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MEROMORPHIC EXTENSION SPACES

by LE MAU HAI and NGUYEN VAN KHUE

The extension of meromorphic maps from a spreaded domain over a Stein manifold to its envelope of holomorphy has been investigated by some authors. This problem for meromorphic functions was proved by Kajiwara and Sakai [9], for meromorphic maps with values in a compact subalgebraic space by Hirschowitz [8].

The extension of meromorphic maps with values in a compact Kahler manifold through an analytic set of codimension $\geqslant 2$ has been established first by P. Griffiths [6] in a particular case and by Siu [17] in general. In the present paper we shall prove the following two theorems are based on ideas of Dloussky [2].

THEOREM 2.2 — *Let $\theta : X \to Y$ be a Hartogs meromorphic extension map. Assume that Y is a Hartogs meromorphic extension space. Then for every meromorphic mapping f from a domain D over a Stein manifold to X, there exists an analytic subset A of codimension at least 2 in $^{\wedge}D$ such that f extends meromorphically to $^{\wedge}D\backslash A$. Moreover, if X is a compact Kahler manifold and Y is a Hartogs meromorphic extension space and θ is a Hartogs meromorphic extension map then X is a meromorphic extension space.*

THEOREM 3.1. — *Let $\theta : X \to Y$ be a finite proper surjective holomorphic map. Then X is a meromorphic extension space if and only if Y has the same property.*

Using Theorem 3.1 we prove that every compact non-singular elliptic Kahler surface is a meromorphic extension space. Moreover using Theorem 2.2 we also prove that every complex Lie group is a meromorphic extension space.

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Key words : Meromorphic map - Meromorphic extension spaces - Branched covering map - Elliptic Kahler surface.
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1. Meromorphic extension spaces.

We first recall that a meromorphic map $f : X \to Y$ is an analytic set $\Gamma(f)$ in $X \times Y$ such that the canonical projection $p(f) : \Gamma(f) \to X$ is proper and there exists an open subset $X(f)$ of X such that $\Gamma(f) \cap (X(f) \times Y)$ is the graph of a holomorphic map from $X(f)$ into Y. $\Gamma(f)$ is called the graph of f. It is known [14] that in the case where X is normal, the indeterminacy locus of f

$$I(f) = \{x \in X : f \text{ is not holomorphic at } x\} = \{x \in X : \dim p(f)^{-1}(x) > 0\}$$

is an analytic set of codimension $\geqslant 2$.

We now give the following

Definition 1.1. — Let X *be a complex space. We say that* X *is a meromorphic extension space if the two following conditions are satisfied :*

H) *every meromorphic map from a spreaded domain* D *over a Stein manifold into* X *can be extended meromorphically to* $^{\wedge}D$, *the envelope of holomorphy of* D.

R) *Every meromorphic map from* $Z \backslash S$ *into* X, *where* Z *is a normal complex space and* S *is an analytic set of codimension* $\geqslant 2$ *in* Z *can be meromorphically extended to* Z.

In the case where only the condition H) (*resp.* R)) *holds,* X *is called a Hartogs* (*resp. Riemann*) *meromorphic extension space.*

We have the following

Proposition 1.2. — *Let* X *be a complex space. Then the following conditions are equivalent :*

(i) *every meromorphic map from a Hartogs domain to* X *can be meromorphically extended to its envelope of holomorphy.*

(ii) μ_R^X *is Stein for every Stein manifold* R, *where* μ_R^X *denotes the spread domain over* R *associated to the sheaf of germs of meromorphic maps on* R *with values in* X.

(iii) X *is a Hartogs meromorphic extension space.*

Proof. — (i) → (ii). By the Docquier-Grauert theorem [3] it suffices to show that μ_R^X is p_7-convex, i.e. every holomorphic embedding $\sigma : H_k(r) \to \mu_R^X$ can be holomorphically extended to Δ^k, where Δ denotes the unit disc in C and $H_k(r)$ is given by

$$H_k(r) = \{(z_1, z_2, \ldots, z_k) \in \Delta^k : |z_j| < r, j=1,2,\ldots,k-1\} \cup \{(z_1, z_2, \ldots, z_k) \in \Delta^k : |z_k| > 1-r\}, \quad 0 < r < 1,$$

$k = \dim R$.

Let $\mathcal{O}_R^X$ denote the spread domain over R associated to the sheaf of germs of holomorphic maps on R with values in X. Obviously $\mathcal{O}_R^X$ is dense open in μ_R^X. Consider the canonical map $e : \mathcal{O}_R^X \to X$ given by

$$e(g_z) = g_z(z) \quad \text{for} \quad z \in R \qquad \text{and} \qquad g_z \in (\mathcal{O}_R^X)_z .$$

It is easy to see that e extends meromorphically by definition μ_R^X. Hence $e\sigma : H_k(r) \to X$ is meromorphic. By hypothesis it is extended to a meromorphic map ${}^\wedge e : \Delta^k \to X$. Let $p : \mu_R^X \to R$ denote the locally biholomorphic canonical map and let $\mathscr{E} : \Delta^k \to R$ be a holomorphic extension of $p\sigma$. Since every hypersurface in Δ^k meets $H_k(r)$, it follows that $\mathscr{E}$ is a locally biholomorphic map. Define now a holomorphic extension ${}^\wedge\sigma : \Delta^k \to \mu_R^X$ by

$${}^\wedge\sigma(z) = (\mathscr{E}(z), {}^\wedge e(\mathscr{E}|_{U_z})^{-1}_{\mathscr{E}(z)}) \quad \text{for} \quad z \in \Delta^k$$

where U_z is a neighbourhood of z in Δ^k on which $\mathscr{E}$ is biholomorphic. Therefore (ii) is proved.

(ii) → (iii). Given a meromorphic map $f : D \to X$, where D is a spread domain over a Stein manifold. Consider D as a spread domain over ${}^\wedge D$ with the canonical map $e : D \to {}^\wedge D$. By $D(f)$ we denote the envelope of meromorphy of f. Then $D(f)$ is a Stein manifold and f has a canonical meromorphic extension $\tilde{f}$ to $D(f)$. By the Steiness of $D(f)$ the canonical map $\beta : D \to D(f)$ can be extended to a holomorphic map ${}^\wedge\beta : {}^\wedge D \to D(f)$. Therefore $\tilde{f}\,{}^\wedge\beta$ is a meromorphic extension of f to ${}^\wedge D$.

(iii) → (i) is trivial.

2. Meromorphic extension maps.

DEFINITION 2.1. — *Let $\theta : X \to Y$ be a holomorphic map between complex space. We say that θ is a Hartogs (resp. Riemann) meromorphic extension map if for each $y \in Y$ there exists a neighbourhood U of y such that $\theta^{-1}(U)$ is a Hartogs (resp. Riemann) meromorphic extension space. If both conditions of Hartogs and Riemann meromorphic extension are satisfied, then θ is called a meromorphic extension map.*

THEOREM 2.2. — *Let $\theta : X \to Y$ be a Hartogs meromorphic extension map. Assume that Y is a Hartogs meromorphic extension space. Then for every meromorphic mapping f from a domain D over a Stein manifold to X, there exists an analytic subset A of codimension at least 2 in ${}^\wedge D$ such that f extends meromorphically to ${}^\wedge D \backslash A$. Moreover if X is a compact Kahler manifold and Y is a Hartogs meromorphic extension space and θ is a Hartogs meromorphic extension map then X is a meromorphic extension space.*

Proof. — (i) Given $f : D \to X$ a meromorphic map, where D is a spread domain over a Stein manifold. By hypothesis we have a following commutative diagram

$$\begin{tikzcd}
D \arrow[rr, "f"] \arrow[dr, "\beta"] \arrow[dd, "e"'] & & X \arrow[dd, "\theta"] \\
& D(f) \arrow[ur, "\tilde{f}"'] \arrow[dl] & \\
{}^\wedge D \arrow[rr, "g"'] & & Y
\end{tikzcd}$$

where g is a meromorphic extension of $\theta \cdot f$.

We show that $\gamma_0 = \gamma|_{D(f)\backslash\gamma^{-1}(I(g))} : D(f)\backslash\gamma^{-1}(I(g)) \to {}^\wedge D\backslash I(g)$ is locally pseudoconvex. Let $z \in {}^\wedge D\backslash I(g)$. Take a neighbourhood V of $g(z)$ in Y and a Stein neighbourhood U of z in ${}^\wedge D\backslash I(g)$ such that $g(U) \subset V$. As in Proposition 1.2 ((i)$\to$(ii)), it follows that $\gamma^{-1}(U)$ is p_7-convex. Therefore $\gamma^{-1}(U)$ is Stein [3], and the local pseudoconvexity of $D(f)\backslash\gamma^{-1}(I(g))$ over ${}^\wedge D\backslash I(g)$ is proved.

We now write $I(g) = \cap Z(h_\alpha)$, where h_α is holomorphic on $^\wedge D$ and vanishes on $I(g)$ and $Z(h_\alpha)$ denotes the zero-set of h_α. Since $\gamma_0 : D(f)\backslash\gamma^{-1}(I(g)) \to {}^\wedge D\backslash I(g)$ is locally pseudoconvex and $^\wedge D\backslash Z(h_\alpha)$ is Stein, $\gamma_0^{-1}(^\wedge D\backslash Z(h_\alpha))$ also is Stein for every α. For each α consider the holomorphic map $\beta_\alpha = \beta|_{D\backslash Z(h_\alpha)} : D\backslash Z(h_\alpha) \to \gamma_0^{-1}(^\wedge D\backslash Z(h_\alpha))$. Then β_α can be extended to a holomorphic map $^\wedge\beta_\alpha : {}^\wedge(D\backslash Z(h_\alpha)) = {}^\wedge D\backslash Z(h_\alpha) \to D(f)$ [2]. By uniqueness the maps $^\wedge\beta_\alpha$ define a holomorphic map $^\wedge\beta : {}^\wedge D\backslash I(g) \to D(f)$ such that $^\wedge\beta \cdot e = \beta$ on $D\backslash e^{-1}(I(g))$. The map $^\wedge f = \tilde{f} \cdot {}^\wedge\beta : {}^\wedge D\backslash I(g) \to X$ is meromorphic and is a meromorphic extension of f.

(ii) Now assume that X is a compact Kahler manifold and Y is a Hartogs meromorphic extension space and θ is a Hartogs meromorphic extension map. By [17] and since (i) it implies that X is a meromorphic extension space.

3. Finite proper holomorphic surjections and meromorphic extension spaces.

The aim of this section is to prove Theorem 3.1 on invariance of meromorphic extendibility under finite proper holomorphic surjections.

THEOREM 3.1. – *Let $\theta : X \to Y$ be a finite proper holomorphic surjective map. Then X is a meromorphic extension space if and only if Y has the same property.*

For the proof of the theorem we need following four lemmas.

LEMMA 3.2. – *Let $\varphi : W \to Z\backslash S$ be unbranched finite covering map, where W, Z are complex manifolds and S is an analytic set in Z of codimension $\geqslant 1$. Then there exists a following commutative diagram*

$$\begin{array}{ccc} W & \xrightarrow{\tilde{e}} & \tilde{W} \\ \varphi \downarrow & & \downarrow \tilde{\varphi} \\ Z\backslash S & \xrightarrow[e]{} & Z \end{array}$$

where $(\tilde{W},\tilde{\varphi},Z)$ is a branched covering map and $\tilde{e}$ is an open embedding.

Proof. – See [5] and [18].

LEMMA 3.3. — *Let $\varphi : G \to D$ be a branched covering map, where G is a normal complex space and D is a spread domain over a Stein manifold such that points of D are separated by holomorphic functions on D. Assume that H is the branch locus of φ and $D_0 = D\setminus H$, $G_0 = G\setminus\varphi^{-1}(H)$, $\varphi_0 = \varphi|_{G_0}$.*

Then there exists an analytic set H' in D contained in H such that ${}^\wedge(D\setminus H') = {}^\wedge D$ and a commutative diagram of normal complex spaces

$$\begin{tikzcd}
 & G\setminus\varphi^{-1}(H') \arrow[drr, "\beta"] \arrow[dd] & & \\
G_0 \arrow[ur] \arrow[rr, "e"] \arrow[dd, "\varphi_0"'] & & {}^\wedge G_0 \arrow[r, "\alpha"] \arrow[dd, "{}^\wedge\varphi_0"] & W \arrow[dd, "4"] \\
 & D\setminus H' & & \\
D_0 \arrow[ur] \arrow[rr] & & {}^\wedge D_0 \arrow[r] & {}^\wedge D
\end{tikzcd}$$

where ${}^\wedge\varphi_0$, 4, $\beta : G\setminus\varphi^{-1}(H') \to \operatorname{Im}\beta$ are branched covering maps, α is an open embedding and $\beta^{-1}(\alpha e(G_0)) = G_0$.

Proof. — Since D and G are normal, it follows that either H is a hypersurface in D or $H = \varnothing$. The case where $H = \varnothing$ is trivial. Therefore we can assume that H is a hypersurface. Then there exists an analytic set ${}^\wedge H$ in ${}^\wedge D$ such that

$${}^\wedge D_0 = {}^\wedge D\setminus {}^\wedge H \quad [2].$$

Observe that ${}^\wedge H \cap D \subset H$. We write $H = ({}^\wedge H \cap D) \cup H'$, where H' is an analytic set in D such that ${}^\wedge(D/H') = {}^\wedge D$. By [11] the map ${}^\wedge\varphi_0 : {}^\wedge G_0 \to {}^\wedge D_0$ is an unbranched covering map and using Lemma 3.2 to ${}^\wedge\varphi_0$, we can construct a commutative diagram

$$\begin{tikzcd}
 & G' \arrow[drr, dashed, "\beta"] \arrow[dd, "\varphi'"] & & \\
G_0 \arrow[ur] \arrow[rr, "e"] \arrow[dd, "\varphi_0"'] & & {}^\wedge G_0 \arrow[r, "\alpha"] \arrow[dd, "{}^\wedge\varphi_0"] & W \arrow[dd, "4"] \\
 & D' \arrow[drr] & & \\
D_0 \arrow[ur] \arrow[rr] & & {}^\wedge D_0 \arrow[r] & {}^\wedge D
\end{tikzcd}$$

where $D' = D\setminus H'$, $G' = G\setminus\varphi^{-1}(H')$ and $\varphi' = \varphi|_G$, in which 4 is a branched covering map of the normal complex space W onto ${}^\wedge D$ and

α is an open embedding. Put $^\wedge\alpha = \alpha e$. We shall prove $^\wedge\alpha$ can be extended to a holomorphic map β from G' to W. Since the Steiness is invariant under finite proper holomorphic surjections [13], W is Stein. Thus by the normality of G' it suffices to show that $^\wedge\alpha$ is locally compact on G', i.e for every $z \in G'$ there exists a neighbourhood U of z such that $^\wedge\alpha(U \cap G_0)$ is relatively compact in W. Assume that $z_0 \in \varphi'^{-1}(H')$ and $\{z_n\} \subset G_0$ converging to z_0.

Then

$$\lim 4\ ^\wedge\alpha(z_n) = \lim \varphi'(z_n) = \varphi_0(z_0) \in D' \hookrightarrow\ ^\wedge D.$$

Thus from property of 4, it follows that $\{^\wedge\alpha(z_n)\}$ is relatively compact in W. This yields the local compactness of $^\wedge\alpha$.

Let $\beta : G' \to W$ be a holomorphic extension of $^\wedge\alpha$. Since φ' and 4 are finite proper maps and D' is contained in $^\wedge D$ as an open subset, it is easy to see that $\beta : G' \to \beta(G')$ is finite proper. Hence by the normality of W and by the equality $\dim G' = \dim W$, it follows that $\beta(G')$ is open in W and $\beta : G' \to \beta(G')$ is a branched covering map. Finally, if $^\wedge\alpha(z_0) = \beta(z_1)$, where $z_0 \in G_0$ and $z_1 \in G'$, then

$$\varphi'(z_1) = 4\beta(z_1) = 4\ ^\wedge\alpha(z_0) = \varphi_0(z_0).$$

This implies $z_1 \in G_0$. Hence $\beta^{-1}(^\wedge\alpha(G_0)) = G_0$.

The lemma is proved.

LEMMA 3.4. – Let X *be a meromorphic extension space and* Z *a normal Stein space. Assume that* H *is a hypersurface of* Z *and* G *is an open subset of* Z *meeting every irreducible branch of* H. *Then every meromorphic map* $f : (D \backslash H) \cup G \to X$ *can be meromorphically extended to* Z.

Proof. – Since Z is normal, $\operatorname{codim} S(Z) \geqslant 2$ [4], where $S(Z)$ denotes the singular locus of Z. We write by the Steiness of $ZS(Z)$ in the form

$$S(Z) = \cap \{Z(h) : h \text{ is holomorphic on } Z, h|_{S(Z)}=0 \text{ and } h \neq 0 \text{ on every irreducible branch of } H\}.$$

From hypothesis, it suffices to show that for every such h the map $f_h = f|_{Z_h \backslash H}$, where $Z_h = Z \backslash Z(h)$, can be meromorphically extended on Z_h. Put $G_h = G \backslash Z(h)$ and $H_h = H \cap Z_h$. Then G_h also meets every irreducible branch of H_h. Consider the meromorphic map $f_h|_{(Z_h \backslash H_h) \cup G_h}$. Since ${}^\wedge((Z_h \backslash H_h) \cup G_h) = Z_h$ [2] it follows that $f_h|_{(Z_h \backslash H_h) \cup G_h}$ can be extended to a meromorphic map ${}^\wedge f_h$ to Z_h.

The lemma is proved.

LEMMA 3.5. — *Let $\pi : Z \to W$ be a banched covering map and $f : Z \to X$ a meromorphic map which can be factorized through $\pi|_{\pi^{-1}(V)}$ for some non-empty open subset V of W. Then f can be factorized through π.*

Proof. — Let H denote the branch locus of π. It is easy to check that there exists a holomorphic map g from $W \backslash (H \backslash \pi(I(f)))$ to X such that $g\pi = f$ on $\pi^{-1}(W \backslash (H \cup \pi(I(f))))$. Since $\pi \times \mathrm{id} : Z \times X \to W \times X$ is proper,

$$\overline{\Gamma(g)} = (\pi \times \mathrm{id})\Gamma(f)$$

is an analytic set in $W \times X$. Hence from property of π and $p(f)$, it follows that $\overline{\Gamma(g)}$ defines a meromorphic map g_1 on W such that $g_1 \cdot \pi = f$.

The lemma is proved.

We now can prove Theorem 3.1.

a) First prove sufficiency of the theorem.

(i) Given $f : D \to X$ a meromorphic map, where D is a spread domain over a Stein manifold. From hypothesis we have a following commutative diagram

$$\begin{tikzcd}
D \arrow[rr, "f"] \arrow[dd, "e"'] \arrow[dr, "\beta"] & & X \arrow[dd, "\theta"] \\
& D(f) \arrow[ur, "\tilde{f}"] \arrow[dl, "\gamma"] & \\
{}^\wedge D \arrow[rr, "g"'] & & Y
\end{tikzcd}$$

where g is a meromorphic extension of $\theta \cdot f$.

As in Theorem 2.2, $\beta|_{D \backslash e^{-1}(I(g))}$ can be extended to a holomorphic map ${}^\wedge\beta : {}^\wedge D \backslash I(g) \to D(f)$. Put $A = (\mathrm{id}^x\, \theta)^{-1}(p(g)^{-1}(I(g))$. Then $\Gamma({}^\wedge f) \subset ({}^\wedge D \backslash I(g)) \times X \subset (D(f) \times X) \backslash A$, where ${}^\wedge f = \tilde{f}\,{}^\wedge\beta$, and is closed in $(\gamma(D(f)) \times X) \backslash A$. Indeed, let $\{(x_n, z_n)\} \subset \Gamma({}^\wedge f)$ converge to

$(x_0, z_0) \in \gamma(D(f)) \times X \backslash A$. Since $(x_0, z_0) \bar{\in} A$, $(\mathrm{id} \times \theta)(x_0, z_0) = (x_0, \theta z_0) \bar{\in} p(g)^{-1}(I(g))$. If $x_0 \bar{\in} I(g)$, then $(x_0, z_0) \in ({}^{\wedge}D \backslash I(g)) \times X$. Hence $(x_0, z_0) \in \Gamma({}^{\wedge}f)$. In the case where $x_0 \in I(g)$, we have $(x_0, \theta z_0) \bar{\in} \Gamma(g)$. This is impossible, because of the relation $\Gamma(g) \supset \{(x_n, z_n)\} \to (x_0, \theta z_0) \in \Gamma(g)$. Therefore $\Gamma({}^{\wedge}f)$ is closed in $(\gamma({}^{\wedge}D(f)) \times X) \backslash A$. Since $\dim \Gamma({}^{\wedge}f) = \dim {}^{\wedge}D > \dim A$, by the Remmert-Stein theorem [7] $\Gamma({}^{\wedge}f)$ is an analytic set in ${}^{\wedge}D \times X$. Since θ is proper, it follows that $\Gamma({}^{\wedge}f)$ defines a meromorphic extension of f to ${}^{\wedge}D$.

(ii) Let now $f: Z \backslash S \to X$, where Z is a normal complex space and S is an analytic set in Z of codimension $\geqslant 2$. From the Riemann meromorphic extendibility of Y we have a following commutative diagram

$$\begin{tikzcd} Z \backslash S \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

Similarly as in (i), where D, ${}^{\wedge}D \backslash I(g)$ and $\tilde{f}{}^{\wedge}\beta$ are replaced by Z, $Z \backslash (I(g) \cup S)$ and f respectively we obtain a meromorphic extension ${}^{\wedge}f$ of f to Z.

b) We now prove necessity of the theorem.

(i) Let f be a meromorphic map from a spread domain D over a Stein manifold to X. By Proposition 1.2 we can assume that D is a Hartogs domain. Consider the commutative diagram

$$\begin{tikzcd} G_1 \arrow[r, "g_1"] \arrow[d, "\varphi_1"'] & X \arrow[d, "\theta"] \\ D_1 \arrow[r, "f_1"'] & Y \end{tikzcd}$$

where $D_1 = D \backslash I(g)$, $G_1 = (D_1 x_Y X)_{\mathrm{red}}$ is the fiber product, $f_1 = f|_{D_1}$ and φ_1, g_1 are canonical projections.

Without loss of generality we may assume that G_1 is normal. Observe that φ_1 is a branched covering map. Let H_1 denote the branched locus of φ_1, Since $\dim H_1 > \dim I(f)$, it follows that $\bar{H}_1$ is an analytic set in D. Using Lemma 3.2 to the unbranched covering map $\varphi_1 : G_1 \backslash \varphi_1^{-1}(H_1 \cup I(f)) \to D_1 \backslash (\bar{H}_1 \cup I(f))$ we have a following commutative diagram

$$\begin{tikzcd}
& G \arrow[dr, dashed, "g"] \arrow[dd, "\varphi" near start] & \\
G_1 \arrow[ur] \arrow[rr, "g_1" near end] \arrow[dd, "\varphi_1"'] & & X \arrow[dd, "\theta"] \\
& D \arrow[dr, "f"] & \\
D_1 \arrow[ur] \arrow[rr, "f_1"'] & & Y
\end{tikzcd}$$

in which φ is a branched covering map and G is normal. Since $\dim G_1 = \dim (\varphi \times \theta)^{-1} \Gamma(f) > \dim (\varphi \times \theta)^{-1} p(f)^{-1}(I(f))$, by the Remmert-Stein theorem [7], $\Gamma(g_1)$ is an analytic set in $G \times X$. Hence by property of θ, it defines a meromorphic extension of g on G such that $\theta g = f$. In notations of Lemma 3.3 we have a following commutative diagram of normal complexe spaces

$$\begin{tikzcd}
& G' \arrow[drr, dashed, "\beta"] \arrow[dd, "\varphi'" near start] & & \\
G_0 \arrow[ur] \arrow[rr, "e"'] \arrow[dd, "\varphi_0"'] & & {}^{\wedge}G_0 \arrow[r, hook, "\alpha"'] \arrow[dd, "{}^{\wedge}\varphi_0"] & W \arrow[dd, "4"] \\
& D' & & \\
D_0 \arrow[ur, hook] \arrow[rr] & & {}^{\wedge}D_0 \arrow[r] & {}^{\wedge}D
\end{tikzcd}$$

in which φ_0 and ${}^{\wedge}\varphi_0$ are unbranched covering maps, φ', 4, $\beta : G' \to \beta(G')$ are branched covering maps. Moreover $G_0 = \beta^{-1}(\alpha e(G_0))$. Thus $g|_G$ can be meromorphically factorized through $\beta : G' \to \beta(G')$. *Let* ${}^{\wedge}g_0$ be a meromorphic extension of $g|_{G_0}$ on ${}^{\wedge}G_0$ and $\tilde{g}$ a meromorphic map on $\beta(G')$ such that $\tilde{g}\beta = g|_{G'}$. Define a meromorphic map g_2 from ${}^{\wedge}G_0 \cup \beta(G')$ into X by

$$g_2 = {}^{\wedge}g_0 \text{ on } {}^{\wedge}G_0 \quad \text{and} \quad g_2 = \tilde{g} \text{ on } \beta(G').$$

Since 4 is finite proper and every irreducible branch of ${}^{\wedge}H$ meets D', it follows that this holds for $4^{-1}({}^{\wedge}H)$ and $\beta(G')$. Thus by Lemma 3.4 we have a meromorphic extension g_3 of g_2 on W. From Lemma 3.5, g_3 can be meromorphically factorized through 4. Hence f is extended to a meromorphic map to ${}^{\wedge}D$.

(ii) Finally we show that Y has the Riemann meromorphic extension property. Given $f: Z\setminus S \to Y$ a meromorphic map, where Z is a normal complex space and S is an analytic set in Z of codimension $\geqslant 2$ which can be assumed to contain the singular locus of Z.

As in (i) we can construct a following commutative diagram of normal complex spaces

$$\begin{tikzcd}
 & G_1 \arrow[dr, "g_1"] \arrow[dd, "\varphi_1"] & \\
G_0 \arrow[ur] \arrow[rr, "g_0"] \arrow[dd, "\varphi_0"'] & & X \arrow[dd, "\theta"] \\
 & Z\setminus S \arrow[dr, "f"] & \\
(Z\setminus S)\setminus I(f) \arrow[ur] \arrow[rr, "f_0"'] & & Y
\end{tikzcd}$$

where φ_0, φ_1 are branched covering maps and g_0, g_1 are meromorphic maps. The problem is local without loss of generality we may assume that there exists a branched covering map $\gamma: Z \to \Delta^n$, $n = \dim Z$. Let H denote the branch locus of φ_1.

Then $\bar{H}$ is an analytic set in Z because of the inequality codim $I(f) \geqslant 2$. Take a hypersurface $\tilde{H}$ in Δ^n containing the branch locus of γ such that $\gamma(S \cup H) \subset \tilde{H}$. Using Lemma 3.3 we give a following commutative diagram

$$\begin{tikzcd}
 & G_1 \arrow[dr, "\beta"] \arrow[dd, "\eta"] & \\
G_1\setminus\eta^{-1}(\tilde{H}) \arrow[ur] \arrow[rr, hook, "\alpha"] \arrow[dd, "\eta"'] & & W \arrow[dd, "4"] \\
 & \Delta^n\setminus\gamma(S) \arrow[dr] & \\
\Delta^n\setminus\tilde{H} \arrow[ur] \arrow[rr] & & \Delta^n
\end{tikzcd}$$

where $\eta = \gamma\varphi_1$, 4, $\beta: G_1 \to \beta(G_1)$ are branched covering maps.

Obviously $\beta^{-1}(\alpha(G_1\setminus\eta^{-1}(\tilde{H}))) = G_1\setminus\eta^{-1}(\tilde{H})$. Thus g_1 can be meromorphically factorized through $\beta: G_1 \to \beta(G_1)$. Hence g_0 and g_1 induce a meromorphic map g_2 on $G_1\setminus\eta^{-1}(\tilde{H}) \cup \beta(G_1)$ with values in X. Since every irreducible branch of $\tilde{H}$ meets $\Delta^n \backslash \gamma(S)$, it follows that this holds for $4^{-1}(\tilde{H})$ and $\beta(G_1)$. By Lemma 3.4, g_2 can be extended to a meromorphic map g_3 on W. Thus from Lemma 3.5 we give a meromorphic extension of f to Δ^n.

Theorem 3.1 is completely proved.

4. Some applications.

We first recall that an elliptic surface is a compact regular surface V equipped with a holomorphic map θ from V onto a non-singular curve C such that $\theta^{-1}(x)$ is an elliptic curve outside a finite set in C.

Using now Theorem 3.1 we prove the following.

THEOREM 4.1. – *Let V be an elliptic Kahler surface. Then V is a meromorphic extension space.*

Proof. – From a result of Siu [17], V is a Riemann meromorphic extension space. Thus it remains to prove that V has the Hartogs meromorphic extension property.

(i) In [12] Kodaira constructed for V a branched covering map α from an elliptic surface $\tilde{V}$ on V such that for each $x \in C$ there exists a sufficiently small disc U_x containing x for which $(\theta\alpha)^{-1}(U_x)$ is biholomorphic to a locally pseudoconvex open subset of a projective surface P_x. Put $\eta = \theta \cdot \alpha$. Given $f: D \to \eta^{-1}(U_x)$ a meromorphic map, where D is a spread domain over a Stein manifold.

Let ${}^{\wedge}f: {}^{\wedge}D \to P_x$ be a meromorphic extension of $f|_{D \backslash I(f)}$. Put

$$G = {}^{\wedge}f_0^{-1}(\eta,{}^{-1}(U_x)), \qquad \text{where} \qquad {}^{\wedge}f_0 = {}^{\wedge}f| \, {}^{\wedge}D \backslash I({}^{\wedge}f).$$

We may suppose that D is a Hartogs domain. Since $D \backslash I(f) \subset G$ we have ${}^{\wedge}G = {}^{\wedge}D$. Let now $G \neq {}^{\wedge}D \backslash I({}^{\wedge}f)$. Then we can find a point $z_0 \in \partial G$ in ${}^{\wedge}D \backslash I({}^{\wedge}f)$ and two Stein neighbourhoods of z_0 and ${}^{\wedge}f_0(z_0)$ in ${}^{\wedge}D \backslash I({}^{\wedge}f)$ and P_x respectively such that ${}^{\wedge}f_0(U) \subset W$ and $z_0 \in {}^{\wedge}(U \cap G)$. Since $W \cap \eta^{-1}(U_x)$ is Stein and ${}^{\wedge}f_0(U \cap G) \subset W \cap \eta^{-1}(U_x)$, it follows that ${}^{\wedge}f_0(z_0) \in W \cap \eta^{-1}(U_x)$. This yields $z_0 \in G$. Hence $G = {}^{\wedge}D \backslash I({}^{\wedge}f)$. On the other hand, since $\alpha \, {}^{\wedge}f_0$ and $\eta \, {}^{\wedge}f_0$ are extended to meromorphic maps $g: {}^{\wedge}D \to V$ and $h: {}^{\wedge}D \to U_x$ respectively. We have $\theta g = h$.

It is easy to see that $\Gamma({}^{\wedge}f_0)$ is contained and closed in $\underline{(\mathrm{id} \times \alpha)}^{-1}\Gamma(g) \backslash (\mathrm{id} \times \alpha)^{-1} p(g)^{-1}(I(g))$, by the Remmert-Stein theorem, $\Gamma({}^{\wedge}f_0)$ defines a meromorphic extension $\tilde{f}$ of f.

From the relation $\eta\tilde{f} = f$, it follows that $\tilde{f}$ induces a meromorphic extension of f with values in $\eta^{-1}(U_x)$.

(ii) Let now f be a meromorphic map from a spread domain D over a Stein manifold into $\tilde{V}$. Consider the following commutative diagram

$$
\begin{tikzcd}
D \arrow[rr, "f"] \arrow[dd] \arrow[dr, "\beta"] & & \tilde{V} \arrow[dd, "\eta"] \\
& D(f) \arrow[ur, "\tilde{f}"] \arrow[dl, "\gamma"] & \\
{}^\wedge D \arrow[rr, "g"'] & & C
\end{tikzcd}
$$

By (i) as in Theorem 2.1 we can find a holomorphic extension ${}^\wedge\beta$ of $\beta|_{D\backslash I(g)}$ on ${}^\wedge D\backslash I(g)$. Let ${}^\wedge f_1 : {}^\wedge D \to V$ be a meromorphic extension of $f_1 = \alpha\tilde{f}\,{}^\wedge\beta : D\backslash I(g) \to V$. Then as in Theorem 2.1, it follows that $\overline{\Gamma(\tilde{f}\,{}^\wedge\beta)}$ defines a meromorphic extension of f. Hence $\tilde{V}$ is a Hartogs meromorphic extension space.

(iii) Given a meromorphic map f from $Z\backslash S$ into $\tilde{V}$, where Z is a normal complex space and S is an analytic set in Z of codimension $\geqslant 2$. Let $g : Z \to V$ be a meromorphic extension of αf. Then as in (i) we infer that $\overline{\Gamma(f)}$ defines a meromorphic extension of f.

(iv) From (ii) and (iii), $\tilde{V}$ is a meromorphic entension space. Hence by Theorem 3.1, V is a meromorphic extension space.

The theorem is proved.

THEOREM 4.2. – *Every complex Lie group is a meromorphic extension space.*

Proof. – Let G be a complex Lie group.

(i) Given $f : D \to G$ a meromorphic map, where D is a spread domain over a Stein manifold. Since $\operatorname{codim} I(f) \geqslant 2$, $f|_{D\backslash I(f)}$ can be holomorphically extended to ${}^\wedge D$ [1]. Thus G is a Hartogs meromorphic extension space.

(ii) Given now f a meromorphic map from $Z\backslash S$ to G, where Z is a normal complex space and S is an analytic set in Z of codimension $\geqslant 2$. Let φ be a plurisubharmonic exhaustion function [10] on G. Since codim $I(f) \geqslant 2$ and codim $S \geqslant 2$, φf is plurisubharmonic on Z [8]. By [19] there exists a holomorphic bundle map θ from G onto a complex torus T such that the fibers of θ are Stein manifolds. Consider the holomorphic map $\theta f|_{(Z\backslash S)\backslash I(f)}$. Then, by the Kahlerness of the torus T, θf is meromorphic on Z [17]. Let $\gamma : {}^{\wedge}Z \to Z$ be the Hironaka singular resolution of Z. By (i), $h = \theta f \gamma$ is holomorphic on ${}^{\wedge}Z$. For each $z_0 \in \gamma^{-1}(S)$ take the two neighbourhoods U and V of z_0 and $h(z_0)$ respectively such that $h(U) \leqslant V$ and $\theta^{-1}(V)$ is a Stein manifold. Then we have $f\gamma(U\backslash^{-1}(S)) \leqslant \theta^{-1}(V)$. By the upper semi-continuity of $\varphi f \gamma$ on ${}^{\wedge}Z$ and since φ is an exhaustion function on G it follows that $f^{\gamma}|_{U\backslash\gamma^{-1}(S)}$ can be extended holomorphically at z_0. Since z_0 is arbitrary $f\gamma$ is extended holomorphically to ${}^{\wedge}Z$. Then $(f\gamma)\gamma^{-1}$ is a meromorphic extension of f.

The theorem is proved.

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