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EMBEDDABILITY OF ABSTRACT CR STRUCTURES AND INTEGRABILITY OF RELATED SYSTEMS

by

M.S. BAOUENDI and L.P. ROTHSCHILD

1. Introduction.

Let M be a smooth real manifold of dimension N, and $\mathcal{V}$ a subbundle of the complex tangent bundle, $\mathbf{C}$TM, with dim $\mathcal{V} = n$. We shall say that $\mathcal{V}$ is *integrable* at a point $p_0 \in M$ if there exists a neighborhood Ω_0 of p_0 and smooth functions $\zeta_1, \ldots, \zeta_{N-n}$ defined on Ω_0 with linearly independent differentials and satisfying

$$\text{(1.1)} \qquad L\zeta_k = 0 \quad \text{in } \Omega_0, \quad k = 1, \ldots, N-n,$$

for all $L \in \mathbf{L}_0$, where $\mathbf{L}_0 = C^\infty(\Omega_0, \mathcal{V})$, the space of smooth sections of $\mathcal{V}$ over Ω_0. In this paper we shall give a criterion for local integrability.

We call $\mathcal{V}$ *formally integrable* if

$$\text{(1.2)} \qquad [\mathcal{V}, \mathcal{V}] \subset \mathcal{V},$$

i.e. if for any sections $L, L' \in \mathbf{L}$, we have $[L, L'] \in \mathbf{L}$, where $\mathbf{L} = C^\infty(M, \mathcal{V})$. The Frobenius theorem then says that formal integrability implies integrability if $\mathcal{V}$ is real (resp. real analytic), i.e. if $\mathbf{L}$ has a basis of real (resp. real analytic) sections. In the general case it is easy to check by dimension that formal integrability is a necessary condition for integrability.

If, in addition, $\mathcal{V}$ satisfies

$$\text{(1.3)} \qquad \mathcal{V} \cap \overline{\mathcal{V}} = (0)$$

then $\mathcal{V}$ is called an *abstract* CR *bundle*, and M an *abstract* CR *manifold.* In this case we have $N = 2n + \ell$ with $\ell \geqslant 0$. We say that $\mathcal{V}$ is of *codimension* ℓ.

Key-words: Embeddability – CR structures – Complex Lie algebre.

A submanifold of $\mathbf{C}^{n+\ell}$ is a *generic* CR *manifold* if it is locally given by $\rho_j = 0\,, j = 1\,,\ldots,\ell\,,$ with ρ_j real valued, smooth, and satisfying $\partial\rho_1\,,\ldots,\partial\rho_\ell$ linearly independent. It can be easily shown that an abstract CR manifold is integrable at p_0 if and only if near p_0, M can be embedded as a generic CR manifold in $\mathbf{C}^{n+\ell}$, with the image of $\mathcal{V}$ equal to the induced CR bundle i.e. the bundle whose sections are tangential, antiholomorphic vector fields.

For this reason an integrable CR structure is also called *embeddable* or *realizable*. The first example of a nonembeddable strictly pseudoconvex abstract hypersurface was given by Nirenberg [8]. (See also Jacobowitz-Treves [5]).

Our main result is the following :

THEOREM. – *Let* M *be a smooth manifold and* $\mathcal{V} \subset \mathbf{C}\mathrm{TM}$ *a subbundle satisfying*

$$[\mathbf{L}\,,\mathbf{L}] \subset \mathbf{L}\,,$$

where $\mathbf{L} = C^\infty(\mathrm{M}\,,\mathcal{V})$. *Then* $\mathcal{V}$ *is locally integrable at* $p_0 \in \mathrm{M}$ *if and only if there exist* $\Omega_0 \subset \mathrm{M}$, *an open neighborhood of* p_0 *in* M, *and smooth complex vector fields* $\mathrm{R}_1\,,\ldots,\mathrm{R}_\ell$ *defined in* Ω_0 *spanning a complex Lie algebra i.e.*

$$[\mathrm{R}_i\,,\mathrm{R}_j] = \sum_{k=1}^{\ell} a_{ijk}\,\mathrm{R}_k\,,\quad a_{ijk} \in \mathbf{C}\,, \tag{1.4}$$

and satisfying

$$[\mathbf{L}_0\,,\mathrm{R}_j] \subset \mathbf{L}_0\,,\quad j = 1\,,\ldots,n\,, \tag{1.5}$$

with $\mathbf{L}_0 = C^\infty(\Omega_0\,,\mathcal{V})$, *and for every* $p \in \Omega_0$

$$\mathcal{V}_p + \overline{\mathcal{V}}_p + \mathcal{R}_p + \overline{\mathcal{R}}_p = \mathbf{C}\mathrm{T}_p\,\Omega_0\,, \tag{1.6}$$

where $\mathcal{V}_p$ *is the fiber of* $\mathcal{V}$ *at* p, *and* $\mathcal{R}_p$ *is the span of the* R_j *at* p. *More precisely, if* $\mathcal{V}$ *is integrable, we may find* R_j *so that* $a_{ijk} = 0$ *for all* i, j, k *and replace (1.6) by*

$$\mathcal{V}_p + \overline{\mathcal{V}}_p \oplus \mathcal{R}_p = \mathbf{C}\mathrm{T}_p\,\Omega_0\,. \tag{1.7}$$

For an integrable structure, the existence of vector fields R_j satisfying conditions similar to (1.4) with $a_{ijk} = 0$, (1.5) and (1.6) was proved and used in Baouendi-Treves [2]. However, the proof we give here is more natural to the embedding and is used to establish the result for the general case.

For the case where $\mathcal{V}$ is an abstract CR structure, the integrability result generalizes a theorem of Jacobowitz [4] where $\mathcal{V}$ is of codimension one, and a theorem of the authors and Treves [1] for the case where the R_j are real independent vector fields. As in [1], the proof of integrability depends, in the CR case, on the Newlander-Nirenberg theorem [6], and in the general case on a corollary of Nirenberg [7], (see also Hörmander [3] and Treves [9]) which states that $\mathcal{V}$ is integrable if $\mathcal{V} + \overline{\mathcal{V}} = \mathbf{C}TM$; we reprove this result by methods in the spirit of this paper.

Remark. – Note that we do not require the vector fields R_j satisfying (1.4), (1.5) and (1.6) to be linearly independent at every point of Ω_0. However, when $\mathcal{V}$ is integrable, we may choose them linearly independent, and such that the subbundle $\mathcal{R}$ whose sections are spanned by them is totally real i.e.

$$\overline{\mathcal{R}} = \mathcal{R}.$$

2. Proof of the existence of the $\mathbf{R}_j$.

We assume first that $\mathcal{V}$ is CR i.e. $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$. Assume M is integrable at p_0, so that M may be regarded as a submanifold of $\mathbf{C}^{n+\ell}$ given by

$$\rho_j = 0\,, \quad j = 1, \dots, \ell \tag{2.1}$$

and $\partial\rho_1, \dots, \partial\rho_\ell$ linearly independent.

By relabeling the coordinates in $\mathbf{C}^{n+\ell}$ we may take $(z, w) \in \mathbf{C}^{n+\ell}$, $w \in \mathbf{C}^\ell$, and assume that

$$\rho_w = \begin{pmatrix} \dfrac{\partial\rho_1}{\partial w_1} & \cdots & \dfrac{\partial\rho_1}{\partial w_\ell} \\ \vdots & & \vdots \\ \dfrac{\partial\rho_\ell}{\partial w_1} & \cdots & \dfrac{\partial\rho_\ell}{\partial w_\ell} \end{pmatrix} \tag{2.2}$$

is invertible near the origin. Similarly, we let

(2.3)
$$\rho_z = \begin{pmatrix} \frac{\partial \rho_1}{\partial z_1} & \cdots & \frac{\partial \rho_1}{\partial z_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial \rho_\ell}{\partial z_1} & \cdots & \frac{\partial \rho_\ell}{\partial z_n} \end{pmatrix}$$

be an $\ell \times n$ matrix. Then a local basis for $C^\infty(M, \mathcal{V})$ is obtained as $(L) = \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix}$, with

$$(L) = \left(\frac{\partial}{\partial \bar{z}}\right) - {}^t\rho_{\bar{z}}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right),$$

where we have written $\left(\frac{\partial}{\partial \bar{z}}\right)$ for $\begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} \\ \vdots \\ \frac{\partial}{\partial \bar{z}_n} \end{pmatrix}$ and similarly for $\frac{\partial}{\partial \bar{w}}$.

We have

(2.4) Proposition. – *Set* $(R) = \begin{pmatrix} R_1 \\ \vdots \\ R_\ell \end{pmatrix}$ *where*

$$(R) = \left(\frac{\partial}{\partial w}\right) - {}^t\rho_w\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right).$$

Then the R_j are tangent to M, commute, and satisfy (1.5), and (1.7).

Proof. – Since $R_j \rho_k = 0$ by construction, the R_j are tangent to M. To prove (1.7) we observe that since $N = 2n + \ell$, and the $L_j, \bar{L}_j$ and R_k are all linearly independent, the result holds by dimension.

For (1.4) and (1.5) we calculate $[L_j, R_k]$ and $[R_j, R_k]$. Each is again tangent to M, and from the form of the $L's$ and $R's$, they contain only $\frac{\partial}{\partial \overline{w}_k}$, and hence are antiholomorphic. Since the L_j form a basis for the tangential antiholomorphic vector fields to M, each $[L_j, R_k]$ and $[R_j, R_k]$ is a linear combination of the $L_j's$ with smooth coefficients. These coefficients must be zero, since neither commutator contains a term of the form $\frac{\partial}{\partial \overline{z}_p}$. This proves (1.4) (with $a_{ijk} = 0$) and (1.5), and hence Proposition (2.4).

□

We now assume that $\mathcal{V}$ is integrable but not necessarily CR. We shall construct the R_j by adding variables in order to reduce to the case of a CR bundle. Let Ω be a small neighborhood of p_0 in M. First choose a basis L_j of $C^\infty(\Omega, \mathcal{V})$ and coordinates (x, y, t, s) in Ω vanishing at p_0,

$$x, y \in \mathbf{R}^r, t \in \mathbf{R}^{n-r}, s \in \mathbf{R}^{\ell}$$

with $\ell = N - n - r$, such that

$$(2.5) \qquad L_j|_{p_0} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \quad 1 \leqslant j \leqslant r,$$

and

$$(2.6) \qquad L_{j+r}|_{p_0} = \frac{\partial}{\partial t_j}, \quad r+1 \leqslant j \leqslant n.$$

We introduce $n - r$ new variables $t'_1, \dots, t'_{n-r}$ and define new vector fields $\widetilde{L}_j$ in $\Omega' = \Omega \times \mathbf{R}^{n-r}$ by

$$\widetilde{L}_j = L_j, \quad 1 \leqslant j \leqslant r,$$

and for $r+1 \leqslant j \leqslant n$, $\widetilde{L}_j$ is obtained from L_j by replacing $\frac{\partial}{\partial t_j}$ by $\frac{\partial}{\partial t_j} + i\frac{\partial}{\partial t'_j}$. Let $\mathcal{V}'$ be the bundle with sections spanned by the $\widetilde{L}_j$ on Ω'. If $\zeta_1, \dots, \zeta_{r+\ell}$ is a set of independent solutions for $\mathcal{V}$, then $\zeta_1, \dots, \zeta_{r+\ell}, t_1 + it'_1, \dots, t_{n-r} + it'_{n-r}$ is a set of independent solutions for $\mathcal{V}'$. Since $\mathcal{V}' \cap \overline{\mathcal{V}'} = \{0\}$, we have proved

(2.7) LEMMA. — *$\mathcal{V}'$ is an integrable* CR *bundle on Ω'.*

Let $\tau_j = t_j + it'_j$, $\tau = (\tau_1, \ldots, \tau_{n-r})$ and $\zeta = (\zeta_1, \ldots, \zeta_{r+\ell})$. The mapping

$$(x, y, t, t', s) \longmapsto (\zeta(x, y, t, s), \tau)$$

is an embedding of Ω' onto a CR generic submanifold of $\mathbf{C}^{n+\ell}$. Therefore there exist real smooth functions $\rho_j(Z, \overline{Z})$ in $\mathbf{C}^{n+\ell}$ so that locally the image of Ω' is given by $\rho_j = 0$, $j = 1, \ldots, \ell$, with $\partial\rho_1, \ldots, \partial\rho_\ell$ linearly independent. Hence we have for $j = 1, \ldots, \ell$

$$\rho_j(\zeta(x, y, t, s), \tau, \overline{\zeta(x, y, t, s)}, \overline{\tau}) \equiv 0 \tag{2.8}$$

in Ω'.

We may assume that $\zeta(0) = 0$. If $Z_1, \ldots, Z_{n+\ell}$ are the variables in $\mathbf{C}^{n+\ell}$, we write τ_k for $Z_{k+r+\ell}$, $k = 1, \ldots, n-r$.

(2.9) Lemma. – *We may assume that the ρ_j are independent of t'_k. Also we have for $j = 1, \ldots, \ell$ and $k = 1, \ldots, n-r$*

$$\frac{\partial \rho_j}{\partial \tau_k}(0) = 0.$$

Proof. – It suffices to differentiate (2.8) with respect to t_k and t'_k, and to use (2.6) and the fact that the ζ_j satisfy the equations

$$L_p \zeta_k = 0 \quad 1 \leqslant p \leqslant n, \quad 1 \leqslant k \leqslant r + \ell.$$

This proves the lemma.

□

Since the ρ_j have independent complex differentials, the matrix

$$\begin{bmatrix} \rho_{1z_1} & \cdots & \rho_{1z_{\ell+r}} & \rho_{1\tau_1} & \cdots & \rho_{1\tau_{n-r}} \\ \rho_{\ell z_1} & \cdots & \rho_{\ell z_{\ell+r}} & \rho_{\ell\tau_1} & \cdots & \rho_{\ell\tau_{n-r}} \end{bmatrix}$$

has rank ℓ, therefore by Lemma (2.9) the submatrix

$$\left[\frac{\partial \rho_j}{\partial z_k}\right]_{1 \leqslant j \leqslant \ell, 1 \leqslant k \leqslant \ell + r}$$

must have rank ℓ at 0. Hence we may find new coordinates $(z, w) \in \mathbf{C}^r \times \mathbf{C}^\ell$ such that the matrix $\left[\dfrac{\partial \rho}{\partial w}\right]$ is invertible at 0. In these coordinates we may find a basis for $\mathcal{V}'$ in the form $(\widetilde{\mathrm{L}}) = \begin{pmatrix} \widetilde{\mathrm{L}}' \\ \widetilde{\mathrm{L}}'' \end{pmatrix}$, where

$$(2.11) \qquad (\widetilde{\mathrm{L}}') = \left(\frac{\partial}{\partial \bar{z}}\right) - {}^t\rho_{\bar{z}}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right),$$

and

$$(2.12) \qquad (\widetilde{\mathrm{L}}'') = \left(\frac{\partial}{\partial \bar{\tau}}\right) - {}^t\rho_{\bar{\tau}}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right),$$

where we use the notation conventions of § 2. Rectricting to $t' = 0$ we find a basis (L) for $\mathcal{V}$ given by $(\mathrm{L}) = \begin{pmatrix} \mathrm{L}' \\ \mathrm{L}'' \end{pmatrix}$:

$$(2.13) \qquad (\mathrm{L}') = \left(\frac{\partial}{\partial \bar{z}}\right) - {}^t\rho_{\bar{z}}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right),$$

and

$$(2.14) \qquad (\mathrm{L}'') = \left(\frac{\partial}{\partial t}\right) - {}^t\rho_{t}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right).$$

Now put

$$(\mathrm{R}) = \left(\frac{\partial}{\partial w}\right) - {}^t\rho_{w}\, {}^t\rho_{\bar{w}}^{-1} \left(\frac{\partial}{\partial \bar{w}}\right)$$

as before. □

3. Proof of Integrability.

We now assume $\{\mathrm{R}_j\}$ exist satisfying (1.4), (1.5), and (1.6) and prove $\mathcal{V}$ is integrable. First we give a new proof of the following result of Nirenberg [7].

(3.1) PROPOSITION. – *If $\mathcal{W}$ is a formally integrable subbundle of* **CTM** *for which*

$$(3.2) \qquad \mathcal{W} + \overline{\mathcal{W}} = \mathbf{C}TM\,,$$

then $\mathcal{W}$ *is locally integrable.*

Proof. – Let Ω be a small neighborhood of $p_0 \in M$, and $V_1, V_2, \ldots, V_n$ be a commuting basis for $C^\infty(\Omega, \mathcal{W})$. After renumbering and multiplication by complex numbers we may assume $V_1, \ldots, V_r$ is a maximal set for which $V_1, \ldots, V_r, \overline{V}_1, \ldots, \overline{V}_r$ is linearly independent at p_0, and that these, together with $\mathrm{Re}\, V_j$, $j > r$, span the section of $\mathbf{C}T\Omega$. Now let $\widetilde{\mathcal{W}}$ be the bundle over $\Omega \times \mathbf{R}^{n-r}$ whose sections are spanned by $\widetilde{V}_j = V_j$, $1 \leqslant j \leqslant r$, and $\widetilde{V}_j = V_j + i\dfrac{\partial}{\partial t_{r-j}}$, $j = r+1, \ldots, n$. Then $\widetilde{\mathcal{W}}$ satisfies the conditions of the Newlander-Nirenberg theorem [6] since

$$\widetilde{\mathcal{W}} \cap \overline{\widetilde{\mathcal{W}}} = (0)\,.$$

Hence there exist n solutions $f_1(u,t), \ldots, f_n(u,t)$ for $\widetilde{\mathcal{W}}$, where (u) is a coordinate system near p_0 in Ω vanishing at p_0, and t is in a neighborhood of 0 in $\mathbf{R}^{n-r}$. We may assume $f_j(0) = 0$, $j = 1, \ldots, n$.

We shall obtain solutions for $\mathcal{W}$ in the form

$$\zeta_k = F_k(f_1, \ldots, f_n)\,,$$

where each $F_k(Z)$ is holomorphic and satisfies

$$(3.3) \qquad \frac{\partial}{\partial t_j}[F_k(f_1(u,t), \ldots, f_n(u,t))] \equiv 0\,, \quad j = 1, \ldots, n-r\,.$$

We shall prove that there exist $F_1, \ldots, F_r$ holomorphic satisfying (3.3) with linearly independent differentials. Indeed, for F holomorphic

$$(3.4) \qquad \frac{\partial}{\partial t_j} F(f_1, \ldots, f_n) = \sum_{p=1}^{n} \frac{\partial f_p}{\partial t_j} \frac{\partial F}{\partial Z_p}(f_1, \ldots, f_n)\,.$$

Since we may choose a basis for $\widetilde{\mathcal{W}}$ taking vector fields with coefficients independent of the t_j, $\dfrac{\partial f_p}{\partial t_j}$ is again a solution for $\widetilde{\mathcal{W}}$. Hence there exists a holomorphic function H_{pj} such that

$$(3.5) \qquad \frac{\partial f_p}{\partial t_j} = \mathrm{H}_{pj}(f_1, \ldots, f_n), \quad 1 \leqslant p \leqslant n, \quad 1 \leqslant j \leqslant n-r.$$

Substituting (3.4) and (3.5) into (3.3) we obtain the system

$$(3.6) \qquad \sum_{p=1}^{n} \mathrm{H}_{pj}(\mathrm{Z}) \frac{\partial \mathrm{F}}{\partial \mathrm{Z}_p}(\mathrm{Z}) = 0, \quad j = 1, \ldots, n-r.$$

Since $df_1, \ldots, df_n, d\overline{f_1}, \ldots, d\overline{f_n}$ are linearly independent we conclude that the matrix

$$\left(\frac{\partial f_p}{\partial t_j}\right), \quad 1 \leqslant p \leqslant n, \quad 1 \leqslant j \leqslant n-r,$$

is of rank $n-r$. Therefore by (3.5) the same is true for the matrix (H_{pj}) at the origin. It follows by the Cauchy-Kovalevsky Theorem that there are $n-(n-r)=r$ linearly independent solution F_k of (3.6) near 0. Hence the functions

$$\zeta_k(u) = \mathrm{F}_k(f_1(u,t), \ldots, f_n(u,t)), \quad 1 \leqslant k \leqslant r,$$

provide a system of solutions for $\mathcal{W}$, proving integrability.

□

We may now complete the proof of the theorem. We assume we are given the R_j satisfying (1.4), (1.5) and (1.6). We let $\mathrm{S}_1, \ldots, \mathrm{S}_\ell$ be a basis for an abstract complex Lie algebra satisfying the same commutation relations as the R_j i.e.

$$(3.7) \qquad [\mathrm{S}_i, \mathrm{S}_j] = \sum_{k=1}^{\ell} a_{ijk} \mathrm{S}_k .$$

By introducing local exponential coordinates on any corresponding connected complex Lie group we may find coordinates in an open neighborhood $\mathcal{O}$ of 0 in $\mathbf{C}^\ell$ near 0 in which we may represent the S_j as holomorphic vector fields with holomorphic coefficients i.e.

$$(3.8) \qquad \mathrm{S}_j = \sum_{k=1}^{\ell} a_{jk}(t) \frac{\partial}{\partial t_k}$$

with $t_k = t'_k + it''_k \in \mathbf{C}$ and the matrix (a_{jk}) is invertible. Now we let $\mathrm{R}'_j = \mathrm{R}_j + \mathrm{S}_j$. We claim that the bundle $\widetilde{\mathcal{V}}$ over $\Omega \times \mathcal{O}$

spanned by $\mathcal{V}$, $\{R'_j\}_{1 \leqslant j \leqslant \ell}$ and $\left\{\frac{\partial}{\partial \bar{t}_k}\right\}_{1 \leqslant k \leqslant \ell}$ satisfies the condition of Proposition (3.1) for integrability.

Indeed, note that the S_j commute with $\frac{\partial}{\partial \bar{t}_j}$, as well as the R_j and $\mathbf{L}_0$. Hence

$$[R_i + S_i, R_j + S_j] = \Sigma\, a_{ijk}\,(R_k + S_k), \tag{3.9}$$

which proves that $\widetilde{\mathcal{V}}$ is formally integrable. Also, the span of the $\widetilde{R}_j$, $\overline{\widetilde{R}}_j$, $\frac{\partial}{\partial t_j}$ and $\frac{\partial}{\partial \bar{t}_j}$ is the same as that of the R_j, $\overline{R}_j$, $\frac{\partial}{\partial t_j}$ and $\frac{\partial}{\partial \bar{t}_j}$. Hence $\widetilde{\mathcal{V}}$ satisfies condition (3.2). By Proposition (3.1) there exist $N - n = N + 2\ell - (n + 2\ell)$ solutions $f_k(u, t', t'')$ which have linearly independent differentials.

Now let $\zeta_k(u) = f_k(u, 0, 0)$, $k = 1, \ldots, N - n$. Since the coefficients of elements of $\mathbf{L}$ are independent of $(t'\, t'')$, it is clear that the ζ_k are solutions of (1.1). It suffices to check that the ζ_k have linearly independent differentials. This will follow if the matrix $\left(\frac{\partial f_i}{\partial u_k}\right)_{\substack{1 \leqslant j \leqslant N-n \\ 1 \leqslant k \leqslant N}}$ has rank $N - n$. By the linear independence of the f_k in the (u, t', t'') variables, it suffices to show that $\frac{\partial f_k}{\partial t'_j}$ and $\frac{\partial f_k}{\partial t''_j}$ are linear combinations of $\frac{\partial f_k}{\partial u_-}$. Since $\frac{\partial f_k}{\partial \bar{t}_j} = 0$ and $(R_j + S_j) f_k = 0$, $1 \leqslant j \leqslant \ell$, this follows, and hence the proof of the theorem is complete.

□

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