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LLUIS ALSEDÀ
J. LLIBRE
M. MISIUREWICZ
C. TRESSER

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PERIODS AND ENTROPY FOR LORENZ-LIKE MAPS

by Ll. ALSEDÀ, J. LLIBRE, M. MISIUREWICZ and C. TRESSER

1. Notation, definitions and statement of results.

In the paper we shall use the following notations. When we write p/q we mean that p, $q \in \mathbf{Z}$ and $q > 0$. If we write $k > 0$ or $k \geqslant 0$, we mean that additionally $k \in \mathbf{Z}$. The greatest common divisor of p and q will be denoted by (p,q). If A is a subset of $\mathbf{N} = \{1,2,3,\dots\}$ then kA will denote the set $\{ka : a \in A\}$. We shall denote by $\mathrm{E}(.)$ the integer part function.

We denote by $e: \mathbf{R} \to \mathbf{S}^1 = \{z \in \mathbf{C} : |z|=1\}$ the natural projection $e(x) = \exp(2\pi i x)$ (here $i=\sqrt{-1}$). A map $F: \mathbf{R} \to \mathbf{R}$ is called *a lifting* of a map $f: \mathbf{S}^1 \to \mathbf{S}^1$ if $e \circ F = f \circ e$ and there is $k \in \mathbf{Z}$ such that $F(x+1) = F(x) + k$ for all $x \in \mathbf{R}$. This k is called the *degree* of F. Note that since we do not say anything about continuity here, every f has liftings of all degrees.

A map $F: \mathbf{R} \to \mathbf{R}$ will be called *old* if $F(x+1) = F(x) + 1$ for all $x \in \mathbf{R}$ (here we follow the terminology of [M3]; old stands for « degree one lifting » with the order of letters changed for mnemonic reasons). It it easy to see that if F is an old map then $F(x+k) = F(x) + k$ for all $x \in \mathbf{R}$ and $k \in \mathbf{Z}$, and that the iterates of an old map are old maps.

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We shall say that a point $x \in \mathbf{R}$ is *periodic (mod. 1) of period q with rotation number p/q* for an old map F if $F^q(x) - x = p$ and $F^i(x) - x \notin \mathbf{Z}$ for $i = 1, 2, \ldots, q-1$. Clearly, if F is a lifting of f then x is periodic (mod. 1) for F if and only if $e(x)$ is periodic for f and their periods are equal, see [M3]. The set of periods of periodic (mod. 1) points of F will be denoted by $\mathrm{Per}(F)$, and the set of periods of periodic (mod. 1) points with rotation number p/q by $\mathrm{Per}_{p/q}(F)$. Also, if $a \notin \mathbf{Q}$ we define $\mathrm{Per}_a(F) = \varnothing$.

Let F be an old map such that F is non-decreasing and continuous on the interval $(0,1)$. For $x \in \mathbf{R}$ we define its rotation number as $\limsup_{n\to\infty} (F^n(x) - x)/n$ and we denote it by $\rho(x)$ or $\rho_F(x)$. Note that if x is a periodic (mod. 1) point of F with rotation number p/q then $\rho(x) = p/q$. We denote by $L(F)$ the set of all rotation numbers of F. From [M3] and [RT] it follows that $L(F) = [a(F), b(F)]$ with $a(F) = \inf_{x\in\mathbf{R}} \liminf_{n\to\infty} (F^n(x)-x)/n$ and $b(F) = \sup_{x\in\mathbf{R}} \limsup_{n\to\infty} (F^n(x)-x)/n$. The interval $[a(F), b(F)]$ will be called the *rotation interval* of F.

We shall denote by $F(x+)$ the $\lim_{y\downarrow x} F(y)$, and by $F(x-)$ the $\lim_{y\uparrow x} F(y)$, if they exist.

In this paper we shall study the class of old maps $F\colon \mathbf{R} \to \mathbf{R}$ such that F is non-decreasing and continuous on the interval $(0,1)$, $0 \leqslant F(0+) < 1$ and $F(1-) \leqslant 2$. This class will be called the class of *Lorenz-like maps* and it will be denoted by $\mathscr{L}$. We remark that if $F \in \mathscr{L}$ then its rotation interval is contained in $[0,1]$.

Let F be an old map such that F is non-decreasing and continuous on the interval $(0,1)$, $0 \leqslant F(0+) < 1$ and $F(1-) > 2$. Clearly there exists $x \in (0,1)$ such that $F(x) = x+1$. If $F([x, x+1]) \subset [x+1, x+2]$ then the study of the dynamics of this map can be reduced to study a map of the class $\mathscr{L}$. Otherwise, it is not difficult to prove that $\mathrm{Per}(F)$ is either $\{1\}$ or $\{1, k, k+1, k+2, \ldots\}$ for some integer $k \geqslant 2$. All this can be done by using the methods of the proof of Proposition 3.

Let $\tilde{\mathscr{L}}$ be the class of maps $\tilde{F}\colon [0,1] \to [0,1]$ such that $\tilde{F}(x) = F(x) - E(F(x))$ for $x \in (0,1)$, $\tilde{F}(0) = F(0+)$ and $\tilde{F}(1) = F(1-) - E(F(1-))$ for some $F \in \mathscr{L}$. Note that such a map $\tilde{F}$ has at most one discontinuity point c such that $\tilde{F}(c-) = 1$ and $\tilde{F}(c+) = 0$. A point $x \in [0,1]$ is *periodic of period q* for $\tilde{F}$ if $\tilde{F}^q(x) = x$ and $\tilde{F}^i(x) \neq x$ for $i = 1, 2, \ldots, q-1$. The set of periods of periodic points of $\tilde{F}$ will be denoted by $\operatorname{Per}(\tilde{F})$. We remark that the study of the class $\mathscr{L}$ is equivalent to study the class $\tilde{\mathscr{L}}$. In fact, if $x \in [0,1]$ is such that $F^n(x) \notin \mathbf{Z}$ for $n \geqslant 0$, then x is a periodic (mod. 1) point of period q with rotation number p/q for $F \in \mathscr{L}$ if and only if x is a periodic point of period q for $\tilde{F} \in \tilde{\mathscr{L}}$ and p is either zero if $\tilde{F}$ is continuous, or $\operatorname{Card}\{n : 1 \leqslant n \leqslant q \text{ and } \tilde{F}^n(x) \geqslant c\}$.

To study the class $\mathscr{L}$ we mainly use the techniques of circle maps of degree one, see for instance [M3] and [GT] (*). However, in some of the proofs we use simultaneously maps of $\mathscr{L}$ and $\tilde{\mathscr{L}}$.

Old maps occur in many branches of dynamics. The simplest ones, $x \to \beta x + \alpha$, furnish an interesting two-parameter family which has been the object of numerous studies in ergodic theory (for a recent contribution, see e.g. [H1]). Notice that the case $\alpha = 0$ gives the famous β-transformations for which an early reference is [R].

In [P], Parry noticed the relation between these classical results and a more recent occurence of old maps in dynamics, namely as reduced Poincaré maps for some flows in $\mathbf{R}^3$. This ocurrence of old maps was first noticed by Lorenz in his seminal paper [L] (in fact Lorenz obtained equivalent unimodal maps). This was formalized by Guckenheimer [G1] when he introduced the geometric Lorenz flows. The most studied Lorenz maps are old maps with slope greather than one. In the study of other flows, the Guckenheimer technique of projecting a Poincaré map along a strong stable invariant foliation yields old maps with one critical point, see [GPTT] and references therein. Obviously, all these maps belong to the class considered in the present paper.

Unfortunately we cannot specify the exact value of $F(0)$. In what follows we consider that $F(0)$ is either $F(0+)$ or $F(0-)$, or both, as necessary. Notice that in the study of periodic (mod. 1) orbits of a map $F \in \mathscr{L}$, because of this ambiguity we shall not be able to control if the zero is a periodic (mod. 1) point and, in this case, its period and rotation number.

Our first result will characterize the set of periods of $f \in \mathscr{L}$. To state it we shall introduce some notation. Let a, $b \in \mathbf{R}$ with $a < b$. Then $M(a,b)$ will denote the set $\{q : \text{there exist } p \text{ such that } a < p/q < b\}$. For $k \in \mathbf{N}$, $k > 1$ we define $K(k) = \{\ell \in \mathbf{N} : \ell \geqslant k\} \cup \{1\}$ and $K(\infty) = \{1\}$.

(*) In [GT] the renormalization scheme does not apply to the general case.

We set $\mathbf{N}^* = (\mathbf{N}\backslash\{1\}) \cup \{\infty\}$. Let $a \in \mathbf{R}$ and $k \in \mathbf{N}^*$ we define the set

$$S(a,k) = \begin{cases} \varnothing & \text{if } \; a \notin \mathbf{Q}, \\ qK(k) & \text{if } \; a = p/q \text{ with } (p,q) = 1. \end{cases}$$

Assume that $a, b \in \mathbf{R}$ with $a < b$ and that $n, m \in \mathbf{N}^*$. In what follows $S(a,n) \cup M(a,b) \cup S(b,m)$ will be denoted by $B(a,b,n,m)$.

THEOREM A. – (a) *Let $F \in \mathscr{L}$. Then* Per(F) *is of one of the following forms :*

either B (where B is either the empty set or a set of the form $B(a,b,n,m)$) ;

or $\{1\}$;

or $\{q_1, q_2, \ldots, q_r\} \cup q_r B$ *where* $r \geq 1$, $q_1 \geq 2$, $q_i < q_{i+1}$ *and* q_i *divides* q_{i+1} *for* $i = 1, 2, \ldots, r-1$;

or $\{q_1, q_2, q_3, \ldots\}$ *where* $q_1 \geq 2$, $q_i < q_{i+1}$ *and* q_i *divides* q_{i+1} *for* $i = 1, 2, \ldots$.

(b) *Let A be one of the sets given in statement* (a). *Then there exists a map $F \in \mathscr{L}$ such that* Per$(F) = A$.

Hofbauer in [H2], by using an oriented graph with infinitely many vertices whose closed paths represent the periodic orbits of the map, obtained a result similar to Theorem A except that he did not characterize completely the set $q_r B$. Note that Theorem A gives the full characterization of the set of periods of the maps $F \in \mathscr{L}$. Moreover, our proof of Theorem A is more geometric. It is based on the use of the rotation interval (for more details see Theorem 4, Proposition 5, Remark 2 and Lemmas 6, 8 and 9), which seems to be simpler than the Hofbauer techniques.

For an old non-decreasing map F, Rhodes and Thompson [RT] showed that $a(F) = b(F)$ and the existence of periodic points with rotation number $a(F)$ when $a(F) \in \mathbf{Q}$ (at this moment we assume that at points of discontinuity f attains both one-sided limits). One can notice that such maps can be suspended to get flows on holed 2-tori and thus are essentially cherry maps (i.e. first return maps on cherry flows). Other maps in $\mathscr{L}$ can be suspended to get semiflows on branched 2-manifolds.

Next we characterize the structure of the set $B(a,b,n,m)$.

We need some definitions. Let p/q be such that $(p,q) = 1$ with $q > 1$. For each $n \in \mathbf{Z}$ there is a unique $\alpha(n) \in \{1,2,\ldots,q\}$ such that $\alpha(n) = -np \pmod{q}$. Then we set $\ell(n) = [\alpha(n)+np]/q$.

We define the *right-hand ordering* $<_r$ associated to p/q with $(p,q) = 1$ and $q > 1$ as follows. For every $m, n \in \mathbf{N}$ with $m \neq n$ we say $m <_r n$ if

either $m/\alpha(m) < n/\alpha(n)$, $(<_r.1)$
or $m/\alpha(m) = n/\alpha(n)$, $(\ell(n),n) \neq 1$, $(\ell(m),m) \neq 1$ and $m < n$, $(<_r.2)$
or $m/\alpha(m) = n/\alpha(n)$ and $(\ell(n),n) = 1$. $(<_r.3)$

This ordering is well defined because the equalities $(\ell(n),n) = 1$ and $(\ell(m),m) = 1$ do not hold simultaneously when $m/\alpha(m) = n/\alpha(n)$.

We define the *left-hand ordering* $<_\ell$ associated to p/q with $(p,q) = 1$ and $q > 1$ as the right-hand one with $\alpha(-k)$ and $[pk - \alpha(-k)]/q$ instead of $\alpha(k)$ and $\ell(k)$, respectively.

In section 4 we give some results which allow to construct easily the $<_r$ and $<_\ell$ orderings (see Lemma 13.*a* and Proposition 11).

Let p/q be such that $q = 1$. Then we define the right-hand and left-hand orderings associated to p/q as the usual ordering of the natural numbers.

If $<_r$ and $<_\ell$ are the right-hand and left-hand orderings associated to p/q, we denote by $R(n)$ the set $\{k \in \mathbf{N} : n <_r k\} \cup \{n\}$ and by $L(n)$ the set $\{k \in \mathbf{N} : n <_\ell k\} \cup \{n\}$.

THEOREM B. — *Let $a, b \in \mathbf{R}$ with $a < b$ and $n, m \in \mathbf{N}^*$ and let p/q be such that $a \leqslant p/q \leqslant b$ with $(p,q) = 1$. We denote by $<_r$ and $<_\ell$ the right-hand and left-hand orderings associated to p/q, respectively. If $p/q < b$ (resp. $a < p/q$) then we set $\rho = \min_{<_r} M(p/q,b) \cup S(b,m)$ (resp. $\lambda = \min_{<_\ell} S(a,n) \cup M(a,p/q)$. Then the following hold.*

(a) *If $a < p/q < b$ then $B(a,b,n,m) = R(\rho) \cup L(\lambda) \cup \{q,2q,3q,\dots\}$.*

(b) *If $a = p/q$ then $B(a,b,n,m) = R(\rho) \cup S(a,n)$.*

(c) *If $b = p/q$ then $B(a,b,n,m) = L(\lambda) \cup S(b,m)$.*

For a given two-parameter family $g_{\mu,\upsilon}$ of maps of $\mathscr{L}$ one can ask for the regions of the parameter space (μ,υ) where the set of periods contains a given element. This study can be done easily by using Theorems A and B, their proofs and Theorem B of [M3]. This kind of study has been done in [GPTT] for the family $g_{\mu,\upsilon} : [-1,1] \to [-1,1]$ defined by

$$g_{\mu,\upsilon}(x) = \begin{cases} \upsilon - x^2 & \text{if } x \leqslant 0, \\ -\mu + x^2 & \text{if } x > 0. \end{cases}$$

They approach the problem by making direct computation for low periods.

Remark 1. — Let p/q be such that $a < p/q < b$ with $(p,q) = 1$ and $q > 1$. Since $\{q, 2q, 3q, \ldots\} \subset B(a,b,n,m)$ for all $n, m \in \mathbf{N}^*$, we have that it is enough to define the $<_r$ and $<_\ell$ orderings in $\mathbf{N} \setminus \{q, 2q, 3q, \ldots\}$ to get statement (a) of Theorem B. Hence, it is sufficient to construct the sets $R(n)$ and $L(n)$ as subsets of $\mathbf{N} \setminus \{q, 2q, 3q, \ldots\}$.

Now, we consider the class $\mathscr{C}$ of old maps $F\colon \mathbf{R} \to \mathbf{R}$ such that F is non-decreasing and continuous on the interval $(0,1)$, and $F(1-) > F(0+) + 1$. The maps from class $\mathscr{C}$ will be called *heavy* (see [M3]; following the graph of a heavy map one can fall down but cannot jump up).

Following the ideas of Misiurewicz and Szlenk and also of Milnor and Thurston for continuous maps (see [MS] and [MT]) we define the growth number of a map of $\mathscr{C}$.

Let F be an old map. We say that F is *piecewise-increasing* if there exist $c_0 = 0 < c_1 < \cdots < c_\ell = 1$ such that the restriction of F to each interval (c_i, c_{i+1}) is non-decreasing and continuous. These intervals will be called *laps* of F if, in addition, $E \circ F|_{(c_i, c_{i+1})}$ is constant and ℓ is the smallest integer satisfying the above conditions. The number $\ell(F) = \ell$ will be called the *lap number* of F.

If $F \in \mathscr{C}$ then there exists the limit of $\ell(F^n)^{1/n}$ as $n \to \infty$. This limit $s(F)$ will be called the *growth number* of F.

By analogy with the continuous maps of the interval we define the *topological entropy* $h(F)$ of $F \in \mathscr{C}$ as the logarithm of its growth number.

Let $a < b$. The equation $\sum_{a<p/q<b} s^{-q} = 1$ has a unique root $\beta_{a,b}$ larger than 1 (see Section 5). From this equality it follows easily that $\beta_{a,b}$ has the following properties:

(1) *If* $c < a < b \leqslant d$ *or* $c \leqslant a < b < d$ *then* $\beta_{c,d} > \beta_{a,b}$.

(2) $\lim_{a \downarrow c, b \uparrow d} \beta_{a,b} = \beta_{c,d}$.

(3) $\beta_{\cdot,d}$ *is continuous at* c *if and only if* c *is irrational.*

(4) $\beta_{c,\cdot}$ *is continuous at* d *if and only if* d *is irrational.*

(5) $\beta_{\cdot,\cdot}$ *is continuous at* (c,d) *if and only if* c *and* d *are irrational.*

From Lemma 14 and Proposition A of [ALMM] it follows that for $a < b$, $\beta_{a,b}$ is the unique root (larger than 1) of

$$z + 1 + (z-1)^{-1} - T_{b-E(a)}(z) - T_{1-a+E(a)}(z) = 0,$$

where $T_c(z) = \sum_{n=0}^{\infty} z^{-E(n/c)}$ if $c > 0$ and $z > 1$, and $T_0(z) \equiv 0$.

The last equality allows us to compute $\beta_{a,b}$ easier than the previous one.

The next result gives the best lower bound of the topological entropy of a map $F \in \mathscr{C}$ depending on its rotation interval.

THEOREM C. — (a) *For* $F \in \mathscr{C}$ *with* $a(F) < b(F)$ *we have* $h(F) \geqslant \log \beta_{a(F),b(F)}$.

(b) *Let* $a, b \in \mathbf{R}$ *such that* $a < b$. *Then there exists* $F \in \mathscr{C}$ *such that* $a(F) = a$, $b(F) = b$ *and* $h(F) = \log \beta_{a,b}$.

Note that $\mathscr{L} \not\subset \mathscr{C}$.

COROLLARY D. — *Theorem C holds with* $\mathscr{L}$ *instead of* $\mathscr{C}$ *and with* $a \geqslant 0$, $b \leqslant 1$.

2. The possible sets of periods (proof of Theorem A. (a)).

We recall that the maps from class $\mathscr{C}$ are called heavy. An old map $F : \mathbf{R} \to \mathbf{R}$ such that F is non-decreasing and continuous on the interval $(0,1)$, and $F(1-) \leqslant F(0+) + 1$ will be called *light*. Note that with this definition a continuous map $F \in \mathscr{L}$ is light (i.e. for technical reasons we do not follow the convention in [M3] making it heavy). Also we note that in general an iterate of a heavy map need not be heavy but this problem does not occur in $\mathscr{L}$.

For a heavy map F we define maps F_ℓ and F_r by

$$F_\ell(x) = \inf \{F(y) : y \geqslant x\},$$
$$F_r(x) = \sup \{F(y) : y \leqslant x\},$$

see [M3], [ALMS] and [CGT].

Let F be an old map, and let $x \in \mathbf{R}$. Then, the set $\{y \in \mathbf{R} : y = F^n(x)$ (mod. 1) for $n = 1,2,\ldots\}$ will be called the (*mod.* 1) *orbit* of x by F. If

x is a periodic (mod. 1) point of F of period q with rotation number p/q, then its (mod. 1) orbit is called *a periodic* (*mod.* 1) *orbit of F of period q with rotation number p/q*.

Let P be a (mod. 1) orbit of an old map F. We say that P is a *twist orbit* if F restricted to P is increasing. If a periodic (mod. 1) orbit is twist then we say that P is a *twist periodic orbit* (from now on *TPO*). Assume that $P = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$ is a *TPO* and that $(x_i, x_{i+1}) \cap P = \varnothing$ for all $i \in \mathbf{Z}$. The fact that $F(x_i) = x_{i+p}$ for all $i \in \mathbf{Z}$, gives a geometrical interpretation of a *TPO* (for more details see Lemma 1 of [ALMS], see also [CGT] and [M2]).

LEMMA 1. — *Assume that $F \in \mathscr{L}$ is heavy. Then the following hold.*

(a) *F_r is an old continuous non-decreasing map.*

(b) *If p/q with $(p,q) = 1$ is the right endpoint of the rotation interval of F, then all periodic* (*mod.* 1) *orbits of F_r have period q, rotation number p/q and are TPO.*

(c) *Under the assumptions of* (b) *at least one of the TPO of F_r of period q and rotation number p/q is a TPO of F with the same period and rotation number.*

Proof. — Statement (*a*) follows from Lemmas 2.2 and 4.1 of [M3].

Now we prove (b) *and* (*c*). By Lemma 3.2 and Corollary 5.2 of [M3] we obtain that p/q is the unique rotation number of F_r. The continuous map $F_r^q - p$ is non-decreasing and therefore all its periodic points are fixed points. Consequently they have period q for F_r. Since F_r is non-decreasing all periodic (mod. 1) orbits of F_r are *TPO*. Hence, (b) is proved.

From the proof of Theorem 2 of [CGT] (or Lemma 3.4 of [M3], Proposition 2.1 and Lemma 2.2 of [M2]), it follows that at least one of the *TPO* of F_r of period q with rotation number p/q does not belong to the open intervals where F_r is locally constant. Therefore, this periodic (mod. 1) orbit is also a *TPO* of F with the same period and rotation number. This proves (c). □

LEMMA 2. — *Assume that $F \in \mathscr{L}$ is heavy and that $F(0+) = 0$. Then* $\operatorname{Per}(F) = K(k)$ *for some $k \in \mathbf{N}^*$.*

Proof. — For each $i \in \mathbf{Z}$, $x \geqslant i$ implies $F(x) \geqslant i$ and $F|_{[i,i+1)}$ is non-decreasing. Therefore all periodic points with rotation number equal to zero are fixed points. Consequently, if F has a periodic (mod. 1) orbit of period $j > 1$ then its rotation interval has the form $[0,d]$ where $d > 0$ and $1/j \in [0,d]$. Hence $1/m \in [0,d]$ for all $m > j$. Therefore, by Theorem A of [M3] the lemma follows. □

PROPOSITION 3. – *Assume that $F \in \mathcal{L}$ is heavy, the rotation interval of F is $[a,b]$, and $p/q \in \{a,b\}$ with $(p,q) = 1$. Then the following hold.*

(a) *There exists a map $G \in \mathcal{L}$ such that* $\mathrm{Per}_{p/q}(F) = \{q\} \cup q\,\mathrm{Per}(G)$.

(b) *If $a < b$ then* $\mathrm{Per}_{p/q}(F) = qK(k)$ *for some* $k \in \mathbf{N}^*$.

Proof. – We may assume that $b = p/q$. If $a = p/q$ the proof is similar.

From Lemma 1.(c) it follows that F has a *TPO* P of period q and rotation number p/q which is also an orbit (mod. 1) of F_r. Let $P = \{x_i\}_{i \in \mathbf{Z}}$, $x_i < x_{i+1}$ and $x_{-1} < 0 \leqslant x_0$. We have $F(x_i) = x_{i+p}$ and $x_{i+q} = x_i + 1$. Since $F \leqslant F_r$ and F_r is a continuous non-decreasing map (see Lemma 1.(a)), we have that $x \leqslant x_i$ implies $F(x) \leqslant F(x_i) = x_{i+p}$.

Set $I = [x_{-1}, x_0]$ and define $H : I \to I$ by

$$H(x) = \begin{cases} F^q(x) - p & \text{if } F^q(x) - p \in I, \\ x_{-1} & \text{otherwise.} \end{cases}$$

Notice that $H(x_{-1}) = x_{-1}$ and $H(x_0) = x_0$.

Let y_0 be a periodic (mod. 1) point of F of period m and rotation number p/q. Set $y_i = F^i(y_0)$ for $i = 1, 2, \ldots, m$. Note that $y_i \neq y_0 + j$ for $i = 1, 2, \ldots, m-1$ and $j \in \mathbf{Z}$ and that $y_m = y_0 + k$ for some $k \in \mathbf{Z}$. Then $m = sq$ and $k = sp$ for some integer $s > 0$. Let j be such that $y_0 \in (x_{j-1}, x_j]$. We claim that we have $y_1 \in (x_{j+p-1}, x_{j+p}], \ldots, y_m \in (x_{j+mp-1}, x_{j+mp}]$. Now we prove the claim. We have $y_0 \leqslant x_j$ and therefore $y_1 \leqslant x_{j+p}, \ldots, y_m \leqslant x_{j+mp}$. If $y_i \leqslant x_{j+ip-1}$ for some $i \leqslant m$ then also $y_m \leqslant x_{j+mp-1}$, which is impossible since $y_m = y_1 + sp > x_{j-1} + sp = x_{j-1+spq} = x_{j-1+mp}$. This completes the proof of our claim.

Among the intervals $(x_{j-1}, x_j], \ldots, (x_{j+(m-1)p-1}, x_{j+(m-1)p}]$ there are exactly s which are of the form $(x_{-1}, x_0] + i$ for some $i \in \mathbf{Z}$. Therefore to the orbit y_0 for F there corresponds a periodic orbit of H of period $s = m/q$. On the other hand, by similar arguments, if z is a periodic point of H of period s then the F-orbit (mod. 1) of z has period sq and rotation number p/q. Consequently, $\mathrm{Per}_{p/q}(F) = q\,\mathrm{Per}(H)$.

Again by similar arguments we see that if $H(x) \neq x_{-1}$ then $F^i(x) \in (x_{ip-1}, x_{ip}]$ for $i = 0, 1, \dots, q$. Since F is continuous and non-decreasing at each $(x_{ip-1}, x_{ip}]$ for $i = 1, 2, \dots, q-1$, the only discontinuity of H can occur at 0, $H(0+) \leqslant H(0-)$ and H is non-decreasing on $[x_{-1}, 0]$ and $[0, x_0]$.

If $x_0 = 0$ then H is continuous and non-decreasing, and thus $\operatorname{Per}(H) = \{1\}$. Then (a) holds with G equal to the identity function and (b) holds with $k = \infty$.

Assume for the rest of the proof that $0 \in (x_{-1}, x_0)$.

If $H(0+) \geqslant 0$ or $H(0-) \leqslant 0$ then clearly $\operatorname{Per}(H) = \{1\}$. In this case (a) and (b) follow as above.

Assume now that $H(0+) < 0 < H(0-)$. Denote $H(0+) = v$, $H(0-) = w$ and $J = [v, w]$. Define $G_1 : J \to J$ as follows:

$$G_1(x) = \begin{array}{lll} H(x) & \text{if} & H(x) \in J, \\ v & \text{if} & H(x) < v, \\ w & \text{if} & H(x) > w. \end{array}$$

We claim that $\operatorname{Per}(H) = \{1\} \cup \operatorname{Per}(G_1)$. If $G(w) = w$ then $x > w$ implies $H(x) \geqslant w$. If $G(w) < w$ then $x < w$ implies $H(x) \leqslant w$. In both cases there are no periodic orbits of H having points in both sides of w. Since H is non-decreasing in $[w, x_0]$, there are no periodic orbits of H, except of period 1, staying to the right of w. Therefore, all periodic orbits of H with period larger than 1 have to be contained in $[x_{-1}, w]$. Analogously, these orbits have to be contained in $[v, x_0]$. Therefore, $\operatorname{Per}(H) \subset \{1\} \cup \operatorname{Per}(G_1)$.

Clearly, $\{1\} \subset \operatorname{Per}(H)$. Let Q be a periodic orbit of G_1 of period larger than 1. Since all the points x of J for which $G_1(x) \neq H(x)$ are mapped to a fixed point of G_1, we have $G_1(y) = H(y)$ for all $y \in Q$. Hence $\operatorname{Per}(G_1) \subset \operatorname{Per}(H)$. This completes the proof of the claim.

Now we renormalize G_1. We define $G_2 : [0,1] \to [0,1]$ by $L^{-1} \circ G_1 \circ L$ where L is the affine transformation $L(x) = (w-v)x + v$. Notice that if $G_1 = (F^q - p)|_J$ then this reduces to the usual renormalization: $G_2 = L^{-1} \circ (F^q - p) \circ L$. We get G from G_2 in the usual way:

$$G(x) = \begin{cases} G_2(x) & \text{if } 0 \leqslant x \leqslant L^{-1}(0), \\ G_2(x) + 1 & \text{if } L^{-1}(0) \leqslant x \leqslant 1, \end{cases}$$

and $G(x) = G(x-\ell) + \ell$ if $x \in [\ell, \ell+1]$ with $\ell \in \mathbf{Z}$. By the construction and in view of the properties of H, $G \in \mathscr{L}$. Clearly $\operatorname{Per}(G_1) = \operatorname{Per}(G)$. This completes the proof of (a).

Assume that $a < b$. Since there are points with rotation numbers smaller than p/q, $F^q(0+) - p < x_{-1}$. Therefore $H(0+) = x_{-1}$ and hence $v = x_{-1}$. Since $H(x_{-1}) = x_{-1}$, we have $G_1(v) = v$ and thus $G(0+) = 0$. Now (b) follows from (a) and Lemma 2. □

Note that if in Proposition 3 we assume that $a = b$ then G can be thought of as a renormalized map F. However, the reader should be aware that if $G_1 \neq (F^q - p)|_J$ (this can happen easily) then, strictly speaking, this is not a renormalization. One may wish to compare this situation with the case of a unimodal map of an interval. There it happens that there is an interval J with $f^q(J) \subset J$ and one can reduce the study of f to study the unimodal map $f^q|_J$ and a relatively simple behaviour of f outside $\bigcup_{i=0}^{q-1} f^i(J)$. Here we reduce the study of F to the study of $G \in \mathscr{L}$ and a very simple behaviour of F outside $\bigcup_{i=0}^{q-1} f^i(J) + \mathbf{Z}$. However, the connection between G and F is not so close.

For a heavy map $F \in \mathscr{L}$ such that $a(F) < b(F)$ we take $n_a \in \mathbf{N}^*$ arbitrarily if $a \notin \mathbf{Q}$; otherwise n_a is defined in such a way that $\operatorname{Per}_{p/q}(F) = qK(n_a)$ if $a = p/q$ with $(p,q) = 1$. When a is rational, by Proposition 3.(*b*), n_a is well defined. Similarly we define n_b by putting b instead of a.

Statement (a) of Theorem A follows from the next theorem.

THEOREM 4. – *Let $F \in \mathscr{L}$ with rotation interval $[a,b]$. If F is light then $a = b$ and* $\operatorname{Per}(F)$ *is either empty if $a \notin \mathbf{Q}$, or $\{q\}$ if $a = p/q$ with $(p,q) = 1$. If F is heavy then the following hold.*

(a) *If $a < b$ then* $\operatorname{Per}(F) = B(a, b, n_a, n_b)$.

(b) *If $a = b \notin \mathbf{Q}$ then* $\operatorname{Per}(F) = \varnothing$.

(c) *If $a = b \in \mathbf{Z}$ then* $\operatorname{Per}(F) = \{1\}$.

(d) *If $a = b = p/q \notin \mathbf{Z}$ where $(p,q) = 1$, then $q \geqslant 2$ and* $\operatorname{Per}(F)$ *is either $\{q_1, q_2 \ldots, q_r\} \cup q_r B$ where $r \geqslant 1$, $q = q_1$, $q_i < q_{i+1}$ and q_i divides q_{i+1} for $i = 1, 2, \ldots, r-1$; or $\{q_1, q_2, q_3, \ldots\}$ with $q = q_1$, $q_i < q_{i+1}$ and q_i divides q_{i+1} for $i = 1, 2, 3, \ldots$; B is the empty set or a set of the form $B(c,d,m,n)$; c, $d \in \mathbf{R}$ with $c < d$ and n, $m \in \mathbf{N}^*$.*

Proof. – Let x, $y \in \mathbf{R}$ with $x < y < x + 1$. If F is light, it is non-decreasing on $\mathbf{R}$ and $F^n(x) \leqslant F^n(y) \leqslant F^n(x) + 1$ for all $n \geqslant 1$. Therefore, $\rho(x) \leqslant \rho(y) \leqslant \rho(x+1) = \rho(x)$. Hence $a = b = \rho(x)$ for all $x \in \mathbf{R}$ (see [M3] and [RT]). Clearly, $\operatorname{Per}(F) = \varnothing$ if $a \notin \mathbf{Q}$. By Theorem 2 of [RT], if $a = p/q$ with $(p,q) = 1$ then there exists $x \in \mathbf{R}$ such that $F^q(x) = x + p$ (recall that we use one-sided limits, if necessary). Hence $\{q\} \subset \operatorname{Per}(F)$.

All periodic (mod. 1) points of F have rotation number p/q and therefore they are periodic points of $F^q - p$. Since $F^q - p$ is non-decreasing, it has no periodic points of period different than 1. Hence $\operatorname{Per}(F) = \{q\}$.

Assume now that F is heavy. From Proposition 3.(b) and Theorem A of [M3] it follows (a). Statement (b) is obvious. Now we assume that $a = b$. If $a = b \in \mathbf{Z}$, since $[a,b] \subset [0,1]$ we have either $a = 0$ or $a = 1$. Hence, all periodic (mod. 1) points of F have rotation number $p \in \{0,1\}$. We define the map $G \in \mathscr{L}$ as follows. If $x \in [0,1]$ then

$$G(x) = \begin{cases} F(x) - p & \text{if } \ F(x) - p \in [0,1], \\ 1 & \text{if } \ F(x) - p > 1, \\ 0 & \text{if } \ F(x) - p < 0; \end{cases}$$

and $G(x) = G(x-\ell) + \ell$ if $x \in [\ell, \ell+1]$ with $\ell \in \mathbf{Z}$. Note that G is light, $a(G) = b(G) = 0$ and $G(y) = y$ for the points y such that $F(y) = y + p$ (note that there exists at least one of these points). From the part already proved it follows that $\operatorname{Per}(G) = \{1\}$. Since $a(F) = b(F) = p$, there are no periodic (mod. 1) points x of F such that $G(x) \neq F(x) - p$. Hence, $\operatorname{Per}(F) = \operatorname{Per}(G)$. Therefore, $\operatorname{Per}(F) = \{1\}$. Hence, statement (c) follows.

At last we assume that $a = b = p/q \notin \mathbf{Z}$. Then clearly $q \geqslant 2$. We set $F_1 = F$. From Proposition 3.(a) there exists $F_2 \in \mathscr{L}$ such that $\operatorname{Per}(F_1) = \{q\} \cup q \operatorname{Per}(F_2)$. We consider five cases.

Case 1. F_2 is light. Then, as we already know, $\operatorname{Per}(F_2)$ is either empty or $\{m\}$ for some $m \in \mathbf{N}$. So $\operatorname{Per}(F_1)$ is either $\{q_1\}$ or $\{q_1, q_2\}$ with $q_1 = q$ and $q_2 = qm$ (if $m = 1$ then $\operatorname{Per}(F_1) = \{q_1\}$).

Case 2. F_2 is heavy and $a(F_2) < b(F_2)$. Then, from (a) it follows that $\operatorname{Per}(F_1) = \{q\} \cup qB(a(F_2), b(F_2), n_{a(F_2)}, n_{b(F_2)})$.

Case 3. F_2 is heavy and $a(F_2) = b(F_2) \notin \mathbf{Q}$. Then, from (b) it follows that $\operatorname{Per}(F_1) = \{q\}$.

Case 4. F_2 is heavy and $a(F_2) = b(F_2) \in \mathbf{Z}$. Then from (c) we have that $\mathrm{Per}(F_1) = \{q\}$.

Case 5. F_2 is heavy and $a(F_2) = b(F_2) = k/m \notin \mathbf{Z}$ with $(k,m) = 1$. By Proposition 3.(a) there exists $F_3 \in \mathscr{L}$ such that

$$\mathrm{Per}(F_1) = \{q, qm\} \cup qm\,\mathrm{Per}(F_3).$$

By iterating this process statement (d) follows. □

3. The effective set of periods (proof of Theorem A. (b)).

PROPOSITION 5. – *Let a, $b \in [0,1]$ with $a < b$ and n, $m \in \mathbf{N}^*$. Then there exists a heavy map $F \in \mathscr{L}$ such that $a(F) = a$, $b(F) = b$ and* $\mathrm{Per}(F) = B(a,b,n,m)$.

Proof. – We define a heavy map $G \in \mathscr{L}$ by $G(x) = 2x - \ell$ if $x \in [\ell, \ell+1]$ with $\ell \in \mathbf{Z}$. Note that $a(G) = 0$ and $b(G) = 1$.

Now we make a construction similar to the one described by Guckenheimer for maps of an inverval in [G2].

For $\mu \in [0,1]$ we define

$$G_\mu(x) = \begin{cases} \mu + \ell & \text{if } \; x \in [\ell, \ell + \mu/2], \\ G(x) & \text{if } \; x \in [\ell + \mu/2, \ell + (\mu+1)/2], \\ \mu + \ell + 1 & \text{if } \; x \in [\ell + (\mu+1)/2, \ell+1], \end{cases}$$

with $\ell \in \mathbf{Z}$. Note that $G_\mu \in \mathscr{L}$ and it is continuous.

From the proof of Theorem 1 and 2 of [CGT] (see also Lemma 3.4 of [M3]), there exists μ_a and μ_b such that $\mu_a < \mu_b$, $a(G_{\mu_a}) = b(G_{\mu_a}) = a$, $a(G_{\mu_b}) = b(G_{\mu_b}) = b$, and if a (resp. b) is rational then G_{μ_a} (resp. G_{μ_b}) has a twist orbit T_a (resp. T_b) with rotation number a (resp. b) which is not contained in the open intervals where G_{μ_a} (resp. G_{μ_b}) is locally constant. We note that $T_a \cap [\ell, \ell+1]$ is contained in $[\ell + \mu_a/2, \ell + (\mu_a+1)/2]$ for all $\ell \in \mathbf{Z}$, and that $G_{\mu_a}|_{T_a} = G|_{T_a}$ and analogously for b instead of a.

Suppose that b is rational. Since $G_{\mu_b}(\mu_b/2) = G_{\mu_b}((\mu_b+1)/2) - 1$, it cannot happen that both $\mu_b/2$ and $(\mu_b+1)/2$ belong to T_b. Hence if $v_b = \min(T_b \cap [0,1])$ and $w_b = \max(T_b \cap [0,1])$ then $w_b - v_b < 1/2$. From this it follows that if $\mu \in I_b$ where $I_b = [2w_b - 1, 2v_b]$, then T_b is also a *TPO* of G_μ and consequently, $\rho(G_\mu) = b$. Notice, that $2v_b > 2w_b - 1$.

Analogously if a is rational we obtain an interval $I_a = [2w_a - 1, 2v_a]$ such that for each λ from it T_a is a *TPO* of G_λ and $\rho(G_\lambda) = a$. If a (resp. b) is irrational, we set $I_a = \{\mu_a\}$ (resp. $I_b = \{\mu_b\}$).

For $0 \leqslant \lambda < \mu \leqslant 1$ we define a heavy map $F_{\lambda\mu} \in \mathscr{L}$:

$$F_{\lambda,\mu}(x) = \begin{cases} \lambda + \ell & \text{if } \; x \in [\ell, \ell + \lambda/2], \\ G(x) & \text{if } \; x \in [\ell + \lambda/2, \ell + (\mu+1)/2], \\ \mu + \ell + 1 & \text{if } \; x \in [\ell + (\mu+1)/2, \ell + 1], \end{cases}$$

with $\ell \in \mathbf{Z}$. Notice that $(F_{\lambda,\mu})_\ell = G_\lambda$, $(F_{\lambda,\mu})_r = G_\mu$.

From Corollary 5.2 of [M3] we have that if $\lambda \in I_a$ and $\mu \in I_b$ then $a(F_{\lambda,\mu}) = a$ and $b(F_{\lambda,\mu}) = b$. Then from Theorem A of [M3] it follows that $\operatorname{Per}(F_{\lambda,\mu}) = M(a,b) \cup \operatorname{Per}_a(F_{\lambda,\mu}) \cup \operatorname{Per}_b(F_{\lambda,\mu})$. To end the proof we have to find λ, μ such that $\operatorname{Per}_a(F_{\lambda,\mu}) = S(a,n)$ and $\operatorname{Per}_b(F_{\lambda,\mu}) = S(b,m)$. We shall see that $\operatorname{Per}_b(F_{\lambda,\mu})$ depends only on the choice of $\mu \in I_b$ and analogously, $\operatorname{Per}_a(F_{\lambda,\mu})$ depends only on the choice of $\lambda \in I_a$.

We shall try to find a suitable μ; the proof for λ is analogous. If $b \notin \mathbf{Q}$ then there is nothing to prove. Assume that $b = p/q$, $(p,q) = 1$. We take H and $I = [x_{-1}, x_0]$ from the proof of Proposition 3 where $P = T_b$ and $F = F_{\lambda,\mu}$. Since $a < b$, we have, as at the end of the proof of Proposition 3, $H(0+) = x_{-1}$. Notice that $x_{-1} = w_b - 1$ and $x_0 = v_b$. Outside the intervals of the form $I + \ell$, $\ell \in \mathbf{Z}$, the map $F_{\lambda,\mu}$

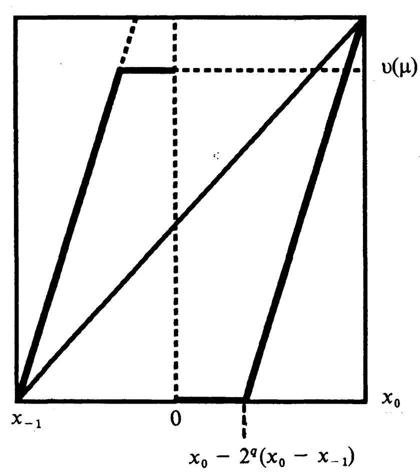

Figure 1.

has a constant slope 2. Therefore the map H has the following form (see Figure 1):

$$H(x) = \begin{cases} 2^q(x-x_{-1}) + x_{-1} & \text{if} \quad x_{-1} \leqslant 2^q(x-x_{-1}) \leqslant \upsilon(\mu), \\ \upsilon(\mu) & \text{if} \quad 2^q(x-x_{-1}) \geqslant \upsilon(\mu),\, x < 0, \\ x_{-1} & \text{if} \quad 0 < x \leqslant x_0 - 2^q(x_0-x_{-1}), \\ x_0 - 2^q(x_0-x) & \text{if} \quad x \geqslant x_0 - 2^q(x_0-x_{-1}), \end{cases}$$

where $x_{-1} \leqslant \upsilon(\mu) \leqslant x_0$ and $\upsilon(\mu)$ depends linearily on μ; when μ varies from $2w_b - 1$ to $2v_b$ then $\upsilon(\mu)$ varies from x_{-1} to x_0.

Clearly H depends only on μ. If $\upsilon(\mu) \leqslant 0$ then we have $\mathrm{Per}(H) = \{1\}$ and consequently $\mathrm{Per}_b(F_{\lambda,\mu}) = \{q\}$. When we go on with the construction from the proof of Proposition 3 for $\upsilon(\mu) \geqslant 0$, we obtain a map G, which depends on μ in a continuous way. By Proposition 3, we have $\mathrm{Per}_b(F_{\lambda,\mu}) = q\,\mathrm{Per}(G) \cup \{q\}$. By [M3], the rotation interval of G varies continuously with μ. It is of the form $[0,d]$, and if $\upsilon(\mu) = 0$ then $d = 0$; if $\upsilon(\mu) = x_0$ then $d = 1$. As in the proof of Lemma 2, we obtain $\mathrm{Per}_b(F_{\lambda,\mu}) = qK(m)$ for some $\mu \in I_b$ (when $d \in [1/m, 1/(m-1))$). □

Remark 2. – If $a \in (0,1)\backslash \mathbf{Q}$ then the map $F(x) = x + a$ belongs to $\mathscr{L}$ and $\mathrm{Per}(F) = \varnothing$.

LEMMA 6. – *Let $a = b \in \{0,1\}$. Then there exists a heavy map $F \in \mathscr{L}$ such that $a(F) = b(F) = a$ and* $\mathrm{Per}(F) = \{1\}$.

Proof. – We define F as follows

$$F(x) = \begin{cases} (x-\ell)/2 + \ell + 1/8 & \text{if} \quad x \in [\ell, \ell+1/2], \\ 7(x-\ell)/4\ell - 1/2 & \text{if} \quad x \in [\ell+1/2, \ell+1), \end{cases}$$

for all $\ell \in \mathbf{Z}$. Since

$$F(\ell+1/4) = \ell + 1/4, \qquad F(\ell+2/3) = \ell + 2/3,$$
$$F([\ell+1/4, \ell+2/3]) = [\ell+1/4, \ell+2/3]$$

and

$$F([\ell+2/3, \ell+5/4]) \subset [\ell+2/3, \ell+5/4],$$

we have that $\mathrm{Per}(F) = \{1\}$ and $L(F) = \{0\}$. The map $G(x) = 2 - F(1-x)$ for all $x \in \mathbf{R}$ satisfies $\mathrm{Per}(G) = \{1\}$ and $L(G) = \{1\}$. □

LEMMA 7. – *Let $G \in \mathscr{L}$ and $p/q \notin Z$ with $(p,q) = 1$. Then there exists a heavy map $F \in \mathscr{L}$ such that $a(F) = b(F) = p/q$ and* $\mathrm{Per}(F) = \{q\} \cup q\,\mathrm{Per}(G)$.

Proof. – Set $\tilde{a} = \inf\{x \in [0,1] : G(x)=1\}$. Since $G \in \mathscr{L}$, we have $0 < \tilde{a} \leqslant 1$. If $\tilde{a} < 1$ then set $a = \tilde{a}$. If $\tilde{a} = 1$ then choose $a \in (0,1)$ arbitrarily. We define an old map $\tilde{G}: \mathbf{R} \to \mathbf{R}$ as follows:

$$\tilde{G}(x) = \begin{cases} G(x) & \text{if } x \in [\ell, \ell+\tilde{a}], \\ G(x) - 1 & \text{if } x \in [\ell+\tilde{a}, \ell+1), \end{cases}$$

for all $\ell \in \mathbf{Z}$. Note that $\operatorname{Per}(G) = \operatorname{Per}(\tilde{G})$.

We shall use the homeomorphism $\tau : \mathbf{R} \to \mathbf{R}$ defined by $\tau(x) = 2qx + a$ for all $x \in \mathbf{R}$. Note that $\tau([\ell - a/(2q), \ell + (1-a)/(2q)]) = [2q\ell, 2q\ell+1]$. We define the map F as follows:

$$F(x) = \begin{cases} x + p/q & \text{if } x \in [\ell + 1/(2q), \ell + 1 - 1/(2q)], \\ (\tau^{-1} \circ \tilde{G} \circ \tau)(x) + p/q & \text{if } x \in [\ell - a/(2q), \ell + (1-a)/(2q)], \end{cases}$$

for all $\ell \in \mathbf{Z}$. Furthermore, we define F to be affine on the intervals $[\ell - 1/(2q), \ell - a/(2q),]$ and $[\ell + (1-a)/(2q), \ell + 1/(2q)]$ for all $\ell \in \mathbf{Z}$ (F has been already defined at their endpoints; it is continuous at them). We remark that $F \in \mathscr{L}$ and it is heavy.

Since

$$F\left(\left[\frac{i}{q} - \frac{a}{2q}, \frac{i}{q} + \frac{1-a}{2q}\right]\right) \subset \left[\frac{i+p}{q} - \frac{a}{2q}, \frac{i+p}{q} + \frac{1-a}{2q}\right]$$

for all $i \in \mathbf{Z}$, we have that

$$(F^q - p)\left(\left[\frac{i}{q} - \frac{a}{2q}, \frac{i+1}{q} - \frac{a}{2q}\right]\right) \subset \left[\frac{i}{q} - \frac{a}{2q}, \frac{i+1}{q} + \frac{1-a}{2q}\right]$$

for all $i \in \mathbf{Z}$. Therefore,

$$(F^q - p)\left(\left[\frac{i}{q} - \frac{a}{2q}, \frac{i+1}{q} + \frac{1-a}{2q}\right]\right) \subset \left[\frac{i}{q} - \frac{a}{2q}, \frac{i+1}{q} + \frac{1-a}{2q}\right]$$

for all $i \in \mathbf{Z}$. Hence, since F is old it follows that $a(F^q - p) = b(F^q - p) = 0$, and consequently $a(F) = b(F) = p/q$.

We note that for all $i \in \mathbf{Z}$ there is a subinterval K_i of $[i/q + (1-a)/(2q), (i+1)/q - a/(2q)]$ such that $F^q|_{K_i}$ is linear and increasing and $(F^q - p)(K_i) = [i/q + (1-a)/(2q), (i+1)/q - a/(2q)]$. Hence, there is a periodic (mod. 1) orbit of F of period q with rotation number p/q contained in $\bigcup_{i \in \mathbf{Z}} K_i$. Furthermore it is easy to see that every periodic (mod. 1) orbit of F contained in $\bigcup_{i \in \mathbf{Z}} [i/q + (1-a)/(2q), (i+1)/q - a/(2q)]$ has period q.

Since $\tilde{G}$ is old we have $F^q(x) = (\tau^{-1} \circ \tilde{G} \circ \tau)(x) + p$ for all $x \in [i/q - a/(2q), i/q + (1-a)/(2q)]$ and $i \in \mathbf{Z}$. Therefore, for such an x we have $F^{kq}(x) = (\tau^{-1} \circ \tilde{G}^{-k} \circ \tau)(x) + kp$ for all $k \geqslant 0$. Hence, $x \in \bigcup_{i \in \mathbf{Z}} [i/q - a/(2q), i/q + (1-a)/(2q)]$ is a periodic (mod. 1) point of F of period kq if and only if $\tau(x)$ is a periodic point of $\tilde{G}$ of period k. This completes the proof. □

LEMMA 8. – *Let $A = \{q_1, q_2, \ldots, q_r\} \cup q_r B$ where $r \geqslant 1$, $q = q_1$, $q_i < q_{i+1}$ and q_i divides q_{i+1} for $i = 1, 2, \ldots, r-1$; $q \geqslant 2$ and B is the empty set or a set of the form $B(c,d,m,n)$ where c, $d \in \boldsymbol{R}$ with $c < d$ and n, $m \in \mathbf{N}^*$. Then, for every k such that $(k,q) = 1$ there exists a heavy map $F \in \mathscr{L}$ such that $a(F) = b(F) = k/q$ and* $\operatorname{Per}(F) = A$.

Proof. – By Proposition 5 and Remark 2 there exists $G \in \mathscr{L}$ such that $\operatorname{Per}(G) = B$. We get the required F by using Lemma 7 r times with q_r/q_{r-1}, $q_{r-1}/q_{r-2}, \ldots, q_2/q_1$, q_1 as q and $1, 1, \ldots, 1, k$ as p respectively. □

LEMMA 9. – *Let $A = \{q_1, q_2, \ldots\}$ where $q_1 \geqslant 2$, $q_i < q_{i+1}$ and q_i divides q_{i+1} for $i = 1, 2, \ldots$. Then, for every k such that $(k,q_1) = 1$ there exists a heavy map $F \in \mathscr{L}$ such that $a(F) = b(F) = k/q_1$ and* $\operatorname{Per}(F) = A$.

Proof. – We set $\bar{q}_1 = q_1$ and $\bar{q}_i = q_i/q_{i-1}$ *for* $i = 2, 3, \ldots$. For a given $r \geqslant 2$ we start with $G_{r,r}$ given by $G_{r,r}(x) = x + 1/\bar{q}_r$ and then use the construction described in the proof of Lemma 7 $r-1$ times with $\bar{q}_{r-1}, \ldots, \bar{q}_2$, $\bar{q}_1$ as q and $1, \ldots, 1, k$ as p. In such a way we get successively heavy maps $G_{r,r-1}, \ldots, G_{r,2}$, $G_{r,1} \in \mathscr{L}$. We have $\operatorname{Per}(G_{r,1}) = \{\bar{q}_1, \bar{q}_1\bar{q}_2, \ldots, \bar{q}_1\bar{q}_2 \ldots \bar{q}_r\} = \{q_1, q_2, \ldots, q_r\}$, and $a(G_{r,1}) = b(G_{r,1}) = k/q_1$.

Let us compare $G_{r,i}$ with $G_{r+m,i}$ for $m \geqslant 1$. For $i = r$ they differ only on the intervals $[\ell - 1/2\bar{q}_i, \ell + 1/2\bar{q}_i]$, $\ell \in \mathbf{Z}$. For both of them $a = (\bar{q}_i - 1)/\bar{q}_i$. Therefore for $i = r-1$ the same is true and additionally $G_{r,i}(\ell+) = G_{r+m,i}(\ell+)$ and $G_{r,i}(\ell-) = G_{r+m,i}(\ell-)$ for $\ell \in \mathbf{Z}$. Consequently, for smaller i we get again the same properties, but the set where $G_{r,i}$ differs from $G_{r+m,i}$ is reduced to two intervals in each $[\ell, \ell+1]$, $\ell \in \mathbf{Z}$; each of them of length at most $(1/2\bar{q}_{r-1})(1/2\bar{q}_{r-2}) \ldots (1/2\bar{q}_i)$ (the first factor is the length of the interval $[\ell, \ell + 1/2\bar{q}_{r-1}]$ or $[\ell + 1 - 1/2\bar{q}_{r-1}, \ell + 1]$; the next ones come from successive applications of τ during the construction). Moreover, these intervals do not depend on m, and if we let r increase, the corresponding intervals will form descending sequences. Therefore the maps $G_{r,1}$ converge as $r \to \infty$ to some map F. Clearly, F is heavy, $F \in \mathscr{L}$, $\mathrm{Per}(F) = A$ and $a(F) = b(F) = k/q_1$. □

The statement (b) of Theorem A follows from Proposition 5, Remark 2, and Lemmas 6, 8 and 9.

4. Structure of the set $B(a,b,n,m)$ (proof of Theorem B).

In this section we use the definitions of $\alpha(n)$, $\ell(n)$, $<_r$ and $<_\ell$ given in the introduction. The following result follows immediately from the definitions.

LEMMA 10. — *Let p/q such that $(p,q) = 1$ with $q > 1$ and let $n > 0$. Then $\ell(n)$ is the smallest positive integer ℓ such that $\ell/n > p/q$.*

PROPOSITION 11. — *Let p/q be such that $(p,q) = 1$ with $q > 1$ and $p/q > 0$, and* let *$<_r$ and $<_\ell$ be the right-hand and left-hand orderings associated to p/q, respectively. Then the following hold.*

(a) *Let $b \in \mathbf{R}$ be such that $p/q < b$ and let $k \in \mathbf{N}^*$. If $m \in M(p/q,b) \cup S(b,k)$ then $n \in M(p/q,b) \cup S(b,k)$ for all n such that $m <_r n$.*

(b) *Let $a \in \mathbf{R}$ be such that $a < p/q$* and let $k \in \mathbf{N}^*$. *If $m \in M(a,p/q) \cup S(a,k)$ then $n \in M(a,p/q) \cup S(a,k)$ for all n such that $m <_\ell n$.*

Proof. — We prove (a); (b) follows in a similar way. We consider two cases.

Case 1: Assume that $m \in M(p/q,b)$. By Lemma 10, we have $p/q < \ell(m)/m < b$. Since $m <_r n$, if $(<_r.1)$ holds then by the definition of $\ell(.)$, $p/q < \ell(n)/n < \ell(m)/n$. Hence, $n \in M(p/q,b)$. If $(<_r.1)$ does not hold then $\ell(n)/n = \ell(m)/m \in (p/q,b)$ and hence $n \in M(p/q,b)$.

Case 2: Assume that $m \notin M(p/q,b)$. By Lemma 10, $\ell(m)/m = b$. If $(<_r.1)$ holds, (a) follows as in Case 1. Suppose $(<_r.1)$ does not hold. Then $\ell(m)/m = \ell(n)/n = t/s$ with $(t,s) = 1$ and s divides m and n. If $(<_r.2)$ holds, then $s \neq m$ and $s \neq n$. Since $m \in S(b,k)$ and $m \neq s$, we have that $m = (k+i)s$ with $i \geqslant 0$. Since $m < n$ and s divides n, if follows that $n = (k+j)s$ with $j > i$. So, $n \in S(b,k)$. If $(<_r.3)$ holds then $n = s$. Hence, $m \in S(b,k)$. □

Proof of Theorem B. – If $q > 1$ then it follows from Proposition 11. If $q = 1$ and $b \in [p+1/k, p+1/(k-1))$ (resp. $a \in (p-1/(k-1), p-1/k]$), then $M(p,b) \cup S(b,m) = \{k, k+1, k+2, \dots\} = R(\rho)$ (resp. $M(a,p) \cup S(a,n) = \{k, k+1, k+2, \dots\} = L(\lambda)$). This completes the proof. □

Let p/q be such that $(p,q) = 1$ with $q > 1$. Let $n_1 <_r n_2 <_r n_3, \dots$ be the set of all natural numbers ordered by $<_r$, where $<_r$ is the right-hand ordering associated with p/q. The next result allows to construct the whole sequence $n_1, n_2, \dots$ starting from the first $q(q+1)/2$ terms.

PROPOSITION 12. – $n_{i+kq(q+1)/2} = n_i + kq\alpha(n_i)$ *for* $k = 0, 1, 2, \dots$.

The same proposition holds with $<_\ell$ instead of $<_r$ and $\alpha(-n_i)$ instead of $\alpha(n_i)$.

To prove Proposition 12 we shall need two lemmas.

For $k \geqslant 0$ we define $B_k = \{n \in \mathbf{N} : kq < n/\alpha(n) < (k+1)q\}$ and $\beta(n) = n - kq\alpha(n)$ where k is such that $n \in B_k$.

LEMMA 13. – (a) Card $B_k = q(q+1)/2$, *for all* k.

(b) *Assume that* $n \in B_k$ *and* $m \in B_j$. *If* $k < j$ *then* $n <_r m$.

(c) $\beta(n) \in B_0$ *and* $\alpha(n) = \alpha(\beta(n))$ *for all* n.

(d) *Let* $n, m \in B_k$. Then, $n/\alpha(n) < m/\alpha(m)$ *if and only if* $\beta(n)/\alpha(\beta(n)) < \beta(m)/\alpha(\beta(m))$.

Proof. – Set $B_k^i = \{n \in B_k : \alpha(n) = i\}$ for $i = 1, 2, \dots, q$. Since $\{\alpha(i) : i = rq+1, rq+2, \dots, rq+q\} = \{1, 2, \dots, q\}$ for all $r \geqslant 0$, and $B_k = \{n \in \mathbf{N} : k\alpha(n)q < n \leqslant k\alpha(n)q + \alpha(n)q\}$, we have that Card $B_k^i = i$. Then Card $B_k = \sum_{i=1}^{q}$ Card $B_k^i = q(q+1)/2$, and (a) follows.

(b) follows from the definition of $<_r$.

The equality $\alpha(n) = \alpha(\beta(n))$ follows from the definitions of α and β. Then we get $\beta(n)/\alpha(\beta(n)) = (n - kq\alpha(n))/\alpha(n) = n/\alpha(n) - kq \in (0, q]$, and (c) follows.

(d) follows immediately from the definition of β. □

For each $n \in \mathbf{N}$ we set $M_n = \{k \in \mathbf{N} : k/\alpha(k) = n/\alpha(n)\}$. By the definition of $\ell(.)$, $M_n = \{k \in \mathbf{N} : \ell(k)/k = \ell(n)/n\}$.

LEMMA 14. – (a) $m \in M_n$ *is the smallest element of* M_n *if and only if* $(\ell(m), m) = 1$.

(b) $(\ell(n), n) = 1$ *if and only if* $(\ell(\beta(n)), \beta(n)) = 1$.

Proof. – Let $m \in M_n$. If $(\ell(m),m) = 1$ then for every $k \in M_n$ we have $\ell(k)/k = \ell(m)/m$. So m divides k and consequently $m \leqslant k$. Assume now that $(\ell(m),m) \neq 1$. Then $\ell(m)/m = \ell/k$ for some ℓ and k, with $(\ell,k) = 1$ and k dividing m. From Lemma 10 applied to m and to k it follows that $\ell(k) = \ell$, and consequently k is the smallest element of M_n. Since $m > k$, this completes the proof of (a).

We have $M_n \subset B_k$ for some k. Therefore $\beta(M_n) \subset M_{\beta(n)}$. Let s be the smallest element of $M_{\beta(n)}$. Then by Lemma 13. (c), $0 < s/\alpha(s) = \beta(n)/\alpha(\beta(n)) = (n-kq\alpha(n))/\alpha(n) = n/\alpha(n) - kq \leqslant q$. Set $m = s + kq\alpha(s)$. Then $\alpha(m) = \alpha(s)$ and $m/\alpha(m) = s/\alpha(s) + kq$. Thus $m \in B_k$. Therefore $s = \beta(m)$ and by Lemma 13. (d) we get that m is the smallest element of M_n. Now (b) follows from (a). $\square$

Proof of Proposition 12. – It follows from Lemma 13 and Lemma 14 (b). $\square$

Now we shall give some examples of the right-hand and left-hand orderings associated with various p/q.

Example 1. – $p/q = 1/2$. From the definition of $<_r$ and Lemma 14. (a), we have that $n_1 = 2$, $n_2 = 1$, $n_3 = 4$ and $q\alpha(n_1) = 4$, $q\alpha(n_2) = 2$, $q\alpha(n_3) = 4$. Therefore, by Proposition 12, we get

$$2 <_r 1 <_r 4 <_r 2 + 4.1 <_r 1 + 2.1 <_r 4 + 4.1 <_r 2 + 4.2 <_r 1 + 2.2 <_r 4 + 4.2 <_r \ldots .$$

Since in this case $\alpha(-n) = \alpha(n)$, for $<_\ell$ we obtain again $<_r$.

Note that if we omit the even numbers and 1 in the previous $<_r$ and $<_\ell$ orderings (see the remark of the introduction), we obtain the following ordering

$$3 <_r 5 <_r 7 <_r 9 <_r \ldots$$

i.e. the beginning of the Sarkovskii ordering.

Example 2. – $p/q = 1/3$. For $<_r$ we have $n_1 = 1$, $n_2 = 3$, $n_3 = 4$, $n_4 = 6$, $n_5 = 2$, $n_6 = 9$ and $q\alpha(n_1) = 6$, $q\alpha(n_2) = 9$, $q\alpha(n_3) = 6$, $q\alpha(n_4) = 9$, $q\alpha(n_5) = 3$, $q\alpha(n_6) = 9$. Hence

$$1 <_r 3 <_r 4 <_r 6 <_r 2 <_r 9 <_r 1 + 6.1 <_r 3 + 9.1 <_r 4 + 6.1 <_r 6 + 9.1 <_r 2 + 3.1 <_r 9 + 9.1 <_r 1 + 6.2 <_r 3 + 9.2 <_r 4 + 6.2 <_r 6 + 9.2 <_r 2 + 3.2 <_r 9 + 9.2 <_r \ldots .$$

For $<_\ell$ we have $n_1 = 2$, $n_2 = 3$, $n_3 = 1$, $n_4 = 6$, $n_5 = 5$, $n_6 = 9$ and $q\alpha(-n_1) = 6$, $q\alpha(-n_2) = 9$, $q\alpha(-n_3) = 3$, $q\alpha(-n_4) = 9$, $q\alpha(-n_5) = 6$, $q\alpha(-n_6) = 9$. Hence,

$2 <_\ell 3 <_\ell 1 <_\ell 6 <_\ell 5 <_\ell 9 <_\ell 2 + 6.1 <_\ell 3 + 9.1 <_\ell 1 + 3.1 <_\ell 6 + 9.1 <_\ell 5 + 6.1 <_\ell 9 + 9.1 <_\ell 2 + 6.2 <_\ell 3 + 9.2 <_\ell 1 + 3.2 <_\ell 6 + 9.2 <_\ell 5 + 6.2 <_\ell 9 + 9.2 <_\ell \ldots .$

Note that if we omit the numbers divisible by 3 and several first numbers in these orderings, we obtain

$$7 <_r 10 <_r 5 <_r 13 <_r 16 <_r 8 <_r \ldots$$

and

$$5 <_\ell 8 <_\ell 4 <_\ell 11 <_\ell 14 <_\ell 7 <_\ell 17 <_\ell \ldots,$$

i.e. the beginnings of the red and green orderings of [ALM] respectively.

Example 3. – $p/q = 2/5$. We obtain

$1 <_r 3 <_r 5 <_r 4 <_r 6 <_r 8 <_r 10 <_r 2 <_r 15 <_r 13 <_r 11 <_r 20 <_r 18 <_r 9 <_r 25 <_r 1 + 15.1 <_r 3 + 20.1 <_r 5 + 25.1 <_r 4 + 10.1 <_r 6 + 15.1 <_r 8 + 20.1 <_r 10 + 25.1 <_r 2 + 5.1 <_r 15 + 25.1 <_r 13 + 20.1 <_r 11 + 15.1 <_r 20 + 25.1 <_r 18 + 20.1 <_r 9 + 10.1 <_r 25 + 25.1 <_r \ldots .$

5. Lower bounds of the topological entropy (proofs of Theorem C and Corollary D).

In this section we shall refer to some results from [ALMM]. However we use their proofs rather than the statements of these results. Thus we shall assume that they are known for the reader and we shall use them freely.

Fix γ, $\delta \in \mathbf{R}$ with $\gamma > 1$. Consider the map $F_{\gamma,\delta}(x) = \gamma x + \delta$ if $x \in (0,1)$, and $F_{\gamma,\delta}(x) = F_{\gamma,\delta}(x-k) + k$ if $x \in (k, k+1)$ for all $k \in \mathbf{Z}$ (if $x \in \mathbf{Z}$, we have two one-sided limits).

Clearly $F_{\gamma,\delta} \in \mathscr{C}$. From [MS] we obtain $h(F_{\gamma,\delta}) = \log \gamma$.

In the same way as Theorem 3.1 of [ALMM], we obtain

PROPOSITION 15. — *Assume that* a, $b \in \mathbf{R}$, $a < b$ *and*

$$\gamma = \beta_{a,b},$$
$$\delta = (\gamma-1)^2 \sum_{n=1}^{\infty} E(na)\gamma^{-n-1}.$$

Then the map $F_{\gamma,\delta}$ *has topological entropy* $\log \beta_{a,b}$, $a(F_{\gamma,\delta}) = a$ *and* $b(F_{\gamma,\delta}) = b$.

In the proof, apart of the obvious change of notations, we have to substitute 1 for U and $(\gamma-1)/\gamma$ for $(a-1)/2a$.

Now Theorem C.(b) follows from Proposition 15.

The proof of Theorem C.(a) is here much simpler than that of the analogous result in [ALMM]. Assume that $a(F) \leqslant a$ and $b(F) \geqslant b$ where a, $b \in \mathbf{Q}$ and $a < b$. Then F has *TPO's* with rotation numbers a and b respectively. Unlike in the situation from [ALMM], their relative position is determined uniquely by the assumption that $F|_{[0,1]}$ is non-decreasing:

LEMMA 16. — *Let* $F \in \mathscr{C}$, $p/q \neq t/s$, $(p,q) = (t,s) = 1$. *Let* $P = \{x_k\}_{k \in \mathbf{Z}}$ *and* $T = \{y_k\}_{k \in \mathbf{Z}}$ *be TPO's with rotation numbers* p/q *and* t/s *respectively, such that* $x_k < x_{k+1}$ *and* $y_k < y_{k+1}$ *for all* $k \in \mathbf{Z}$; $x_{-1} < 0 < x_0$ *and* $y_{-1} < 0 < y_0$ *(we can allow also non-sharp inequalities ; then we take one-sided limits). Then to determine whether* $x_i < y_j$ *or* $x_i > y_j$ *it is enough to know* i, j, p, q, t, s.

Proof. — We have $F(x_k) = x_{k+p}$, $F(y_k) = y_{k+t}$, $x_{k+q} = x_k + 1$, $y_{k+s} = y_k + 1$. Since $p/q \neq t/s$, there exists $m \geqslant 0$ such that $E(x_{i+mp}) \neq E(y_{j+mt})$ (here if we take one-sided limits, we define $E(\ell-) = \ell-1$). Take such smallest m. Since F is increasing on each $[\ell,\ell+1] \cap (P \cup T)$, $\ell \in \mathbf{Z}$, we have $x_i < y_j$ if $E(x_{i+mp}) < E(y_{j+mt})$ and $x_i > y_j$ if $E(x_{i+mp}) > E(y_{j+mt})$. However, $E(x_k) = E(k/q)$ and $E(y_k) = E(k/s)$, so the above criterion involves only i, j, p, q, t, s. $\square$

Analogously to Lemma 4.5 and Remark 4.6 of [ALMM] we know that the orbits of $0+$ and $1-$ for $F_{\gamma,\delta}$ (where γ and δ are from Proposition 15) are *TPO's* with rotation numbers a and b respectively. Then in view of Lemma 16, we get immediately by the usual consideration of Markov graphs that $h(F) \geqslant \beta_{\gamma,\delta}$, and by the properties of $\beta_{\cdot,\cdot}$ we get $h(F) \geqslant \beta_{a(F),b(F)}$. This completes the proof of Theorem C.

Proof of Corollary D. – Let $F \in \mathscr{L}$ with $a(F) < b(F)$. From Theorem 4 we have that F is heavy (that is $F \in \mathscr{C}$). By Theorem C.(*a*), $h(F) \geqslant \log \beta_{a(F),b(F)}$.

Now, assume that $0 \leqslant a < b \leqslant 1$. By Theorem C.(*b*), there exists $F \in \mathscr{C}$ such that $a(F) = a$, $b(F) = b$ and $h(F) = \log \beta_{a,b}$.

To prove that $F \in \mathscr{L}$, it is enough to show that $F(0+) \geqslant 0$ and $F(1-) \leqslant 2$. We shall prove $F(1-) \leqslant 2$; the other inequality follows in the similar way.

If $b = 1$ then we know from the construction (see Lemma 4.5 of [ALMM]) that the orbit of $1-$ is a *TPO* with rotation number 1. But this means that $F(1-) = 2$. If $b < 1$ then for all $x \in \mathbf{R}$ we have $F_r(x) < x + 1$ (see Lemma 3.3 of [M3]). In particular, $F(1-) \leqslant F_r(1) < 2$. □

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Ll. ALSEDÀ,

Departament d'Economia i Història Econòmica
Facultat de Ciències Econòmiques
Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona (Spain).

J. LLIBRE,

Departament de Matemàtiques,
Facultat de Ciències
Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona (Spain).

M. MISIUREWICZ,

Instytut Matematyki,
Uniwersytet Warszawski
Pałac Kultury i Nauki IX
P. 00-901 Warszawa (Poland).

C. TRESSER,

I.B.M. Corporation
T.J. Watson Research Center
P.O. Box 218
Yorktown Heights N.Y. 10598 (U.S.A.).