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BASE POINTS OF POLAR CURVES

by Eduardo CASAS-ALVERO

Introduction.

In a previous paper ([2]) we have determined the singularities of the generic polar curves of a sufficiently general irreducible algebroid curve γ with prescribed characteristic exponents : we gave the (effective) multiplicities of the generic polar curves at the (ordinary and infinitely near) points of γ. In this paper we determine the whole set of infinitely near base points of the system of polar curves of a such γ: this set contains not only the multiple points of γ, but also a set of simple and free base points lying outside of γ.

We give also two corollaries: firstly, we find a lower bound for the Tjurina number τ of irreducible curves. Secondly, many families of continuous analytical (or formal) invariants of γ are obtained from the set of base points of the polar curves.

0. Notations and conventions.

Throughout this paper we are placed under the conventions and general hypothesis of [2]. In particular we deal with algebroid curves which

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always will be assumed to be (locally) plane curves defined over the field $\mathbb{C}$ of complex numbers. Irreducible (reduced) algebroid curves will be called branches for short. A branch γ will be said to have *general type* if and only if its generic polars go through $\widetilde{\partial(\gamma)}$ with effective multiplicities equal to the virtual ones and have no singularities outside of $\widetilde{\partial(\gamma)}$ ([2] 11.4); such polar curves will be called *polars with general behaviour.*

Let p be a point on a smooth algebraic surface S and denote by $\mathcal{O}_p$ the complete local ring of p on S. A family $\mathcal{L}$ of algebroid curves with origin at p will be called a linear system if and only if their equations describe the set of non zero elements of an ideal of $\mathcal{O}_p$. We say that a point q infinitely near to p is a *ν-fold base point* of a linear system $\mathcal{L}$ if and only if q belongs with (effective) multiplicity ν to the generic curves of $\mathcal{L}$. The ν-fold base points with $\nu > 0$ will be also called *base points.*

1. The branches of a polar with general behaviour.

Let γ be a branch with origin at p and assume that γ has general type. We shall use for γ the notations of [2] §11, in particular we write $\mathcal{M} = \{m_i/n\}_{i=1,\dots,r}$ for the system of characteristic exponents of γ and

$$S(x) = \sum_{i \in I(\mathcal{M})} a_i x^{i/n}$$

for its Puiseux series. We assume furthermore that the local coordinates x, y are chosen so that the y-axis is non tangent to γ, i.e., $m_1/n > 1$, and take

$$S_k(x) = \sum_{\substack{i \in I(\mathcal{M}) \\ i < m_k}} a_i x^{i/n}.$$

If $n_0^k = \text{g.c.d.}\ (n, m_1, \dots, m_{k-1})$, let us write m_k/n_0^k as a continued fraction of even order

$$\frac{m_k}{n_0^k} = \frac{m_{k-1}}{n_0^k} + b_0^k + \cfrac{1}{b_1^k + \cfrac{1}{\ddots \; \cfrac{1}{b_{2t(k)}^k}}}$$

In other words, with respect to the notations of [2], §11 we take, $b_i^k = h_i^k$ for $i < s(k)$, $2t(k) = s(k)$ and $b_{2t(k)}^k = h_{s(k)}^k$ if $s(k)$ is even, and $2t(k) = s(k)+1$, $b_{2t(k)-1}^k = h_{s(k)}^k - 1$ and $b_{2t(k)}^k = 1$ if $s(k)$ is odd.

Then we will consider the reduced fractions of the former continued fractions :

$$\frac{u_i^k}{v_i^k} = \frac{m_{k-1}}{n_0^k} + b_0^k + \cfrac{1}{b_1^k + \cfrac{1}{\ddots \; \cfrac{1}{b_i^k}}} \quad , \quad (u_i^k, v_i^k) = (1).$$

We have :

1.1. Proposition. — *If ζ is a polar of γ with general behaviour, then the branches of ζ are as follows :*

For each $k = 1, \dots, r$ and each $t = 1, \dots, t(k)$, ζ has b_{2t}^k different branches $\zeta_{t,j}^k$ $j = 1, \dots, b_{2t}^k$ with Puiseux series

$$S_k(x) + \alpha_{t,j}^k x^{\frac{n_0^k}{n} \frac{u_{2t-1}^k}{v_{2t-1}^k}} + \dots$$

where there are no characteristic exponents bigger than $n_0^k u_{2t-1}^k / n v_{2t-1}^k$ and

$$(\alpha_{t,j}^k)^{v_{2t-1}^k} \neq (\alpha_{t,j'}^k)^{v_{2t-1}^k}$$

for $j' \neq j$.

Proof. — We use for the infinitely near points on γ the notations introduced in [2], §11. By hypothesis ζ goes through $\widetilde{\partial(\gamma)}$ with effective multiplicities equal to the virtual ones and no singularities outside of $\widetilde{\partial(\gamma)}$. Then, by using the proximity equalities ([2] 1.4.1, for instance) it is easy to find, for each $q \in \widetilde{\partial(\gamma)}$, the number of (necessarily simple and free) points on ζ in the first neighbourhood of q and outside of $\widetilde{\partial(\gamma)}$; there are no such points except for the following cases :

a) $q = p_{2t-1, h_{2t-1}^k}^k$, $t < (s(k)+1)/2$ where we find h_{2t}^k points, and

b) $q = p_{s(k), h_{s(k)}^k - 1}^k$ and $s(k)$ odd, where we find a single point.

Each of one these points corresponds to a single branch of ζ that contains it as a simple point. Since we know the infinitely near points each one of these branches is going through, the claim follows from the standard relation between Puiseux series and infinitely near points on branches ([3], IV.I.7 or [4], XI.6.1).

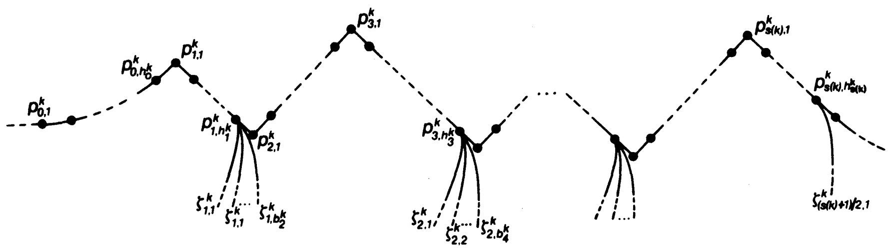

Figure 1 : the branches $\zeta^k_{t,j}$ for $s(k)$ odd.

Notice that $n^k_0 u^k_{2t-1}/n v^k_{2t-1}$ is the last characteristic exponent of $\zeta^k_{t,j}$ if $v^k_{2t-1} > 1$. In the sequel, since no confusion may be made, we will refer to $n^k_0 u^k_{2t-1}/n v^k_{2t-1}$ (resp. to $\alpha^k_{t,j}$) as the last characteristic exponent (resp. coefficient) of $\zeta^k_{t,j}$ even in the case in which $v^k_{2t-1} = 1$.

2. Base points.

Let γ and ζ as before. We have :

2.1. THEOREM. — *The base points of the system of polars of γ not belonging to γ are, for each branch $\zeta^k_{t,j}$ of ζ, the first*

$$u^k_{2t-1} - \frac{n}{n^k_0} v^k_{2t-1} - 1$$

infinitely near points on $\zeta^k_{t,j}$ which do not belong to γ. All of them are simple base points.

Proof. — Let us denote by $\widehat{\partial(\gamma)}$ the cluster defined by adding to $\widetilde{\partial(\gamma)}$ the points we claim to be base points of the system of polars, all virtually counted once. To see that these points are base points it is enough to see that the polar curves go through $\widehat{\partial(\gamma)}$: in that case, since ζ goes through $\widehat{\partial(\gamma)}$ with effective multiplicities equal to the virtual ones, by [2] 3.1, the same does a generic polar (and in fact any polar with general behaviour as one can see directly).

Assume that the coordinates and the equation f of γ are chosen in such a way that the polar ζ has equation $\partial f/\partial y$ and denote by ξ the polar defined by $\partial f/\partial x$. Since the condition of going through a cluster is linear ([2] 2.4), the system of polar curves is defined by the jacobian ideal $(\partial f/\partial x, \partial f/\partial y, f)$ and ζ obviously goes through $\widetilde{\partial(\gamma)}$, we need only to see that ξ and γ also go through $\widetilde{\partial(\gamma)}$.

Let us denote by ν_q the virtual multiplicity in $\widetilde{\partial(\gamma)}$ of any point $q \in \widetilde{\partial(\gamma)}$. We know that ξ and γ go through $\widetilde{\partial(\gamma)}$, hence, by using inductively the "virtual Noether's formula" [2] 8.3 we need only to see that, for any branch $\zeta^k_{t,j}$ of ζ,

$$(\zeta^k_{t,j}.\xi)_p \geq \sum_{q\in\widetilde{\partial(\gamma)}} e_q(\zeta^k_{t,j})\nu_q + u^k_{2t-1} - \frac{n}{n^k_0}v^k_{2t-1} - 1 \tag{1}$$

and

$$(\zeta^k_{t,j}.\gamma)_p \geq \sum_{q\in\widetilde{\partial(\gamma)}} e_q(\zeta^k_{t,j})\nu_q + u^k_{2t-1} - \frac{n}{n^k_0}v^k_{2t-1} - 1. \tag{2}$$

We will do with γ first. Since $\zeta^k_{t,j}$ and γ do not share points outside of $\widetilde{\partial(\gamma)}$, Noether's formula ([2] 1.3.1, for instance) gives

$$(\zeta^k_{t,j}.\gamma)_p = \sum_{q\in\widetilde{\partial(\gamma)}} e_q(\zeta^k_{t,j})e_q(\gamma),$$

so that (2) is equivalent to

$$\sum_{q\in\widetilde{\partial(\gamma)}} e_q(\zeta^k_{t,j})(e_q(\gamma) - \nu_q) \geq u^k_{2t-1} - \frac{n}{n^k_0}v^k_{2t-1} - 1. \tag{3}$$

Both multiplicities $e_q(\gamma)$ and ν_q are given in [2] §11 for all $q \in \widetilde{\partial(\gamma)}$ from which we obtain

$$e_q(\gamma) - \nu_q = 0$$

for $q = p^{k'}_{i,\ell}$, i odd and furthermore $\ell < h^{k'}_{s(k)}$ if $i = s(k')$, and

$$e_q(\gamma) - \nu_q = 1$$

otherwise. Thus, the first member of (3) is

$$\sigma = \sum_q e_q(\zeta^k_{t,j})$$

where the summation runs on the points $q = p^{k'}_{i,\ell}$ with $k' < k$ and either i even or $i = s(k')$, $s(k')$ odd and $\ell = h^{k'}_{s(k')}$, and also on the points $q = p^k_{i,\ell}$

with i even and $i < 2t$. A straightforward computation using the proximity relations gives

$$\sigma = u^k_{2t-1} - 1$$

so that (3), and hence (2), are satisfied.

In order to prove (1), recall first that former computation gives

$$(\zeta^k_{t,j}.\gamma)_p = \sum_{q \in \widetilde{\partial(\gamma)}} e_q(\zeta^k_{t,j})\nu_q + u^k_{2t-1} - 1. \tag{4}$$

Then assume that $x = z^\rho$, $y = y(z)$ is a Puiseux parameterization of $\zeta^k_{t,j}$, $\rho = nv^k_{2t-1}/n^k_0$ being the order of $\zeta^k_{t,j}$. We have

$$\frac{d}{dz} f(z^\rho, y(z)) = \rho \frac{\partial f}{\partial x}(z^\rho, y(z)) z^{\rho-1} + \frac{\partial f}{\partial y}(z^\rho, y(z)) \frac{dy}{dz}$$

where in fact

$$\frac{\partial f}{\partial y}(z^\rho, y(z)) \equiv 0$$

because $\zeta^k_{t,j}$ is a branch of ζ. Thus, by equating the orders in z, we obtain

$$(\zeta^k_{t,j}.\gamma)_p - 1 = (\zeta^k_{t,j}.\xi)_p + \rho - 1,$$

this is,

$$(\zeta^k_{t,j}.\xi)_p = (\zeta^k_{t,j}.\gamma)_p - nv^k_{2t-1}/n^k_0$$

which with (4) proves that (1) is satisfied (and is in fact an equality).

Lastly we must show that there are no more base points outside of γ. For this let us compute first the total number of base points we have found:

$$N_1 = \sum_{k=1}^{r} \sum_{t=1}^{t(k)} b^k_{2t} \left(u^k_{2t-1} - \frac{n}{n^k_0} v^k_{2t-1} - 1 \right)$$

$$= m_r - n + 1 - \sum_{k=1}^{r} \sum_{t=1}^{t(k)} b^k_{2t},$$

by using the standard properties of continued fractions.

On the other hand, let us compute

$$N_2 = \sum_{q \in \widetilde{\partial(\gamma)}} \nu_q^2$$

$$= \sum_{k=1}^{r} \left(\sum_{i \le s(k)/2} h^k_{2i}(n^k_{2i} - 1)^2 + \sum_{i < s(k)/2} h^k_{2i+1}(n^k_{2i+1})^2 - \varepsilon_k(2n^k_{s(k)} - 1) \right)$$

where we use the values of ν_q given in [2], §11 and $\varepsilon_k = 0$ if $s(k)$ is even, $\varepsilon_k = 1$ if $s(k)$ is odd. We easily obtain

$$N_2 = \sum_{k=1}^{r}\sum_{i=1}^{s(k)} h_i^k (n_i^k - 1) n_i^k - m_r + n - 1 + \sum_{k=1}^{r}\sum_{t=1}^{t(k)} b_{2t}^k$$

which introducing Milnor's μ, gives

$$N_2 = \mu(\gamma) - m_r + n - 1 + \sum_{k=1}^{r}\sum_{t=1}^{t(k)} b_{2t}^k.$$

Now assume that there is some base point outside of γ besides that described in the claim. By the Noether's formula any pair of generic polars, and hence any pair of polars, must have intersection multiplicity strictly bigger than $N_1 + N_2 = \mu(\gamma)$ against the well known equality $(\zeta.\xi)_p = \mu(\gamma)$. This ends the proof.

Notice that it follows in particular from 2.1 that the base points do not depend on the polar ζ. Let us state, for further reference :

2.2. Corollary (of the proof). — *The number of base points of the system of polar curves of γ which do not belong to γ is*

$$m_r - n + 1 - \sum_{k=1}^{r}\sum_{t=1}^{t(k)} b_{2t}^k.$$

3. A bound for $\tau(\gamma)$.

As customary, denote by τ the Tjurina number

$$\tau = \tau(\gamma) = \dim_{\mathbb{C}} \mathcal{O}_p/(\partial f/\partial x, \partial f/\partial y, f),$$

where $\mathcal{O}_p$ is the complete local ring of S at p, and f is an equation of γ. It is well known that $\tau(\gamma) \leq \mu(\gamma)$, and that the equality $\tau(\gamma) = \mu(\gamma)$ characterizes quasihomogeneous branches (cf. [5]). We have :

3.1. Corollary. — *For any branch γ with system of characteristic exponents $\mathcal{M}$,*

$$\tau(\gamma) \geq \frac{1}{2}\mu(\gamma) + m_r - \sum_{k=1}^{r}\sum_{t=1}^{t(k)} b_{2t}^k$$

Proof. — Since the claimed bound depends only on the topological type of γ, it is enough to give a proof for branches γ with general type. Thus assume that γ has general type, and let $H = H_{\widetilde{\partial(\gamma)}}$ be the ideal described by the equations of the curves going through $\widetilde{\partial(\gamma)}$. Theorem 2.1 says that $H \supset (\partial f/\partial x, \partial f/\partial y, f)$, so that,

$$\tau \geq \dim_{\mathbb{C}} \mathcal{O}_p/H.$$

The last integer being given by [2] 6.1, using 2.2 we have

$$\tau \geq \sum_{q \in \widetilde{\partial(\gamma)}} \frac{\nu_q(\nu_q+1)}{2} + m_r - n + 1 - \sum_{k=1}^{r} \sum_{t=1}^{t(k)} b_{2t}^k$$

from which the claim follows after a computation like that made for N_2 in the proof of 2.1.

It must be noticed that, although we can find non trivial examples of branches for which the bound is reached (namely that of branches with general type and characteristic exponent 10/3), the equality is false in general, even for curves with general type : the curves with single characteristic exponent 5/4 have $\tau = 12$ or $\tau = 11$ (cf. [6] V.2) whereas our bound is 9.

4. Continuous invariants.

Assume again that γ has general type and that ζ is a polar of γ with general behaviour, its branches and corresponding Puiseux series being described in 1.1.

4.1. COROLLARY. — *Choose k and t such that $1 \leq k \leq r$, $1 \leq t \leq t(k)$). Assume furthermore that $t > 1$ if $b_1^k = 1$ and that $(k,t) \neq (1,1)$ if $b_0^1 = 1$. Then the v_{2t-1}^k-powers of the ratios between the last characteristic coefficients of the branches $\zeta_{t,j}^k$, namely*

$$\left(\frac{\alpha_{t,j}^k}{\alpha_{t,j'}^k}\right)^{v_{2t-1}^k}$$

for $j \neq j'$, $j, j' = 1, \ldots, b_{2t}^k$, are analytic (or formal) invariants of γ.

Proof. — For $s(k)$ odd we have $b_{2t(k)}^k = 1$, i.e., a single branch $\zeta_{t(k),j}^k$, and in this case the claim is empty. Hence we may assume that $t < t(k)$ for $s(k)$ odd.

An easy computation shows that the number of base points on $\zeta^k_{t,j}$ is positive unless $k = 1$, $t = 1$ and $b^1_0 = 1$, which is excluded in the claim.

Thus, under our hypothesis, the first point on each branch $\zeta^k_{t,j}$ outside of γ, say $q^k_{t,j}$, is a base point of the system of polars. All these points, for $j = 1, \dots, b^k_{2t}$, belong to the first neighbourhood E^k_t of $p^k_{2t-1,h^k_{2t-1}}$, the last point shared by γ and the branches $\zeta^k_{t,j}$.

Since we assume $t < t(k)$ for $s(k)$ odd, we have $h^k_{2t-1} = b^k_{2t-1}$ and hence, in particular, $p^k_{2t-1,h^k_{2t-1}} \neq p^k_{1,1}$ because of the hypothesis. Then the point $p^k_{2t-1,h^k_{2t-1}}$ is always a satellite point and we know ([1] §4) that an absolute projective coordinate may be taken in its first neighbourhood E^k_t in such a way that :

a) The two satellite points in E^k_t have coordinates 0 and ∞ (depending on which point besides $p^k_{2t-1,h^k_{2t-1}}$ each of them is proximate to), and

b) each point $q^k_{t,j}$ has coordinate $(\alpha^k_{t,j})^{v^k_{2t-1}}$.

On the other hand ([1] §5), any analytic (or formal) transformation φ defined at p induces a linear projectivity between E^k_t and the first neighbourhood of $\varphi(p^k_{2t-1,h^k_{2t-1}})$ under which the points with coordinate 0 are correspondent, as well as that with coordinate ∞. This projectivity must take the form $\alpha \longrightarrow \lambda\alpha$ for certain fixed λ if we use in both first neighbourhoods the absolute coordinate mentioned above. Since the base points of the system of polars of γ must be transformed in the base points of the system of polars of $\varphi(\gamma)$, the claim follows.

Remark. — For $k = 1$, using the way on which the characteristic coefficients $\alpha^1_{t,j}$ may be computed from the Newton polygon of the polar curve ζ, one can easily find an equivalent set of analytic invariants which are rational functions of the coefficients of the equation f of γ. Let us show this by means of an example :

Example. — Consider branches with characteristic exponent 12/5 and equation

$$f = \sum_{5i+12j \geq 60} a_{i,j} x^i y^j .$$

For generic coefficients the x-polar has two branches with characteristic exponent 5/2 and its Newton polygon has a single side. The monomials on this side give rise to the equation

$$5a_{0,5}\alpha^4 + 3a_{5,3}\alpha^2 + a_{10,1} = 0$$

whose roots give the characteristic coefficients of the branches. It follows from 4.1 that $a_{10,0}a_{0,5}/a_{5,3}^2$ is an invariant.

BIBLIOGRAPHIE

[1] E. Casas-Alvero, Moduli of algebroid plane curves, Algebraic Geometry, La Rábida 1981, Lecture Notes in Math 961, Springer Verlag, 1982, 32-83.

[2] E. Casas-Alvero, Infinitely near singularities and singularities of polar curves, Math. Ann., 287 (1990), 429-454.

[3] F. Enriques, O. Chisini, Teoria Geometrica delle Equazioni e delle Funzioni Algebriche, N. Zanichelli, Bologna, 1915.

[4] J.G. Semple, G.T. Kneebone, Algebraic Curves, Oxford University Press, London, 1959.

[5] O. Zariski, Characterization of plane curves whose module of differentials has maximal torsion, Proc. Nat. Acad. of Science, U.S.A.,56,3 (1966), 781-786.

[6] O. Zariski, Le probleme des modules pour les branches planes, Centre de Math., École Polytechnique, Paris, 1973.

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