ANNALES DE L'INSTITUT FOURIER

MICHEL BOILEAU
D. J. COLLINS
H. ZIESCHANG

Genus 2 Heegaard decompositions of small Seifert manifolds

Annales de l'institut Fourier, tome 41, nº 4 (1991), p. 1005-1024
<http://www.numdam.org/item?id=AIF_1991__41_4_1005_0>

Ann. Inst. Fourier, Grenoble
41, 4 (1991), 1005-1024.

GENUS 2 HEEGAARD DECOMPOSITIONS OF SMALL SEIFERT MANIFOLDS

by M. BOILEAU, D. J. COLLINS and H. ZIESCHANG

The Heegaard splittings and decompositions of genus 2 of Seifert manifolds over S^2 with three exceptional fibres are classified with respect to isotopies and homeomorphisms. In general there are three different isotopy classes of Heegaard splittings and six different isotopy classes of decompositions. Moreover, we determine when an isotopy class is also a homeomorphism class.

1. Introduction.

Let M^3 be a closed connected orientable 3-manifold. A *Heegaard decomposition* $(M^3; V)$ *of genus* g *of* M^3 is (defined by) a handlebody V of genus g embedded in M^3 such that $W = \overline{M^3 - V}$ is also a handlebody. A *Heegaard splitting of genus* g of M^3 is (defined by) a closed orientable surface F_g^2 of genus g separating M^3 into two handlebodies (of genus g). Two Heegaard decompositions or splittings of M^3 are called *isotopic* or *homeomorphic* if there exists an isotopy or homeomorphism, respectively, mapping one decomposition or splitting, respectively, to the other.

Heegaard splittings were introduced to construct and classify 3-manifolds. In this context there arises the classification problem for Heegaard splittings. Moreover, they can be used to study homeomorphisms of 3-manifolds and to compute the mapping class group of some special 3-manifolds, for example, lens spaces [7] or some « small » Seifert manifolds [3]. In particular, the classification of Heegaard splittings is the main tool to show that every homeomorphism of the Poincaré sphere is isotopic to the identity [3].

Key-words : Heegaard decomposition - Seifert manifolds - Generating system - Nielsen equivalence.

A.M.S. Classification : 57H25.

In general, the classification of Heegaard decompositions or splittings of a given manifold M^3 is difficult; it is known only for S^3, $S^1 \times S^2$, the lens spaces and $S^1 \times S^1 \times S^1$, see [20], [4], [8]. All these spaces are Seifert manifolds and have, up to isotopy, a unique Heegaard splitting of a given genus.

M^3 is a *Seifert manifold* if it admits a foliation into circles; this foliation is called a *Seifert fibration.* The quotient space obtained by identifying each fibre (leaf) to a point is a surface F^2, called *basis of the Seifert fibration.* Let us assume that F^2 is closed and orientable. Under these assumptions M^3 admits a S^1-action where the orbits are the fibres and F^2 is the orbifold. In fact, all such S^1-actions on 3-manifolds are obtained in this way. The pojection $\pi : M^3 \to F^2$ does, in general, *not* define a fibre bundle in the usual sense, but the restriction does when we exclude a finite number of points $x_1, \ldots, x_m$ of F^2 and the corresponding « exceptional » fibres $\varepsilon_1, \ldots, \varepsilon_m$ of M^3. (See 2.1.)

In this article we classify up to isotopy (or homeomorphism, respectively) the genus 2 Heegaard splittings for the Seifert spaces with basis S^2 and three exceptional fibres. It turns out that there are in general *three different* isotopy or homeomorphism classes; in some cases there are two classes, while in others any two Heegaard splittings are isotopic or homeomorphic, respectively. The classification with respect to isotopy differs in a few cases from the classification up to homeomorphism. These results have been announced in [2]. Furthermore we classify the genus 2 Heegaard decompositions for the above manifolds.

The results on Heegaard splittings have independently been obtained by Y. Moriah [13], except for those cases where two exceptional fibres have the same invariant β_i/α_i up to sign. In this case the methods used in [13] do not lead to a definitive answer. On the other hand, our methods cannot be applied in the form given here when $\alpha_i = 2$ for one of the exceptional fibres. Although the basic small cancellation techniques we employ are still available in this situation, a more elaborate argument would be necessary. Since a simple argument from the method of Moriah [13] deals with this case, we have not attempted to give a separate argument here. However the results stated do give a full classification up to isotopy or homeomorphism of all genus 2 Heegaard splittings and decompositions of the manifolds considered. We are grateful to Y. Moriah for making the manuscript of [13] available to us and to the referee for his valuable suggestions.

2. Heegaard splittings of Seifert manifolds $S(0\,;e_0\,;\ \beta_1/\alpha_1,\ \beta_2/\alpha_2,\ \beta_3/\alpha_3)$.

Let M be a closed orientable Seifert manifold with basis the 2-sphere S^2 and three exceptional fibres ε_1, ε_2, ε_3. Let us consider first in more detail the situation at the i-th exceptional fibre.

2.1. *Exceptional fibres.* – The projection

$$\pi : \pi^{-1}(F^2 - \{x_1, x_3, x_3\}) \to (F^2 - \{x_1, x_2, x_3\})$$

is a locally trivial fibration. Every point x_i has a disc neighbourhood D_i^2 such that $T_i = \pi^{-1}(D_i^2)$ is a solid torus $D_i^2 \times S^1$, the core of which is mapped to x_i. If we use polar coordinates (r, φ) for D_i^2 and ψ for S^1 then the fibres are

$$\left\{\left(r, \varphi + \frac{\gamma_i}{\alpha_i}\cdot\psi, \psi\right)\middle| 0 \leqslant \psi \leqslant 2\pi\right\}, \text{ where } \alpha_i > 1,\ \alpha_i,\ \gamma_i \in \mathbf{Z},\ \gcd(\alpha_i, \gamma_i) = 1\,.$$

Choose β_i, δ_i such that $\alpha_i\delta_i - \beta_i\gamma_i = 1$. When ψ runs from 0 to 2π every « ordinary » fibre f with $r > 0$ is traversed exactly once, but the central fibre with $r = 0$ is traversed α_i times. The latter is called an *exceptional fibre of type* $\beta_i/\alpha_i \bmod 1$.

Let (the homotopy classes of) the meridian m_i and the longitude ℓ_i of T_i be defined by the coordinates φ and ψ, respectively. Here $\ell_i \simeq \varepsilon_i$ in T_i. The curve m_i is uniquely determined up to isotopy and reversing of orientation, while ℓ_i can be replaced by a curve of a class $m_i^k \ell_i^{\pm 1}$, $k \in \mathbf{Z}$. For the ordinary fibre f we have $f \simeq m_i^{\gamma_i}.\ell_i^{\alpha_i}$ on ∂T_i. Moreover there is a simple closed curve $s_i \subset \partial T_i$ intersecting any fibre of ∂T_i exactly once such that $s_i \simeq m_i^{-\delta_i}.\ell_i^{-\beta_i}$ on ∂T_i, where $\alpha_i\delta_i - \beta_i\gamma_i = 1$. Hence, $m_i \simeq s_i^{-\alpha_i}.f^{-\beta_i}$ and $\ell_i \simeq s_i^{\gamma_i}.f^{\delta_i}$ on ∂T_i.

Now it is easy to see that we obtain the following presentation of the fundamental group:

2.2. $\pi_1 M = \langle f, s_1, s_2, s_3 \,|\, [s_i, f] = s^{\alpha_i} f^{\beta_i} = s_1 s_2 s_3 f^e = 1, i = 1, 2, 3\rangle$

where the integer e is the usual Euler class representing the obstruction to extend a section given on the boundary components of regular neighbourhoods of the exceptional fibres to the complement.

Following Thurston, one introduces the rational number

$$e_0 = e - \frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2} - \frac{\beta_3}{\alpha_3}$$

(called the rational Euler class) and the manifold M is usually denoted by $S\left(0; e_0; \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3}\right)$, where 0 is the genus of the basis of the fibration. These numbers determine the topological type of the manifolds (also the type of the Seifert fibration). For details see [15], [17], [18].

2.3. *The Heegaard splitting $HS(i,j)$ of M^3.* On M^3 we construct three Heegaard splittings of genus 2 as follows : take two exceptional fibres ε_i, ε_j, $1 \leqslant i \neq j \leqslant 3$ and connect them by an arc projected to a simple arc on the base S^2. A regular neighbourhood $V(i,j)$ of the graph obtained is a handlebody of genus 2. The closure $W(i,j)$ of the complement is also a handlebody of genus 2 and one obtains a Heegaard splitting $HS(i,j) = (M^3; \partial V(i,j))$. This is called a *vertical Heegaard splitting*, see [5], [6]. The pairs $(M^3; V(i,j))$ and $(M^3; W(i,j))$ are genus 2 *Heegaard decompositions of M^3*.

According to [3], with only a few exceptions every Heegaard splitting of genus 2 is isotopic to a vertical splitting :

2.4. PROPOSITION [3]. — *Every Heegaard splitting of $S(0; e_0; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ of genus 2 is isotopic to a vertical splitting, except in the following cases :*

(i) $V(2,3,a) = \{z \in \mathbf{C}^3 \mid z_1^2 + z_2^3 + z_3^a = 0, \|z\| = 1\}$ *with* $3 \nmid a$, $a \geqslant 7$,

(ii) $W(2,4,b) = \{z \in \mathbf{C}^3 \mid z_1^2 + (z_2^2 + z_3^b) z_2 = 0, \|z\| = 1\}$ *with* $2 \nmid b$, $b \geqslant 5$.

($V(2,3,a)$ is a Brieskorn manifold.) In these exceptional cases M^3 admits a further Heegaard splitting of genus 2 obtained by presenting M^3 as a double covering of S^3 branched along a 3-bridge presentation of an algebraic link. □

Hence, to determine the classes of Heegaard splittings of genus 2 of M^3 it suffices to classify the vertical Heegaard splittings. Our final result is :

2.5. THEOREM. — *Let $M^3 = S(0; e_0; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ be a Seifert manifold fibred over S^2 with three exceptional fibres.*

(A) *If* $\beta_i \not\equiv \pm 1 \bmod \alpha_i$ *for* $i = 1, 2, 3$ *then* M^3 *admits, up to isotopy, exactly three Heegaard splittings of genus* 2, *namely* $HS(1,2)$, $HS(2,3)$, $HS(3,1)$.

(B) $M = S(0; e_0; \beta_1/\alpha_1, \beta_2/\alpha_2, \pm 1/\alpha_3)$ *with* $\beta_i \not\equiv \pm 1 \bmod \alpha_i$ *for* $i = 1, 2$ *has, up to isotopy, exactly two Heegaard splittings of genus* 2, *namely* $HS(1,2)$ *and* $HS(3,1)$ *and the latter is isotopic to* $HS(2,3)$.

(C) *Let* $M = S(0; e_0; \beta_1/\alpha_1, \pm 1/\alpha_2, \pm 1/\alpha_3)$. *Then* M^3 *admits, up to isotopy, a single Heegaard splitting of genus* 2 *except when* M^3 *is one of the algebraic varieties* $V(2,3,a)$, $W(2,4,b)$, $\gcd(a,3) = \gcd(b,2) = 1$, $a \geqslant 7$, $b \geqslant 5$. *In each exceptional case* M^3 *admits, up to isotopy, an additional Heegaard splitting of genus* 2 *which is not vertical, see* [1], [3] (1).

2.6. Corollary. — (a) *Two vertical Heegaard splittings* $HS(i,j)$, $HS(j,k)$, $k \neq i$, $1 \leqslant i, j, k \leqslant 3$ *are isotopic if and only if* $\beta_j \equiv \pm 1 \bmod \alpha_j$ *or if* $\beta_i \equiv \pm 1 \bmod \alpha_i$ *and* $\beta_k \equiv \pm 1 \bmod \alpha_k$.

(b) *If two of these Heegaard splittings are not isotopic then they are homeomorphic if and only if* $\beta_i/\alpha_i \equiv \beta_k/\alpha_k \bmod 1$.

Moreover, we will distinguish between the Heegaard decompositions corresponding to a splitting:

2.7. Corollary (Classification of Heegaard Decompositions). — *In case* (A) *of Theorem* 2.5 *there are six different Heegaard decompositions up to homeomorphism* (*and isotopy*). *In case* (B) *there are four Heegaard decompositions up to isotopy. There are three Heegaard decompositions up to homeomorphism if and only if* $\beta_1/\alpha_1 \equiv \beta_2/\alpha_2 \bmod 1$. *In the non-exceptional cases of* (C) *there are two different decompositions up to homeomorphism except when all* $\beta_i \equiv \pm 1 \bmod \alpha_i$ *when there is only one Heegaard decomposition up to isotopy. In the exceptional cases there is only one additional Heegaard decomposition of genus* 2.

---

(1) The exceptional manifolds have the following Seifert invariants:

$$V(2,3,a) = S\left(0; -\frac{1}{6a}; \frac{1}{2}, \frac{(-a)^{-1}}{3}, \frac{(6)^{-1}}{a}\right) \quad \text{if a is odd,}$$
$$V(2,3,2a') = S\left(0; -\frac{1}{3a}; \frac{(-a')^{-1}}{3}, \frac{(-a')^{-1}}{3}, \frac{(3)^{-1}}{a'}\right),$$
$$W(2,4,b) = S\left(0; -\frac{1}{4b}; \frac{1}{2}, \frac{(-b)^{-1}}{4}, \frac{(4)^{-1}}{b}\right)$$

where (n) and $(n)^{-1}$ denote the number n or its inverse reduced modulo the denominator to a non-negative integer smaller than the denominator.

3. The commutator invariant $\mathcal{K}(M^3; V)$ of a Heegaard decomposition.

Let F_n be the free group of rank n and G an arbitrary group. Two systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of elements of G are called *Nielsen equivalent* if there exists a homomorphism $\sigma : F_n \to G$ such that the two systems are the images under σ of two free bases of F_n. It is important to bear in mind that the homomorphism σ is the *same* for both systems. Clearly, using the known generators of Aut (F_n), it follows that the above definition is equivalent to the original idea of calling two systems Nielsen equivalent if one can be transformed into the other by a sequence of replacements of x_1 by $x_1 x_2^{\pm 1}$ or x_1^{-1} and of permutations of the elements in question.

Given a Heegaard decomposition $(M; V)$ of a 3-manifold M^3 of genus n, let $i : V \hookrightarrow M^3$ be the inclusion. A system of free generators of $\pi_1 V \cong F_n$ is mapped by $i_\# : \pi_1 V \to \pi_1 M^3$ to a system of generators of $\pi_1 M^3$ which will be called *geometric*. Recall that every system of free generators of $\pi_1 V$ can be represented by a system of simple closed paths on ∂V where different curves have only the basepoint in common; hence, a geometric system of generators of $\pi_1 M^3$ is obtained from a « nice » system of curves on the surface $\partial V = \partial W$. Moreover, any system Nielsen equivalent to a system defined by V is itself defined by V and, thus, is also geometric. We denote the Nielsen equivalence class by $\mathcal{N}(M^3; V)$ and call it *adjoined to the Heegaard decomposition.* Changing the sides of the Heegaard decomposition, i.e. considering $(M; W)$, we obtain the class $\mathcal{N}(M^3; W)$ which is in general different from $\mathcal{N}(M^3; V)$. A crucial observation is the second statement of the following proposition.

3.1. PROPOSITION. — (a) *Any two geometric systems defined by the same Heegaard decomposition $(M^3; V)$ are Nielsen equivalent.*

(b) *If two Heegaard decompositions $(M^3; V)$ and $(M^3; V')$ of M^3 are isotopic, then $\mathcal{N}(M^3; V) = \mathcal{N}(M^3; V')$ and $\mathcal{N}(M^3; W) = \mathcal{N}(M^3; W')$.*

(c) *If there is a homeomorphism $f : (M^3, V) \to (M^3, V')$ then, for every system $(x_1, \ldots, x_n) \in \mathcal{N}(M^3; V)$, one has*

$$(f_\#(x_1), \ldots, f_\#(x_n)) \in \mathcal{N}(M^3; V'),$$

that is, we can write $f_\#(\mathcal{N}(M^3; V)) = \mathcal{N}(M^3; V')$. In addition, $f_\#(\mathcal{N}(M^3; W)) = \mathcal{N}(M^3; W')$.

Proof. — (b) follows from the fact that all inner automorphisms of $\pi_1 M^3$ are induced by inner automorphisms of $\pi_1 V$, i.e. by Nielsen processes.

Given two Heegaard decompositions $(M^3; V)$ and $(M^3; V')$ and a homeomorphism $f : (M^3; V) \to (M^3; V')$ one obtains the following commutative diagram :

$$\begin{array}{ccc} \pi_1 V & \xrightarrow{(f_{|V})_\#} & \pi_1 V' \\ {\scriptstyle i_\#}\big\downarrow & & \big\downarrow{\scriptstyle i'_\#} \\ \pi_1 M^3 & \xrightarrow{f_\#} & \pi_1 M^3 . \end{array}$$

Therefore, if $(x_1, \ldots, x_n) \in \mathcal{N}(M^3; V)$ then

$$(f_\#(x_1), \ldots, f_\#(x_n)) \in \mathcal{N}(M^3; V') .$$

Hence we can write $f_\#(\mathcal{N}(M^3; V)) = \mathcal{N}(M^3; V')$ and, similarly, $f_\#(\mathcal{N}(M^3; W)) = \mathcal{N}(M^3; W')$. In particular, when f is *homotopic* to the identity then $\mathcal{N}(M^3; V) = \mathcal{N}(M^3; V')$ and $\mathcal{N}(M^3; W) = \mathcal{N}(M^3; W')$. □

To show that two Heegaard decompositions $(M^3; V)$ and $(M^3; V')$ are not homeomorphic, one must be able :

i) to conclude that the two generating systems adjoined to the Heegaard decompositions are not Nielsen equivalent, and

ii) to determine the images of one of the two generating systems under all automorphisms of $\pi_1 M^3$ (induced by self-homeomorphisms of M^3) and to compare these with the other generating system according to i).

If two Heegaard decompositions are to be distinguished with respect to isotopy one only has to solve problem i), but already this is not decidable in general. The problem ii) is even more difficult.

If one wants to distinguish two Heegaard splittings up to isotopy (or homeomorphism) one has to solve problem i) at least twice and possibly four times in comparing the Heegaard decomposition $(M^3; V)$ with $(M^3; V')$ and $(M^3; W')$.

For genus 2 Heegaard decompositions problem i) is much easier to handle than in general due to the following result.

3.2. LEMMA [14]. — *Let $F\langle s,t\rangle$ be the free group of rank 2 and x, $y \in F\langle s,t\rangle$. Then (x,y) generates $F\langle s,t\rangle$ if and only if the commutator $[x,y]$ is conjugate to $[s,t]^{\pm 1}$.* □

Hence, $\mathcal{N}(M^3; V)$ defines the conjugacy class of one element of $[\pi_1 M^3, \pi_1 M^3]$ or its inverse as an invariant of $(M^3; V)$, up to isotopy. Let us denote the union of these conjugacy classes by $\bar{\mathcal{K}}(M^3; V)$. Next we determine this invariant for the Heegaard decompositions $(M^3; V(i,j))$ and $(M^3; W(i,j))$ of the Seifert fibre manifold $M^3 = S(0; e_0; \beta_1/\alpha_1, \beta_2/\alpha_2; \beta_3/\alpha_3)$ described above. By 2.2, the fundamental group has the following symmetric presentation according to the construction of M^3 as Seifert fibration:

$$\pi_1 M^3 = \langle s_1, s_2, s_3, f \mid [s_i, f] = s_i^{\alpha_i} f^{\beta_i} = s_1 s_2 s_3 f^e = 1\,,\ i = 1,2,3\rangle.$$

The handle T_i with core ε_i in $(M^3; V(i,j))$, $i \neq j$ admits a meridian from the homotopy class $s_i^{\alpha_i} f^{\beta_i}$ (now considered as element of $\pi_1(\partial T_i)$). To it corresponds a generator of the class $s_i^{\gamma_i} f^{\delta_i}$. Instead of using the subset $\bar{\mathcal{K}}(M^3; V)$ of $\pi_1 M^3$ we consider its image $\mathcal{K}(M^3; V)$ in the triangle group $\Delta(\alpha_1, \alpha_2, \alpha_3) = \pi_1 M^3/\langle f\rangle$ obtained from $\pi_1 M^3$ by factoring out the centre of $\pi_1 M^3$. (We denote the image of an element x by X.) Hence, to $(M; V(i,j))$ there are associated the elements $S_i^{\gamma_i}$, $S_j^{\gamma_j}$ and $\mathcal{K}(M^3; V(i,j))$ is represented by the commutator $[S_i^{\gamma_i}, S_j^{\gamma_j}]$. By [5, section 7] (see proof of Proposition 2.8), the handlebody $W(i,j)$ of $HS(i,j)$ has as generator the class s_i (or s_j) plus the generator belonging to the third exceptional fibre ε_k, $i \neq k \neq j$, that is $s_k^{\gamma_k}$, and here we deduce that the commutator $[S_i, S_k^{\gamma_k}]$ represents $\mathcal{K}(M^3; W(i,j))$; another representative is $[S_j, S_k^{\gamma_k}]$.

The result is:

3.3. PROPOSITION. — *The subsets $\mathcal{K}(M^3; V(i,j))$ and $\mathcal{K}(M^3; W(i,j))$ of*

$$\Delta(\alpha_1, \alpha_2, \alpha_3) = \langle S_1, S_2, S_3 \mid S_1^{\alpha_1} = S_2^{\alpha_2} = S_3^{\alpha_3} = S_1 S_2 S_3 = 1\rangle$$

are represented by $[S_i^{\gamma_i}, S_j^{\gamma_j}]$ and $[S_i, S_k^{\gamma_k}]$, respectively, where $i \neq k \neq j$. □

Remarks. — (a) The commutator was first used in Moriah's thesis [12], see also [13]. It can be used as a tool only for genus 2 Heegaard splittings. An invariant for higher genus has been constructed by Lustig [9] and has been applied to classify Heegaard splittings of genus more than 2 in [10].

(b) In the case at hand it turns out that the weaker invariant $\mathcal{K}(M^3; V)$ determines $\mathcal{N}(M^3; V)$ and the isotopy class of $(M^3; V)$.

4. Generating systems for triangle groups.

To prove 2.5 and 2.6 we decide which commutators of type $[S_i^{\gamma_i}, S_j^{\gamma_j}]^{\pm 1}$ or $[S_i, S_k^{\gamma_k}]^{\pm 1}$ are conjugate one to another. To do so we use the notion of cancellation diagram discussed in [11, V] in particular the idea of a conjugacy diagram given in pp. 252-259. In the following we study a triangle group

$$\Delta(p,q,r) = \langle S, T \mid S^p = T^q = (ST)^r = 1 \rangle .$$

We summarize exactly what we need from [11]. Let $\langle X \mid R \rangle$ be a symmetrized group presentation. Thus R is a subset of the free group $F(X)$, with basis X, such that (a) every element of R is cyclically reduced, (b) if r lies in R then all cyclic permutations of r and r^{-1} also lie in R. A (reduced) word x is a *piece* relative to R if there are distinct elements r_1 and r_2 of R which can be written as reduced products $r_1 = xr'_1$, $r_2 = xr'_2$. We say that R *satisfies* $C'(1/4)$ if, whenever $r \in R$ and $r = xr'$ with x a piece, then $|x| < \frac{1}{4}|r|$ – here $|w|$ denotes the length of w relative to X. Further we say that R *satisfies* $T(4)$ if for any r_1, r_2, $r_3 \in R$ at least one of the words r_1r_2, r_2r_3, r_3r_1 either cancels to the identity in $F(X)$ or is reduced.

4.1. Lemma. – *If* p, $q \geqslant 5$ *and* $r \geqslant 3$ *then the presentation* $\langle S, T \mid S^{\pm p}, T^{\pm q}, (ST)^{\pm r}, (TS)^{\pm r} \rangle$ *of* $\Delta(p,q,r)$ *satisfies* $C'(1/4)$ *and* $T(4)$.

Proof. – Inspection shows that a piece is always of the form $S^{\pm 1}$, $T^{\pm 1}$, whence the lemma is immediate. □

Let $\langle X \mid R \rangle$ be a symmetrized group presentation. A *cancellation diagram* over $\langle X \mid R \rangle$ is a pair (K, φ) satisfying the following conditions:

(i) K is a finite, connected 2-complex embedded in the plane;

(ii) φ is a function that assigns to each oriented edge (i.e. 1-cell) e of K an element $\varphi(e) \in F(X)$ (not just a generator or its inverse), with the requirement that if $\bar{e}$ is the oppositely oriented edge to e, then $\varphi(\bar{e}) = \varphi(e)^{-1}$;

(iii) if D is any region (i.e. 2-cell) of K and $\alpha = e_1e_2 \ldots e_n$ is a (complete) boundary cycle for D, then the product $\varphi(\alpha)$ defined by $\varphi(\alpha) = \varphi(e_1)\varphi(e_2) \ldots \varphi(e_n)$ is a reduced product which gives an element of R.

A cancellation diagram (K,φ) is called *reduced* unless there exist two (not necessarily distinct) regions with complete (oriented) boundary cycles of the form αe and $\bar{e}\beta$ such that $\varphi(\alpha) = \varphi(\beta)^{-1}$.

A *conjugacy diagram* over $\langle X|R\rangle$ is a cancellation diagram (K,φ) such that the complement of K has precisely two components. Given a conjugacy diagram (K,φ), let $\alpha = e_1e_2 \ldots e_m$ and $\beta = f_1f_2 \ldots f_n$ be positively oriented complete boundary cycles for the two components of the complement of K. By [11, Lemma V.5.1], the words $u = \varphi(\alpha) = \varphi(e_1) \ldots \varphi(e_m)$ and $v = \varphi(\beta) = \varphi(f_1) \ldots \varphi(f_n)$ represent conjugate elements of the group G defined by $\langle X|R\rangle$. In this situation say that (K,φ) is a *conjugacy diagram for u and v*. Furthermore, given any two cyclically reduced words u and v, which represent elements conjugate in G, by [11, Lemma V.5.2], there is a reduced conjugacy diagram for u and v. Moreover, if u and v are not conjugate in $F(X)$ then such a diagram must contain at least one region.

We now describe three types of reduced conjugacy diagrams which may arise in dealing with a presentation satisfying both $C'(1/4)$ and $T(4)$. Here we are essentially summarizing Theorems V.5.3 and V.5.5 of [11]. In what follows (K,φ) denotes a reduced conjugacy diagram and α, β positively oriented complete boundary cycles for the outer (i.e. infinite) component, respectively inner component, of the complement of K. We say that a region D of K has an edge of α, respectively β, in its boundary if some complete boundary cycle of D contains an edge of α or the inverse of an edge of α (respectively β).

Type I:

(i) The paths α and β are disjoint and simple.

(ii) The boundary of every region of K contains an edge of α or of β but no region contains edges of both α and β in its boundary.

(iii) Every region of K has precisely three edges not in α or β.

(iv) Every interior vertex of K has degree (or valency) 4.

These conditions mean that K is an annulus with two « layers » as illustrated in fig. 1.

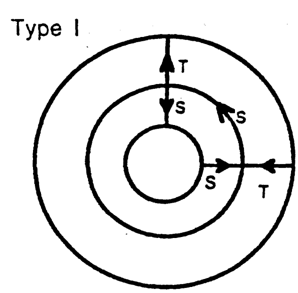

Figure 1.

Type II :

(i) The paths α and β are simple. (They may be disjoint or not.)

(ii) The boundary of every region of K has one edge in α and one edge in β.

(iii) The boundary of a region of K has at most two edges not in α or β and the boundary of some region has at least one edge not in α or β.

(iv) There are no interior vertices in K.

If, in fact, α and β are disjoint then K consists of an annulus, with a single layer, and the boundary of any region of K has exactly two edges not in α or β, see fig. 2. If α and β are not disjoint then the annulus becomes « degenerate » in the sense that parts of its inner and outer boundaries coincide. Here some regions will have only one edge in α or β, see fig. 2.

Type III :

(i) The paths α and β are simple but not disjoint.

(ii) The boundary of every region of K consists of one edge in α and one edge in β.

Then K is again a « degenerate » annulus with a single layer, but having all its edges in α or β, see fig. 3.

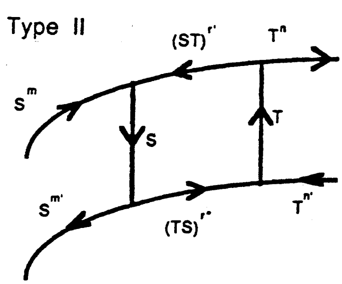

Figure 2.

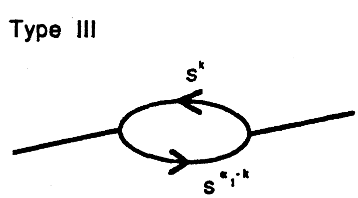

Figure 3.

4.2. THEOREM. – *Let $G = \langle X | R \rangle$ satisfy $C'(1/4)$ and $T(4)$ and let u, v be cyclically reduced words which represent conjugate elements of G but are not conjugate in $F(X)$. If neither u nor v contains a subword which constitutes more than half an element of R then there is a reduced conjugacy diagram for u and v of one of the types I-III.*

Proof. – This is essentially the content of Theorems V.5.3 and V.5.5 of [11]. Theorem V.5.3 gives the diagram of type I while Theorem V.5.5 gives types II and III. We have subdivided the description in Theorem V.5.5 because we need to treat the two cases differently. $\square$

4.3. PROPOSITION. – *Let* $\Delta = \langle S, T | S^{\pm p}, T^{\pm q}, (ST)^{\pm r}, (TS)^{\pm r}\rangle$ *where* p, $q \geqslant 5$ *and* $r \geqslant 3$. *Let* $u = [S^a, T^b]$, $v = [S^c, T^d]$ *where* $0 < a$, $c < p/2$, $0 < b$, $d < q/2$. *Then* u *and* $v^{\pm 1}$ *are conjugate in* Δ *if and only if* $(a,b) = (c,d)$.

Proof. – One implication is trivial. For the other we suppose $a \neq c$ or $b \neq d$ and derive a contradiction. In particular we note that u and $v^{\pm 1}$ are not then conjugate in $F(S,T)$. We observe that neither u nor $v^{\pm 1}$ has a subword constituting more than half of one of the relators of Δ. Hence, by Theorem 4.2, there is a reduced conjugacy diagram (K,φ) for u and v of one of the types I-III. Since the diagram is reduced, it follows that:

(1) The label on any edge not in the boundary of K is just $S^{\pm 1}$, $T^{\pm 1}$.

(2) If two regions of K have a common edge in their boundaries, then one of the two regions is assigned a label $(ST)^{\pm r}$ by φ and the other $S^{\pm p}$ or $T^{\pm q}$.

For a diagram of type I, (2) means that some region D has label $(ST)^{\pm r}$. From (1) the three edges of D not in the boundary of K contribute exactly three letters – namely $(STS)^{\pm 1}$ or $(TST)^{\pm 1}$ – to the label of D. The remaining fourth edge has a label of length $2r - 3$ which contradicts the fact that neither u nor $v^{\pm 1}$ contains a subword of the form $(STS)^{\pm 1}$ of $(TST)^{\pm 1}$.

For a diagram of type II, again (2) means that some region D of K has label $(ST)^{\pm r}$. The edges in the boundary of D that are not in the boundary of K contribute at most two letters to this label and therefore the remaining two edges in the boundary of D carry, between them, at least $2r - 2$ letters. If $r \geqslant 4$ this again contradicts the fact that neither u nor $v^{\pm 1}$ contains $(STS)^{\pm 1}$ or $(TST)^{\pm 1}$.

In the case when $r = 3$ the fact that $(STS)^{\pm 1}$ (or $(TST)^{\pm 1}$ respectively) does not appear in u and $v^{\pm 1}$, together with (1), means that the boundary of D has four edges. Moreover, the edge in the outer boundary α will carry label $(ST)^{\varepsilon}$, $\varepsilon = \pm 1$, the edge running from α to the inner boundary β will carry S^{ε} and the edge running from β to

α will carry T^{ε}. (Or the same holds with the roles of S and T interchanged.) For definiteness, let $\varepsilon = 1$. This means that there are regions adjacent to D carrying (positively oriented) labels S^{-p} and T^{-q} and therefore, in traversing α, we encounter a word $T^{-n}STS^{-m}$, n, $m > 0$ and in traversing β we encounter $T^{n'}S^{-1}T^{-1}S^{m'}$, n', $m' > 0$. This implies $a = b = c = d = 1$ which, at this point in the argument constitutes a desired contradiction.

There remains the diagram of type III. In this case no region can have label $(ST)^{\pm r}$ by φ because the diagram is reduced. Since u and $v^{\pm 1}$ are not conjugate in $F(S,T)$, K has at least one region and therefore one of u and $v^{\pm 1}$ contains say $S^{\varepsilon k}$, $\varepsilon = \pm 1$, and the other contains $S^{-\varepsilon(p-k)}$. Since either $|k| \geqslant p/2$ or $|p-k| \geqslant p/2$, our assumptions on u and v are contradicted. (A similar argument applies for $T^{\varepsilon k}$.) □

Direct consequences are the following three statements.

4.4. Corollary. – *Let p, $q \geqslant 5$ and $r \geqslant 3$ and let*

$$\Delta(p,q,r) = \langle S,T \mid S^p = T^q = (ST)^r = 1 \rangle$$

be a triangle group. By [16], any system $(x_1, \ldots, x_n)$ of generators of $\Delta(p,q,r)$ is Nielsen equivalent to a system $(S^a, T^b, 1, \ldots, 1)$. If $n > 2$, any two generating systems are Nielsen equivalent. Two generating systems (S^a, T^b) and (S^α, T^β) are Nielsen equivalent if and only if $\alpha \equiv \pm a \bmod p$ and $\beta \equiv \pm b \bmod q$.

We point out the following special cases not decided in [13].

4.5. Corollary. – *(S^a, T) or (S, T^b) are not Nielsen equivalent to (S,T) or to one another except when $a \equiv \pm 1 \bmod p$ or $b \equiv \pm 1 \bmod q$, respectively.*

4.6. Corollary. – *The assumptions are the same as in 4.3. Let $u = [S^a, T^b]$ and $w = [S, (ST)^c]$ with $0 < a < p/2$, $0 < b < q/2$, $0 < c < r/2$. If u is conjugate to $w^{\pm 1}$ in Δ then $a = b = c = 1$. If u is the image of $w^{\pm 1}$ under some automorphism of Δ then either*

$$a = 1,\ q = r \text{ and } b = c, \qquad \text{or} \qquad b = 1,\ p = r \text{ and } a = c.$$

Proof. – When u is conjugate to $w^{\pm 1}$, the argument is similar to that for Proposition 4.3. The second statement follows from the fact that all automorphisms of Δ are inner except when two of p, q and r coincide. Then, for example, if $p = q$, the only non-inner automorphism is given by $S \mapsto T$, $T \mapsto STS^{-1}$ and the conclusion follows. □

4.7. *The excluded cases.* – There are only a few cases excluded which are of interest for us. The above solution of the conjugacy problem cannot be applied if two of the numbers p, q, r are smaller than 5 or if one of the numbers equals 2. Since in the first case every number relatively prime to $p \in \{2,3,4\}$ is congruent to $\pm 1 \bmod p$ there is only one vertical Heegaard splitting of genus 2 and nothing has to be proved. In the case $r = 2$ and $5 \leqslant p < q$ the problem has been solved by Moriah [13] using a representation and considering traces. (In principle small cancellation methods can also be applied in this case since the group is of type $C(4)$ and $T(4)$ (see [11], chap. IV) but a more extended analysis would be required and we do not attempt it here.) The trace argument does not apply when $p = q$ and it is not clear from it whether there are in fact two non isotopic or homeomorphic Heegaard decompositions. This can be obtained as follows.

4.8. Proposition. – *Let* $\Delta = \langle S, T \mid S^p = T^p = (ST)^2 = 1 \rangle$, *where* $p \geqslant 5$. *For* $\gcd(m,p) = \gcd(n,p) = 1$, $0 < m$, $n < p/2$, *the elements* $[S^m, T]$ *and* $[S, T^n]^\varepsilon$, $\varepsilon = \pm 1$, *are conjugate if and only if* $m = n = 1$.

Proof. – Consider the epimorphism

$$\varphi : \langle S, T \mid S^p = T^p = (ST)^2 = 1 \rangle \to \mathbf{Z}_p, \qquad S \mapsto 1,\ T \mapsto -1.$$

Take S^i, $0 \leqslant i < p$ as coset representatives. The Reidemeister-Schreier generators for the kernel $\ker \varphi$ of φ are:

$$S_{p-1} = S^{p-1}S = S^p,\ T_0 = TS^{-p+1},\ T_i = S^iTS^{-(i-1)} \qquad \text{for} \qquad 1 \leqslant i \leqslant p-1;$$

defining relators are:

$$\begin{aligned}
1 &= S^p = S_{p-1},\\
1 &= T^p = T_0T_{p-1}T_{p-2} \ldots T_1,\\
1 &= S^j(ST)^2S^{-j} = T_{j+1}^2 \qquad \text{for} \qquad 0 \leqslant j < p-1,\\
1 &= S^{p-1}(ST)^2S^{-p+1} = S_{p-1}T_0S_{p-1}T_0.
\end{aligned}$$

Hence, $\ker \varphi = \langle T_0, \ldots, T_{p-1} \mid T_j^2 = 1, 0 \leqslant j < p, T_{p-1}T_{p-2} \ldots T_1T_0 = 1 \rangle$. For the above and the following calculations it is convenient to use the following equation:

$$T^n = T_pT_{p-1} \ldots T_{p-n+1} \cdot S^{p-n};$$

here the subscripts are taken modulo p. A conjugation with S moves the subscripts ahead by 1; e.g. $ST^nS^{-1} = T_{p+1}T_pT_{p-1} \ldots T_{p-n+2} \cdot S^{p-n}$.

We obtain the following equations in $\ker \varphi$:

$$\begin{aligned}[S,T^n] = ST^nS^{-1}\cdot T^{-n} &= T_{p+1}T_pT_{p-1}\ \dots\ T_{p-n+2}\cdot S^{p-n}\cdot T^{-n}\\ &= T_{p+1}T_pT_{p-1}\ \dots\ T_{p-n+2}\cdot (T_pT_{p-1}\dots T_{p-n+1})^{-1}\\ S^i[S^m,T]S^{-i} &= S^{i+m}TS^{-(i+m-1)}\cdot S^{i-1}T^{-1}S^{-i}\\ &= T_{m+i}T_i^{-1}.\end{aligned}$$

Since

$$\ker \varphi = \langle T_{p-1},\dots,T_2 | T_{p-1}^2=\dots=T_2^2=1\rangle *_A \langle T_1,T_0 | T_1^2=T_0^2=1\rangle,$$

with $A = \langle T_{p-1}\dots T_2\rangle = \langle (T_1T_0)^{-1}\rangle$, we can apply the solution of the conjugacy problem for free products with amalgamated subgroups, see [11, IV Theorem 2.8]. According to this theorem two elements (in cyclically reduced form) are conjugate if and only if one is obtained from the other by a cyclic permutation and a conjugation by an element of the amalgamated subgroup. In particular, they have the same cyclic length. If $n \geqslant 3$ the cyclic length of $T_{p+1}T_pT_{p-1}\ \dots\ T_{p-n+2}\cdot(T_pT_{p-1}\dots T_{p-n+1})^{-1}$ is 4, while the cyclic length of $(T_{m+i}T_i^{-1})^{-1}$ is at most 2. Therefore these elements are not conjugate in $\ker\varphi$. If $n = 2$ the first word is conjugate to $(T_0^{-1}T_1T_0)T_{p-1}$; hence, it has cyclic length 2. According to the solution of the conjugacy problem the critical case occurs for $i = 1$ when the second word is $T_1T_{m+1}^{-1}$. Since

$$m+1 < \frac{p}{2}+1 < p-1$$

the two words are not conjugate. Therefore $[S,T^n]$ and $[S^m,T]^{+1}$ are not conjugate in Δ. □

A similar argument can be given in the case when

$$\Delta = \langle S,T | S^{mp}=T^{mq}=(ST)^2=1\rangle$$

with $m \geqslant 2$, p, $q \geqslant 1$.

5. Classification of the Heegaard splittings and decompositions.

Proposition 2.4 gives the basic fact that, if we forget the exceptional manifolds $V(2,3,a)$, $W(2,4,b)$, we only have to classify the vertical Heegaard splittings (and decompositions). For the cases (B) and (C) of Theorem 2.5 we have to show that some of the splittings are isotopic and we do this next using geometric arguments.

5.1. PROPOSITION. — *Assume that one of the following conditions is fulfilled :*

(*a*) $\beta_j \equiv \pm 1 \bmod \alpha_j$,

(*b*) $\beta_i \equiv \pm 1 \bmod \alpha_i$ *and* $\beta_k \equiv \pm 1 \bmod \alpha_k$.

Then the Heegaard splittings $HS(i,j)$ *and* $HS(j,k)$ *are isotopic. More precisely, in case* (*a*) *the Heegaard decompositions* $(M^3; V(i,j))$ *and* $(M^3; W(j,k)) = (M^3; \overline{M^3 - V(j,k)})$ *are isotopic and in case* (*b*) $(M^3; V(i,j))$ *and* $(M^3; V(j,k))$ *are isotopic.*

Proof. — As in [5, section 7], it is easy to see that the handlebody $W(i,j)$ is a regular neighbourhod of the graph Γ formed by the union of a (parallel to a) section in the boundary of a regular neighbourhood of the exceptional fibre ε_j and the third exceptional fibre ε_k, $i \neq k \neq j$, joined by an arc which projects to an embedded arc on the basis.

Now the condition $\beta_j \equiv \pm 1 \bmod \alpha_j$ implies that any such section in the regular neighbourhood of an exceptional fibre of type β_j/α_j is isotopic in the regular neighbourhood to the exceptional fibre itself. Therefore Γ is isotopic to $\Gamma_{i,k}$ and this proves the assertion for case (a). To obtain case (b) it suffices to apply (a) twice, namely first to $(M^3; V(i,j))$ and $(M^3; W(i,k))$ and then to $(M^3; V(j,k))$ and $(M^3; W(i,k))$. □

A different proof can be based on [5, Corollary 5.8].

5.2. *Isotopy classification of Heegaard splittings* (*proof of Theorem* 2.5). — According to 3.3 the invariants $\mathscr{K}(M^3; V(i,j))$ and $\mathscr{K}(M^3; W(i,j))$, represented by $[S_i^{\gamma_i}, S_j^{\gamma_j}]$ and $[S_i, S_k^{\gamma_k}]$ (or $[S_j, S_k^{\gamma_k}]$), respectively, with $i \neq k \neq j$, are associated to the Heegaard splitting $HS(i,j)$.

Case (A) : The assumptions imply that $\alpha_i \geqslant 5$ for $i = 1, 2, 3$ and the situation becomes symmetric in S_1, S_2, S_3. Let $\{i,j,k\} = \{1,2,3\}$. The handlebody $W(i,k)$ is represented by $[S_i, S_j^{\gamma_j}]$ or $[S_k, S_j^{\gamma_j}]$. By 4.3, $[S_i, S_j^{\gamma_j}]$ is not conjugate to $[S_i^{\gamma_i}, S_j^{\gamma_j}]^{\pm 1}$ and $[S_k, S_j^{\gamma_j}]$ is not conjugate to $[S_j, S_k^{\gamma_k}]^{\pm 1}$. Hence, $(M^3; W(i,k))$ is neither isotopic to $(M^3; V(i,j))$ nor to $(M^3; W(i,j))$.

Case (B) : Now α_1, $\alpha_2 \geqslant 5$ since $\beta_i \not\equiv \pm 1 \bmod \alpha_i$, $i = 1, 2$. By 5.1, $\beta_3 \equiv \pm 1 \bmod \alpha_3$ implies that $HS(1,3)$ and $HS(2,3)$ are isotopic. Assume that $\alpha_3 \geqslant 3$. The Heegaard decomposition $(M^3; W(1,2))$ is represented by

$$[S_1, S_3^{\gamma_3}] = [S_1, S_3^{\pm 1}] = [S_1, (S_1 S_2)^{\mp 1}] \doteq [S_1, S_2^{\mp 1}]$$

which, by 4.3, is not conjugate to the invariants $[S_1^{\gamma_1}, S_3^{\gamma_3}]^{\pm 1} \doteq [S_1^{\gamma_1}, S_3^{\pm 1}]^{\pm 1}$ and $[S_1, S_2^{\gamma_2}]^{\pm 1}$ of $(M^3; V(1,3))$ and $(M^3; W(1,3))$, respectively. (Here $\doteq$ denotes conjugacy.)

For $\alpha_3 = 2$ the trace argument of Moriah [12] using a representation of $\Delta(\alpha_1, \alpha_2, 2)$ in $PSL_2(\mathbf{R})$ applies when $\alpha_1 \neq \alpha_2$. The case $\alpha_1 = \alpha_2$ is treated in 4.8.

Case (C) immediately follows from 5.1. □

One can reformulate the main result of the proof as follows.

5.3. Corollary. – *The pairs of commutator invariants $\{\mathscr{K}(M^3; V(i,j)), \mathscr{K}(M^3; W(i,j))\}$ give a complete isotopy classification of vertical Heegaard splittings.* □

5.4. *Classification up to homeomorphism* (*proof of* 2.6). – The statement (a) is already proved in 5.2. Claim (b) is obtained as follows. If $\beta_i/\alpha_i \equiv \beta_k/\alpha_k \bmod 1$ then there is a fibre preserving homeomorphism of M^3 interchanging the exceptional fibres ε_i and ε_k and, hence, mapping $HS(i,j)$ to $HS(j,k)$.

Conversely, let $f : M^3 \to M^3$ be a homeomorphism mapping $HS(i,j)$ to $HS(j,k)$. By assumption, these Heegaard splittings are not isotopic. Now it follows from 5.3 that the corresponding pairs of commutator invariants are different. Thus f does not induce an inner automorphism of $\Delta(\alpha_1, \alpha_2, \alpha_3)$ nor an automorphism sending each generator to a conjugate of its inverse. By [20], the automorphism $f_\#$ can be realized by a fibre preserving homeomorphism $g : M^3 \to M^3$ which non-trivially interchanges the exceptional fibres.

If $\beta_i/\alpha_i \not\equiv \beta_k/\alpha_k \bmod 1$ then neither g nor g^2 maps ε_i to ε_k. Since g preserves orientation, see Lemma 5.5, g maps ε_k or ε_i to itself, say ε_k, and interchanges the other two, i.e. ε_i and ε_j. We may assume, possibly after some isotopy, that g maps $HS(i,k)$ to $HS(j,k)$ and preserves $HS(i,j)$. Then $(f \circ g)(HS(i,j)) = HS(j,k)$. Moreover, $(f \circ g)_\# = g_\#^2$ which by construction induces an inner automorphism of $\Delta(\alpha_1, \alpha_2, \alpha_3)$, contradicting, by 5.2, the assumption that $HS(i,j)$ and $HS(j,k)$ are not isotopic. Hence, $\beta_i/\alpha_i \equiv \beta_k/\alpha_k \bmod 1$. □

5.5. Lemma. – *A Seifert fibre space $S(g; e_0; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$ with an odd number of exceptional fibres does not admit an orientation reversing self-homeomorphism.*

Proof. – Under an orientation reversing homeomorphism one obtains the « normal form » $S(g; -e_0; -\beta_1/\alpha_1, \ldots, -\beta_m/\alpha_m)$ by [17]. By the classification theorem of Seifert manifolds ([15], [18]), there is a permutation $\sigma \in S_m$ such that $\beta_{\sigma(i)}/\alpha_{\sigma(i)} \equiv -\beta_i/\alpha_i \bmod 1$ and $-e_0 = e_0$. Let m' be the number of i with $\alpha_i = 2$. Then $m - m'$ is even and

$$\mathbf{Z} \ni e = e_0 + \sum_{i=1}^{m} \beta_i/\alpha_i \equiv \sum_{\{i : \alpha_i = 2\}} \beta_i/\alpha_i \equiv \frac{m'}{2} \bmod 1 ;$$

hence, m' and m are even. □

Now we distinguish Heegaard decompositions.

5.6. Proposition. – *For $M^3 = S(0; e_0; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ let $HS(i,j)$ be a vertical Heegaard splitting. Let k be the subscript belonging to the third exceptional fibre, i.e. $\{i,j,k\} = \{1,2,3\}$. Then*

(a) $(M^3; V(i,j))$ *is* isotopic *to* $(M^3; W(i,j))$ *if and only if*

$$\beta_n \equiv \pm 1 \bmod \alpha_n \qquad \text{for } n = 1, 2, 3.$$

(b) $(M^3; V(i,j))$ *is* homeomorphic *to* $(M^3; W(i,j))$ *if and only if*

$$\beta_i \equiv \pm 1 \bmod \alpha_i \qquad \text{and} \qquad \frac{\beta_j}{\alpha_j} \equiv \frac{\beta_k}{\alpha_k} \bmod 1, \text{ or}$$

$$\beta_j \equiv \pm 1 \bmod \alpha_j \qquad \text{and} \qquad \frac{\beta_i}{\alpha_i} \equiv \frac{\beta_k}{\alpha_k} \bmod 1.$$

Proof. – (a) Proposition 5.1 implies that the conditions are sufficient. That they are necessary follows from Proposition 4.6: to the decompositions $(M^3; V(i,j))$ and $(M^3; W(i,j))$ there are associated the commutators $[S_i^{\gamma_i}, S_j^{\gamma_j}]^{\pm 1}$ and $[S_i, S_k^{\gamma_k}]^{\pm 1} = [S_i, (S_i S_j)^{\gamma_k}]^{\mp 1}$, respectively. If they are conjugate then, by 4.6, $\gamma_n \equiv \pm 1 \bmod \alpha_n$ for $n = 1, 2, 3$. The claim follows from $\gamma_n \beta_n \equiv -1 \bmod \alpha_n$.

(b) Assume that

$$\beta_i \equiv \pm 1 \bmod \alpha_i \qquad \text{and} \qquad \beta_j/\alpha_j \equiv \beta_k/\alpha_k \bmod 1.$$

According to Proposition 5.1 the first condition implies that $(M^3; V(i,j))$ is isotopic to $(M^3; W(i,k))$. The second ensures the existence of a homeomorphism of $(M^3; V(i,j))$ to $(M^3; W(i,j))$.

Assume now that $(M^3; V(i,j))$ and $(M^3; W(i,j))$ are homeomorphic. A consequence of 3.1 (c) is that the commutator invariant $[S_i^{\gamma_i}, S_j^{\gamma_j}]$ is mapped to $[S_i, (S_i S_j)^{\gamma_k}]^{\pm 1}$. By 4.6, either $\gamma_i \equiv \pm 1 \bmod \alpha_i$ and $\gamma_j/\alpha_j \equiv \gamma_k/\alpha_k \bmod 1$ or $\gamma_j \equiv \pm 1 \bmod \alpha_j$ and $\gamma_i/\alpha_i \equiv \gamma_k/\alpha_k \bmod 1$. □

5.7. *Proof of Corollary* 2.7. – For vertical Heegaard decompositions Corollary 2.7 is a direct consequence of Theorem 2.5 and Proposition 5.6.

In the exceptional case of Case (C) only one Heegaard decomposition corresponds to the additional (non-vertical) Heegaard splitting. This follows from the fact that this Heegaard splitting is obtained from a double covering of S^3 branched along a 3-bridge presentation of an algebraic link ([1], [3]) and that the two sides of the 3-bridge presentation can be exchanged by an orientation preserving involution of S^3 inverting the link (i.e. respecting each component of the link while reversing its orientation); see fig. 4, 5. □

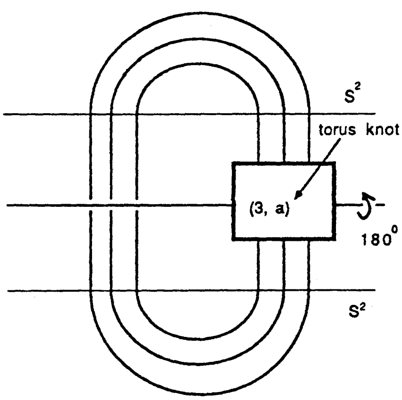

Figure 4. — $V(2,3,a)$ is a 2-fold covering of S^3 branched along the torus knot (3,a)

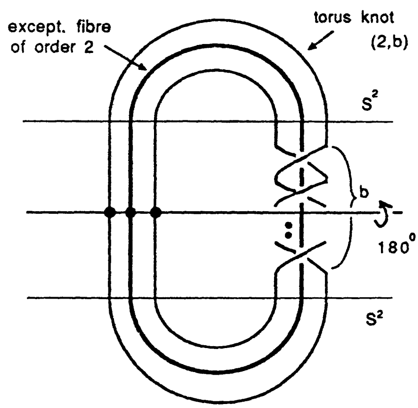

Figure 5. — $W(2,4,b)$ is a 2-fold covering of S^3 branched along the above link

BIBLIOGRAPHY

[1] J. S. Birman, F. Gonzáles-Acuña, J. M. Montesinos, Heegaard splittings of prime 3-manifolds are not unique, Mich. Math., J., 23 (1976), 97-103.

[2] M. Boileau, D. J. Collins, H. Zieschang, Scindements de Heegaard des petites variétés de Seifert, C.R. Acad. Sci. Paris, 305-I (1987), 557-560.

[3] M. Boileau, J.-P. Otal, Groupes des difféotopies de certaines variétés de Seifert. C.R. Acad. Sci., Paris, 303-I (1986), 19-22.
Groupes d'homéotopies et scindements de Heegaard des petites variétés de Seifert, Invent. Math., 106 (1991), 85-107.

[4] M. Boileau, J.-P. Otal, Sur les scindements de Heegaard du tore T^3, J. Diff. Geom., 32 (1990), 209-233.

[5] M. Boileau, M. Rost, H. Zieschang, On Heegaard decompositions of torus knot exteriors and related Seifert fibre spaces, Math. Ann., 279 (1988), 553-581.

[6] M. Boileau, H. Zieschang, Heegaard genus of closed orientable Seifert 3-manifolds, Invent. Math., 76 (1984), 455-468.

[7] F. Bonahon, Difféotopies des espaces lenticulaires, Topology, 22 (1983), 305-314.

[8] F. Bonahon, J.-P. Otal, Scindements de Heegaard des espaces lenticulaires, Ann. Scient. Éc. Norm. Sup., 16 (1983), 451-466.

[9] M. Lustig, Nielsen equivalence and simple homotopy type, Proc. London Math. Soc. (to appear).

[10] M. Lustig, Y. Moriah, Nielsen equivalence in Fuchsian groups and Seifert fibre spaces, Topology (to appear).

[11] R. C. Lyndon, P. E. Schupp, Combinatorial Group Theory, Berlin-Heidelberg, New York, Springer, 1977.

[12] Y. Moriah, Heegaard splittings and group presentations, Thesis, University of Texas, Austin, 1986.

[13] Y. Moriah, Heegaard splittings of Seifert fibered spaces, Invent. Math., 91 (1988), 465-481.

[14] J. Nielsen, Die Isomorphismen der allgemeinen, unendlichen Gruppe mit zwei Erzeugenden, Math. Ann., 78 (1918), 385-397.

[15] P. Orlik, E. Vogt, H. Zieschang, Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten, Topology, 6 (1967), 49-64.

[16] G. Rosenberger, All generating pairs of all two generator Fuchsian groups, Arch. Math., 46 (1986), 198-204.

[17] H. Seifert, Topologie dreidimensionaler gefaserter Räume, Acta Math., 60 (1933), 147-238.

[18] F. Waldhausen, Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, Invent. Math., 3 (1967), 308-333 ; II, Invent. Math., 4 (1967), 87-117.

[19] F. Waldhausen, Heegaard-Zerlegungen der 3-Sphäre, Topology, 7 (1968), 195-203.

[20] H. Zieschang, B. Zimmermann, Über Erweiterungen von Z und Z $*$ Z durch nichteuklidische kristallographische Gruppen, Math. Ann., 259 (1982), 29-51.

Manuscrit reçu le 11 juin 1990,
révisé le 29 août 1991.

M. Boileau,
Laboratoire Géométrie et Topologie
Université Paul Sabatier
Département de Mathématiques
118, route de Narbonne
31062 Toulouse (France).

D. J. Collins,
School of Mathematical Studies
Queen Mary College
Mile End Road
London E1 4NS (United Kingdom).

H. Zieschang,
Fakultät für Mathematik
Ruhr-Universität Bochum
Postfach 10 2148
4630 Bochum 1 (Germany).