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ON THE EXISTENCE OF WEIGHTED BOUNDARY LIMITS OF HARMONIC FUNCTIONS

by Yoshihiro MIZUTA

1. Introduction.

In this paper we are concerned with the existence of boundary limits of functions u which are harmonic in a bounded open set $G \subset R^n$ and satisfy a condition of the form :

$$\int_G \Psi(|\operatorname{grad} u(x)|)\omega(x)\,dx < \infty\,,$$

where $\Psi(r)$ is a nonnegative nondecreasing function on the interval $[0,\infty)$ and ω is a nonnegative measurable function on G. In case G is a Lipschitz domain, $\Psi(r) = r^p$ and $\omega(x) = \rho(x)^\beta$, many authors studied the existence of (non) tangential boundary limits; see, for example, Carleson [2], Wallin [10], Murai [7], Cruzeiro [3] and Mizuta [5], [6]. Here $\rho(x)$ denotes the distance of x from the boundary ∂G. In this paper, we assume that Ψ is of the form $r^p\psi(r)$, where ψ is a nonnegative nondecreasing function on the interval $[0,\infty)$ such that $\psi(2r) \leqslant A_1\psi(r)$ for any $r > 0$, with a positive constant A_1. In case G is a Lipschitz domain and $\omega(x)$ is of the form $\lambda(\rho(x))$, where λ is a positive and nondecreasing function on the interval $(0,\infty)$ such that $\lambda(2r) \leqslant A_2\lambda(r)$ for any $r > 0$ with a positive constant A_2, our first aim is to find a positive function $\kappa(r)$ such that $[\kappa(\rho(x))]^{-1}u(x)$ tends to zero as x tends to the boundary ∂G; when κ is bounded, u is shown to be extended to a continuous function on $G \cup \partial G$.

Key-words : Harmonic functions - Tangential boundary limits - Bessel capacity Hausdorff measure.

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It is known (see [5]) that if u is a harmonic function on the unit ball B satisfying

$$\int_B |\operatorname{grad} u(x)|^p (1-|x|^2)^\beta \, dx < \infty, \qquad \beta \geqslant p-n,$$

then $u(x)$ has a finite limit as $x \to \xi$ along $T_\alpha(\xi,a) = \{x \in B\,;\, |x-\xi|^\alpha < a\rho(x)\}$ for any $a > 0$ and any $\xi \in \partial G$ except those in a suitable exceptional set, where $\alpha \geqslant 1$. Further it is known that this fact is best possible as to the size of the exceptional sets. We shall show in Theorem 1 that if u is a harmonic function on B satisfying the stronger condition :

$$\int_B \Psi_p(|\operatorname{grad} u(x)|)(1-|x|^2)^{p-n} \, dx < \infty$$

and if ψ is of logarithmic type (see condition (ψ_1) below) and $\int_0^1 [\psi(t^{-1})]^{-1/(p-1)} t^{-1} \, dt < \infty$, then u is extended to a function which is continuous on $B \cup \partial B$.

Next let us consider the case where

$$G = G_\alpha \equiv \{x=(x',x_n) \in R^{n-1} \times R^1;\, |x'|^\alpha < x_n < 1\}.$$

In case $\alpha < 1$, G_α is not a Lipschitz domain. However, we will also find a positive function $\kappa(r)$ such that $[\kappa(|x|)]^{-1} u(x)$ tends to zero as $x \to 0$, $x \in G_\alpha$; when κ is bounded, u is shown to have a finite limit at the origin.

Further, we study the existence of (tangential) boundary limits

$$\lim_{x \to \xi,\, x \in T_\alpha(\xi,a,b)} u(x)$$

at $\xi \in \partial G$ except those in a suitable exceptional set, where $T_\alpha(\xi,a,b) = \{\xi + \Xi_\xi x;\, x_n > a|x'| + b|x'|^\alpha\}$ with $a \geqslant 0$, $b \geqslant 0$ and an orthogonal transformation Ξ_ξ. We note here that if G is a Lipschitz domain, then for any $\xi \in \partial G$, there exist a_ξ, $b_\xi \geqslant 0$, $r_\xi > 0$ and an orthogonal transformation Ξ_ξ such that $T_\alpha(\xi, a_\xi, b_\xi) \cap B(\xi, r_\xi) \subset G$, where $B(x,r)$ denotes the open ball with center at x and radius r. If $\alpha = 1$, then our results will imply the usual angular limit theorem.

2. Weighted boundary limits.

Throughout this paper, let ψ be a nonnegative nondecreasing function on the interval $(0,\infty)$ satisfying the following condition:

(ψ_1) There exists $A > 1$ such that $A^{-1}\psi(r) \leqslant \psi(r^2) \leqslant A\psi(r)$ whenever $r > 0$.

By condition (ψ_1), we see that ψ satisfies the so-called (Δ_2) condition, that is, we can find $A_1 > 1$ such that

$$(\Delta_2) \qquad A_1^{-1}\psi(r) \leqslant \psi(2r) \leqslant A_1\psi(r) \quad \text{for any } r > 0.$$

For $p > 1$, set $\Psi_p(r) = r^p\psi(r)$. Since $\Psi_p(r) \to 0$ as $r \to 0$, we may assume that $\Psi_p(0) = 0$.

If η is a positive measurable function on the interval $(0,\infty)$, then we define

$$\kappa_\eta(r) = \left(\int_r^1 s^{p'(1-n/p)}\eta(s)^{-p'/p}s^{-1}\,ds\right)^{1/p'},$$

where $1/p + 1/p' = 1$.

In this paper, let $M_1, M_2, \ldots$ denote various constants independent of the variables in question. Further, we denote by $B(x,r)$ the open ball with radius r and center at x.

Our first aim is to establish the following result, which gives a generalization of Theorem 1 in [6].

THEOREM 1. – *Let λ be a nonnegative monotone function on the interval $(0,\infty)$ satisfying the (Δ_2) condition, and let ψ be a nonnegative nondecreasing function on the interval $(0,\infty)$ satisfying condition (ψ_1). Set $\eta(r) = \psi(r^{-1})\lambda(r)$. Suppose u is a function harmonic in a bounded Lipschitz domain G in R^n and satisfying*

$$(1) \qquad \int_G \Psi_p(|\operatorname{grad} u(x)|)\lambda(\rho(x))\,dx < \infty.$$

If $\kappa_\eta(0) = \infty$, then $\lim_{x\to\partial G}[\kappa_\eta(\rho(x))]^{-1}u(x) = 0$; if $\kappa_\eta(0) < \infty$, then u has a finite limit at each boundary point of G.

Remark. — If $\lambda(r) = r^{p-n}$ and ψ satisfies the additional condition:

$$(\psi_2) \qquad \int_0^1 [\psi(r^{-1})]^{-1/(p-1)} r^{-1}\, dr < \infty,$$

then $\kappa_\eta(0) < \infty$.

For a proof of Theorem 1, we need the following lemma (see [6], Lemma 1).

LEMMA 1. — *Let G be a bounded Lipschitz domain in R^n. Then for each $\xi \in \partial G$, there exist $r_\xi > 0$ and $c_\xi > 0$ with the following properties:*

i) *if $0 < r < r_\xi$, then there exist $x_r \in G \cap B(\xi, r)$ and $\sigma_r > 0$ such that*

$$E(x, x_r) = \bigcup_{0 \leqslant t \leqslant 1} B(X(t), c_\xi \rho(X(t))) \subset G \cap B(\xi, 2r)$$

whenever $x \in G \cap B(\xi, \sigma_r)$, where $X(t) = (1-t)x + tx_r$;

ii) $\rho(x) + |x-y| < M_1 \rho(y)$ *whenever* $y \in E(x, x_r)$;

iii) *if u is a function harmonic in G, then*

$$|u(x) - u(x_r)| \leqslant M_2 \int_{E(x, x_r)} |\text{grad}\, u(y)| \rho(y)^{1-n}\, dy$$

for any $x \in G \cap B(\xi, \sigma_r)$. Here M_1 and M_2 are positive constants independent of x, r and u.

Proof of Theorem 1. — Let u be as in the theorem, and let $\xi \in \partial G$. For a sufficiently small $r > 0$, by Lemma 1, we find that

$$|u(x) - u(x_r)| \leqslant M_1 \int_{E(x, x_r)} |\text{grad}\, u(y)| \rho(y)^{1-n}\, dy$$

for any $x \in G \cap B(\xi, \sigma_r)$. Let $0 < \delta < 1$. By condition (ψ_1), we can find a constant $A_\delta > 1$ such that

$$(2) \qquad A_\delta^{-1} \psi(r) \leqslant \psi(r^\delta) \leqslant A_\delta \psi(r) \quad \text{whenever } r > 0.$$

Hence, from Hölder's inequality we derive

$$|u(x) - u(x_r)| \leqslant M_1 \left(\int_{\{y \in E(x, x_r);\, f(y) > \rho(y)^{-\delta}\}} \rho(y)^{p'(1-n)} \psi(f(y))^{-p'/p} \right.$$

$$\left. \times\, \lambda(\rho(y))^{-p'/p}\, dy \right)^{1/p'} F(r) + M_1 \int_{E(x, x_r)} \rho(y)^{1-n-\delta}\, dy$$

$$\leqslant M_2\left(\int_0^{3r}(\rho(x)+t)^{p'(1-n/p)-1}[\psi((\rho(x)+t)^{-1})]^{-p'/p}\right.$$

$$\left.\times\ \lambda(\rho(x)+t))^{-p'/p}\,dt\right)^{1/p'} F(r) + M_2\int_{B(x,2r)}|x-y|^{1-\delta-n}\,dy$$

$$\leqslant M_3\kappa_\eta(\rho(x))F(r) + M_3 r^{(1-\delta)/n},$$

where $f(y) = |\mathrm{grad}\, u(y)|$ and $F(r) = \left(\int_{G\cap B(\xi,2r)}\Psi_p(f(y))\lambda(\rho(y))\,dy\right)^{1/p}$.

Consequently, if $\kappa_\eta(0) = \infty$, then we obtain

$$\limsup_{x\to\xi}\kappa_\eta(\rho(x))^{-1}|u(x)| \leqslant M_3\left(\int_{G\cap B(\xi,2r)}\Psi_p(f(y))\lambda(\rho(y))\,dy\right)^{1/p}.$$

Condition (1) implies that the right hand side tends to zero as $r\to 0$, so that the left hand side is equal to zero.

On the other hand, if $\kappa_\eta(0) < \infty$, then we see that $\sup_{x\in G\cap B(\xi,\sigma_r)}|u(x)-u(x_r)|$ tends to zero as $r\to 0$, which implies that $u(x)$ has a finite limit at ξ. Thus Theorem 1 is established.

3. The case $G = G_\alpha$ with $\alpha < 1$.

If $\alpha < 1$, then G_α is not a Lipschitz domain. However, we study the existence of boundary limits for u satisfying condition (1).

For simplicity, set

$$\kappa_{\eta,\alpha}(r) = \left(\int_r^1 s^{p'(1-n/p)}[\eta(s)]^{-p'/p}s^{\alpha-2}\,ds\right)^{1/p'}$$

and

$$K_{\eta,\alpha}(x) = \kappa_\eta(\rho(x)) + \kappa_{\eta,\alpha}(x_n^{1/\alpha}) \qquad \text{for } x = (x', x_n).$$

THEOREM 2. — *Let λ, ψ and η be as in Theorem 1. Let u be a function harmonic in G_α and satisfying condition (1). If $0 < \alpha < 1$ and $K_{\eta,\alpha}(x)\to\infty$ as $x\to 0$, then*

$$\lim_{x\to 0,\, x\in G_\alpha}[K_{\eta,\alpha}(x)]^{-1}u(x) = 0;$$

and if $K_{\eta,\alpha}(x)$ is bounded, then $u(x)$ has a finite limit as $x\to 0$, $x\in G_\alpha$.

Proof. – For $r > 0$, let $X(r) = (0, \dots, 0, r)$ and $B_r = B(X(r), \rho(X(r)))$. If $E(x, X(r)) \subset B_r$, then, in view of Lemma 1, we have

$$|u(x) - u(X(r))| \leqslant M_1 \int_{B_r} |\operatorname{grad} u(y)|\, \rho(y)^{1-n}\, dy.$$

As in the proof of Theorem 1, by use of Hölder's inequality we establish

$$(3) \quad |u(x) - u(X(r))| \leqslant M_2 \kappa_\eta(\rho(x), 2\rho(X(r)))\, U(r) + M_2[m_n(B_r)]^{(1-\delta)/n},$$

where $0 < \delta < \alpha < 1$, $\kappa_\eta(t, r) = \left(\int_t^r s^{p'(1-n/p)}\, \eta(s)^{-p'/p} s^{-1}\, ds\right)^{1/p'}$ and

$$U(r) = \left(\int_{B_r} \Psi_p(|\operatorname{grad} u(y)|)\, \lambda(\rho(y))\, dy\right)^{1/p}.$$

For a large integer $j(\geqslant j_0)$, set $r_j = M_3\, j^{-\alpha/(1-\alpha)}$, where j_0 and $M_3 > 0$ are chosen so that $r_j - r_{j+1} < \rho(X(r_j))$. Now we define

$$F_j = \{x = (x', x_n) \in G_\alpha\, ;\, |x_n - r_j| < \rho(X(r_j))\}.$$

We shall show the existence of $N > 0$ such that the number of F_m with $F_m \cap F_j \neq 0$ is at most N for any j. Letting a and b be positive numbers, we assume that $r_j - a r_j^{1/\alpha} \leqslant r_{j+k} + b(r_{j+k})^{1/\alpha}$. Then

$$j[1 - (j/(j + k))^{\alpha/(1-\alpha)}] \leqslant M_3^{(1-\alpha)/\alpha}[a + b(j/(j + k))^{1/(1-\alpha)}].$$

Since $M_4 = \inf_{0<t<1}(1 - t^{\alpha/(1-\alpha)})/(1 - t) > 0$, we derive

$$jk/(j + k) \leqslant M_5 \quad \text{with} \quad M_5 = [M_3^{(1-\alpha)/\alpha}(a + b)]/M_4,$$

so that

$$k \leqslant M_5 j/(j - M_5) \quad \text{when} \quad j > M_5.$$

From this fact we can readily find $N > 0$ with the required property. Thus $\{F_\ell\}$ is shown to satisfy the above condition.

By (3) we have

$$\begin{aligned} |u(X(r_j)) - u(X(r_{j+k}))| &\leqslant |u(X(r_j)) - u(X(r_{j+k}))| \\ &\quad + |u(X(r_{j+1})) - u(X(r_{j+2}))| + \cdots + |u(X(r_{j+k-1})) - u(X(r_{j+k}))| \\ &\leqslant M_6 \left(\sum_{\ell=j}^{j+k-1} U(r_\ell)^p\right)^{1/p} \left(\sum_{\ell=j}^{j+k-1} \rho(X(r_\ell))^{p'(1-n/p)}\, [\eta(\rho(X(r_\ell)))]^{p'}\right)^{1/p'} \\ &\quad + M_2 \sum_{\ell=j}^{\infty} [m_n(B_{r_\ell})]^{(1-\delta)/n}. \end{aligned}$$

We note here that

$$\sum_{\ell=j}^{\infty} [m_n(B_{r_\ell})]^{(1-\delta)/n} \leqslant M_7 \sum_{\ell=j}^{\infty} \ell^{-(1-\delta)/(1-\alpha)} < \infty$$

since $\delta < \alpha$, and, by setting $\sigma(j) = j^{-1/(1-\alpha)}$ for simplicity,

$$\begin{aligned} \sum_{\ell=j}^{j+k-1} \rho(X(r_\ell))^{p'(1-n/p)} [\eta(\rho(X(r_\ell)))]^{-p'/p} &\leqslant M_8 \sum_{\ell=j}^{j+k-1} [\ell^{-1/(1-\alpha)}]^{p'(1-n/p)} [\eta(\ell^{-1/(1-\alpha)})]^{-p'/p} \\ &\leqslant M_9 \int_j^{j+k} [t^{-1/(1-\alpha)}]^{p'(1-n/p)} [\eta(t^{-1/(1-\alpha)})]^{-p'/p}\, dt \\ &= M_{10} \int_{\sigma(j+k)}^{\sigma(j)} s^{p'(1-n/p)} [\eta(s)]^{-p'/p} s^{\alpha-2}\, ds \\ &\leqslant M_{10} [\kappa_{\eta,\alpha}(\sigma(j+k))]^{p'} \leqslant M_{11} [\kappa_{\eta,\alpha}(\rho(X(r_{j+k})))]^{p'}. \end{aligned}$$

First suppose $K_{\eta,\alpha}(x) \to \infty$ as $x \to 0$. Then, since $\{F_\ell\}$ meets mutually at most N times, we obtain

$$\limsup_{k\to\infty} [K_{\eta,\alpha}(X(r_{j+k}))]^{-1} |u(X(r_{j+k}))| \leqslant M_6 [M_{11}]^{1/p'} \left(\int_{\bigcup_{\ell \geqslant j} F_\ell} \Psi_p(|\operatorname{grad} u(y)|)\, \lambda(\rho(y))\, dy \right)^{1/p}$$

for any j. Thus it follows that the left hand side is equal to zero. We also see from (3) that

$$\lim_{r\to 0} [\sup_{x \in B_r \cap G_\alpha} [K_{\eta,\alpha}(x)]^{-1} |u(x) - u(X(r))|] = 0.$$

Since B_r contains some $X(r_j)$, it follows that

$$\lim_{x\to 0, x\in G_\alpha} [K_{\eta,\alpha}(x)]^{-1} u(x) = 0.$$

If $K_{\eta,\alpha}(x)$ is bounded, then we see that

$$\lim_{j\to\infty} \sup_{k \geqslant j} |u(X(r_j)) - u(X(r_k))| = 0$$

and

$$\lim_{r\downarrow 0} \sup_{x\in B_r} |u(x) - u(X(r))| = 0.$$

These facts imply that u has a finite limit at the origin.

Here we give a result, which is a generalization of Theorem 2.

PROPOSITION 1. — *Let λ_1 and λ_2 be nonnegative monotone functions on the interval $(0, \infty)$ satisfying the (Δ_2) condition, and let ψ be a nonnegative nondecreasing function on the interval $(0, \infty)$ satisfying condition (ψ_1). Suppose u is a function harmonic in G_α and satisfying*

$$\int_{G_\alpha} \Psi_p(|\mathrm{grad}\, u(x)|)\lambda_1(\rho(x))\lambda_2(|x|^{1/\alpha})\, dx < \infty.$$

Set $\eta_1(r) = \psi(r^{-1})\lambda_1(r)$, $\eta(r) = \psi(r^{-1})\lambda_1(r)\lambda_2(r)$ *and*

$$K(x) = \kappa_{\eta_1}(\rho(x))[\lambda_2(x_n^{1/\alpha})]^{-1/p} + \kappa_{\eta,\alpha}(x_n^{1/\alpha}).$$

If $K(0)\,(= \lim_{x\to 0} K(x)) = \infty$, *then* $[K(x)]^{-1}u(x) \to 0$ *as* $x \to 0$, $x \in G_\alpha$; *if $K(x)$ is bounded, then $u(x)$ has a finite limit as* $x \to 0$, $x \in G_\alpha$.

Proof. — As in the proof of Theorem 2, for $x \in B_r$, we see that

$$\begin{aligned} |u(x) - u(X(r))| &\leqslant M_1 r^{1-\delta} + M_1\kappa_{\eta_1}(\rho(x))\left(\int_{B_r} \Psi_p(f((y))\lambda_1(\rho(y))\, dy\right)^{1/p} \\ &\leqslant M_1 r^{1-\delta} + M_2\kappa_{\eta_1}(\rho(x))\lambda_2(r^{1/\alpha})^{-1/p} \\ &\qquad \times \left(\int_{B_r} \Psi_p(f(y))\lambda_1(\rho(y))\lambda_2(|y|^{1/\alpha})\, dy\right)^{1/p} \end{aligned}$$

and

$$\begin{aligned} |u(X(r_j)) - u(X(r_{j+k}))| &\leqslant M_3 j^{-(1-\delta)/(1-\alpha)} + M_3\kappa_{\eta,\alpha}(\rho(X(r_{j+k}))) \\ &\qquad \times \left(\int_{(\Delta_{j+k,j})} \Psi_p(f(y))\lambda_1(\rho(y))\lambda_2(|y|^{1/\alpha})\, dy\right)^{1/p}, \end{aligned}$$

where $f(y) = |\mathrm{grad}\, u(y)|$ and $\Delta_{k,j} = \bigcup_{l=j}^{k} Br_l$. Thus the remaining part of the proof is similar to the proof of Theorem 2.

Next, for $0 < a < 1$, let $G_\alpha(a) = \{x = (x', x_n) \in R^{n-1} \times R^1;\ 0 < x_n < 1,\ |x'|^\alpha < ax_n\}$. Then the following result can be proved similarly.

PROPOSITION 2. — *Let λ, ψ and η be as in Theorem 1. Let u be a function harmonic in G_α and satisfying*

$$\int_{G_\alpha} \Psi_p(|\mathrm{grad}\ u(x)|)\lambda(|x|^{1/\alpha})\, dx < \infty. \tag{4}$$

If $0 < \alpha < 1$ *and* $\kappa_{\eta,\alpha}(0) = \infty$, *then*

$$\lim_{x \to 0,\, x \in G_\alpha(a)} [\kappa_{\eta,\alpha}(\rho(x))]^{-1} u(x) = 0$$

for any a such that $0 < a < 1$; *and if* $\kappa_{\eta,\alpha}(r)$ *is bounded, then* $u(x)$ *has a finite limit as* $x \to 0$, $x \in G_\alpha(a)$, *for any a such that* $0 < a < 1$.

Remark. — Proposition 2 is best possible as to the order of infinity in the following sense: if $\varepsilon > 0$, $\beta > \alpha p - \alpha - 1$ and D is the half plane $\{(x, y); x > 0\}$, then we can find a harmonic function u on D which satisfies condition (4) with $\lambda(r) = r^\beta$ and

$$\lim_{x \to 0} x^{-\varepsilon}[\kappa_{\eta,\alpha}(x^{1/\alpha})]^{-1} u(x, 0) = \infty. \tag{5}$$

For this purpose, consider $u(x, y) = r^{-a} \cos a\theta$, where $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$. Then u is harmonic in D. Since $\lambda(r) = r^\beta$, we see that

$$M_1 \psi(r^{-1})^{-1/p} r^{-a_0} \leqslant \kappa_{\eta,\alpha}(r) \leqslant M_2 \psi(r^{-1})^{-1/p} r^{-a_0}$$

with $a_0 = (2 - p + \beta)/\alpha p + (1 - \alpha)/\alpha p'$. If $0 < a < a_0$, then

$$\int_{G_\alpha} \Psi_p(|\text{grad } u(z)|)\, \lambda(\rho(z))\, dz < \infty.$$

If a is taken so large that $-\varepsilon + a_0 < a < a_0$, then we see that u also satisfies (5).

4. Removability of the origin.

In this section we are concerned with the removability of the origin for harmonic functions satisfying condition (1) with $G = B(0, a) - \{0\}$, $a > 0$.

THEOREM 3. — *Let* λ, ψ *and* η *be as in Theorem* 1, *and let u be a function which is harmonic in* $B(0, r_0) - \{0\}$ *and satisfies*

$$\int_{B(0,r_0) - \{0\}} \Psi_p(|\text{grad } u(x)|)\, \lambda(|x|)\, dx < \infty.$$

If $\limsup_{r \downarrow 0} N(r)^{-1} \kappa_\eta(r) < \infty$, *then u can be extended to a function harmonic in* $B(0, r_0)$, *where* $N(r) = \log(1/r)$ *in case* $n = 2$ *and* $N(r) = r^{2-n}$ *in case* $n \geqslant 3$.

Proof. — For $\varepsilon > 0$ and $x \in B(0, r_0/2) - \{0\}$, let $x_\varepsilon = \varepsilon x/|x|$. Then Lemma 1 gives

$$|u(x) - u(x_\varepsilon)| \leqslant M\kappa_\eta(|x|)\left(\int_{B(0,2\varepsilon)} \Psi_p(|\operatorname{grad} u(y)|)\, \lambda(|x|)\, dx\right)^{1/p} + M\int_{B(0,2\varepsilon)} |y|^{1-\delta-n}\, dy,$$

where $0 < \delta < 1$. Consequently, it follows that $\lim_{x \to 0} N(|x|)^{-1} u(x) = 0$. Now our result is a consequence of a result in [1], p. 204.

5. Limits at infinity.

In this section, we discuss the existence of limits at infinity for harmonic functions on a tube domain $T_\ell = \{x = (x', x'') \in R^\ell \times R^{n-\ell};\ |x''| < 1\}$. This T_ℓ is not generally obtained, by inversion, from G_α.

THEOREM 4. — *Let u be a harmonic function on* T_ℓ *satisfying*

$$\int_{T_\ell} \Psi_p(|\operatorname{grad} u(x)|)\, \rho(x)^{p-n}\, \lambda(|x|)\, dx < \infty,$$

where λ *is a positive monotone function on* $(0, \infty)$ *satisfying the* (Δ_2) *condition. Set*

$$\tilde{\psi}(r) = \left(\int_0^r [\psi(t^{-1})]^{-p'/p}\, t^{-1}\, dt\right)^{1/p'}$$

and

$$\kappa(r) = \left(\int_1^r [\tilde{\psi}(t)\, \lambda(t)^{-1/p}]^{p'}\, dt\right)^{1/p'},$$

$r > 1$. *If* $\kappa(r) \to \infty$ *as* $r \to \infty$, *then* $[\kappa(|x|)]^{-1} u(x) \to 0$ *as* $|x| \to \infty$, $x \in T_\ell$; *and if* $\kappa(r)$ *is bounded, then* $u(x)$ *has a finite limit at infinity.*

For the study of the behavior at infinity, we do not think it necessary to replace $\rho(x)^{p-n}$ by a more general function $\lambda_1(\rho(x))$. The proof of this theorem is similar to the proofs of Theorem 2 and Proposition 1 ; but we give a proof for the sake of completeness.

Proof of Theorem 4. – For $x \in T_\ell$, take $x_0 \in T_\ell$ such that $E(x, x_0) \subset B(x_0, 1)$. Then, by Lemma 1, we have

$$|u(x) - u(x_0)| \leqslant M_1 \int_{E(x,x_0)} f(y)\,\rho(y)^{1-n}\,dy,$$

where $f(y) = |\operatorname{grad} u(y)|$. Hence Hölder's inequality implies that

$$\begin{aligned} |u(x) - u(x_0)| \leqslant M_1 &\left(\int_{\{y \in E(x,x_0);\, f(y) \geqslant \alpha\rho(y)^{-\delta}\}} \Psi_p(f(y))\,\rho(y)^{p-n}dy\right)^{1/p} \\ &\times \left(\int_{\{y \in E(x,x_0)\,;\, f(y) \geqslant \alpha\rho(y)^{-\delta}\}} \rho(y)^{p'(1-n)}\,[\psi(f(y))\,\rho(y)^{p-n}]^{-p'/p}dy\right)^{1/p'} \\ &+ \alpha \int_{E(x,x_0)} \rho(y)^{1-n-\delta}dy \\ &\geqslant M_1 \left(\int_{B(x_0,1)} \Psi_p(f(y))\,\rho(y)^{p-n}dy\right)^{1/p} \\ &\times \left(\int_{E(x,x_0)} [\psi(\alpha\rho(y)^{-\delta})]^{-p'/p}\rho(y)^{-n}dy\right)^{1/p'} + M_2\alpha, \end{aligned}$$

where $\alpha > 0$ and $0 < \delta < 1$. If we note that

$$\begin{aligned} &\left(\int_{E(x,x_0)} [\psi(\alpha\rho(y)^{-\delta})]^{-p'/p}\rho(y)^{-n}dy\right)^{1/p'} \\ &\qquad \leqslant M_3 \left(\int_0^2 [\psi(\alpha r^{-\delta})]^{-p'/p} r^{-1}dr\right)^{1/p'} \leqslant M_4\,\tilde{\psi}(\alpha^{-1}), \end{aligned}$$

then

$$|u(x) - u(x_0)| \leqslant M_5 \left(\int_{B(x_0,1)} \Psi_p(f(y))\,\rho(y)^{p-n}dy\right)^{1/p} \tilde{\psi}(\alpha^{-1}) + M_2\alpha.$$

Taking $\alpha = |x|^{-2}$, we have

$$\begin{aligned} |u(x) - u(x_0)| \leqslant M_6 &\left(\int_{B(x_0,1)} \Psi_p(f(y))\,\rho(y)^{p-n}\lambda(|y|)dy\right)^{1/p} \\ &\times \tilde{\psi}(|x|)\,\lambda(|x|)^{-1/p} + M_2|x|^{-2}. \end{aligned}$$

For $x = (x', x'')$, let k be the nonnegative integer such that $k \leqslant |x'| < k + 1$. Put $x_j = j(x',0)/|x'|$ for $j = 0, 1, \ldots, k$ and

$x_{k+1} = (x',0)$. Then

$$
\begin{aligned}
|u(x) - u(x_{j_0})| &\leqslant |u(x) - u(x_{k+1})| + |u(x_{k+1}) - u(x_k)| + \cdots \\
&\qquad\qquad + |u(x_{j_0+1}) - u(x_{j_0})| \\
&\leqslant M_6\left(\int_{\Delta(x,x_{j_0})} \Psi_p(f(y))\,\rho(y)^{p-n}\lambda(|y|)dy\right)^{1/p} \\
&\quad \times \left(\sum_{j=j_0}^{k+1} [\tilde{\psi}(j)\lambda(j)^{-1/p}]^{p'}\right)^{1/p'} + M_2\left(\sum_{j=j_0}^{k+1} j^{-2}\right) \\
&\leqslant M_7\left(\int_{\Delta(x,x_{j_0})} \Psi_p(f(y))\,\rho(y)^{p-n}\lambda(|y|)dy\right)^{1/p} \kappa(|x|) + M_7 j_0^{-1},
\end{aligned}
$$

where $\Delta(x,x_{j_0}) = \bigcup_{j_0 \leqslant j \leqslant k+1} B(x_j,1)$. If $\kappa(r)$ is not bounded, then it follows that

$$
\limsup_{|x'|\to\infty,\, x\in T_\ell} [\kappa(|x|)]^{-1}|u(x)| \leqslant M_7\left(\int_{T_\ell - B(0,j_0-1)} \Psi_p(f(y))\,\rho(y)^{p-n}\lambda(|y|)dy\right)^{1/p}
$$

for any j_0, which implies that the left hand side equals zero.

If $\kappa(r)$ is bounded, then $u(x)$ is shown to have a finite limit at infinity.

6. Global boundary behavior.

In this section we are concerned with the global existence of tangential boundary limits of harmonic functions u on G satisfying (1). Our aim is to give generalizations of the author's results [5], [6]. We consider the sets

$$
E_0 = \left\{\xi \in \partial G \,;\, \int_{G\cap B(\xi,1)} |\xi - y|^{1-n}|\operatorname{grad} u(y)|dy = \infty\right\}
$$

and

$$
E_h = \left\{\xi \in \partial G; \limsup_{r\downarrow 0} h(r)^{-1} \int_{G\cap B(\xi,r)} \Psi_p(|\operatorname{grad} u(y)|)\,\lambda(\rho(y))dy > 0\right\},
$$

where h is a positive nondecreasing function on the interval $(0,\infty)$. From condition (1) it follows that $H_h(E_h) = 0$; moreover, in case $\lambda(r) = r^\beta$, $B_{1-\beta/p,p}(E_0) = 0$. Here H_h denotes the Hausdorff measure with the measure function h and $B_{\alpha,p}$ denotes the Bessel capacity of index (α,p) (see Meyers [4]). As to the size of E_0, we shall give a precise evaluation in Proposition 3 below, after discussing the Ψ_p norm inequality of singular integrals.

Further, let φ be a positive nondecreasing function on the interval $(0,\infty)$ such that $\lim_{r\downarrow 0} \varphi(r) = 0$, $\varphi(r)/r$ is nondecreasing on $(0,\infty)$ and $\varphi(2r) \leqslant M\varphi(r)$ for any $r > 0$ with a positive constant M. For $a > 0$ and $\xi \in \partial G$, set

$$S_\varphi(a) = \{x = (x',x_n) \in R^{n-1} \times R^1 ; \varphi(|x-\xi|) < ax_n\}$$

and

$$T_\varphi(\xi,a) = \{\xi + \Xi_\xi x ; x \in S(a)\}$$

with an orthogonal transformation Ξ_ξ.

THEOREM 5. — *Let G be a Lipschitz domain in R^n, and let u be a harmonic function on G satisfying condition* (1). *If $\xi \in \partial G - E_0 \cup E_h$, $T_\varphi(\xi,a) \subset G$ and $\kappa_\eta(\rho(x)) \leqslant M(a)\, h(|\xi-x|)^{-1/p}$ on $T_\varphi(\xi,a)$, with a positive constant $M(a)$, then $u(x)$ has a finite limit as $x \to \xi$, $x \in T_\varphi(\xi,a)$.*

Proof. — In view of Lemma 1, we can find $\{r_j\}$, $\{x_j\}$ and $c > 0$ (in Lemma 1) with the following properties:

i) $0 < r_{j+1} < r_j < 1/j$.

ii) $x_j \in G \cap B(\xi,r_j)$.

iii) If $x \in G \bigcap B(\xi,r_{j+1})$, then $E(x,x_j) \subset G \bigcap B(\xi,r_j)$, $\rho(x) + |x-y| \leqslant M_1\rho(y)$ for any $y \in E(x,x_j)$ and

$$|u(x) - u(x_j)| \leqslant M_1 \int_{E(x,x_j)} f(y)\, \rho(y)^{1-n}\, dy,$$

where $f(y) = |\mathrm{grad}\, u(y)|$. Hence, as in the proof of Theorem 1, we obtain

$$\begin{aligned} |u(x) - u(x_j)| \leqslant{}& M_1 \int_{E(x,x_j) - B(\xi,2|x-\xi|)} f(y)\, \rho(y)^{1-n}\, dy \\ &+ M_1 \int_{\{y \in G \cap B(\xi,2|x-\xi|)\, ; f(y) < \rho(y)^{-\delta}\}} \rho(y)^{1-\delta-n}\, dy \\ &+ M_2\kappa_\eta(\rho(x)) \left(\int_{G \cap B(\xi,2|\xi-x|)} \Psi_p(f(y))\, \lambda(\rho(y))\, dy \right)^{1/p} \\ \leqslant{}& M_3(I_1 + I_2 + I_3), \end{aligned}$$

where $0 < \delta < 1$. If $y \in E(x,x_j)$ and $|y-\xi| \geqslant 2|x-\xi|$, then $\rho(y) \geqslant M_1^{-1}|x-y| \geqslant M_1^{-1}(|y-\xi| - |x-\xi|) \geqslant (2M_1)^{-1}|y-\xi|$, so that

$$I_1 \leqslant M_4 \int_{E(x,x_j) - B(\xi,2|x-\xi|)} f(y)|\xi-y|^{1-n}\,dy.$$

Moreover, $I_2 \leqslant M_5|x-\xi|^{1-\delta}$ and $\kappa_\eta(\rho(x)) \leqslant M(a)\,h(|x-\xi|)^{-1/p}$ for $x \in T_\varphi(\xi,a)$ by our assumption. Consequently, if $\xi \in \partial G - (E_0 \cup E_h)$, then $\{u(x_\ell)\}_{\ell \geqslant j+1}$ is bounded, so that we can find a subsequence $\{u(x_{j_k})\}$ which converges to a number u_0 as $k \to \infty$. Hence, since

$$\lim_{j\to\infty}[\limsup_{x\to\xi,\, x\in T_\varphi(\xi,a)} |u(x) - u(x_j)|] = 0,$$

it follows that $u(x) \to u_0$ as $x \to \xi$ along $T_\varphi(\xi,a)$.

For a, $b \geqslant 0$ and $\alpha > 1$, set

$$S_\alpha(a,b) = \{x = (x',x_n)\,;\, x_n > a|x'| + b|x'|^\alpha\}.$$

If G is a Lipschitz domain, then, for each $\xi \in \partial G$ we can find a_ξ, $b_\xi \geqslant 0$, $r_\xi > 0$ and an orthogonal tranformation Ξ_ξ such that

$$\{\xi + \Xi_\xi x\,;\, x \in S_\alpha(a_\xi,b_\xi)\} \cap B(\xi,r_\xi) \subset G.$$

For $b > b_\xi$, put

$$T_\alpha(\xi,b) = T_\alpha(\xi,\Xi_\xi,b) \equiv \{\xi + \Xi_\xi x\,;\, x \in S_\alpha(a_\xi,b)\} \cap B(\xi,r_\xi).$$

COROLLARY – *Let G be a Lipschitz domain. For $\alpha > 1$, let $\{T_\alpha(\xi,b)\,;\, \xi \in \partial G,\ b > b_\xi\}$ be given as above. If u is a function which is harmonic in G and satisfies*

$$\int_G \Psi_p(|\mathrm{grad}\, u(x)|)\rho(x)^\beta\,dx < \infty$$

for $\beta > p - n$, then there exists a set $E \subset \partial G$ such that

i) *$H_h(E) = 0$ for $h(r) = \inf_{t \geqslant r} t^{\alpha(n-p+\beta)}\psi(t^{-1})$;*

ii) *$u(x)$ has a finite limit as $x \to \xi$ along $T_\alpha(\xi,b)$ whenever $\xi \in \partial G - E$ and $b > b_\xi$.*

Proof. – First note that for $\varepsilon > 0$, $r^\varepsilon\psi(r^{-1}) \geqslant M_1 s^\varepsilon\psi(s^{-1})$ whenever $0 < s < r$, on account of condition (ψ_1). Hence, since $\rho(x) \geqslant M_1|x-\xi|^\alpha$

for $x \in T_\alpha(\xi,b)$,

$$\kappa_\eta(\rho(x)) \leqslant \left(\int_{M_1 r^\alpha}^1 [s^{n-p+\beta}\psi(s^{-1})]^{-p'/p} s^{-1}\, ds \right)^{1/p'}$$

$$\leqslant M_2 [r^{\alpha(n-p+\beta-\delta)}\psi(r^{-1})]^{-1/p} \left(\int_{M_1 r^\alpha}^1 s^{-\delta p'/p-1}\, ds \right)^{1/p'}$$

$$\leqslant M_3 h(r)^{-1/p},$$

where $0 < \delta < n - p + \beta$ and $r = |x-\xi|$. Let $E = E_0 \cup E_h$ in the notation given in Theorem 5. Since $B_{1-\beta/p,p}(E_0) = 0$ implies that E_0 has Hausdorff dimension at most $n - p + \beta$, on account of [4], Theorem 22. Since $\alpha > 1$ and $n - p + \beta > 0$, $\lim_{r\to 0} h(r)/r^{n-p+\beta} = 0$, so that we see that $H_h(E_0) = 0$. Hence $H_h(E) = 0$, and the Corollary follows from Theorem 5.

Remark 1. — In case $\psi(r) \equiv 1$, $\lambda(r) = r^\beta$ with $p - n \leqslant \beta < p - 1$ and $\varphi(r) = r^\alpha$ with $\alpha > 1$, we can take h so that $h(r) = r^{\alpha(n-p+\beta)}$ if $n - p + \beta > 0$ and $h(r) = [\log(2 + r^{-1})]^{1-p}$ if $n - p + \beta = 0$. Hence, Theorem 5 and its Corollary give the usual T_α-limit theorem (see [5]).

Remark 2. — Nagel, Rudin and Shapiro [8] proved the existence of T_α-limits of harmonic functions represented as Poisson integrals in a half space.

7. Singular integrals.

Here we establish the following result.

THEOREM 6. — *Let f be a function on R^n such that*

$$\int (1+|y|)^{1-n} |f(y)| dy < \infty$$

and $\int \Psi_p(|f(y)||y_n|^{\beta/p})\, dy < \infty$, *where* $-1 < \beta < p - 1$. *If we set* $u(x) = \int |x-y|^{1-n} f(y)\, dy$, *then*

$$\int \Psi_p(|\mathrm{grad}\, u(x)||x_n|^{\beta/p})\, dx \leqslant M \int \Psi_p(|f(y)||y_n|^{\beta/p})\, dy$$

with a positive constant M independent of f.

Proof. – Without loss of generality, we may assume that $f \geqslant 0$ on R^n. First we consider the case $\beta = 0$. We note, by the well-known fact from the theory of singular integral operators, that

$$\begin{aligned}\lambda(a) &\equiv H_n(\{x;\ |\operatorname{grad} u(x)| > a\}) \\ &\leqslant M_1 a^{-1} \int_{\{y;\, f(y) \geqslant a/2\}} U(y)\, dy + M_1 a^{-q} \int_{\{y;\, f(y) < a/2\}} U(y)^q\, dy \\ &= M_1 \mu_1(a) + M_1 \mu_2(a),\end{aligned}$$

where H_n denotes the n-dimensional Lebesgue measure, $q > p$ and $U(y) = |\operatorname{grad} u(y)|$. Hence we have

$$\begin{aligned}&\int \Psi_p(|\operatorname{grad} u(x)|)\, dx = \int_0^\infty \lambda(a)\, d\Psi_p(a) \\ &\leqslant M_1 \int_0^\infty \mu_1(a)\, d\Psi_p(a) + M_1 \int_0^\infty \mu_2(a)\, d\Psi_p(a) \\ &\leqslant M_1 \int U(y) \left(\int_0^{2f(y)} a^{-1}\, d\Psi_p(a) \right) dy + M_1 \int U(y)^q \left(\int_{2f(y)}^\infty a^{-q}\, d\Psi_p(a) \right) dy \\ &\leqslant M_2 \int \Psi_p(U(y))\, dy.\end{aligned}$$

In case $\beta \neq 0$, set $g(y) = |y_n|^{\beta/p} U(y)$ and

$$v(x) = \int |x-y|^{1-n} g(y)\, dy.$$

For $j = 1, 2, \ldots, n$, we see that

$$||x_n|^{\beta/p} (\partial/\partial x_j) u(x) - (\partial/\partial x_j) v(x)| \leqslant M_3 \int K_\beta(x_n, y_n)\, (P_{|x_n - y_n|} g)\, (x', x_n)\, dy_n,$$

where $K_\beta(x_n, y_n) = |1 - |x_n/y_n|^{\beta/p}| / |x_n - y_n|$ and P denotes the Poisson kernel in the upper half space $D = \{x = (x', x_n) \in R^{n-1} \times R^1;\, x_n > 0\}$. By [9], Theorem 1, (*a*) in Chap. III and Theorem 1, (*c*) in Chap. I, we have for $q \geqslant 1$

$$\int [P_t g(x', x_n)]^q\, dx' \leqslant M_4 \int g(y', y_n)^q\, dy'.$$

Hence, by using Minkowski's inequality (cf. [9], Appendix A.1), we establish

$$\int\left(\int K_\beta(x_n,y_n)\,(P_{|x_n-y_n|}g)\,(x',x_n)\,dy_n\right)^q dx$$
$$\leqslant M_4\int\left(\int K_\beta(x_n,y_n)\left(\int g(y',y_n)^q\,dy'\right)^{1/q} dy_n\right)^q dx_n.$$

Let q_1 and q_2 be positive numbers such that $\beta < q_1 - 1$ and $1 < q_1 < p < q_2$. Applying Appendix A.3 in Stein's book [9], we see that

$$\lambda(a) \equiv H_n(\{x;\ ||x_n|^{\beta/p}(\partial/\partial x_j)u(x)-(\partial/\partial x_j)v(x)| > a\})$$
$$\leqslant M_5(\mu_1(a)+\mu_2(a)),$$

where

$$\mu_1(a) = a^{-q_1}\int_{\{y;\,g(y)\geqslant a/2\}} g(y)^{q_1}\,dy$$

and

$$\mu_2(a) = a^{-q_2}\int_{\{y;\,g(y)< a/2\}} g(y)^{q_2}\,dy.$$

Consequently, by the above considerations, we see that

$$\int \Psi_p(||x_n|^{\beta/p}(\partial/\partial x_j)u(x)-(\partial/\partial x_j)v(x)|) \leqslant M_6\int \Psi_p(g(y))\,dy.$$

Thus it follows that

$$\int \Psi_p(|x_n|^{\beta/p}(\partial/\partial x_j)u(x)|)\,dx \leqslant M_7\int \Psi_p(g(y))\,dy,$$

or

$$\int \Psi_p(|x_n|^{\beta/p}|\operatorname{grad} u(x)|)\,dx \leqslant M_8\int \Psi_p(g(y))\,dy < \infty.$$

Remark. – Consider the functions

$$u_j(x) = \int (x_j-y_j)|x-y|^{-n}f(y)\,dy.$$

Then the same inequality as in Theorem 6 still holds for each u_j.

For $\beta > 0$ and $E \subset R^n$, we define

$$C_{\beta,\Psi_p}(E) = \inf \int \Psi_p(f(y))\, dy,$$

where the infimum is taken over all nonnegative measurable functions f on R^n such that $\int_{B(x,1)} |x-y|^{\beta-n} f(y)\, dy \geqslant 1$ for every $x \in E$.

PROPOSITION 3. – *Let f be a nonnegative measurable function on a Lipschitz domain G such that $\int_G \Psi_p(f(y))\rho(y)^\beta\, dy < \infty$, and set $E = \{\xi \in \partial G\,;\, \int_{G\cap B(\xi,1)} |\xi-y|^{1-n} f(y)\, dy = \infty\}$. If $-1 < \beta < p-1$, then $C_{1-\beta/p,\Psi_p}(E) = 0$.*

Proof. – By a change of variables, we may assume that G is the half space D and f vanishes outside some ball $B(0,N)$. Let $u(x) = \int_D |x-y|^{1-n} f(y)\, dy$ for a nonnegative measurable function f on D such that $\int_D \Psi_p(f(y))y_n^\beta\, dy < \infty$. Here note that

$$\begin{aligned}\int \Psi_p(f(y)y_n^{\beta/p})\, dy &\leqslant \int_{\{y\in D;\, f(y)^\varepsilon \geqslant y_n^{\beta/p}\}} \Psi_p(f(y)y_n^{\beta/p})\, dy \\ &\quad + \int_{\{y\in D;\, f(y)^\varepsilon \leqslant y_n^{\beta/p}\}} \Psi_p(f(y)y_n^{\beta/p})\, dy \\ &\leqslant \int_D y_n^\beta f(y)^p \psi(f(y)^{1+\varepsilon})\, dy \\ &\quad + \int_{\{y\in D;\, f(y)>0\}} \Psi_p(y_n^{(1+\varepsilon^{-1})\beta/p})\, dy < \infty,\end{aligned}$$

if $\varepsilon > 0$ and $\beta(1+\varepsilon^{-1}) > -1$. Hence, from Theorem 6, it follows that $\int \Psi_p(|\mathrm{grad}\, u(x)|\,|x_n|^{\beta/p})\, dx < \infty$. Since $|\mathrm{grad}\, u(x)| = O(|x|^{-n})$ as $|x| \to \infty$, we see that $\int_{R^n - B(0,a)} \Psi_p(|\mathrm{grad}\, u(x)|)|x_n|^\beta\, dx < \infty$ for a sufficiently large a. Moreover, we have, by letting $U(x) = |\operatorname{grad} u(x)|$,

$$\int_{B(0,a)} \Psi_p(U(x))|x_n|^\beta dx \leqslant \int_{\{x\in B(0,a);\, U(x)\geqslant |x_n|^{-(1+\delta^{-1})\beta/p}\}} \Psi_p(U(x))|x_n|^\beta dx$$
$$+ \int_{\{x\in B(0,a);\, U(x)< |x_n|^{-(1+\delta^{-1})\beta/p}\}} \Psi_p(U(x))|x_n|^\beta dx$$
$$\leqslant \int \psi([U(x)|x_n|^{\beta/p}]^{1+\delta})\, U(x)^p|x_n|^\beta dx$$
$$+ \int_{B(0,a)} \Psi_p(|x_n|^{-(1+\delta^{-1})\beta/p})|x_n|^\beta dx < \infty,$$

if $\delta > 0$ and $\delta > \beta$. Thus $\int \Psi_p(U(x))|x_n|^\beta dx < \infty$.

Consider the set

$$E^* = \{x \in \partial D\,;\, \int_D |x-y|^{1-\beta/p-n}\,[U(y)y_n^{\beta/p}]\,dy = \infty\}.$$

Then, by definition, $C_{1-\beta/p,\Psi_p}(E^*) = 0$. If $\xi \in \partial D - E^*$ and $a > 0$, then

$$\int_{\Gamma(\xi,a)} |\xi-y|^{1-n}|\operatorname{grad} u(y)|dy < \infty,$$

where $\Gamma(\xi,a) = \{x\in D\,;\, |x-\xi|<ax_n\}$. It follow that

$$\int_0^{r_0} |\operatorname{grad} u(\xi+r\theta)|dr < \infty \quad \text{for almost every } \theta\in\partial B(0,1),$$

which implies that $u(\xi+r\theta)$ has a finite limit for almost every $\theta\in\partial B(0,1)$. If $\xi\in E$, then $\liminf_{r\to 0} u(\xi+rx) \geqslant u(\xi) = \infty$ for any $x\in D$ by the lower semicontinuity of potentials. Thus $\xi\in\partial D - E$. Hence $E\subset E^*$, or $C_{1-\beta/p,\Psi_p}(E) = 0$.

8. Best possibility.

Here we deal with the best possibility of Theorem 1 as to the order of infinity. Let D be the upper half space, that is, $D = \{x=(x',x_n)\in R^{n-1}\times R^1\,;\, x_n>0\}$.

PROPOSITION 4. — *Let* λ, ψ *and* η *be as in Theorem* 1. *Suppose* $\kappa_\eta(0) = \infty$ *and* $r^\delta \eta(r)^{-1}$ *is bounded above on* $(0,1]$ *for some* $\delta > 1 - n$. *If* $a(r)$ *is a nonincreasing positive function on the interval* $(0,\infty)$ *such that* $\lim_{r\downarrow 0} a(r) = \infty$, *then there exists a nonnegative measurable function* f *such that* $f = 0$ *outside* $B(0,1)$,

$$\int_{R^n} \Psi_p(f(y))\lambda(|y_n|)\, dy < \infty$$

and

$$\limsup_{r\downarrow 0} a(r)\kappa_\eta(r)^{-1}u(r\xi) = \infty \quad \textit{for any } \xi \in D,$$

where $u(x) = \displaystyle\int_{R^n - D} (x_n - y_n)|x-y|^{-n} f(y)\, dy$.

Remark. — By the Remark after Theorem 6, if $\lambda(r) = r^\beta$ with $-1 < \beta < p - 1$, then

$$\int \Psi_p(|\text{grad}\, u(x)|)|x_n|^\beta\, dx < \infty .$$

Proof of Proposition 4. — Let $\{r_j\}$ be a sequence of positive numbers such that $r_j < r_{j-1}/2$ and

$$\kappa_\eta(r_j) \leqslant 2\left(\int_{r_j}^{r_{j-1}} [s^{n-p}\eta(s)]^{-p'/p} s^{-1} ds\right)^{1/p'} .$$

Further take a sequence $\{b_j\}$ of positive numbers such that $\lim_{j\to\infty} b_j a(r_j) = \infty$ and $\sum_{j=1}^{\infty} b_j^p < \infty$. Let $\Gamma(c)$ be the cone $S_\varphi(c)$ with $\varphi(r) \equiv r$, and set $\hat{\Gamma}(c) = \{x \in R^n ; -x \in \Gamma(c)\}$. Now we define

$$f(y) = b_j \kappa_\eta(r_j)^{-p'/p} [|y|^{n-1}\eta(|y|)]^{-p'/p}$$

if $y \in \hat{\Gamma}_j \equiv \hat{\Gamma}(1) \cap B(0,r_{j-1}) - B(0,r_j)$ and $f = 0$ otherwise, and consider the function u defined as in Proposition 4. If

$$x \in \Gamma(c) \cap B(0,2r_j) - B(0,r_j),$$

then

$$\begin{aligned} u(x) &\geqslant M_1 b_j \kappa_\eta(r_j)^{-p'/p} \int_{\hat{\Gamma}_j} |y|^{1-n} [|y|^{n-1}\eta(|y|)]^{-p'/p}\, dy \\ &\geqslant M_2 b_j \kappa_\eta(r_j), \end{aligned}$$

so that

$$\lim_{x\to 0,\, x\in\Delta(c)} a(|x|)\,\kappa_\eta(|x|)^{-1}\,u(x) = \infty$$

with $\Delta(c) = \bigcup_{j=1}^{\infty}\{x\in\Gamma(c)\,;\, r_j < |x| < 2r_j\}$. On the other hand, since $r^\delta\eta(r)^{-1}$ is bounded above by our assumption, $f(y) \leqslant M_3|y|^{-p'(n-1+\delta)/p}$, so that $\psi(f(y)) \leqslant M_4\psi(|y|^{-1})$ by (2). Hence we establish

$$\int_{R^n}\Psi_p(f(y))\,\lambda(|y|)\,dy \leqslant M_5\sum_{j=1}^{\infty} b_j^p\,\kappa_\eta(r_j)^{-p'}\int_{\Gamma_j}|y|^{p'(1-n)}\,\eta(|y|)^{1-p'}\,dy$$

$$\leqslant M_6\sum_{j=1}^{\infty} b_j^p < \infty\,.$$

Thus f satisfies all the required assertions.

The Corollary to Theorem 5 is best possible as to the size of the exceptional sets, in the following sense.

PROPOSITION 5. — *Let ψ, λ and η be as in Theorem 1. Let φ be a nonnegative nondecreasing function on $(0,\infty)$ such that $\varphi(r) \leqslant Mr$ for any $r > 0$, with a positive constant M, and set*

$$\varphi^*(r) = \int_{\varphi(r)}^{2Mr}[t^{n-p}\eta(t)]^{-p'/p}\,t^{-1}\,dt\,.$$

Suppose further that the following assertions hold :

i) *$r^{\delta_1}\lambda(r)^{-1}$ is nondecreasing on $(0,\infty)$ for some $\delta_1 > 1/p - n$.*

ii) *$r^{\delta_2}\lambda(r)$ is nondecreasing on $(0,\infty)$ for some $\delta_2 < 1$.*

iii) *$\varphi^*(r) \to \infty$ as $r \to 0$.*

iv) *$\varphi^*(r) \leqslant M^*\varphi^*(s)$ whenever $0 < s < r$, with a positive constant M^*.*

We now define $h(r) = \inf_{s\geqslant r}[\varphi^*(s)]^{-p/p'}$. Then, for a compact set $K \subset \partial D$ such that $H_h(K) = 0$, there exists a nonnegative measurable function f on R^n such that

$$\int \Psi_p(f(y))\,\lambda(|y_n|)\,dy < \infty$$

and $Uf(x) \equiv \int_{R^n - D}(x_n - y_n)\,|y-y|^{-n}f(y)\,dy$ does not have a finite limit as $x \in T_\varphi(\xi,1) \to \xi$ at any $\xi \in K$, where $T_\varphi(\xi,1) \equiv \{x+\xi\,;\, x\in S_\varphi(1)\}$.

Proof. – For the construction of such f, we take, for each positive integrer m, a finite family $\{B(x_{j,m}, r_{j,m})\}$ of balls such that $x_{j,m} \in \partial D$, $r_{j,m} < 1/m, \sum_j h(r_{j,m}) < 2^{-m}/m$ and $\bigcup_j B(x_{j,m}, r_{j,m}) \supset K$. Setting

$$B_{i,j} = B(x_{i,j}, 2Mr_{i,j}) - B(x_{i,j}, \varphi(r_{i,j})),$$

we define

$$f_{m,j}(y) = m^{1/p}[h(r_{j,m})]^{p'/p}[|x_{j,m} - y|^{n-1}\eta(|x_{j,m} - y|)]^{-p'/p}$$

for $y \in B_{m,j}$ and $f_{m,j}(y) = 0$ elsewhere. Consider the function $f(y) = \sup_{m,j} f_{m,j}(y)$. Since $f_{m,j}(y) \leqslant M_1|x_{j,m} - y|^{-\gamma}$, where

$$\gamma = 1/p + p'(n-1+\delta_1)/p > 0,$$

we see that $\psi(f_{m,j}(y)) \leqslant M_2\psi(|x_{j,m} - y|^{-1})$ on account of (2). Since $r^{\delta_2}\lambda(r)$ is nondecreasing and $\varphi^*(r) \leqslant M_3[h(r)]^{-p'/p}$, we establish

$$\int_{R^n - D} \Psi_p(f(y))\,\lambda(|y_n|)\,dy \leqslant M_4 \sum_m m\Bigg(\sum_j [h(r_{j,m})]^{p'} \int_{B_{j,m}} |x_{j,m} - y|^{p'(1-n)}$$
$$\times [\eta(|x_{j,m} - y|)]^{p'}\,\psi(|x_{j,m} - y|^{-1})[|x_{j,m} - y|^{\delta_2}\lambda(|x_{j,m} - y|)]\,|y_n|^{-\delta_2} dy\Bigg)$$
$$\leqslant M_5 \sum_m m\Bigg(\sum_j [h(r_{j,m})]^{p'}\,\varphi^*(r_{j,m})\Bigg)$$
$$\leqslant M_6 \sum_m m\Bigg(\sum_j h(r_{j,m})\Bigg) \leqslant M_6 \sum_m 2^{-m} < \infty.$$

Further,

$$\begin{aligned} Uf(x) &\geqslant \int (x_n - y_n)\,|x - y|^{-n} f_{m,j}(y)\,dy \\ &\geqslant M_7 m^{1/p}[h(r_{j,m})]^{p'/p} \int_{\varphi(r_{i,j})}^{2Mr_{i,j}} r^{p'(1-n)}\,[\eta(r)]^{-p'/p}\,r^{-1}\,dr \\ &\geqslant M_7 m^{1/p} \end{aligned}$$

for any $x \cap D \bigcap B(x_{j,m}, \varphi(r_{j,m}))$. If $\xi \in K$, then for each m there exists $j(m)$ such that $\xi \in B(x_{j(m),m}, r_{j(m),m})$. Since

$$B(x_{j(m),m}, \varphi(r_{j(m),m})) \cap T_\varphi(\xi, 1) \neq \varnothing,$$

if follows that

$$\limsup_{x \to \xi, x \in T_\varphi(\xi,1)} Uf(x) = \infty.$$

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