Annales de l'institut Fourier

E. S. Katsoprinakis
V. N. Nestoridis

Partial sums of Taylor series on a circle

Annales de l'institut Fourier, tome 39, nº 3 (1989), p. 715-736
<http://www.numdam.org/item?id=AIF_1989__39_3_715_0>

Ann. Inst. Fourier, Grenoble
39, 3 (1989), 715-736

PARTIAL SUMS OF TAYLOR SERIES ON A CIRCLE

by E.S. KATSOPRINAKIS and V.N. NESTORIDIS

1. Introduction.

In connection with a theorem of Marcinkiewicz and Zygmund (see [4], [5] Vol. II, p. 178 or [1]) S.K. Pichorides suggested to the first author to examine, as a thesis problem (see [2]), the power series

$$\sum_{n=0}^{\infty} c_n z^n,$$

with the following special property (a) :

(a) : *For every* z *in a nondenumerable subset* E *of the unit circle* T , *all partial sums*

$$s_n(z) = \sum_{k=0}^{n} c_k z^k$$

lie on the union of a finite number of circles, $C_1(z), C_2(z), \ldots, C_{M(z)}(z)$, *in the complex plane.*

In [3], which contains the main results of [2], the following characterization has been obtained :

THEOREM A. — *Let*

$$\sum_{n=0}^{\infty} c_n z^n$$

Key-words : Power series – Rational functions – Taylor expansion.
A.M.S. Classification : 30B10 – 42A99.

be a power series with complex coefficients. Then, this series has the property (a), if and only if, (b) holds :

(b) : The above series has a representation of the form :

$$\sum_{n=0}^{\infty} c_n z^n = G(e^{it}z) + (e^{it}z)^{\mu} F(e^{it}z) \sum_{m=0}^{\infty} (e^{it}z)^{m\rho},$$

where t is a real number, μ, ρ are integers, $\mu \geq 0$, $\rho \geq 1$, and G, F are polynomials satisfying $\deg G < \mu$ *or* $G \equiv 0$, *and* $\deg F < \rho$ *or* $F \equiv 0$.

It is easy to check that a power series, which has a representation of the form (b), is $(C, 1)$ summable to a finite sum $\sigma(z)$, for all, but a finite number of z, $z \in T$; further, all its partial sums $s_n(z)$ lie on the union of a finite number of concentric circles with center $\sigma(z)$. Moreover, the angular distribution of the sequence $\{s_n(z)\}$ around $\sigma(z)$, is uniform, for all, but denumerable many z, z in T.

The difficult part of theorem A is the implication (a) $\Rightarrow$ (b). The effort is to control the Taylor coefficients c_n and establish a kind of periodicity among them. The proof uses the full hypothesis, that *all* partial sums lie on the union of a finite number of circles.

J.-P. Kahane asked whether it is possible to obtain the same result using only *one* circle $C(z)$ containing infinitely many partial sums, but *not all* of them. For instance, one can suppose that $C(z)$ contains all $s_\nu(z)$, with ν in an infinite or finite arithmetic progression; what is then the conclusion?

The above question led us to introduce the notion of "continuation" of a polynomial with respect to a family of circles of special type. More precisely, we consider any family of circles $C(z)$, $z \in T$, with centers $B(z) + z^\lambda [A(z)/Q(z)]$ and radii $|z^\lambda A(z)/Q(z)|$, where $\lambda \geq 1$ is an element of the set Z of integers and B, A, Q are polynomials. We suppose that A and Q do not have common factors, $A(0)Q(0) \neq 0$, $\deg A < \deg Q$ and $\deg B < \lambda$ or $B \equiv 0$, where $\deg Y$ denotes the degree of the polynomial Y. In particular, the family of circles, defined by three different partial sums of any power series, is of this type. Further, if P is any polynomial, then we call the polynomial $R(z) \equiv B(z) + z^\lambda P(z)$ a "continuation" of B with respect to $C(z)$, if $R(z)$ lies on $C(z)$ for infinitely many z in T. Then the main results of this paper are given by the following theorems B, C. For simplicity we write $\sum$ instead of $\sum_0^\infty$.

THEOREM B. — *Let λ, A, B, Q and $C(z)$ be as above. Then every continuation of B with respect to $C(z)$ is a partial sum of the Taylor development $\sum b_n z^n$ of the "center function" $g(z) = B(z) + z^\lambda[A(z)/Q(z)]$ of $C(z)$. It follows that the set of continuations of B with respect to $C(z)$ is at most countable.*

THEOREM C. — *Let λ, A, B, Q and $C(z)$ be as above. Then the following (i), (ii), (iii) are equivalent :*

(i) *The set of continuations of B with respect to $C(z)$ is infinite.*

(ii) *$Q(z)$ is a non constant factor of a polynomial of the form $1-(e^{it}z)^\rho$, where $t \in R$ and $\rho \in Z$, $\rho \geq 1$.*

(iii) *There is a power series $\sum b_n z^n$ with the following two properties :*

(*) *$\sum b_n z^n$ is not a polynomial.*

(**) *There is an infinite subset S of $\{0, 1, \ldots\}$, such that, for all ν in S, the partial sums s_ν of $\sum b_n z^n$ are continuations of B with respect to $C(z)$.*

If a series $\sum b_n z^n$ satisfies (*) and (**), then this series is unique and coincides with the Taylor development of $g(z) = B(z) + z^\lambda[A(z)/Q(z)]$; moreover, we have :

$$\sum_{n=0}^{\infty} b_n z^n \equiv B(z) + (e^{it}z)^\lambda F(e^{it}z) \sum_{m=0}^{\infty} (e^{it}z)^{m\rho},$$

where $t \in R$, $\rho \in Z$, $\rho \geq 1$ and F is a non identically zero polynomial with $\deg F < \rho$. Further, any continuation of B with respect to $C(z)$ is a partial sum of this series.

Theorem C answers in the affirmative part of the question of J.-P. Kahane. We prove theorems B and C in §2. The methods of proof are different than the methods in [3]. We use factorization and thus, we deal with the zeros of certain polynomials instead of their coefficients.

In §3 we derive stronger versions of theorem A and give complete answer to the question of J.-P. Kahane (see prop. 8 and prop. 9). In particular proposition 9 is a finite version of theorem A. The proof of proposition 8 could be shortened by avoiding the notion of continuation. We did not follow this approach in order to obtain also the results of proposition 9 and study in more detail continuations, which we think that they present some interest in themselves. Section 4 contains remarks, examples and open questions.

2. Proof of the main results.

Three complex numbers w_1, w_2, w_3 do not lie on a straight line, if and only if, the system $|k|^2 = |w_2 - w_1 - k|^2 = |w_3 - w_1 - k|^2$ has a unique complex solution $k \neq 0$. In this case, w_1, w_2, w_3 determine a unique circle containing them, with center $w_1 + k$ and radius $|k|$. The above system takes the form of a linear system with unknows k and $\overline{k}$, as follows :

$$(\overline{w}_2 - \overline{w}_1)k + (w_2 - w_1)\overline{k} = (w_2 - w_1)(\overline{w}_2 - \overline{w}_1)$$
$$(\overline{w}_3 - \overline{w}_1)k + (w_3 - w_1)\overline{k} = (w_3 - w_1)(\overline{w}_3 - \overline{w}_1).$$

Let ν_1, ν_2, ν_3 be three integers, such that, $0 \leq \nu_1 < \nu_2 < \nu_3$. If the partial sums of a power series, with indices ν_1, ν_2, ν_3, are different as polynomials, then, for all, but finitely many z, $z \in T$, the complex numbers $s_{\nu_1}(z), s_{\nu_2}(z), s_{\nu_3}(z)$ do not lie on a straight line. So, they define a unique circle $C(z)$ containing them. In order to see this, we can set :

$$s_{\nu_1}(z) \equiv P_1(z), \qquad s_{\nu_2}(z) \equiv P_1(z) + z^{\lambda}P_2(z),$$

and

$$s_{\nu_3}(z) \equiv P_1(z) + z^{\lambda}P_2(z) + z^{\lambda+\mu+q}P_3(z),$$

where λ, μ, q are integers, $\lambda > \nu_1$, $\mu = \deg P_2$, $q \geq 1$ and P_1, P_2, P_3 are polynomials, with $P_2(0)P_3(0) \neq 0$. We also denote $\nu = \deg P_3$. Since $\overline{z} = 1/z$ for z in T, it follows that $z^{\mu}\overline{P}_2(z)$, $z^{\nu}\overline{P}_3(z)$, are restrictions on T of two polynomials with non-zero constant terms and degrees μ and ν, respectively. After this notation we have to examine the following system :

$$z^{-\mu-\lambda}[z^{\mu}\overline{P}_2(z)]k(z) + z^{\lambda}P_2(z)\overline{k}(z) = z^{-\mu}P_2(z)[z^{\mu}\overline{P}_2(z)]$$

$$z^{-q-\mu-\nu-\lambda}[z^{\nu}\overline{P}_3(z)]k(z) + z^{q+\mu+\lambda}P_3(z)\overline{k}(z) = z^{-\nu}P_3(z)[z^{\nu}\overline{P}_3(z)] + z^{q}P_3(z)[z^{\mu}\overline{P}_2(z)] + z^{-q-\mu-\nu}P_2(z)[z^{\nu}\overline{P}_3(z)].$$

The determinant $D(z)$ of this system is the restriction on T of a non identically zero rational function; more precisely, we have :

$$D(z) \equiv z^{-q-\mu-\nu}\{z^{2q+\mu+\nu}P_3(z)[z^{\mu}\overline{P}_2(z)] - P_2(z)[z^{\nu}\overline{P}_3(z)]\}$$

and

$$\deg[z^{q+\mu+\nu}D(z)] = 2(q + \mu + \nu) \geq 2q \geq 2.$$

For each z in T, such that $D(z) \neq 0$, the unique solution $k(z)$ is :

$$k(z) = z^{\lambda}[A_1(z)/Q_1(z)],$$

where

$$A_1(z) = P_2^2(z)[z^\nu \overline{P}_3(z)] + z^{q+\mu} P_2(z) P_3(z)[z^\nu \overline{P}_3(z)],$$

$$Q_1(z) = P_2(z)[z^\nu \overline{P}_3(z)] - z^{2q+\mu+\nu} P_3(z)[z^\mu \overline{P}_2(z)].$$

We observe that A_1, Q_1 are restrictions on T of two polynomials, which, for simplicity, we denote again by A_1, Q_1, respectively. Further, we have :

$$A_1(0)Q_1(0) \neq 0\ , \qquad \deg A_1 = q + 2\mu + 2\nu$$

and

$$\deg Q_1 = q + \deg A_1 \geq 1 + \deg A_1 > \deg A_1.$$

Since $A_1 D$ is a non identically zero rational function, the set $\Omega = \{z \in T : A_1(z)D(z) = 0\}$ is finite. Then, for every z in $T - \Omega$, the system considered above has a unique non-zero solution $k(z)$; thus, for each $z \in T - \Omega$, the complex numbers $s_{\nu_1}(z), s_{\nu_2}(z), s_{\nu_3}(z)$ do not lie on a straight line and they define a unique circle $C(z)$ containing them, with center : $s_{\nu_1}(z) + k(z) = s_{\nu_1}(z) + z^\lambda[A_1(z)/Q_1(z)]$ and radius $|k(z)| \neq 0$. Since $\lambda > \nu_1$ and $s_{\nu_2}(z) = s_{\nu_1}(z) + z^\lambda P_2(z)$, we see that $s_{\lambda-1}(z) \equiv s_{\nu_1}(z)$. Further, we can write $A_1(z)/Q_1(z) \equiv A(z)/Q(z)$, where A, Q are polynomials without common factors; then, $A(0)Q(0) \neq 0$ and $\deg Q - \deg A = \deg Q_1 - \deg A_1 = q \geq 1$. Since $k(z)$ is known and non zero on the infinite set $T - \Omega$, it has at most one rational extension $W \neq 0$, which in turn has a unique decomposition $W(z) = z^\lambda[A(z)/Q(z)]$, where λ is an integer and the polynomials A, Q do not have common factors and satisfy $A(0) \neq 0$ and $Q(0) = 1$. Thus, we have proved the following :

LEMMA 1. — *Let*

$$\sum_{n=0}^{\infty} c_n z^n$$

be a power series with complex coefficients and ν_1, ν_2, ν_3 *be three integers, such that* $0 \leq \nu_1 < \nu_2 < \nu_3$. *If the partial sums of the above series, with indices* ν_1, ν_2, ν_3, *are different as polynomials, then there exist a finite subset* Ω *of the unit circle* T, *an integer* $\lambda > \nu_1$ *and two polynomials* A, Q *without common factors, which satisfy* $A(0) \neq 0$, $Q(0) = 1$ *and* $\deg A < \deg Q$, *such that the following holds :*

For every z *in* $T - \Omega$ *we have* $A(z)Q(z) \neq 0$ *and the complex numbers* $s_{\nu_1}(z), s_{\nu_2}(z), s_{\nu_3}(z)$ *do not lie on a straight line; they define a unique circle* $C(z)$ *containing them, with center* $s_{\nu_1}(z) + z^\lambda[A(z)/Q(z)]$ *and radius* $|z^\lambda[A(z)/Q(z)]| \neq 0$.

The polynomials A, Q and the integer λ with the above properties are uniquely determined by $s_{\nu_1}, s_{\nu_2}, s_{\nu_3}$. The integer λ is the least element of the set Z of integers, such that, $\lambda > \nu_1$ and $c_\lambda \neq 0$. Thus, we have $s_{\lambda-1} \equiv s_{\nu_1}$.

Lemma 1 leads us to consider polynomials B, A, Q with $A(0) \neq 0$, $Q(0) = 1$, $\deg A < \deg Q$, and an integer $\lambda \geq 1$, such that, $\lambda > \deg B$ or $B \equiv 0$. We suppose that A, Q do not have common factors. For every z in T, such that $A(z)Q(z) \neq 0$, we denote by $C(z)$ the circle with center $g(z) = B(z) + z^\lambda[A(z)/Q(z)]$ and radius $|z^\lambda A(z)/Q(z)|$. We also consider the factorization :

$$A(z) = c\prod_{j\in I}(1 + \bar{a}_j z),$$

where c is a non-zero complex number, I is a finite set and a_j are non-zero complex numbers, for all j in I. The following definition will be useful for our purposes :

DEFINITION 2. — *Let λ, A, B, Q and $C(z)$ be as above. If P is any polynomial, then the polynomial $R(z) \equiv B(z) + z^\lambda P(z)$ is called "a continuation of B with respect to $C(z)$", if $R(z)$ lies on $C(z)$ for infinitely many z in T.*

Now, by an application of the reflection principle, we prove our basic lemma :

LEMMA 3. — *Let λ, A, B, Q and $C(z)$ be as above. If R is any polynomial, then, (i), (ii), (iii) are equivalent :*

(i) *$R(z) \in C(z)$ holds for every z in T with $A(z)Q(z) \neq 0$.*

(ii) *$R(z) \in C(z)$ holds for infinitely many z in T.*

(iii) *There exist $\gamma \in C$, $|\gamma| = 1$, $k \in Z$ and $J \subset I$ with $|a_j| \neq 1$ for all j in J, such that, the following identity of rational functions holds :*

$$\frac{[R(z) - B(z)]Q(z) - z^\lambda A(z)}{z^\lambda A(z)} = \gamma z^k \prod_{j\in J}\frac{z + a_j}{1 + \bar{a}_j z}.$$

Proof. — (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). For every z in an infinite subset E of T we have $R(z) \in C(z)$, which implies that

$$|R(z) - B(z) - z^k[A(z)/Q(z)]| = |z^\lambda A(z)/Q(z)| \ ;$$

thus, the function $\varphi(z) \equiv \dfrac{[R(z) - B(z)]Q(z) - z^\lambda A(z)}{z^\lambda A(z)}$ satisfies $|\varphi(z)| = 1$, for all $z \in E$. It follows that the rational function $f(z) \equiv \varphi(z) - [\overline{\varphi}(\overline{z}^{-1})]^{-1}$ vanishes on the infinite set $E \subset T$. Thus, $f(z) \equiv 0$ and $\varphi(z) \equiv [\overline{\varphi}(\overline{z}^{-1})]^{-1}$; it follows that the map $z \to (\overline{z})^{-1}$ induces a bijective correspondence among zeros and poles of φ, preserving multiplicities. From the definition of φ we see that, if $b \neq 0, \infty$ is a pole of φ with multiplicity m, then b is a zero of A with multiplicity $m' \geq m$. Since

$$A(z) = c \prod_{j \in I} (1 + \overline{a}_j z),$$

it follows that,

$$\varphi(z) = \gamma z^k \prod_{j \in J} \frac{z + a_j}{1 + \overline{a}_j z},$$

with $\gamma \in C$, $|\gamma| = 1$, $k \in Z$ and $J \subset I$. If for some $j \in J$ we have $|a_j| = 1$ then the factor $(z + a_j)/(1 + \overline{a}_j z)$ equals a_j and can be absorbed in the constant γ : so, the particular j can be deleted from J. In this way we have $|a_j| \neq 1$ for all j in J, as requested. We also notice that the function φ is the quotient of two finite Blaschke products.

(iii) $\Rightarrow$ (i). Since $\left| \gamma z^k \prod_{j \in J} \dfrac{z + a_j}{1 + \overline{a}_j z} \right| = 1$ for all z in T, we have $|[R(z) - B(z)]Q(z) - z^\lambda A(z)| = |z^\lambda A(z)|$ on T ; this implies

$$|R(z) - B(z) - z^\lambda [A(z)/Q(z)]| = |z^\lambda A(z)/Q(z)|,$$

for all z in T, such that $Q(z) \neq 0$. This gives (i). □

Lemma 3 yields the following :

PROPOSITION 4. — *Let λ, A, B, Q and $C(z)$ be as in definition 2. If P is any polynomial, then the following are equivalent :*

(i) *$B(z) + z^\lambda P(z) \in C(z)$ holds for all z in T with $A(z)Q(z) \neq 0$.*

(ii) *$B(z) + z^\lambda P(z)$ is a continuation of B with respect to $C(z)$.*

(iii) *There exist $\gamma \in C$, $|\gamma| = 1$, $k \in Z$ and $J \subset I$ with $|a_j| \neq 1$ for all j in J, such that :*

$$P(z) \equiv \frac{A(z)}{Q(z)} + \gamma z^k \cdot \frac{L_J(z)}{Q(z)},$$

where

$$L_J(z) \equiv c \prod_{j \in J} (z + a_j) \prod_{j \in I - J} (1 + \overline{a}_j z).$$

Further, $k > \deg P$ *and* $P(0) \neq 0$ *if* $P \not\equiv 0$; *if* $P \equiv 0$, *then* $k = 0$.

Proof. — A straightforward application of lemma 3 to the polynomial $R(z) = B(z) + z^\lambda P(z)$ gives the equivalence of (i), (ii), (iii). Since $\deg L_J(z) = \operatorname{card} I = \deg A(z) < \deg Q(z)$, we see that $k = \deg P + \deg Q - \deg A > \deg P \geq 0$ and $P(0) = A(0)/Q(0) \neq 0$, when $P \not\equiv 0$. If $P \equiv 0$, then we have $\deg A = k + \deg L_J$, which gives $k = 0$. □

Now, with the aid of proposition 4, we can prove that every continuation of B with respect to $C(z)$ is a partial sum of the Taylor development of the center function $g(z) = B(z) + z^\lambda[A(z)/Q(z)]$. More precisely we have :

THEOREM 5. — *Let* λ, A, B, Q *and* $C(z)$ *be as in definition 2. Then, every continuation of* B *with respect to* $C(z)$ *is a partial sum of the Taylor development* $\sum b_n z^n$ *of the center function* $g(z) = B(z) + z^\lambda[A(z)/Q(z)]$. *It follows that the set of continuations of* B *with respect to* $C(z)$ *is at most countable and if* R_1, R_2 *are two continuations with* $\deg R_1 \leq \deg R_2$ *or* $R_1 \equiv 0$, *then* R_1 *is an initial part of* R_2.

Proof. — Let $R(z) \equiv B(z) + z^\lambda P(z)$ be a continuation of B with respect to $C(z)$, where P is a polynomial. If $P \equiv 0$, then, $R \equiv B$ is a partial sum of the Taylor development of $g(z)$, because $A(0)Q(0) \neq 0$, $\lambda \geq 1$ and $\lambda > \deg B$ or $B \equiv 0$. If $P \not\equiv 0$, then,

$$P(z) \equiv \frac{A(z)}{Q(z)} + \gamma z^k \cdot \frac{L_J(z)}{Q(z)},$$

according to proposition 4. Since $\deg P < k$ and $L_J(0)Q(0) \neq 0$, we see that P is a partial sum of the Taylor development of $\frac{A}{Q}$. This implies the result and completes the proof. □

If the set of continuations of B with respect to $C(z)$ if *finite*, then, according to theorem 5, there is a partial sum s_N of the Taylor development of $B(z) + z^\lambda[A(z)/Q(z)]$ with the following two properties :

(α) s_N is a continuation of B with respect to $C(z)$.

(β) every continuation of B with respect to $C(z)$ is an initial part of s_N.

Obviously, s_N is *unique*.

Next, we consider the case of infinitely many continuations of B with respect to $C(z)$.

THEOREM 6. — *Let λ, A, B, Q and $C(z)$ be as in definition 2. Then, the following (i), (ii), (iii) are equivalent :*

(i) *The set of continuations of B with respect to $C(z)$ is infinite.*

(ii) *$Q(z)$ is a non constant factor of a polynomial of the form $1-(e^{it}z)^\rho$, where $t \in R$ and $\rho \in Z$, $\rho \geq 1$.*

(iii) *There is a power series $\sum b_n z^n$ with the following two properties :*

- (*) *$\sum b_n z^n$ is not a polynomial.*
- (**) *There is an infinite subset S of $\{0, 1, \ldots\}$, such that, for all ν in S, the partial sums s_ν of $\sum b_n z^n$ are continuations of B with respect to $C(z)$.*

If a series $\sum b_n z^n$ satisfies (*) and (**), then this series is unique and coincides with the Taylor development of $B(z) + z^\lambda[A(z)/Q(z)]$; moreover, we have :

$$\sum_{n=0}^{\infty} b_n z^n \equiv B(z) + (e^{it}z)^\lambda F(e^{it}z) \sum_{m=0}^{\infty} (e^{it}z)^{m\rho},$$

where $t \in R$, $\rho \in Z$, $\rho \geq 1$ and F is a non identically zero polynomial with $\deg F < \rho$. Further, any continuation of B with respect to $C(z)$ is a partial sum of this series.

Proof. — (i) $\Rightarrow$ (ii). Since the number of subsets of the finite set I is $2^{\operatorname{card} I} = 2^{\deg A}$ and we have infinitely many continuations $B(z) + z^\lambda P(z)$ of B with respect to $C(z)$ (in fact $1 + 2^{\deg A}$ different continuations are sufficient), there are two distinct continuations with the same $J \subset I$ (see prop. 4). Then, there are γ, δ in C, $|\gamma| = |\delta| = 1$, κ, μ in Z, $0 \leq \kappa \leq \mu$, and *one* subset J of I, such that, $B(z) + z^\lambda P_1(z)$, $B(z) + z^\lambda P_2(z)$ are continuations of B with respect to $C(z)$ and the polynomials P_1 and P_2 satisfy $\deg P_1 \neq \deg P_2$ (see theorem 5) and

$$P_1(z) \equiv \frac{A(z)}{Q(z)} + \gamma z^\kappa \cdot \frac{L_J(z)}{Q(z)}, \qquad P_2(z) \equiv \frac{A(z)}{Q(z)} + \delta z^\mu \cdot \frac{L_J(z)}{Q(z)}.$$

Therefore, $A(z)[1-(\delta/\gamma)z^{\mu-\kappa}] \equiv [P_2(z) - (\delta/\gamma)z^{\mu-\kappa}P_1(z)]Q(z) : (I)$. But, A and Q do not have common factors; thus, if $\kappa < \mu$, then $Q(z)$ is a factor of $1-(e^{it}z)^\rho$, where $\rho = \mu - \kappa \geq 1$, $\rho \in Z$, $t \in R$ and $\delta/\gamma = e^{i(\mu-\kappa)t}$. Since $\deg Q > \deg A \geq 0$, the polynomial Q is a non constant factor of $1-(e^{it}z)^\rho$. If $\kappa = \mu$, then, $\deg P_1 \neq \deg P_2$ and (I) imply $\deg A > \deg Q$, which is in contradiction with $\deg A < \deg Q$.

(ii) $\Rightarrow$ (iii). We consider the power series :

$$\sum_{n=0}^{\infty} b_n z^n \equiv B(z) + z^{\lambda}[A(z)/Q(z)].$$

Since $A \not\equiv 0$ and Q is a factor of $1-(e^{it}z)^{\rho}$, we find a polynomial $F \not\equiv 0$, such that $z^{\lambda}[A(z)/Q(z)] \equiv (e^{it}z)^{\lambda}F(e^{it}z)/[1-(e^{it}z)^{\rho}]$. We observe that $\deg A < \deg Q$ yields $\deg F < \rho$. Thus, we have :

$$\sum_{n=0}^{\infty} b_n z^n \equiv B(z) + z^{\lambda}[A(z)/Q(z)] \equiv B(z) + (e^{it}z)^{\lambda}F(e^{it}z)\sum_{m=0}^{\infty}(e^{it}z)^{m\rho}.$$

Since $F \not\equiv 0$ and $\deg F < \rho$ this power series is not a polynomial; it follows that, for every $\nu = \lambda - 1 + n\rho$, $n = 0, 1, 2, \ldots$, we have :

$$s_{\nu}(z) = B(z) + (e^{it}z)^{\lambda} \cdot \frac{F(e^{it}z)}{1-(e^{it}z)^{\rho}}[1-(e^{it}z)^{n\rho}] \equiv B(z) + z^{\lambda}P(z),$$

where P is the polynomial $P(z) \equiv A(z)[1-(e^{it}z)^{n\rho}]/Q(z) \equiv e^{i\lambda t}F(e^{it}z)[1-(e^{it}z)^{n\rho}]/[1-(e^{it}z)^{\rho}]$. It follows that s_{ν} are continuations of B with respect to $C(z)$, for all $\nu = \lambda - 1 + n\rho$, $n = 0, 1, 2, \ldots$. Therefore, $\sum b_n z^n$ satisfies $(*)$ and $(**)$.

(iii) $\Rightarrow$ (i). Let $\sum b_n z^n$ be a power series satisfying $(*)$ and $(**)$. Since $\sum b_n z^n$ has infinitely many non-zero coefficients there is an infinite subset S' of S, such that, $s_{\nu} \neq s_{\mu}$, for all ν, μ in S', $\nu \neq \mu$. According to $(**)$, s_{ν} is a continuation of B with respect to $C(z)$, for each ν in S' ; thus, there are infinitely many different continuations. This gives (i).

Suppose now that $\sum b_n z^n$ is a power series satisfying $(*)$ and $(**)$. Then, according to theorem 5, $\sum b_n z^n$ has infinitely many partial sums, which are simultaneously partial sums of the Taylor development of the center function $g(z) = B(z) + z^{\lambda}[A(z)/Q(z)]$. Since $\sum b_n z^n$ is not a polynomial, we see that $\sum b_n z^n$ coincides with the above Taylor development. Further, theorem 5 implies that every continuation of B with respect to $C(z)$ is a partial sum of $\sum b_n z^n$. Finally, we have already seen, in the proof of (ii) $\Rightarrow$ (iii), that :

$$\sum_{n=0}^{\infty} b_n z^n \equiv B(z) + (e^{it}z)^{\lambda}F(e^{it}z)\sum_{m=0}^{\infty}(e^{it}z)^{m\rho}.$$

This completes the proof of theorem 6. □

3. Further results.

In this section, we use the previous results to derive stronger versions of theorem A. We first prove :

PROPOSITION 7. — *Let* $\sum_{n=0}^{\infty} c_n z^n$ *be a power series with complex coefficients. Then, (i), (ii), (iii) are equivalent :*

(i) *There exist an infinite subset* S *of* $\{0,1,2,\ldots\}$ *and a family of infinite subsets* E_M, $M = (m_1, m_2, m_3, m_4) \in S^4$, *of the unit circle* T, *such that, for every* $M = (m_1, m_2, m_3, m_4)$ *in* S^4 *and every* z *in* E_M, *the complex numbers* $s_{m_1}(z), s_{m_2}(z), s_{m_3}(z), s_{m_4}(z)$ *lie on a circle* $C_M(z)$.

(ii) *There exist* $t \in R$, $\rho \in Z$, $\rho \geq 1$, $\mu \in Z$, $\mu \geq 0$ *and polynomials* G, F *satisfying* $\mu > \deg G$ *or* $G \equiv 0$ *and* $\rho > \deg F$ *or* $F \equiv 0$, *such that :*

$$\sum_{n=0}^{\infty} c_n z^n \equiv G(e^{it}z) + (e^{it}z)^{\mu} F(e^{it}z) \sum_{m=0}^{\infty} (e^{it}z)^{m\rho}.$$

(iii) *There exist* $t \in R$ *and* r *in* $\{0,1,2,\ldots\}$, *such that, the sequence* $\{d_n\}$, $n \geq 0$, *defined by* $\sum c_n z^n \equiv \sum d_n (e^{it}z)^n$, *is periodic for* $n \geq r$.

Proof. — (i) $\Rightarrow$ (ii). If $\sum c_n z^n$ is a polynomial, then obviously (ii) holds. Therefore, we assume that infinitely many coefficients c_n are non-zero. Thus, if $\nu_1 = \min S$, we can find $\nu_2, \nu_3 \in S$, such that, $\nu_1 < \nu_2 < \nu_3$ and the partial sums of $\sum c_n z^n$, with indices ν_1, ν_2, ν_3, are different as polynomials. We fix two such indices ν_2 and ν_3 ; then, according to lemma 1, there is a finite subset Ω of T, such that, for every z in $T - \Omega$, the complex numbers $s_{\nu_1}(z), s_{\nu_2}(z), s_{\nu_3}(z)$ define a circle $C(z)$; further, there are λ, A, Q and $B \equiv s_{\nu_1}$, which determine the center and the radius of $C(z)$, as in lemma 1. By the same lemma we know that λ is the least integer greater than ν_1, such that $c_\lambda \neq 0$; it follows that for every $n > \nu_1$, the partial sum s_n is of the form $s_n = s_{\nu_1} + z^{\lambda} P_n$, where P_n is a polynomial. We observe that, for every ν_4 in S and every z in the infinite subset $E_N - \Omega$ of $T - \Omega$, where $N = (\nu_1, \nu_2, \nu_3, \nu_4) \in S^4$, we have :

$$s_{\nu_1}(z) \neq s_{\nu_2}(z) \neq s_{\nu_3}(z) \quad \text{and} \quad s_{\nu_1}(z), s_{\nu_2}(z), s_{\nu_3}(z) \in C(z) \cap C_N(z).$$

It follows that the circles $C(z)$ and $C_N(z)$ coincide, for every z in $E_N - \Omega$ and every ν_4 in S. Therefore, for each $\nu \in S$, we have $s_\nu \in C(z)$ for infinitely many z in T. Since $\nu \geq \nu_1$ and $s_\nu = s_{\nu_1} + z^{\lambda} P_\nu$, we see that s_ν is a continuation of B with respect to $C(z)$. Thus, our series satisfies $(*)$ and $(**)$ of theorem 6. Now, the same theorem assures that :

$$\sum_{n=0}^{\infty} c_n z^n \equiv B(z) + (e^{it}z)^{\lambda} F(e^{it}z) \sum_{m=0}^{\infty} (e^{it}z)^{m\rho},$$

with $\deg F < \rho$. This gives the result with $\mu = \lambda$ and G the polynomial defined by $G(e^{it}z) \equiv B(z) \equiv s_{\nu_1}(z)$.

(ii) $\Rightarrow$ (iii). Since $\deg F < \rho$, we can write :

$$F(w) \equiv \sum_{n=0}^{\rho-1} \alpha_n w^n.$$

We observe that $d_{\mu+\kappa\rho+q} = \alpha_q$ for all $\kappa \in Z$, $\kappa \geq 0$ and $q = 0, 1, \ldots, \rho - 1$. We set $r = \mu$; then $\{d_n,\ n \geq r\}$ is periodic with period ρ.

(iii) $\Rightarrow$ (i). If $\sum c_n z^n$ is a polynomial, then, there is $n_0 \in Z^+$, such that, $c_n = 0$ for all $n > n_0$. Then (i) is valid with $S = \{n_0, n_0 + 1, \ldots\}$ and $E_M = T$, $M \in S^4$. Therefore, we assume that our series is not a polynomial. If $\rho \in Z$, $\rho \geq 1$, is a period of $\{d_n\}$ $n \geq r$, then the polynomial $H(w) = d_r + d_{r+1}w + \cdots + d_{r+\rho-1}w^{\rho-1}$ is non identically zero. It follows that the set

$$E = \{z \in T : (e^{it}z)^{\rho} \neq 1, \quad H(e^{it}z) \neq 0\}$$

is infinite. Let $S = \{r + k\rho - 1 : k = 1, 2, 3, \ldots\}$. By a straightforward calculation, we see that, for every ν in S and every z in E, the partial sums $s_\nu(z)$ lie on the circle $C(z)$ with center :

$$\sum_{n=0}^{r-1} d_n (e^{it}z)^n + (e^{it}z)^r \frac{H(e^{it}z)}{1 - (e^{it}z)^{\rho}}$$

and radius :

$$\left| \frac{H(e^{it}z)}{1 - (e^{it}z)^{\rho}} \right| \neq 0.$$

This gives (i) with $E_M = E$ and $C_M(z) = C(z)$, for all $M \in S^4$. The proof is complete, now. □

The following proposition is an immediate corollary of proposition 7.

PROPOSITION 8. — *Let*

$$\sum_{n=0}^{\infty} c_n z^n$$

be a power series with complex coefficients and $E \subset T$ and $S \subset \{0, 1, 2, \ldots\}$ be infinite sets. We suppose that, for every z in E, there is a circle $C(z)$, such *that, $s_\nu(z) \in C(z)$ for all ν in S. Then the above series has a representation of the form :*

$$\sum_{n=0}^{\infty} c_n z^n = G(e^{it}z) + (e^{it}z)^{\mu} F(e^{it}z) \sum_{m=0}^{\infty} (e^{it}z)^{m\rho},$$

where t is a real number, $\mu, \rho \in Z$, $\mu \geq 0$, $\rho \geq 1$, and G, F are polynomials satisfying $\deg G < \mu$ or $G \equiv 0$ and $\deg F < \rho$ or $F \equiv 0$. Further, G, F, μ and t can be chosen so that $\deg G \leq \min S$ or $G \equiv 0$.

Proof. — It suffices to apply proposition 7 with $E_M = E$, for all $M = (m_1, m_2, m_3, m_4) \in S^4$. □

Now, we give a finite version of proposition 8 :

PROPOSITION 9. — *Let*

$$R(z) = \sum_{n=0}^{q} c_n z^n$$

be a polynomial with complex coefficients and E an infinite subset of T. We suppose that three initial parts

$$s_{\nu_k}(z) = \sum_{n=0}^{\nu_k} c_n z^n,$$

$k = 1, 2, 3$, $0 \leq \nu_1 < \nu_2 < \nu_3 < q$, of R are different as polynomials. We also suppose that, for all z in E, $R(z)$ lies on the circle $C(z)$ determined by $s_{\nu_1}(z), s_{\nu_2}(z), s_{\nu_3}(z)$. Then, the coefficients c_n, $\nu_3 < n \leq \deg R(z)$, are uniquely determined from the coefficients c_n, $\nu_1 < n \leq \nu_3$, and the integers ν_1, ν_2, ν_3. More precisely, $R(z)$ is a partial sum of the Taylor development $\sum b_n z^n$ of the unique rational function g, which gives the center of $C(z)$. We write :

$$s_{\nu_1}(z) \equiv P_1(z), \quad s_{\nu_2}(z) \equiv P_1(z) + z^{\lambda} P_2(z),$$

and

$$s_{\nu_3}(z) \equiv P_1(z) + z^{\lambda} P_2(z) + z^{\lambda+\mu+q} P_3(z),$$

where λ, μ, q are integers, $\mu = \deg P_2$, $q \geq 1$, P_1, P_2, P_3 are polynomials, with $P_2(0)P_3(0) \neq 0$, and λ is the least integer satisfying $\lambda > \nu_1$ and $c_\lambda \neq 0$; we also denote by ν the degree of P_3. Then,

$$g(z) \equiv P_1(z) + z^{\lambda} \cdot \frac{A_1(z)}{Q_1(z)},$$

where, the polynomials A_1, Q_1 are defined by the relations :

$$A_1(z) = P_2^2(z)[z^{\nu}\overline{P}_3(z)] + z^{q+\mu} P_2(z) P_3(z) [z^{\nu}\overline{P}_3(z)],$$

$$Q_1(z) = P_2(z)[z^\nu \overline{P}_3(z)] - z^{2q+\mu+\nu} P_3(z)[z^\mu \overline{P}_2(z)],$$

for all z in T.

Proof. — According to lemma 1, the circle $C(z)$, $|z| = 1$, defined by $s_{\nu_1}(z), s_{\nu_2}(z), s_{\nu_3}(z)$, has center :

$$g(z) \equiv P_1(z) + z^\lambda \cdot \frac{A_1(z)}{Q_1(z)},$$

where λ, P_1, A_1, Q_1 are as in the statement. Further, A_1/Q_1 and λ are uniquely determined from the coefficients c_n, $\nu_1 < n \leq \nu_3$ and the numbers ν_1, ν_2, ν_3. Since R is a continuation of s_{ν_1} with respect to $C(z)$, theorem 5 yields the result. □

Now, we prove a lemma which, combined with proposition 7, yields another version of theorem A.

LEMMA 10. — *Let*

$$\sum_{n=0}^{\infty} c_n z^n$$

be a power series with complex coefficients, $m \geq 1$ be an integer and let E be an infinite subset of the unit circle T. We suppose that, for every z in E, there are m circles $C_1(z), C_2(z), \ldots, C_m(z)$, such that, for all $n = 0, 1, 2, 3, \ldots$, we have

$$s_n(z) \in \bigcup_{j=1}^{m} C_j(z).$$

Then, there exist an integer j_0 in $\{1, 2, \ldots, m\}$, an infinite subset S of $\{0, 1, 2, \ldots\}$ and a decreasing family of infinite subsets E_ν of E, $\nu \in S$, such that $s_\nu(z) \in C_{j_0}(z)$, for all ν in S and z in E_ν.

Proof. — For $n = 0, 1, 2, \ldots$, and z in E, we set :

$$t(n, z) = \min\{j \in \{1, 2, \ldots, m\} : s_n(z) \in C_j(z)\}.$$

Since E is infinite and the set $\{1, 2, \ldots, m\}$ is finite, there is an infinite subset E_0 of E, such that, the map :

$$E \ni z \to t(0, z) \in \{1, 2, \ldots, m\}$$

is constant on E_0. Let t_0 be the constant value of this map restricted on E_0. Then we have $s_0(z) \in C_{t_0}(z)$, for all z in E_0. Suppose that we have defined $E_0, E_1, \ldots, E_k$, infinite subsets of E and $t_0, t_1, \ldots, t_k$ elements of

$\{1,2,\ldots,m\}$, such that, $E_{\lambda+1} \subset E_\lambda$, for all $\lambda = 0, 1, \ldots, k-1$, and $s_\lambda(z) \in C_{t_\lambda}(z)$, for all z in E_λ and $\lambda = 0,1,\ldots,k$. Since E_k is infinite and $\{1,2,\ldots,m\}$ is finite, there is an infinite subset E_{k+1} of E_k, such that, the map : $E_k \ni z \to t(k+1,z) \in \{1,2,\ldots,m\}$ is constant on E_{k+1}. We denote by t_{k+1} the constant value of this map on E_{k+1} and we have $s_{k+1}(z) \in C_{t_{k+1}}(z)$ for all $z \in E_{k+1}$.

By induction, we obtain a sequence $\{t_n\}$, $t_n \in \{1,\ldots,m\}$, $n = 0,1,2,3,\ldots$, and a decreasing sequence of infinite sets, $E \supset E_0 \supset E_1 \supset \cdots \supset E_n \supset \cdots$, such that, $s_n(z) \in C_{t_n}(z)$ for all z in E_n and $n = 0,1,2,\ldots$. Since the sequence $\{t_n\}$ takes values in the finite set $\{1,2,\ldots,m\}$, there is an infinite subset S of $\{0,1,2,\ldots\}$, such that, for all ν in S, t_ν takes the same value; say $t_\nu = j_0 \in \{1,2,\ldots,m\}$ for all ν in S. It follows

$$s_\nu(z) \in C_{j_0}(z) \text{ for all } z \text{ in } E_\nu \text{ and all } \nu \text{ in } S,$$

as requested. □

The following proposition has been announced without proof in [3].

PROPOSITION 11. — *Let*

$$\sum_{n=0}^{\infty} c_n z^n$$

be a power series with complex coefficients, $m \geq 1$ be an integer and let E be an infinite subset of the unit circle T. We suppose that, for every z in E, there are $M(z)$, $1 \leq M(z) \leq m$, circles $C_1(z), C_2(z), \ldots, C_{M(z)}(z)$, such that :

$$s_n(z) \in C_1(z) \cup C_2(z) \cup \ldots \cup C_{M(z)}(z),$$

for all $n = 0,1,2,3,\ldots$. Then, the above series has a representation of the form :

$$\sum_{n=0}^{\infty} c_n z^n = G(e^{it}z) + (e^{it}z)^\mu F(e^{it}z) \sum_{m=0}^{\infty} (e^{it}z)^{m\rho},$$

where t is a real number, μ, ρ are integers, $\mu \geq 0$, $\rho \geq 1$, and G, F are polynomials satisfying $\deg G < \mu$ or $G \equiv 0$ and $\deg F < \rho$ or $F \equiv 0$.

Proof. — If $M(z) < m$ for some z in E, then, we set $C_j(z) = C_{M(z)}(z)$ for all j, $M(z) < j \leq m$. Thus, we have :

$$s_n(z) \in C_1(z) \cup C_2(z) \cup \ldots \cup C_m(z),$$

for all $n = 0, 1, 2, \ldots$, and all z in E. By lemma 10 there exist an element j_0 of $\{1, 2, \ldots, m\}$, an infinite subset S of $\{0, 1, 2, 3, \ldots\}$ and a decreasing family E_ν, $\nu \in S$, of infinite subsets of T, such that $s_\nu(z) \in C_{j_0}(z)$, for all z in E_ν, $\nu \in S$.

For $N = (\nu_1, \nu_2, \nu_3, \nu_4) \in S^4$, we set $E_N = E_{\nu_1} \cap E_{\nu_2} \cap E_{\nu_3} \cap E_{\nu_4} = E_{\nu_4}$ which is an infinite subset of T. We see that, for every z in E_N, the complex numbers $s_{\nu_1}(z), s_{\nu_2}(z), s_{\nu_3}(z), s_{\nu_4}(z)$ lie on the circle $C_{j_0}(z)$. Now, proposition 7 yields the result. □

Finally, we prove theorem A of §1.

THEOREM 12. — *Let*

$$\sum_{n=0}^{\infty} c_n z^n$$

be a power series with complex coefficients, and E' be an infinite nondenumerable subset of the unit circle T. We suppose that, for every $z \in E'$, there are an integer $M(z)$, $M(z) \geq 1$, and $M(z)$ circles $C_1(z), C_2(z), \ldots, C_{M(z)}(z)$, such that :

$$s_n(z) \in C_1(z) \cup C_2(z) \cup \ldots \cup C_{M(z)}(z),$$

for all $n = 0, 1, 2, 3, \ldots$. Then the above series has a representation of the form :

$$\sum_{n=0}^{\infty} c_n z^n = G(e^{it}z) + (e^{it}z)^\mu F(e^{it}z) \sum_{m=0}^{\infty} (e^{it}z)^{m\rho},$$

where t is a real number, μ, ρ are integers, $\mu \geq 0$, $\rho \geq 1$, and G, F are polynomials satisfying $\deg G < \mu$ or $G \equiv 0$, and $\deg F < \rho$ or $F \equiv 0$.

Proof. — We consider the map $z \to M(z)$ from the nondenumerable set E' into the denumerable set $\{1, 2, 3, \ldots\}$. It follows that there is a nondenumerable subset E of E', such that $M(z)$ is constant on E ; say $M(z) = m \in \{1, 2, 3, \ldots\}$, for all z in E. Since E is an infinite subset of T, proposition 11 yields the result. □

4. Remarks and examples.

A. — The cardinality of the set E' in theorem 12 can not be supposed denumerable without any other supplementary hypothesis. This can easily be seen by the example :

$$\sum_{n=0}^{\infty} e^{\text{int}} z^{(2^n)}$$

with $E = \{\exp(2\pi i/2^k) : k = 0, 1, 2, \ldots\}$ and $t \in R$ (see also [3]).

The cardinality of the set E in propositions 8 and 11 can not be supposed finite. More precisely, for any finite cardinality $N < \infty$, there is a power series and a finite set $E \subset T$ with $\operatorname{card} E = N$, such that, this series is not of the form of propositions 8 or 11, but, for every z in E, a circle $C(z)$ contains all its partial sums. Such an example is given by the set $E = E_N = \{z \in T : z^N = 1\}$ and a series of the form :

$$\sum_{k=0}^{\infty} a_k z^{kN}, \text{ where } a_k \text{ satisfy } \left|\sum_{k=0}^{n} a_k\right| = 1, \text{ for all } n = 0, 1, 2, 3, \ldots ,$$

and the set $\{|a_k| : k = 0, 1, 2, \ldots\}$ is infinite. For instance, we can set $a_0 = 1$ and define inductively a_k by the relations :

$$a_0 + a_1 + \cdots + a_n = (a_0 + a_1 + \cdots + a_{n-1}) \exp(i/n).$$

B. — Lemma 1 implies that, if three partial sums $s_k(z)$,$s_n(z)$,$s_m(z)$, with $k < n < m$, of a power series lie on a straight line $\varepsilon(z)$, for every z in an infinite subset E of T, then $s_k \equiv s_n$ or $s_n \equiv s_m$. This observation shows that, if in propositions 7, 8, 11 and theorem 12 we replace circles by straight lines, then the power series in question are polynomials.

By the term "generalized circle" we mean any subset of the extended plane $C \cup \{\infty\}$, which is a circle or a straight line extended with the point at infinity. It is easy to check that propositions 7, 8, 11 and theorem 12 remain valid, if we replace circles by generalized circles.

Further, by a straightforward calculation, one can check that the converses of propositions 8, 11 and theorem 12 also hold (see proposition 9 and [3]).

C. — Let λ, A, B, Q and $C(z)$ be as in definition 2. Using lemma 2, one can easily see that, every polynomial R, which is not a continuation of B with respect to $C(z)$, but satisfies $R(z) \in C(z)$ for infinitely many z in T, is of the form :

$$R(z) \equiv B(z) + \sum_{n=k}^{p} \beta_n z^n,$$

where $p = \lambda + \deg A - \deg Q < \lambda$, $0 \le k \le p$ and $\sum_{-\infty}^{p} \beta_n z^n$ is the Laurent development of $z^\lambda[A(z)/Q(z)]$ around ∞. It follows that there are at most finitely many such polynomials R. In particular, if $\lambda < \deg Q - \deg A$, there is no such polynomial.

Let us consider the particular case, where B is the opposite of the regular part of the Laurent development $\sum_{-\infty}^{p} \beta_n z^n$ of $z^\lambda A(z)/Q(z)$, *i.e.* $B(z) = -\sum_{n=0}^{p} \beta_n z^n$. Then all polynomials R, which are not continuations of B with respect to $C(z)$, but satisfy $R(z) \in C(z)$ for infinitely many z in T, are initial parts of B. It follows that, in this particular case, all polynomials R satisfying $R(z) \in C(z)$ for infinitely many z in T (which may be continuations or not), are partial sums of one power series, the Taylor development of $B(z) + z^\lambda[A(z)/Q(z)]$. This shows that the above polynomials R, may be seen as continuations after some minor modifications.

In the proof of theorem 6, we see that, if two different continuations correspond to the same polynomial L_J (see prop. 4), then $Q(z)$ is of the form (ii) of theorem 6. A similar argument shows that if two different polynomials R_1 and R_2 (not necessary continuations of B with respect to $C(z)$) satisfy $R_1(z) \in C(z)$ and $R_2(z) \in C(z)$, for infinitely many $z \in T$, and if R_1, R_2 correspond to the same $J \subset I$ in lemma 3, then Q is of the form (ii) of theorem 6.

The questions, which and how many polynomials L_J can appear in proposition 4 and how many continuations exist, are not yet completely investigated. In [3] one can find information about the number of the circles appearing in theorem 12. This number is related with the absolute values of the polynomials obtained by circular permutations of the coefficients of the polynomial F. The same question has not yet been examined in connection with the polynomials L_J, in proposition 4.

D. — In connection with the not yet investigated problem of the number of continuations we give examples illustrating some of the possibilities which can arise.

Let λ, A, B, Q and $C(z)$ be as in definition 2 and γ, k, J as in proposition 4.

(i) Let $B \equiv 0$, $\lambda = 1$, $A(z) \equiv c \prod_{j \in I}(1 + \bar{a}_j z) = 1 + 2z$ and $Q(z) = z^2 - 3z + 2$. Then $I = \{1\}$, $a_1 = 2$, and there is only one continuation $R \equiv B \equiv 0$ of B with respect to $C(z)$, which corresponds to $\gamma = -1$, $k = 0$ and $J = \emptyset$.

(ii) Let $B \equiv 0$, $\lambda = 1$, $A(z) = (1-3z)(2-9z)$ and $Q(z) = z^3-2z^2-z+2$. Then $I = \{1,2\}$, $a_1 = -3$, $a_2 = -9/2$ and there are exactly two continuations R_1 and R_2 of B with respect to $C(z)$:

$R_1 \equiv B \equiv 0$ corresponds to $\gamma = -1$, $k = 0$ and $J = \emptyset$.

$R_2(z) \equiv B(z)+z(1-7z+11z^2-2z^3) = z-7z^2+11z^3-2z^4$ corresponds to $\gamma = -1$, $k = 4$ and $J = \{1,2\} = I$.

(iii) Let $B \equiv 0$, $\lambda = 1$, $A(z) = (1 + a_1 z)(1 + a_2 z)(1 + a_3 z)$ and $Q(z) = 2z^4 - 5z^3 + 5z - 2$; thus $I = \{1,2,3\}$. Then :

(a) If $a_1 = 3$, $a_2 = -11/2$, $a_3 = -51/22$, there are exactly three continuations R_1, R_2, R_3 of B with respect to $C(z)$:

$R_1 \equiv B \equiv 0$ corresponds to $\gamma = -1$, $k = 0$ and $J = \emptyset$.

$R_2(z) \equiv B(z)+z\left(-\frac{1}{2}+\frac{51}{22}z\right) = -\frac{1}{2}z+\frac{51}{22}z^2$ corresponds to $\gamma = -1$, $k = 2$ and $J = \{1,2\}$.

$R_3(z) \equiv B(z)+z\left(-\frac{1}{2}+\frac{51}{22}z+\frac{12}{11}z^2+\frac{11}{4}z^3\right) = -\frac{1}{2}z+\frac{51}{22}z^2+\frac{12}{11}z^3+\frac{11}{4}z^4$ corresponds to $\gamma = -1$, $k = 4$ and $J = \{1,3\}$.

(b) If $a_1 = -3$, $a_2 = -9/2$, $a_3 = -5/2$, there are exactly four continuations R_1, R_2, R_3, R_4 of B with respect to $C(z)$:

$R_1 \equiv B \equiv 0$ corresponds to $\gamma = -1$, $k = 0$ and $J = \emptyset$.

$R_2(z) \equiv -\frac{1}{4}z(2-9z)(1-3z) = \frac{1}{4}(-2z+15z^2-27z^3)$ corresponds to $\gamma = -1$, $k = 3$ and $J = \{3\}$.

$R_3(z) \equiv \frac{1}{4}z(5z-2)(z^2-5z+1) = \frac{1}{4}(-2z+15z^2-27z^3+5z^4)$ corresponds to $\gamma = -1$, $k = 4$ and $J = \{1,2\}$.

$R_4(z) \equiv \frac{1}{4}(-2z+15z^2-27z^3+5z^4-27z^5+15z^6-2z^7)$ corresponds to $\gamma = -1$, $k = 3$ and $J = \{1,2,3\} = I$.

(c) If $a_1 = -5/2$, $a_2 = -21/10$, $a_3 = -85/42$, then, the number of continuations of B with respect to $C(z)$ is equal to the number of subsets of I, that is $2^3 = 8$. According to proposition 5, these eight continuations are partial sums of the Taylor development of the center function $g(z) \equiv B(z)+z[A(z)/Q(z)]$. We write an initial part of this Taylor series :

$$0 - 420z + 1732z^2 - 1785z^3 + 1050z^4 - 2125z^5 + 882z^6 - 2205z^7 + 850z^8 - 2205z^9 + 882z^{10} - 2125z^{11} + 1050z^{12} - 1785z^{13} + 1732z^{14} - 420z^{15} + 4462.5z^{16} + 5041.25z^{17} + \cdots ;$$

then, the above mentioned eight continuations are $s_0, s_3, s_5, s_7, s_8, s_{10}, s_{12}$ and s_{15}.

(iv) Let $B \equiv 0$, $\lambda = 11$, $A(z) = (3/2)(1+a_1 z)(1+a_2 z)(1+a_3 z)^3$ and $Q(z) = z^6 + 6az^5 - 6az - 1$, where $a = (2\sqrt{6})^{-1}$, $a_1 = 6a$, $a_2 = -2a$, $a_3 = a_4 = a_5 = 4a$.

In this case Q is not of the form (ii) of theorem 6, $I = \{1,2,3,4,5\}$ and the number of all possible polynomials L_J is sixteen (see prop. 4 and lemma 3); however, only eight of them lead to polynomials R satisfying $R(z) \in C(z)$, for infinitely many z in T. Five of these polynomials R are continuations of B with respect to $C(z)$ and correspond to $\gamma = -1$, $k = 0,3,5,8,13$ and $J = \emptyset, I, \{1\}, \{2,3\}, \{1,2,3\}$, respectively. These five continuations are the following :

$$R_1 \equiv B \equiv 0, \quad R_2(z) = -(3/2)z^{11} - 15az^{12} - (3/2)z^{13},$$
$$R_3(z) = -(3/2)z^{11} - 15az^{12} - (3/2)z^{13} + 2az^{14} + (1/3)z^{15},$$
$$R_4(z) = -(3/2)z^{11} - 15az^{12} - (3/2)z^{13} + 2az^{14} + (1/3)z^{15} - 9az^{16} - 3z^{17} - 6az^{18},$$
$$R_5(z) = -(3/2)z^{11} - 15az^{12} - (3/2)z^{13} + 2az^{14} + (1/3)z^{15} - 9az^{16} - 3z^{17} - 6az^{18} + (1/2)z^{19} + az^{20} - (13/6)z^{21} - 14az^{22} - z^{23},$$

respectively. The other three polynomials R_6, R_7, R_8 are not continuations of B with respect to $C(z)$ and correspond to $\gamma = -1$, $k = -2,-5,-10$ and $J = \{2,3,4,5\}, \{1,4,5\}, \{4,5\}$. More explicitely we have :

$$R_6(z) = -(1/3)z^9 - 2az^{10}, \quad R_7(z) = 6az^6 + 3z^7 + 9az^8 - (1/3)z^9 - 2az^{10},$$

and

$$R_8(z) = z + 14az^2 + (13/6)z^3 - az^4 - (1/2)z^5 + 6az^6 + 3z^7 + 9az^8 - (1/3)z^9 - 2az^{10}.$$

(v) In [3] it has been proved that the power series (which are not polynomials), such that, all partial sums are contained in one circle $C(z)$, are exactly the series of the form :

$$[a + b(e^{it}z)^{\rho}] \sum_{n=0}^{\infty} (e^{it}z)^{2n\rho},$$

with $\bar{a}b \in R$. We can arrive to the same characterization using the method of the present paper, as well. Then the subsets J of I that can appear are $J = \emptyset$ and $J = I$. In the case $b = 0$ the only J that appears is $J = \emptyset$. If $b \neq 0$, then $J = \emptyset$ appears for n even and $J = I$ appears for n odd.

We also mention that in [3] one can find a characterization of the power series, such that, all partial sums lie on exactly two circles.

E. — In connection with proposition 8, S. Argyros, A. Bernard and S. Pichorides asked whether the set S may depend on the point z, $|z| = 1$. We prove the following :

PROPOSITION 8A. — *Let $\sum_0^\infty c_n z^n$ be a power series and $E \subset T$ a nondenumerable set. We suppose that for every $z \in E$ there are a circle $C(z)$ and an infinite set $I_z \subset \{0, 1, 2, \ldots\}$, such that $s_\nu(z) \in C(z)$ for all $v \in I_z$. Then there are $t \in R$ and n_0, such that the sequence $c_n e^{\mathrm{int}}$, $n \geq n_0$, is periodic.*

For the proof we consider the function $z \to \min I_z$, $z \in E$, $\min I_z$ in $\{0, 1, 2, \ldots\}$. Since E is nondenumerable, we find $E_1 \subset E$ nondenumerable and $\nu_1 \in \{0, 1, 2, \ldots\}$, such that $\min I_z = \nu_1$ for all $z \in E_1$; obviously $\nu_1 \in I_z$ for all $z \in E_1$. By induction we find a decreasing sequence of nondenumerable sets $E \supset E_1 \supset E_2 \supset E_3 \supset \ldots$ and an increasing sequence of integers $0 \leq \nu_1 < \nu_2 < \ldots$, so that $\nu_k \in I_z$ for all $z \in E_k$ and $k = 1, 2, \ldots$; obviously $s_{\nu_k} \in C(z)$ for all $z \in E_k$ and $k = 1, 2, \ldots$.

We set $S = \{\nu_k : \kappa = 1, 2, \ldots\}$. Then, for any four indices m_1, m_2, m_3, m_4 in S and any $z \in E_{m_1} \cap E_{m_2} \cap E_{m_3} \cap E_{m_4}$, we have $s_{m_1}(z), s_{m_2}(z), s_{m_3}(z), s_{m_4}(z) \in C(z)$. Since the set $E_{m_1} \cap E_{m_2} \cap E_{m_3} \cap E_{m_4}$ is infinite, proposition 7 yields the result.

Finally the example $E = \{\exp \frac{2\pi i}{k} : k = 1, 2, \ldots\}$, $I_{\exp \frac{2\pi i}{k}} = \{k, k + 1, \ldots\}$, $\sum_0^\infty c_n z^n = \sum_0^\infty i^m z^{m!}$ shows easily that the result of proposition 8A fails in general, if the set E is countable.

BIBLIOGRAPHY

[1] J.-P. KAHANE, Sur la structure circulaire des ensembles de points limites des sommes partielles d'une série de Taylor, Acta Sci. Math. (Szeged), 45, n° 1–4 (1983), 247–251.

[2] E.S. KATSOPRINAKIS, Characterization of power series with partial sums on a finite number of circles (in Greek), Ph. D. thesis 1988, Dept. of Mathematics, University of Crete, Iraklion, Greece.

[3] E.S. KATSOPRINAKIS, On a Theorem of Marcinkiewicz and Zygmund for Taylor series, Arkiv for Matematik, to appear.

[4] J. MARCINKIEWICZ and A. ZYGMUND, On the behavior of Trigonometric series and power series, T.A.M.S., 50 (1941), 407–453.

[5] A. ZYGMUND, Trigonometric Series, 2nd edition, reprinted, Vol. I. II, Cambridge : Cambridge University Press, 1979.

Manuscrit reçu le 10 novembre 1988.

E.S. KATSOPRINAKIS & V.N. NESTORIDIS,
Department of Mathematics
University of Crete
P.O. Box 470
Iraklion – Crete (Greece).