta-i)| \geqslant 4\mathrm{Im}\, \zeta/(1+|\zeta|^2)$, hence

$$t \log \left| \frac{ti+\zeta}{ti-\zeta} \right| = t \log \left| \frac{i+\zeta/t}{i-\zeta/t} \right| \geqslant 2\mathrm{Im}\, \zeta/(1+|\zeta/t|^2) \geqslant 2\mathrm{Im}\, \zeta/(1+|\zeta|^2), \quad t \geqslant 1,\ \zeta \in \mathbf{C}_+,$$

so it remains to prove that

$$\log \frac{|\bar{z}-\zeta|}{\max(r, |z-\zeta|)} \leqslant 2C(z,r) \frac{\mathrm{Im}\, \zeta}{1+|\zeta|^2}, \quad \zeta \in \mathbf{C}_+.$$

The left-hand side is harmonic when $|z-\zeta| \neq r$ and the right-hand side is superharmonic, so it suffices to verify this when $|z-\zeta| = r$ and on $\mathbf{R}$ and at ∞. Thus the inequality follows with

$$C(z,r)=\log\Big(\frac{r+2\mathrm{Im}\,z}{r}\Big)\frac{1+2r^2+2|z|^2}{\mathrm{Im}\,z-r}.$$

PROPOSITION 2.6. — *Let H be a supporting function in $\mathbf{R}^n$ and let j be a plurisubharmonic function in $\mathbf{C}^n$ such that*

$$j(\zeta)\leqslant H(\mathrm{Im}\,\zeta),\quad \zeta\in\mathbf{C}^n;\quad j(\zeta)=H(\mathrm{Im}\,\zeta),\quad \zeta\in\mathbf{C}\mathbf{R}^n. \tag{2.8}$$

If $\eta,\theta\in\mathbf{R}^n$ are linearly independent and H is flat at η in the direction θ in the sense that

$$\varliminf_{(\tau,\tilde\theta)\to(0,\theta)}(H(\eta+\tau\tilde\theta)+H(\eta-\tau\tilde\theta)-2H(\eta))/\tau^2=0, \tag{2.9}$$

then for $\xi\in\mathbf{R}^n$

$$j(t\theta+\xi+z\eta)=H(\mathrm{Im}\,z\eta),\quad \textit{if } t\in\mathbf{R},\ \mathrm{Im}\,z>0,\ \textit{and } j(\xi+i\eta)=H(\eta). \tag{2.10}$$

In particular,

$$j(t\theta+z\eta)=H(\mathrm{Im}\,z\eta),\quad \textit{if } t\in\mathbf{R},\ \mathrm{Im}\,z>0, \tag{2.11}$$

which is also true if instead of (2.9) *we only have*

$$\varliminf_{(\tau,\tilde\eta,\tilde\theta)\to(0,\eta,\theta)}(H(\tilde\eta+\tau\tilde\theta)+H(\tilde\eta-\tau\tilde\theta)-2H(\tilde\eta))/\tau^2=0. \tag{2.9$'$}$$

Proof. — Assuming first that (2.9) is valid, we choose sequences $\tau_\nu\downarrow 0$ and $\theta_\nu\to\theta$ such that

$$H(\eta+\tau_\nu\theta_\nu)+H(\eta-\tau_\nu\theta_\nu)-2H(\eta)\leqslant\tau_\nu^2/\nu,\quad \nu=1,2,\dots. \tag{2.12}$$

Since $\tau\mapsto H(\eta+\tau\theta_\nu)$ is convex, we can choose $c_\nu\in\mathbf{R}$ such that

$$H(\eta+\tau\theta_\nu)-H(\eta)\geqslant c_\nu\tau,\quad \tau\in\mathbf{R},$$

and then it follows that

$$0\leqslant H(\eta\pm\tau_\nu\theta_\nu)-H(\eta)\mp c_\nu\tau_\nu\leqslant\tau_\nu^2/\nu. \tag{2.12$'$}$$

With $t_\nu=1/\tau_\nu$ and $\xi\in\mathbf{R}^n$ set

$$F_\nu(w,z)=t_\nu(j(w\theta_\nu+\xi+t_\nu z\eta)-t_\nu\mathrm{Im}\,zH(\eta)-c_\nu\mathrm{Im}\,w),\quad (w,z)\in\mathbf{C}^2.$$

By (2.8) and (2.12)′ we have if $\operatorname{Im} z > 0$

$$\begin{aligned} F_\nu(w,z) &\leqslant t_\nu(H(\operatorname{Im} w\theta_\nu + t_\nu \operatorname{Im} z\eta) - t_\nu \operatorname{Im} zH(\eta) - c_\nu \operatorname{Im} w) \\ &= t_\nu^2 \operatorname{Im} z\big(H((\tau_\nu \operatorname{Im} w/\operatorname{Im} z)\theta_\nu + \eta) - H(\eta) - c_\nu\tau_\nu \operatorname{Im} w/\operatorname{Im} z\big) \\ &\leqslant \nu^{-1}(\operatorname{Im} w)^2/\operatorname{Im} z = \operatorname{Im} z/\nu, \qquad \operatorname{Im} w = \pm \operatorname{Im} z. \end{aligned} \tag{2.13}$$

Assume as in (2.10) that $j(\xi+i\eta) = H(\eta)$. This means that $F_\nu(0, i/t_\nu) = 0$. Since $F_\nu(0,z) = t_\nu(j(\xi + t_\nu z\eta) - t_\nu \operatorname{Im} zH(\eta)) \leqslant 0$ by (2.8), it follows that $F_\nu(0,z) = 0$ for every z in the upper half plane $\mathbf{C}_+$. Fix $z = i$ for the sake of convenience. Since $F_\nu(0,i) = 0$ and the upper bound (2.13) remains valid in $\mathbf{C}_z = \{w \in \mathbf{C}; |\operatorname{Im} w| < \operatorname{Im} z\}$ by the three line theorem, there is a subsequence $F_{\nu_k}(w,i)$ converging in $\mathcal{D}'(\mathbf{C}_i)$, and the limit is a subharmonic function $f(w)$ with $f(0) = 0$ and $f(w) \leqslant 0$, $w \in \mathbf{C}_i$. But this implies that $f \equiv 0$ in $\mathbf{C}_i$. The uniqueness of the limit proves that it was not necessary to pass to a subsequence, so $F_\nu(w,i) \to 0$ in $\mathcal{D}'(\mathbf{C}_i)$. Hence $\varlimsup_{\nu\to\infty} F_\nu(w,i) = 0$ for $w \in \mathbf{R} \setminus E$ where E has measure zero (in fact exterior capacity zero). For the non-positive subharmonic functions

$$\varphi_\nu(z) = j(w\theta_\nu + \xi + z\eta) - \operatorname{Im} zH(\eta), \quad z \in \mathbf{C}_+,$$

we have $\varlimsup_{\nu\to\infty} t_\nu\varphi_\nu(t_\nu i) = \varlimsup_{\nu\to\infty} F_\nu(w,i) = 0$ if $w \in \mathbf{R} \setminus E$. Hence we conclude using Lemma 2.5 that the upper limit of every circular mean value of φ_ν in the upper half plane is equal to 0. Since j is upper semicontinuous and $\varphi_\nu \leqslant 0$, Fatou's lemma proves that the mean values of $z \mapsto j(w\theta + \xi + z\eta) - \operatorname{Im} zH(\eta)$ are equal to 0 when $w \in \mathbf{R} \setminus E$. Thus $j(w\theta + \xi + z\eta) = \operatorname{Im} zH(\eta)$ for $z \in \mathbf{C}_+$ and all $w \in \mathbf{R} \setminus E$. By the upper semi-continuity of j and (2.8) we conclude that this is true for $w \in \mathbf{R}$, which completes the proof of (2.10).

If we only assume that (2.9)′ holds, we can choose sequences $\eta_\nu \to \eta$, $\theta_\nu \to \theta$, $\tau_\nu \downarrow 0$, and c_ν such that (2.12) and (2.12)′ hold with η replaced by η_ν. With $t_\nu = 1/\tau_\nu$ and

$$F_\nu(w,z) = t_\nu(j(w\theta_\nu + t_\nu z\eta_\nu) - t_\nu \operatorname{Im} zH(\eta_\nu) - c_\nu \operatorname{Im} w),$$

we obtain as before

$$F_\nu(w,z) \leqslant \operatorname{Im} z/\nu, \quad |\operatorname{Im} w| \leqslant \operatorname{Im} z.$$

By (2.8) we have $F_\nu(0,z) = 0$ when $\operatorname{Im} z > 0$; in particular when $z = i$. Hence the same argument as before shows that $\varlimsup_{\nu\to\infty} F_\nu(w,i) = 0$ for

$w \in \mathbf{R} \setminus E$, where E has measure 0. Applying Lemma 2.5 to the non-positive subharmonic functions

$$\varphi_\nu(z) = j(w\theta_\nu + z\eta_\nu) - \operatorname{Im} zH(\eta_\nu), \quad z \in \mathbf{C}_+,$$

as in the first part of the proof, we can now conclude that $j(w\theta + z\eta) = \operatorname{Im} zH(\eta)$ for arbitrary $z \in \mathbf{C}_+$ and $w \in \mathbf{R}$, and this completes the proof.

COROLLARY 2.7. — *Let u be a hyperfunction with* $\operatorname{supp} u \subset B$, *where $B \subset \mathbf{R}^n$ is a ball, let $x^0 \in \partial B \cap \operatorname{supp} u$ and let η be the exterior conormal of B at x^0. If H is the supporting function of* $\operatorname{supp} u$ *then*

$$(x^0, \xi + t\theta) \in WF_A(u), \tag{2.14}$$

if $(x^0, \xi) \in WF_A(u)$, (2.9) is valid, and $t \in \mathbf{R}$, $\xi + t\theta \neq 0$. Moreover,

$$(x^0, \theta) \in WF_A(u), \quad \textit{if } (2.9)' \textit{ is valid.} \tag{2.15}$$

Proof. — This follows from Proposition 2.6 by the arguments given after Proposition 2.4.

If $(x^0, \xi^0) \in N_e(\operatorname{supp} u)$ we can always localize and change coordinates so that the fiber of $WF_A(u)$ at x^0 can be analyzed using Corollary 2.7. However, to state the result we must examine to what extent it depends on the local coordinates chosen. In the following preliminary lemma we have chosen $x^0 = 0$ and $\eta = (0, \dots, 0, 1)$; later on K will be $\operatorname{supp} u$.

LEMMA 2.8. — *Let $K \subset \mathbf{R}^n$ be a compact set such that $0 \in K$ and $x_n \leqslant -c|x'|^2$ for some $c > 0$ when $x = (x', x_n) \in K$; here $x' = (x_1, \dots, x_{n-1})$. Then*

$$0 \leqslant H_K(\theta', 1) = \sup_{x \in K}(\langle x', \theta' \rangle + x_n) \leqslant |\theta'|^2/4c, \quad \theta' \in \mathbf{R}^{n-1}; \tag{2.16}$$

if $x \in K$ and $\langle x', \theta' \rangle + x_n > 0$ then $|x' - \theta'/2c|^2 < |\theta'|^2/4c^2$, hence $|x'| < |\theta'|/c$, and $0 \geqslant x_n > -|\theta'|^2/c$. We have the Lipschitz continuity

$$H_K(\theta_1', 1) \leqslant H_K(\theta_2', 1) + |\theta_1' - \theta_2'||\theta_1'|/c, \quad \theta_1', \theta_2' \in \mathbf{R}^{n-1}. \tag{2.17}$$

Proof. — Since $0 \in K$ we have $H_K(\theta', 1) \geqslant 0$, and $H_K(\theta', 1) \leqslant |\theta'|^2/4c$ since

$$\langle x', \theta' \rangle - c|x'|^2 = |\theta'|^2/4c - c|x' - \theta'/2c|^2.$$

If $\langle x', \theta'\rangle + x_n > 0$ and $x_n \leqslant -c|x'|^2$ then $|x' - \theta'/2c|^2 < |\theta'|^2/4c^2$, hence $|x'| < |\theta'|/c$, and $x_n > -\langle x', \theta'\rangle > -|\theta'|^2/c$. We also obtain

$$\langle x', \theta'\rangle - \langle x', \theta'_2\rangle \leqslant |\theta' - \theta'_2||\theta'|/c,$$

which proves (2.17).

From (2.17) it follows at once that if $\theta_\nu \to \theta$, $\tau_\nu \to 0$ and $H_K(\tau_\nu\theta'_\nu, 1+\tau_\nu\theta_{\nu n})/\tau_\nu^2 \to 0$, then $H_K(\tau_\nu\theta', 1)/\tau_\nu^2 \to 0$. Under the hypotheses of the lemma the condition (2.9) is therefore not strengthened if we take $\tilde{\theta} = \theta$ fixed $= (\theta', 0)$.(1) With the notation in the lemma and $\theta = (\theta', 0)$ the condition then becomes

$$\underline{\lim}_{\tau\to 0}(H_K(\tau\theta', 1) + H_K(-\tau\theta', 1))/\tau^2 = 0. \tag{2.9$''$}$$

Before examining the invariance of (2.9)$''$ under changes of coordinates we shall discuss the stronger condition

$$\lim_{\tau\to 0}(H(\eta + \tau\theta) + H(\eta - \tau\theta) - 2H(\eta))/\tau^2 = 0, \tag{2.9$'''$}$$

which is much easier to do. Under the hypotheses in Lemma 2.8, for $\eta = (0, \dots, 0, 1)$, this condition is equivalent to

$$\lim_{\tau\to 0} H(\tau\theta', 1)/\tau^2 = 0, \tag{2.9$''''$}$$

or explicitly, that for every $\varepsilon > 0$ we can find $\delta_\varepsilon > 0$ such that

$$H(\tau\theta', 1) \leqslant \varepsilon\tau^2, \quad |\tau| < \delta_\varepsilon.$$

Thus $\langle x', \tau\theta'\rangle + x_n \leqslant \varepsilon\tau^2$ if $x \in K$ and $|\tau| < \delta_\varepsilon$, which implies that

$$x_n \leqslant -\langle x', \theta'\rangle^2/4\varepsilon, \quad \text{if } x \in K \text{ and } |\langle x', \theta'\rangle| < 2\varepsilon\delta_\varepsilon.$$

Conversely, if this is true then $\langle x', \tau\theta'\rangle + x_n \leqslant \varepsilon\tau^2$ if $x \in K$ and $|\langle x', \theta'\rangle| < 2\varepsilon\delta_\varepsilon$. If $x \in K$ and $|\langle x', \theta'\rangle| \geqslant 2\varepsilon\delta_\varepsilon$ then $|x'| \geqslant 2\varepsilon\delta_\varepsilon/|\theta'|$, so $\langle x', \tau\theta'\rangle + x_n \leqslant 0$ by Lemma 2.8 if $2\varepsilon\delta_\varepsilon/|\theta'| \geqslant |\tau\theta'|/c$. Thus $H(\tau\theta', 1) \leqslant \varepsilon\tau^2$ if $|\tau| < 2c\varepsilon\delta_\varepsilon/|\theta'|^2$, so we have proved that (2.9)$'''$ is true if and only if for every $\kappa > 0$ there is a neighborhood U_κ of the origin such that

$$x_n \leqslant -\kappa\langle x', \theta'\rangle^2, \quad x \in K \cap U_\kappa. \tag{2.18}$$

(1) Lemma 2.5 is then not needed in the proof of the first part of Proposition 2.6. It suffices to note that $\overline{\lim}_{t\to\infty} t\varphi(ti) = 0$ implies $\varphi = 0$ if $\varphi \leqslant 0$ is subharmonic in $\mathbf{C}_+$.

(Note that x_n has a negative upper bound in $K \cap \complement U_\kappa$ so (2.18) is valid for every $x \in K$ with $|\langle x', \theta' \rangle|$ small enough.)

We shall now give this condition an invariant form.

PROPOSITION 2.9. — *Let F be a closed subset of a C^2 manifold X, and let $(x^0, \xi^0) \in N_e(F)$. Then the set G_{x^0,ξ^0} of all $f \in C^2(X)$ such that $f(x^0) = 0, f'(x^0) = \xi^0$, and (1.1) holds for some open neighborhood U of x^0, depending on f, is a convex set. If $f_0 \in G_{x^0,\xi^0}$ then*

$$Q_{x^0,\xi^0} = \{(f - f_0)''(x^0); f \in G_{x^0,\xi^0}\} \tag{2.19}$$

is a convex set in the symmetric tensor product $S^2(T^*_{x^0}(X))$ consisting of quadratic forms on $T_{x^0}(X)$, and it is independent of the choice of f_0 apart from a translation. The asymptotic cone $\widehat{Q}_{x^0,\xi^0}$ is a closed convex cone independent of the choice of f_0, and it contains all negative semidefinite forms. The positive semidefinite forms of maximal rank in $\widehat{Q}_{x^0,\xi^0}$ have the same radical, and the annihilator $\varrho(x^0, \xi^0)$ in $T^*_{x^0}$ contains ξ^0 and $-\xi^0$.

Proof. — If $f_0, f_1 \in G_{x^0,\xi^0}$ then it is obvious that $\lambda f_0 + (1 - \lambda) f_1 \in G_{x^0,\xi^0}$, and we have

$$Q_{x^0,\xi^0} = \{(f - f_1)''(x^0); f \in G_{x^0,\xi^0}\} + (f_1 - f_0)''(x^0).$$

(Note that the second differential is invariantly defined when the first differential vanishes.) If $f \in G_{x^0,\xi^0}$ and g is a negative semidefinite quadratic form in $x - x^0$, then $f + g \in G_{x^0,\xi^0}$ which proves that $Q_{x^0,\xi^0} + g'' \subset Q_{x^0,\xi^0}$. In particular, Q_{x^0,ξ^0} has interior points. We can choose f_0 so that the origin is an interior point. Then

$$\widehat{Q}_{x^0,\xi^0} = \{\gamma; \mathbf{R}_+\gamma \subset Q_{x^0,\xi^0}\}, \tag{2.20}$$

which is a closed convex cone with $\gamma_1 + \widehat{Q}_{x^0,\xi^0} \subset Q_{x^0,\xi^0}$ for every interior point γ_1 in Q_{x^0,ξ^0}. In fact, if $\gamma \in \widehat{Q}_{x^0,\xi^0}$ then

$$\gamma + \gamma_1 = \varepsilon(\gamma/\varepsilon) + (1 - \varepsilon)(\gamma_1/(1 - \varepsilon)) \in Q_{x^0,\xi^0},$$

if $\varepsilon > 0$ is so small that $\gamma_1/(1 - \varepsilon) \in Q_{x^0,\xi^0}$. In particular, $\widehat{Q}_{x^0,\xi^0} + B \subset Q_{x^0,\xi^0}$ if $B \subset Q_{x^0,\xi^0}$ is an open ball with center at 0, so the closure of $\widehat{Q}_{x^0,\xi^0}$ is a subset of Q_{x^0,ξ^0}. Hence $\widehat{Q}_{x^0,\xi^0}$ is closed and independent of the choice of interior point $f_0 \in Q_{x^0,\xi^0}$. If $\gamma_1, \gamma_2 \in \widehat{Q}_{x^0,\xi^0}$ are positive semidefinite then $\gamma = \gamma_1 + \gamma_2 \in \widehat{Q}_{x^0,\xi^0}$, and the radical of γ is the intersection of those of γ_1 and of γ_2. If γ_1 has maximal rank, then the radical of γ_1 is contained in that of γ_2. Since $f \in G_{x^0,\xi^0}$ implies $f + \kappa f^2 \in G_{x^0,\xi^0}$ for every κ if $f(x^0) = 0$, it follows that $(df(x^0))^2 \in \widehat{Q}_{x^0,\xi^0}$, so the radical of γ_j is orthogonal to ξ^0, which means that $\pm\xi^0 \in \varrho(x^0,\xi^0)$. (Note that $\widehat{Q}_{x^0,\xi^0}$ contains all quadratic forms with radical containing the annihilator of $\varrho(x^0,\xi^0)$.)

PROPOSITION 2.10. — *If $(x^0,\xi_j^0) \in N_e(F)$ for $j = 1,2$, then $\widehat{Q}_{x^0,\xi_1^0}+\widehat{Q}_{x^0,\xi_2^0} \subset \widehat{Q}_{x^0,\xi_1^0+\xi_2^0}$ and $\varrho(x^0,\xi_j^0) \subset \varrho(x^0,\xi_1^0+\xi_2^0)$, $j=1,2$. When ξ^0 is in the relative interior of the convex set $N_e(F)_{x^0}$ it follows that $\varrho(x^0,\xi) \subset \varrho(x^0,\xi^0)$ for every $\xi \in N_e(F)_{x^0}$. In particular, $N(F)_{x^0} \subset \varrho(x^0,\xi^0)$.*

Proof. — Since $Q_{x^0,\xi_1^0+\xi_2^0} \supset Q_{x^0,\xi_1^0}+Q_{x^0,\xi_2^0}$, the statement is obvious.

DEFINITION 2.11. — *Let F be a closed subset of a C^2 manifold X. Then the effective conormal set $N_{\text{eff}}(F)$ is defined as the set of all (x^0,ξ) such that $x^0 \in F$ and $0 \neq \xi \in \varrho(x^0,\xi^0)$ if ξ^0 is in the relative interior of the convex set $N_e(F)_{x^0}$ and ϱ is defined as in Proposition 2.9.*

Recall that the relative interior of a convex subset of a finite dimensional affine space is the interior of the set considered as a subset of the smallest such affine space. By Proposition 2.10 the definition is independent of the choice of ξ in the relative interior of $N_e(F)_{x^0}$. It follows from the definition that if $F_1 \subset F$ and $x^0 \in F_1$ then

$$(x^0,\xi) \in N_{\text{eff}}(F) \Longrightarrow (x^0,\xi) \in N_{\text{eff}}(F_1),$$

for the set G_{x^0,ξ^0} in Proposition 2.9 does not decrease if F is replaced by F_1.

We are now ready to improve Theorem 2.1:

THEOREM 2.12. — *If u is a hyperfunction in a real analytic manifold X, then*

$$(x^0,\theta) \in N_{\text{eff}}(\operatorname{supp} u),\ (x^0,\xi) \in WF_A(u) \Longrightarrow (x^0,\xi+t\theta) \in WF_A(u), \quad \text{if } t \in \mathbf{R},\ \xi+t\theta \neq 0.$$

In particular,

$$N_{\text{eff}}(\operatorname{supp} u) \subset WF_A(u). \tag{2.21}$$

Proof. — Choose (x^0,ξ^0) as in Definition 2.11, and take f_0 as in Proposition 2.9 with $F = \operatorname{supp} u$ so that 0 is an interior point of Q_{x^0,ξ^0}. Replacing f_0 by the second order Taylor polynomial, we may assume that

f_0 is analytic. We choose local coordinates so that $x^0 = 0$ and $f_0(x) = x_n$, thus $\xi^0 = (0, \dots, 0, 1)$. That 0 is interior in Q_{x^0,ξ^0} means that there is some $f \in G_{x^0,\xi^0}$ with $(f - f_0)''$ positive definite, that is, $f(x) \geqslant f_0(x) + c|x|^2$ in a neighborhood U of 0 for some $c > 0$. Thus

$$x_n \leqslant -c|x|^2 \leqslant -c|x'|^2, \quad x \in U \cap \operatorname{supp} u, \tag{2.22}$$

where $x' = (x_1, \dots, x_{n-1})$. If u is a distribution we can multiply u be a suitable cutoff function equal to 1 in a neighborhood of the origin, and when u is a hyperfunction we can use the flabbiness of the hyperfunction sheaf to make sure that u has compact support in $\mathbf{R}^n$ and that $x_n \leqslant -c|x'|^2$ in $\operatorname{supp} u$, while u is not changed in some neighborhood of 0. Let $A(x)$ be a positive semi-definite form of maximal rank in $\widehat{Q}_{x^0,\xi^0}$. If $0 \neq \theta = (\theta', 0) \in \varrho(x^0, \xi^0)$, then θ is orthogonal to the radical of A so $\langle x', \theta' \rangle^2 \leqslant CA(x)$, hence $x \mapsto \langle x', \theta' \rangle^2$ is also in $\widehat{Q}_{x^0,\xi^0}$. This means that for any $\kappa > 0$ there is some $f \in G_{x^0,\xi^0}$ with $f(x) - f_0(x) - \kappa\langle x', \theta' \rangle^2 \geqslant 0$ in a neighborhood U_κ of the origin. Thus we have

$$x_n \leqslant -\kappa\langle x', \theta' \rangle^2, \quad x \in U_\kappa \cap \operatorname{supp} u. \tag{2.23}$$

This is condition (2.18), so we conclude that (2.9)$'''$ is fulfilled with $\eta = \xi^0$, and the theorem follows now from Corollary 2.7.

It is clear from the proof that Theorem 2.12 gives a complete expression of (2.14) with (2.9) replaced by the stronger condition (2.9)$'''$. We shall now examine the weaker condition (2.9) in Proposition 2.6 and Corollary 2.7 from the point of view of invariance. Again we assume that the hypotheses of Lemma 2.8 are fulfilled and that $\eta = (0, \dots, 0, 1)$. Since (2.9) is invariant under affine transformations it suffices to consider a diffeomorphism ψ defined in a neighborhood of K such that $\psi(0) = 0$ and $\psi'(0)$ is the identity. In addition we assume that $x_n \leqslant -c_1|x'|^2$ for some other constant $c_1 > 0$ when $x \in \psi(K)$. Set $\psi(x) = (\psi^1(x), \psi_n(x))$. When examining

$$H_{\psi(K)}(\theta', 1) = \sup_{x \in K} \langle \psi(x), (\theta', 1) \rangle$$

it is by Lemma 2.8 legitimate to assume that $|\psi^1(x)| < |\theta'|/c_1$ and $0 \geqslant \psi_n(x) > -|\theta'|^2/c_1$, which implies $|x| = O(|\theta'|)$ and $x_n = O(|\theta'|^2)$. Then we have

$$\langle \psi(x) - x, (\theta', 0) \rangle = O(|\theta'|^3), \quad \psi_n(x) - x_n = q(x') + O(|\theta'|^3)$$

where $q(x')$ is the second order polynomial in the Taylor expansion of $\psi_n(x',0)$. Hence

$$H_{\psi(K)}(\theta',1) = h_q(\theta',1) + O(|\theta'|^3), \quad \text{where}$$
$$h_q(\theta',1) = \sup_{x\in K}(\langle x',\theta'\rangle + q(x') + x_n),$$

which means that (2.9)″ with K replaced by $\psi(K)$ is equivalent to

$$\underline{\lim}_{\tau\to 0}(h_q(\tau\theta',1) + h_q(-\tau\theta',1)/\tau^2 = 0. \tag{2.24}$$

Since h_q increases with q it is clear that this condition remains valid if q is replaced by a smaller quadratic form. Now we claim that if $q_\lambda(x') = -\lambda|x'|^2$, corresponding to the diffeomorphism $\psi(x) = (x', x_n - \lambda|x'|^2)$ with $c_1 = c+\lambda$, then (2.24) is independent of λ when $\lambda > -c$. We have already observed that if (2.24) is valid for q_λ with one value of λ then it is valid for every larger value. Now (2.24) with $q = q_\lambda$ means that there exist sequences $\tau_\nu \to 0$ and $\varepsilon_\nu \to 0$ such that

$$\pm\langle x',\tau_\nu\theta'\rangle - \lambda|x'|^2 + x_n \leqslant \varepsilon_\nu\tau_\nu^2, \quad x \in K.$$

Replacing τ_ν by $\lambda\tau'_\nu$ and ε_ν by ε'_ν/λ we see if $\lambda > 0$ that this is equivalent to the existence of sequences $\varepsilon'_\nu \to 0$ and $\tau'_\nu \to 0$ such that

$$x_n \leqslant \lambda(|x'|^2 + \varepsilon'_\nu\tau'^2_\nu \mp \langle x',\tau'_\nu\theta'\rangle), \quad x \in K.$$

If $\lambda > 0$, this inequality is always valid when the parenthesis on the right is $\geqslant 0$, and when it is negative the condition becomes more restrictive the larger λ is. Hence (2.24) for $q = q_\lambda$ is independent of λ when $\lambda > 0$. If we start from $q_{-\mu}$ with $0 < \mu < c$ we conclude that this is true for all $\lambda > -c$. Hence (2.24) is equivalent to (2.9)″ for an arbitrary $q \geqslant 0$, and we have proved:

PROPOSITION 2.13. — *Let F be a closed set in a C^∞ manifold and let $(x^0,\xi^0) \in N_e(F)$. Then the condition (2.9) on $\theta \in T^*_{x^0}$ is either valid for the supporting function of a compact neighborhood K of x^0 in F in any local coordinate system $\subset \mathbf{R}^n$ for which K is a compact subset of the coordinate patch and $K \subset B$ where B is a ball with $x^0 \in \partial B$ and exterior conormal $\xi^0 = \eta$ at x^0, or else this is false for all such local coordinates.*

DEFINITION 2.14. — *The linear subspace of $T^*_{x^0}$ generated by all θ satisfying the condition in Proposition 2.13, for some ξ^0 such that $(x^0,\xi^0) \in N_e(K)$ will be denoted by $\widetilde{N}(F)$.*

As for $N_{\mathrm{eff}}(F)$ we note that if $F_1 \subset F$ and $x^0 \in F_1$, then

$$(x^0,\xi) \in \widetilde{N}(F) \Longrightarrow (x^0,\xi) \in \widetilde{N}(F_1).$$

This is clear from (2.9), for if H is replaced by another supporting function $H_1 \leqslant H$ with $H_1(\eta) = H(\eta)$ then (2.9) remains valid since

$$0 \leqslant H_1(\eta+\tau\tilde{\theta}) + H_1(\eta-\tau\tilde{\theta}) - 2H_1(\eta) \leqslant H(\eta+\tau\tilde{\theta}) + H(\eta-\tau\tilde{\theta}) - 2H(\eta).$$

However, we shall see later that this is not the case for condition (2.9)$'$ where $\tilde{\eta}$ may differ from η.

Since Definition 2.14 is an expression of the condition (2.9) while $N_{\mathrm{eff}}(F)$ expresses the stronger condition (2.9)$'''$, we have

$$N_{\mathrm{eff}}(F) \subset \widetilde{N}(F).$$

That the two sides may differ is easily seen when $F \subset \mathbf{R}^2$ is defined by $x_2 \leqslant -f(x_1)$ where f is an even convex function with $f(0) = 0$ such that for the Legendre transform $\tilde{f}(\theta) = \sup(x_1\theta - f(x_1))$ we have $0 = \underline{\lim}_{\theta\to 0} \tilde{f}(\theta)/\theta^2 < \overline{\lim}_{\theta\to 0} \tilde{f}(\theta)/\theta^2$, or equivalently if $\tilde{f} \in C^2$,

$$0 = \underline{\lim}_{\theta\to 0} \int_0^1 (1-t)\tilde{f}''(\theta t)\,dt < \overline{\lim}_{\theta\to 0} \int_0^1 (1-t)\tilde{f}''(\theta t)\,dt.$$

Such a function is obtained by taking $\tilde{f}''$ equal to 0 and equal to 1 in every other interval bounded by the points 2^{-n^2}, $n = 1,2,\ldots$. By Corollary 2.7 we can improve Theorem 2.12 as follows:

THEOREM 2.12$'$. — *If u is a hyperfunction in a real analytic manifold X, then*

$$(x^0,\theta) \in \widetilde{N}(\operatorname{supp} u),\ (x^0,\xi) \in WF_A(u) \Longrightarrow (x^0,\xi+t\theta) \in WF_A(u), \quad \text{if } t \in \mathbf{R},\ \xi + t\theta \neq 0.$$

In particular,

$$\widetilde{N}(\operatorname{supp} u) \subset WF_A(u). \tag{2.21$'$}$$

Theorem 2.12$'$ is an improvement of Kashiwara's Theorem 2.1 and thereby of (1.2). We could also use (2.15) to give a further extension of (1.2). The condition (2.9)$'$ is much weaker than condition (2.9) (or (2.9)$''$); with the notation in the example above it is fulfilled unless $\tilde{f}'' \geqslant c$

for some positive constant c in a neighborhood of the origin. Without giving a complete geometric interpretation of (2.9)′ we shall now give some consequences of the fact that (2.9)′ implies (2.11).

If $\theta \in \mathbf{R}^n$, $\eta \in \mathbf{R}^n$ and there exists a plurisubharmonic function j satisfying (2.8) and some $t \in \mathbf{R}$, $z \in \mathbf{C}$ with $\operatorname{Im} z > 0$ such that

$$j(t\theta + z\eta) < H(\operatorname{Im} z\eta), \tag{2.25}$$

then it follows from Proposition 2.6 that (2.9)′ is not valid. Moreover, since

$$j(t(\theta + s\eta) + (z - ts)\eta) < H(\operatorname{Im} z\eta)$$

for every $s \in \mathbf{R}$, it follows that (2.9)′ does not hold with θ replaced by $\theta + s\eta$ either. The negation of (2.9)′ means that there are constants $c > 0$, $\delta > 0$, such that

$$H(\tilde{\eta}+\tau\tilde{\theta}) + H(\tilde{\eta}-\tau\tilde{\theta}) - 2H(\tilde{\eta}) \geqslant c\tau^2, \quad \text{if } |\tilde{\eta}-\eta| < \delta,\ |\tilde{\theta}-\theta| < \delta,\ |\tau| < \delta.$$

If ε is sufficiently small this remains true with H replaced by H_ε,

$$H_\varepsilon(\xi) = H(\xi) - \varepsilon|\xi|,$$

and c replaced by 0. This means that $\tau \mapsto H_\varepsilon(\tilde{\eta} + \tau\tilde{\theta})$ is convex when $|\tilde{\eta} - \eta| < \delta$, $|\tilde{\theta} - \theta| < \delta$ and $|\tau| < \delta$. If this is true for every $\theta \in \mathbf{R}^n \setminus \mathbf{R}\eta$, and $\Pi = \{\theta; \langle\theta, \eta\rangle = 0\}$ is the hyperplane orthogonal to η, we can by the Borel-Lebesgue lemma find $\delta > 0$ and $\varepsilon > 0$ such that $\tau \mapsto H_\varepsilon(\tilde{\eta} + \tau\theta)$ is convex when $|\tilde{\eta}-\eta| < \delta$, $\theta \in \Pi$, $|\theta| = 1$ and $|\tau| < \delta$. Thus $\Pi \ni \theta \mapsto H_\varepsilon(\eta+\theta)$ is convex when $|\theta| < \delta$, and by the homogeneity of H_ε it follows that H_ε is convex in the cone

$$\Gamma = \{\xi; |\xi - \langle\xi, \eta\rangle\eta/|\eta|^2| < \delta\langle\xi, \eta\rangle/|\eta|^2\}.$$

Let $K = \{x; \langle x, \xi\rangle \leqslant H(\xi) \text{ when } \xi \in \mathbf{R}^n\}$ be the convex compact set with supporting function H, and let

$$K_\xi = \{x \in K; \langle x, \xi\rangle = H(\xi)\} = \{x \in K; H(\xi) \leqslant \langle x, \xi\rangle\}$$

be the convex 'face' $\subset \partial K$ of K where ξ is an exterior normal. If $x \in K_\xi$ then $H(\cdot) - \langle x, \cdot\rangle$ is non-negative in $\mathbf{R}^n$ and vanishes at ξ. To interpret this geometrically we need an elementary lemma:

LEMMA 2.15. — *Let f be a non-negative convex function in $\mathbf{R}^n$ which is positively homogeneous of degree 1, and assume that $f(\xi_1, 0) = 0$ when*

$\xi_1 > 0$ and that $f(\xi) - \varepsilon|\xi|$ is convex when $\sqrt{\xi_2^2 + \ldots + \xi_n^2} < \delta\xi_1$. Then it follows that

$$f(\xi) \geqslant r(|\xi| - \xi_1), \quad r = \tfrac{1}{2}\varepsilon(1 - 1/\sqrt{1+\delta^2}). \tag{2.26}$$

Proof. — It suffices to prove (2.26) when $n = 2$ and f is smooth and strictly convex except in the radial direction. Then f is the supporting function of a convex compact set $k \subset \mathbf{R}^2$ with smooth boundary Γ, and the radius of curvature is $\geqslant \varepsilon$ when the direction φ of the exterior unit normal satisfies $|\sin\varphi| < \delta\cos\varphi$, thus $|\tan\varphi| < \delta$ or $|\varphi| < \varphi_0$ where $\cos\varphi_0 = 1/\sqrt{1+\delta^2}$, $0 < \varphi_0 < \frac{1}{2}\pi$. If s is the arc length on Γ then $d\varphi/ds \leqslant 1/\varepsilon$, and since $dx_1/ds = -\sin\varphi$ we have

$$\int_0^{\varphi_0} dx_1 = -\int_0^{\varphi_0} \sin\varphi\, ds \leqslant -\varepsilon\int_0^{\varphi_0} \sin\varphi\, d\varphi = -\varepsilon(1-\cos\varphi_0) = -2r,$$

and similarly $\int_{-\varphi_0}^0 dx_1 \geqslant 2r$. Hence the disc of radius $-r$ with center at $(-r, 0)$ is contained in k, for the boundary cannot cut through Γ which has smaller curvature when $x_1 > -2r$. The proof is complete.

From Lemma 2.15 applied to $H(\cdot) - \langle x, \cdot\rangle$, where $x \in K_\xi$, we conclude, if H_ε is convex in the open cone Γ and $\xi \in \Gamma$, that ∂K is differentiable at x and that K contains a ball with x on the boundary. The radius of the ball has a positive lower bound when ξ belongs to a compact subset of Γ. If B is the closed unit ball it follows that we can choose $r > 0$ such that if

$$K_{(r)} = \{y; \{y\} + rB \subset K\},$$

which is a compact convex subset of K, then the compact convex subset $K_{(r)} + rB$ of K contains K_ξ for all ξ in a neighborhood of η, so it contains a neighborhood of K_η in K. Hence

$$K = (K_{(r)} + rB) \cup K'_{(r)}$$

where the supporting function of $K'_{(r)}$ is strictly smaller than H in a neighborhood of η. Let χ_r be the characteristic function of rB, and let u_r, u'_r be positive measures with support equal to $K_{(r)}$ and $K'_{(r)}$ respectively. Then the support of $u = \chi_r * u_r + u'_r$ is equal to K, and since $j_{\hat{\chi}_r}(\zeta) < r|\mathrm{Im}\,\zeta|$ if $\zeta \in \mathbf{C}^n \setminus \mathbf{CR}^n$, it follows that $j_{\hat{u}}(\zeta) < H(\mathrm{Im}\,\zeta)$ if $\chi \in \mathbf{C}^n \setminus \mathbf{CR}^n$ and $\mathrm{Im}\,\zeta$ is sufficiently close to η. Hence we have proved the following theorem:

THEOREM 2.16. — *Let K be a convex compact subset of $\mathbf{R}^n$ with supporting function H, and let $\eta \in \mathbf{R}^n \setminus \{0\}$. Then the following conditions are equivalent:*

(i) *There is a positive measure u with $\operatorname{supp} u = K$ such that $j_{\hat{u}}(\theta + z\eta) < H(\operatorname{Im} z\eta)$ for every $z \in \mathbf{C}$ with $\operatorname{Im} z > 0$ and every $\theta \in \mathbf{R}^n \setminus \mathbf{R}\eta$.*

(ii) *For every $\theta \in \mathbf{R}^n \setminus \mathbf{R}\eta$ there is a plurisubharmonic function j satisfying (2.8) such that $j(\theta + i\eta) < H(\eta)$.*

(iii) *(2.9)′ is not valid for any $\theta \in \mathbf{R}^n \setminus \mathbf{R}\eta$.*

(iv) *$H(\xi) - \varepsilon|\xi|$ is convex in a neighborhood of η for some $\varepsilon > 0$.*

Proof. — That (i) implies (ii) is trivial, that (ii) implies (iii) follows from Proposition 2.6, and we have just proved that (iii) implies (iv) and that (iv) implies (i). (That (iv) implies (i) was essentially proved in [14, Prop. 3.8]; what is new here is that (i) implies (iv).)

However, already our definition of $\widetilde{N}(F)$ above was rather cumbersome, and examining the invariance properties of condition (2.9)′ for fixed θ, η seems much more complicated still, so we shall not give a general invariant formulation of (2.15). However, we shall discuss an instructive example based on the following consequence of Corollary 2.7:

COROLLARY 2.17. — *Let u be a distribution or hyperfunction in $\mathbf{R}^3$ such that $0 \in \operatorname{supp} u$ and $x_3 \leqslant -q(x_1, x_2)$ if $x \in \operatorname{supp} u$, where $q \geqslant 0$. Set*

$$\Gamma_\delta = \{x' = (x_1, x_2) \in \mathbf{R}^2; |x_2| < \delta x_1\},$$

and assume that there is a constant $c > 0$ such that for every $\delta > 0$

$$q(x_1, x_2)/x_1^2 \to \infty \quad \text{when } x' \in \Gamma_c \setminus \Gamma_\delta \text{ and } x' \to 0. \tag{2.27}$$

Then $(0, \xi) \in WF_A(u)$ when $\xi_1 = 0$ unless

$$x_3/|x'|^2 \to -\infty \quad \text{when } x \in \operatorname{supp} u,\ x' \in \Gamma_c \text{ and } x \to 0. \tag{2.28}$$

Proof. — Assuming that (2.28) does not hold, that is, that

$$m = \varliminf_{x \in \operatorname{supp} u, x' \in \Gamma_c, x \to 0} |x_3|/x_1^2 < \infty$$

we choose a sequence $x^\nu \to 0$ for which the lower limit is attained. With $\varepsilon_\nu = x_1^\nu$ we have $x_2^\nu/\varepsilon_\nu \to 0$ by (2.27), and $x_3^\nu/\varepsilon_\nu^2 \to -m$, as $\nu \to \infty$. We change coordinates introducing

$$v(x) = u(x', x_3 + \tfrac{1}{2}A|x'|^2)$$

for some large A to be chosen later. We have $(0, \xi) \in WF_A(u)$ if and only if $(0, \xi) \in WF_A(v)$. Without restriction we may assume that v has compact support. The supporting function H is

$$H(\xi', 1) = \sup_{x \in \operatorname{supp} v} (\langle x', \xi' \rangle + x_3),$$

and we shall first prove that

$$\varlimsup_{\nu \to \infty} H(\varepsilon_\nu \xi', 1)/\varepsilon_\nu^2 \leqslant \max(\xi_1^2/(2A + 4m), (|\xi'|^2 - d_c(\xi')^2)/2A), \tag{2.29}$$

where $d_c(\xi')$ is the distance from ξ' to $\complement\Gamma_c$. For the proof we first note that since $x_3 \leqslant -\frac{1}{2}A|x'|^2$ when $x \in \operatorname{supp} v$, it follows that

$$(\langle x', \varepsilon_\nu \xi' \rangle + x_3)/\varepsilon_\nu^2 \leqslant \langle y', \xi' \rangle - \tfrac{1}{2}A|y'|^2 = (|\xi'|^2 - |\xi' - Ay'|^2)/2A, \; y' = x'/\varepsilon_\nu.$$

This is bounded by $(|\xi'|^2 - d_c(\xi')^2)/2A$ if $y' \notin \Gamma_c$ and by 0 if $|\xi' - Ay'| > |\xi'|$. To prove (2.29) it remains to find a bound when $y' \in \Gamma_c$ and $A|y'| \leqslant 2|\xi'|$. For arbitrary $\delta > 0$ and $M > 0$ we obtain using (2.27) if $x' \in \Gamma_c \setminus \Gamma_\delta$ and ν is large enough

$$(\langle x', \varepsilon_\nu \xi' \rangle + x_3)/\varepsilon_\nu^2 \leqslant \langle y', \xi' \rangle - M|y'|^2 \leqslant |\xi'|^2/4M,$$

which is a better bound than (2.29) for sufficiently large M. If $x' \in \Gamma_\delta$ and $m_1 < m$ we obtain for large ν by the definition of m

$$\begin{aligned}(\langle x', \varepsilon_\nu \xi' \rangle + x_3)/\varepsilon_\nu^2 &\leqslant \langle y', \xi' \rangle - \tfrac{1}{2}A|y'|^2 - m_1 y_1^2 \\ &\leqslant |y_1|(|\xi_1| + \delta|\xi_2|) - (\tfrac{1}{2}A + m_1)y_1^2 \leqslant (|\xi_1| + \delta|\xi_2|)^2/(2A + 4m_1).\end{aligned}$$

When $m_1 \uparrow m$ and $\delta \downarrow 0$ the estimate (2.29) follows. When $\xi_2 = 0$ and $\xi_1 > 0$ we have $|\xi'|^2 - d_c(\xi')^2 = \xi_1^2/(1 + c^2)$. If we choose A so large that $A + 2m < A(1 + c^2)$, that is, $A > 2m/c^2$, it follows that the right-hand side of (2.29) is equal to $\xi_1^2/(2A + 4m)$ in a neighborhood of the positive ξ_1 axis.

Next we note that

$$\begin{aligned}H(\varepsilon_\nu \xi_1, 0, 1)/\varepsilon_\nu^2 &\geqslant (x_1^\nu \varepsilon_\nu \xi_1 + x_3^\nu - \tfrac{1}{2}A({x_1^\nu}^2 + {x_2^\nu}^2))/\varepsilon_\nu^2 \\ &\to \xi_1 - m - \tfrac{1}{2}A = \xi_1^2/(2A + 4m)\end{aligned}$$

if $\xi_1 = A + 2m$. Hence it follows from (2.29) in view of the convexity that

$$\lim_{\nu\to\infty} H(\varepsilon_\nu \xi', 1)/\varepsilon_\nu^2 = \xi_1^2/(2A+4m) = \tfrac{1}{2}A + m,$$

provided that $\xi_1 = A + 2m$ and $|\xi_2|$ is sufficiently small. Since the limit is independent of ξ_2, hence flat, we conclude that $(2.9)'$ is fulfilled with

$$\tilde{\eta} = (\varepsilon_\nu \xi_1, 0, 1),\ \tilde{\theta} = (0, \xi_2, 0),\ \tau = \varepsilon_\nu.$$

Hence Corollary 2.7 applied to v proves that $(0,\xi) \in WF_A(u)$ if $\xi_1 = 0$.

Example. — If $0 \in \operatorname{supp} u \subset \mathbf{R}^3$ and

$$x_3 \leqslant -x_1^2 x_2^2/|x'|^\mu \quad \text{when } x \in \operatorname{supp} u,$$

where $\mu > 2$, it follows from Corollary 2.17 that $(0,\xi) \in WF_A(u)$ either for all ξ with $\xi_1 = 0$, or for all ξ with $\xi_2 = 0$, or else (2.28) is valid with Γ_c replaced by $\mathbf{R}^2 \setminus 0$. Then we have $(0,\xi) \in WF_A(u)$ for all $\xi \neq 0$ by Theorem 2.12. If we assume that the origin is not an isolated point in the intersection of $\operatorname{supp} u$ and the x_1 or the x_2 axes then $(0,\xi) \in WF_A(u)$ when $\xi_1\xi_2 = 0$. The examples $u_j = \delta(x_j)Y(-x_3)$, $j = 1, 2$, and $u_1 + u_2$ show that these conclusions are optimal. (Here Y is the Heaviside function.) It is not clear if $(0,\xi) \in WF_A(u)$ for every ξ when $\operatorname{supp} u$ is *equal* to the set where $x_3 \leqslant -x_1^2 x_2^2/|x'|^\mu$, in a neighborhood of the origin. However, the example shows that we may have $\widetilde{N}(F)|_{x^0} \neq WF_A(u)|_{x^0}$ for every u with $x^0 \in \operatorname{supp} u \subset F$, although the intersection of $WF_A(u)|_{x^0}$ for all such u is $\widetilde{N}(F)|_{x^0}$.

3. A general property of the conormal set.

The conormal bundle of a smooth manifold is always Lagrangian, that is, the dimension equals that of the manifold and the symplectic form vanishes there. In particular, it is involutive which means that the Hamilton field of any function vanishing on it is tangential. The general conormal set introduced in Definition 1.1 has a weak form of this property.

THEOREM 3.1 (Sjöstrand). — *Let F be a closed subset of a C^∞ manifold X and assume that $p \in C^\infty(T^*(X) \setminus 0)$ is real valued and vanishes on $N_e(F)$. If $(x^0, \xi^0) \in N_e(F)$, it follows that $N_e(F)$ contains* *a neighborhood of* (x^0, ξ^0) *on the bicharacteristic strip* Γ *for* p *through* (x^0, ξ^0) *defined by the Hamilton equations*

$$dx/dt = \partial p(x,\xi)/\partial \xi,\ d\xi/dt = -\partial p(x,\xi)/\partial x;\ (x,\xi) = (x^0,\xi^0) \ \text{ when } t = 0.$$

If $p'_\xi(x^0, \xi^0) \neq 0$ *there is a function* $\Phi \in C^\infty(X)$ *such that for some* $\varepsilon > 0$

$$\Phi(x(t)) = 0, \quad d\Phi(x(t)) = \xi(t), \quad \text{if } |t| < \varepsilon,$$

and $\Phi(x) < 0$ *when* x *is in a neighborhood of* $\{x(t); |t| < \varepsilon\}$ *in* F *but* $x \notin \Gamma$.

This is [7, Theorem 8.5.9], and by [7, Corollary 8.5.10] the following improvement of earlier results due to J.-M. Bony and the author follows:

THEOREM 3.2. — *Let* F *be a closed subset of* X, *and set*

$$\mathcal{N}_F = \{p \in C^\infty(T^*(X) \setminus 0); p = 0 \text{ on } N_e(F)\}.$$

Then $\mathcal{N}_F$ *is an ideal in* $C^\infty(T^*(X) \setminus 0)$ *which is closed under the Poisson bracket*

$$\{p, q\} = \sum(\partial p/\partial \xi_j \partial q/\partial x_j - \partial p/\partial x_j \partial q/\partial \xi_j).$$

If $p_1, \ldots, p_k \in \mathcal{N}_F$ *are real valued with linearly independent differentials at* $(x^0, \xi^0) \in N_e(F)$, *then* $k \leqslant n$, *there is a* k *dimensional manifold through* (x^0, ξ^0) *contained in* $N_e(F)$ *in a neighborhood, and the symplectic form vanishes on it.*

4. Solutions of analytic differential equations.

If P is a linear differential equation with real analytic coefficients, then (1.3) is valid for all solutions of the equation $Pu = 0$. There are many equations for which better results are known; for example, one can replace the characteristic set by the empty set when P is subelliptic. We shall not review the extensive literature on this matter here but just denote by $C(P)$ the smallest subset of $\operatorname{Char} P$ such that

$$Pu = 0 \Longrightarrow WF_A(u) \subset C(P). \tag{4.1}$$

An immediate combination of (2.21)′ and (4.1) gives of course

$$\widetilde{N}(\operatorname{supp} u) \subset C(P) \cap \check{C}(P) \quad \text{if } Pu = 0.$$

Here $\check{C}(P) = \{(x,\xi); (x,-\xi) \in C(P)\}$. However, Theorem 3.2 often allows one to make a much stronger conclusion:

THEOREM 4.1. — *Let P be a differential operator with real analytic coefficients and let $\mathcal{N}(P)$ be the smallest subset of $C^\infty(T^*(X)\setminus 0)$ which contains all C^∞ functions vanishing on $C(P)\cap\check{C}(P)$ and is closed under Poisson brackets. Then it follows that*

$$Pu = 0 \Longrightarrow \widetilde{N}(\operatorname{supp} u) \subset \widetilde{C}(P) = \{(x,\xi) \in T^*(X)\setminus 0; \tag{4.2}$$
$$q(x,\xi) = 0 \ \forall\, q \in \mathcal{N}(P)\}.$$

With $N(\operatorname{supp} u)$ instead of $\widetilde{N}(\operatorname{supp} u)$ this is essentially [7, Theorem 8.6.6]. The inclusion (4.2) may be much stricter than (4.1); in [7, p. 310] an example in $\mathbf{R}^3$ is given where $\widetilde{C}(P) = \varnothing$ but $C(P)$ is of dimension 3.

The inclusion (4.2) gives a great deal of information on the support of a solution of $Pu = 0$, such as the absence of sharp corners and even points of infinite curvature.

5. A non-analytic differential equation.

The proof of the Holmgren theorem which we have discussed shows in a very convincing way that the analyticity of the coefficients is essential, for otherwise one can hardly expect microlocal analyticity of all solutions as in (1.3). Indeed, there are very strong counterexamples in the non-analytic case. For example, if $\square = D_0^2 - \sum_1^n D_j^2$ is the wave operator in $\mathbf{R}^{1+n}$, with coordinates denoted $(x_0,\dots,x_n)$, and if $n \geqslant 2$, then one can find $u, a_j \in C^\infty(\mathbf{R}^{1+n})$, $j = 0,\dots,n$ such that

$$\square u + \sum_0^n a_j D_j u = 0 \quad \text{in } \mathbf{R}^{1+n}, \operatorname{supp} u = \operatorname{supp} a_j = \{x; x_1 \geqslant 0\},$$
$$j = 0,\dots,n.$$

Thus there is no unique continuation at all across the timelike plane where $x_1 = 0$ although the coefficients are smooth and the leading ones are constant. It was therefore very surprising that Robbiano [12] was able to prove a uniqueness theorem when the coefficients are independent of t.

(This result has applications in control theory.) In [8] we gave the following improvement of Robbiano's result:

THEOREM 5.1. — *Let*

$$P = \sum_{|\alpha|\leqslant 2} a_\alpha(x) D^\alpha$$

be a second order hyperbolic differential operator in an open set $X \subset \mathbf{R}^{1+n}$, such that a_α is locally Lipschitz continuous when $|\alpha| = 2$ and $a_\alpha \in L^\infty_{\mathrm{loc}}$ for all α. Assume that there is a timelike vector θ such that $a_\alpha(x+t\theta) = a_\alpha(x)$ if $x \in X$ and $x + t\theta \in X$. Denote by p the real valued principal symbol of P,

$$p(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha, \quad x \in X,\ \xi \in \mathbf{R}^{1+n}, \tag{5.1}$$

denote the corresponding symmetric bilinear form by $p(x,\xi,\eta)$, and denote by $\theta^\flat$ the covector defined by $p(x,\theta^\flat,\cdot) = \langle\theta,\cdot\rangle$. Then it follows that

$$\overline{N}(\operatorname{supp} u) \subset \{(x,\xi); 0 \geqslant p(x,\theta^\flat,\theta^\flat)p(x,\xi,\xi) \geqslant (1-K^2)p(x,\theta^\flat,\xi)^2\}, \tag{5.2}$$

if $u \in K^{\mathrm{loc}}_{(1)}(X)$ satisfies the equation $Pu = 0$ in X. Here K is a universal constant with optimal value $\in [1, \sqrt{27/23}]$.

The translation invariance can be stated invariantly by letting θ be a $C^{1,1}$ real vector field with $[P,\theta] = 0$. When $K = 1$ the right-hand side of (5.2) becomes the characteristic set. Holmgren's uniqueness theorem states that (5.2) is valid with $K = 1$ for arbitrary real analytic coefficients. It is not known if K can be taken equal to 1 in Theorem 5.1. In any case it shows that there are hypotheses other than analyticity which lead to inclusions of the form (1.4) not much weaker than that in Holmgren's uniqueness theorem although they cannot be obtained via an analyticity result such as (1.3). Another result of this form is the Calderón uniqueness theorem and its later refinements; we refer to [7, Theorem 28.1.8] for a general form of it stated along the lines of this paper. (The other results proved in [7, Chapter 28] involve very precise relations between curvature and operator so they are quite different in spirit from those which follow in the analytic case from Theorems 2.12 and 2.12′.)

6. Miscellaneous remarks.

So far we have only discussed differential operators, for they are microlocally the most convenient ones since they cannot enlarge the analytic wave front set (if the coefficients are analytic) and are microlocally bijective at non-characteristic points. However, the splitting of the Holmgren theorem discussed here is also useful in connection with other Fourier integral operators such as the Radon transform

$$(Rf)(H) = \int_H f\, dS_H$$

where H is an affine hyperplane in $\mathbf{R}^n$, dS_H is the Euclidean surface measure in H, and f is a continuous function in $\mathbf{R}^n$ such that $f(x) = O(|x|^{-n})$ as $x \to \infty$. Writing

$$H = \{x; \langle x, \omega\rangle + \omega_{n+1} = 0\}, \quad \text{where } \omega \in \mathbf{R}^n,\ |\omega|^2 + \omega_{n+1}^2 = 1,$$

we can regard H as a point in the projective space P^n, obtained by identifying antipodal points in the unit sphere $S^n \subset \mathbf{R}^{n+1}$. Writing

$$F(X) = |X_{n+1}|^{-n} f(x), \quad \text{if } x \in \mathbf{R}^n,\ X = \pm(x, 1)/\sqrt{|x|^2 + 1},$$

we obtain a bounded even function on S^n, defined when $X_{n+1} \neq 0$, thus a function on P^n. An elementary calculation (see Boman [1]) gives

$$(Rf)(H) = c \int F(X)\delta(\langle X, \Omega\rangle)\, dS = c(\widetilde{R}F)(\Omega),$$

where Ω is one of the points in S^n defining H, c is a constant, dS is the Euclidean surface measure on S^n, and $\Omega_{n+1} \neq \pm 1$. The analytic wave front set of the distribution $\delta(\langle X, Y\rangle)$ on $S^n \times S^n$ is

$$\{(X, Y; \tau Y, \tau X); 0 \neq \tau \in \mathbf{R},\ X, Y \in S^n, \langle X, Y\rangle = 0\}.$$

(Note that a cotangent vector $\neq 0$ of S^n at X can be identified with an orthogonal vector in $\mathbf{R}^{n+1}$.) Thus $(X, \tau Y)$, where $\langle X, Y\rangle = 0$, is not in $WF_A(\widetilde{R}F)$ if $(\pm Y, \mp\tau X) \notin WF_A(F)$. These two conditions are equivalent when F is even, and since the symbol of the conormal distribution δ is non-zero, we can then reverse the implication: if $(X, \tau Y) \notin WF_A(\widetilde{R}F)$ and $\langle X, Y\rangle = 0$, then $(\pm Y, \mp\tau X) \notin WF_A(F)$. In particular,

$$X \notin \operatorname{supp} \widetilde{R}F \Longrightarrow (Y, \tau X) \notin WF_A(F) \ \text{ if } \tau \neq 0 \text{ and } \langle X, Y\rangle = 0,$$

which is an analogue of (1.3). (Recall that $\langle X, Y\rangle = 0$ means that Y is in the hyperplane defined by X.) If $f(x) = O(|x|^{-N})$ for every N as $x \to \infty$, then F vanishes of infinite order on the plane at infinity $\Pi = (0, \dots, 0, \pm 1)$ defined by $X_{n+1} = 0$. If in addition $\widetilde{R}F(\Omega) = 0$ for every hyperplane which does not intersect a compact set $K \subset \mathbf{R}^n$, then $\Pi \notin \operatorname{supp} \widetilde{R}F$, so the conormal of Π does not intersect $WF_A(F)$. As observed by Boman [1] it follows, if a is a polynomial in $\mathbf{R}^{n+1}$, that the integral of aF over the sphere $\{X \in S^n; X_{n+1} = t\}$ is an analytic function of t in a fixed interval $I \ni 0$, vanishing of infinite order when $t = 0$. Hence it is identically 0, and since a is arbitrary it follows that $F = 0$ in $\{X \in S^n; X_{n+1} \in I\}$. (The argument is here again the classical one of Holmgren.) One can now conclude using (1.2) that F vanishes in a neighborhood of every plane in the component of Π in the set of planes which do not intersect K, hence that $\operatorname{supp} F \subset K$ if K is convex. This is Helgason's support theorem.

Boman [3] has also proved a localized version of Helgason's theorem where he only assumes that f vanishes of infinite order at ∞ in the directions of an open cone Γ. The proof then requires a variant of (1.2), proved in Boman [2]. The inclusion (1.2) means that a hyperfunction u vanishing on one side of an analytic surface S must vanish in a neighborhood of any point $x \in S$ such that $(x, \xi) \notin WF_A(u)$ for some conormal ξ of S at x. Under the stronger assumption that $(x, \pm\xi) \notin WF_A(u)$ for both signs, there is a neighborhood V of x such that the restriction of any derivative $D^\alpha u$ to $S \cap V$ is well defined, and Boman [2] proved that if they all vanish and u is a *distribution* then $u = 0$ in a neighborhood of x. This conclusion is not valid if only one of the conormal directions is absent, even if $u \in C^\infty$ so that the restrictions have an obvious and elementary meaning. It is also false for hyperfunctions by an example of Sato presented in Kaneko [10, Note 3.3].

Only the analytic wave front set has been discussed above. However, (1.2) can be extended to

$$N(\operatorname{supp} u) \subset WF_L(u) \tag{6.1}$$

if L is quasi-analytic and u is a distribution. (For notation and definitions we refer to [7, Section 8.4].) The proof of (1.2) given in [7, Theorem 8.5.6′] extends at once if we prove the following substitute for [7, Corollary 8.4.16]:

THEOREM 6.1. — *If $u \in \mathcal{D}'(X)$ where X is an interval on $\mathbf{R}$, and if $x^0 \in X$ is a boundary point of* $\operatorname{supp} u$, *then $(x^0, \pm 1) \in WF_L(u)$ for every quasi-analytic class L.*

Proof. — Since $WF_L(u)$ is closed it suffices to prove the statement when u vanishes in a onesided neighborhood of x^0, that is, prove that if $u = 0$ in $(x^0 - \varepsilon, x^0)$ or $(x^0, x^0 + \varepsilon)$ for some $\varepsilon > 0$ and $(x^0, 1) \notin WF_L(u)$, then L is non-quasianalytic. We may of course assume that $x^0 = 0$. From [7, Theorem 8.4.15], which is very elementary in the one dimensional case, we conclude that in a neighborhood $(-2\delta, 2\delta)$ of 0 we can write $u = u_+ + u_-$ where $u_+ \in C^L$ and u_- is the boundary value of a function U_- analytic in the lower half plane. Thus we have for some constants C and N

$$|u_+^{(j)}(x)| \leqslant C^{j+1}L_j^j, \quad j = 0, 1, \ldots, \ |x| < 2\delta, \tag{6.2}$$

$$|U_-(x + iy)| \leqslant C|y|^{-N}, \quad |x| < 2\delta, \ -1 < y < 0, \tag{6.3}$$

where (6.3) is a consequence of the hypothesis that $u \in \mathcal{D}'$. Now we compose u with the increasing function

$$f(x) = \delta x/\sqrt{1 + x^2}. \tag{6.4}$$

Set $v_\pm = u_\pm \circ f$. From the proof of [7, Proposition 8.4.1] and (6.2) we obtain with a new constant C

$$|v_+^{(j)}(x)| \leqslant C^{j+1}L_j^j, \quad j = 0, 1, \ldots, \ x \in \mathbf{R}. \tag{6.2$'$}$$

We have $f'(x) = \delta(1 + x^2)^{-\frac{3}{2}} > 0$ and $f''(z) = O(\delta(1 + |z|^2)^{-2})$ when $|\operatorname{Im} z| \leqslant \frac{1}{2}$, hence

$$\operatorname{Im} f(x + iy) \leqslant \delta\big((1 + x^2)^{-\frac{3}{2}}y + Cy^2(1 + x^2)^{-2}\big) < \tfrac{1}{2}\delta(1 + x^2)^{-\frac{3}{2}}y$$

if $-c < y < 0$, for sufficiently small $c \in (0, 1)$, and we have $|\operatorname{Re} f(x + iy)| < 2\delta$ then. Hence $V_-(z) = U_-(f(z))$ is analytic when $-c < \operatorname{Im} z < 0$, and

$$|V_-(x + iy)| \leqslant C(1 + |x|)^{3N}|y|^{-N}, \quad -c \leqslant y < 0. \tag{6.3$'$}$$

Now we choose an analytic convergence factor such as e^{-x^2} and set $w_\pm(x) = e^{-x^2}v_\pm(x)$. Then $w = w_+ + w_-$ vanishes on a half axis by hypothesis, and since

$$\hat{w}_-(\xi) = \int e^{-i(x+iy)\xi}e^{-(x+iy)^2}V_-(x + iy)\,dx, \quad -c < y < 0,$$

it follows from (6.3)$'$ that

$$|\hat{w}_-(\xi)| \leqslant Ce^{-c\xi} \quad \text{when } \xi > 0. \tag{6.5}$$

From (6.2)$'$ we obtain using the proof of [7, Proposition 8.4.1] that

$$|w_+^{(j)}(x)| \leqslant C^{j+1}L_j^j e^{-x^2/2}, \quad j = 0, 1, \ldots,$$

which implies

$$|\hat{w}_+(\xi)| \leqslant C^{j+1}(L_j/|\xi|)^j, \quad j = 0, 1, \dots, \ \xi \in \mathbf{R}.$$

When $\xi > 0$ the same estimate follows for w_- from (6.5), so with still another constant C we have

$$|\hat{w}(\xi)| \leqslant C^{j+1}(L_j/|\xi|)^j, \quad j = 0, 1, \dots, \ \xi > 0. \tag{6.6}$$

Since w is supported by a half axis, the Fourier-Laplace transform is analytic in a half plane and bounded by a polynomial, so

$$\int_{-\infty}^{\infty} |\log|\hat{w}(\xi)||\, d\xi/(1+\xi^2) < \infty, \tag{6.7}$$

for by hypothesis w is not identically 0. Combining (6.6) and (6.7) we obtain

$$\int_0^{\infty} \log\Big(\sup_{j\geqslant 0}(\xi/L_j)^j\Big)\, d\xi/(1+\xi^2) < \infty,$$

and it is classical (Carleman [4, p.50]) that this is equivalent to $\sum 1/L_k < \infty$. The proof is complete.

The proof is not only valid for distributions. We could have allowed in the proof a bound $e^{y^{-\kappa}}$ instead of (6.2) provided that we use a convergence factor $e^{-x^{2N}}$ with $2N > 3\kappa$. Thus the proof extends to distributions in the dual of any Gevrey class.

It is clear that (6.1) is false if L is not quasi-analytic, for $\operatorname{supp} u$ can then be the closure of any open set while $u \in C^L$. A very weak analogue for wave front sets is given in the following:

THEOREM 6.2. — *If V is a C^∞ submanifold of a C^∞ manifold X and $u \in \mathcal{D}'(X)$, $\operatorname{supp} u \subset V$, then $N_V^*|_{\operatorname{supp} u} \subset WF(u)$.*

Proof. — The statement is local and invariant so we may assume that $X \subset \mathbf{R}^n$ is a neighborhood of 0 and that $x_1 = 0$ in V; it suffices to prove that $(0, e_1) \in WF(u)$ if $0 \in \operatorname{supp} u$ and $e_1 = (1, 0, \dots, 0)$. Shrinking X if necessary we can assume that

$$u = \sum_0^N \delta^{(j)}(x_1) \otimes a_j$$

where a_j are distributions in $\mathbf{R}^{n-1}$ near 0 with $0 \in \operatorname{supp} a_N$. Then $x_1^N u = (-1)^N N! \delta(x_1) \otimes a_N$. Choose $\varphi \in C_0^\infty(\mathbf{R}^{n-1})$ with support close to the origin and $\varphi a_N \neq 0$. Then the Fourier transform of $x_1^N \varphi u$ is $(-1)^N N! \widehat{\varphi a_N}(\xi_2, \dots, \xi_n)$. We can choose $\xi_2, \dots, \xi_n$ so that this is not equal to 0. When $\xi_1 \to \infty$ the Fourier transform does not tend to 0 and we conclude that $(0, e_1) \in WF(u)$.

It is not clear if Theorem 6.2 remains true if V is just a C^1 submanifold.

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