

R. G. M. BRUMMELHUIS

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*Annales de l'institut Fourier*, tome 37, n° 2 (1987), p. 139-150

[http://www.numdam.org/item?id=AIF\\_1987\\_\\_37\\_2\\_139\\_0](http://www.numdam.org/item?id=AIF_1987__37_2_139_0)

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## AN F. AND M. RIESZ THEOREM FOR BOUNDED SYMMETRIC DOMAINS

by R. G. M. BRUMMELHUIS (\*)

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*to the memory of my mother*

### 1. Introduction.

In [11] J. H. Shapiro has given new proofs of the classical F. and M. Riesz theorem for the circle group  $\mathbf{T} = \{z \in \mathbf{C} : |z|=1\}$  and of Bochner's generalization of F. and M. Riesz to the torus  $\mathbf{T} \times \mathbf{T}$ . These proofs were based on a study of the duals of certain subspaces of  $L^p(\mathbf{T})$ , respectively  $L^p(\mathbf{T} \times \mathbf{T})$  for  $p$ 's between 0 and 1.

In this paper Shapiro's methods are generalized to arbitrary compact groups. As a result, we obtain in section 3 a general F. and M. Riesz theorem for compact groups whose center contains a circle group.

A typical special case of our F. and M. Riesz theorem is the unit sphere  $\mathbf{S}$  in  $\mathbf{C}^n$ :  $\mathbf{S} = \mathbf{S}_{2n-1} = U(n)/U(n-1)$ , where  $U(n)$  is the unitary group. For the formulation we have to recall some definitions from harmonic analysis on  $\mathbf{S}$ , cf. [7], chapter 12. Let  $\mathbf{H}(p,q)$  be the set of restrictions to  $\mathbf{S}$  of harmonic polynomials in  $z$  and  $\bar{z}$  which are homogeneous of degree  $p$  in  $z$  and of degree  $q$  in  $\bar{z}$ . Let  $\sigma$  denote the  $U(n)$ -invariant measure on  $\mathbf{S}$  with total mass 1. The spaces  $\mathbf{H}(p,q)$  span  $L^2(\mathbf{S},\sigma)$  and are pairwise orthogonal. Let  $\pi_{pq}$  denote the orthogonal projection of  $L^2(\mathbf{S},\sigma)$  onto  $\mathbf{H}(p,q)$ . The map  $f \rightarrow (\pi_{pq}f)(z) (z \in \mathbf{S})$  can be represented as the inner product in  $L^2$  of  $f$  with an element  $\mathbf{K}_z$  in  $\mathbf{H}(p,q)$ . Hence we can define  $\pi_{pq}\mu \in \mathbf{H}(p,q)$  for any finite Borel measure  $\mu$  on  $\mathbf{S}$ . Let  $\text{spec } \mu = \{(p,q) \in \mathbf{N} \times \mathbf{N} : \pi_{pq}\mu \neq 0\}$ .

(\*) Author supported by the Netherlands organization for the advancement of pure research (Z.W.O.).

*Key-words:* F. and M. Riesz theorem - Measures on compact groups - Absolute continuity - Non locally convex  $L^p$ -spaces ( $p < 1$ ) -  $H^p$  theory - Bounded symmetric domains - Spherical harmonics.

**1.1. THEOREM.** — *Let  $\Delta \subseteq \mathbf{N} \times \mathbf{N}$  satisfy the following two conditions :*

- (i) *For each  $m \in \mathbf{Z}$  the set  $\{(p,q) \in \Delta : p-q=m\}$  is finite.*
- (ii) *The set  $\{p-q : (p,q) \in \Delta\}$  is bounded from below (or above).*

*Let  $\mu$  be a finite Borel measure on  $S$  such that  $\text{spec } \mu \subseteq \Delta$ . Then  $\mu$  is absolutely continuous with respect to  $\sigma$ . □*

Examples of sets  $\Delta$  which satisfy conditions (i) and (ii) of 1.1 are the sets  $\Delta_\alpha = \{(p,q) \in \mathbf{N} \times \mathbf{N} : q \leq \alpha p\}$  for  $\alpha < 1$ . The singular measures  $\tau_m$  defined by

$$\int_S f d\tau_m = \int_{-\pi}^{\pi} f(e^{i\theta}\zeta) e^{im\theta} d\theta, \quad f \in C(S),$$

( $\zeta \in S$  fixed) show that condition 1.1 (ii) by itself is not sufficient. Similarly, the existence of a singular pluriharmonic measure  $\mu$  (that is,

$$\text{spec } \mu \subseteq \mathbf{N} \times \{0\} \cup \{0\} \times \mathbf{N},$$

cf. Aleksandrov [1] or Rudin [8]) shows that some condition on the set  $\{p-q : (p,q) \in \text{spec } \mu\}$  is necessary. Cf. also remark 3.4 below.

Another application of our F. and M. Riesz theorem is made to the Bergman-Shilov boundary  $S$  of a bounded symmetric domain  $\Omega$ : we get another proof of the known result that an  $H^1$  function on  $\Omega$  can be written as the Poisson integral of an  $L^1$  function on  $S$ . Finally, our F. and M. Riesz theorem contains the classical results of the Riesz brothers and of Bochner as special cases.

Kanjin [5] has proved an F. and M. Riesz theorem for zonal (i.e.  $U(n-1)$ -invariant) measures on  $S$ : such a measure  $\mu$  is absolutely continuous with respect to  $\sigma$  if  $\text{spec } \mu \subseteq \{(p,q) \in \mathbf{N} \times \mathbf{N} : \min(p,q) \leq N\}$  for some  $N \in \mathbf{N}$ . I do not know if Kanjin's result can be proved (and extended) by the methods in this paper.

As in the classical case, if  $\mu$  is a measure such that  $\text{spec } \mu$  satisfies 1.1 (i) and (ii) then not only is  $\mu$  absolutely continuous with respect to  $\sigma$  but  $\sigma$  is absolutely continuous with respect to  $\mu$  as well. This will be shown in the final section of this paper.

*Acknowledgement.* I would like to thank dr. T. H. Koornwinder and prof. J. Korevaar for helpful comments.

**2. Absolute continuity and the existence of  $L^p$  continuous linear functionals,  $p < 1$ .**

**2.1. Notations.** — If  $X$  is a compact topological space, let  $C(X)$  denote the space of complex valued continuous functions on  $X$ , with the sup norm.  $M(X)$  denotes the dual of  $C(X)$ , the space of finite Borel measures on  $X$ .

Throughout this paper,  $K$  will denote a compact group with a countable basis of neighborhoods at  $e$ . By  $dk$  we denote the Haar measure on  $K$ , normalized to total mass 1;  $L^p(K, dk) = L^p(K)$  and  $\|f\|_p$  ( $0 < p < \infty$ ) have their usual meaning. If  $\mu \in M(K)$  we write (as usual)  $\mu \ll dk$ ,  $\mu \perp dk$  for «  $\mu$  is absolutely continuous with respect to  $dk$  », respectively, «  $\mu$  is singular with respect to  $dk$  ».

**2.2 Fourier transform on  $K$ .** — Let  $\hat{K}$  be the unitary dual of  $K$ , i.e.  $\hat{K}$  is the set of (equivalence classes of) irreducible unitary continuous representations of  $K$ . For  $\tau \in \hat{K}$ , let  $H(\tau)$  denote the representation space of  $\tau$ , and  $d_\tau$  the complex dimension of  $H(\tau)$ , the degree of  $\tau$ . The Fourier transform  $\hat{\mu}$  of  $\mu \in M(K)$  is defined as the following (operator valued) function on  $K$ :

$$\hat{\mu}(\tau) = \int_K \tau(x^{-1}) d\mu(x).$$

Let  $T(K)$  be the space of trigonometric polynomials on  $K$ ; i.e.  $T(K)$  is the set of finite linear combinations of functions  $k \rightarrow (\tau(k)v, w)$  where  $v, w \in H(\tau)$ ,  $\tau \in \hat{K}$  and  $(., .)$  is the inner product of  $H(\tau)$ . If  $\chi_\tau$  denotes the character of  $\tau$  then for  $F \in T(K)$

$$F(k) = \sum_{\tau} d_{\tau}(\chi_{\tau} * F)(k) = \sum_{\tau} d_{\tau} \text{Tr} \{ \hat{F}(\tau) \tau(k) \}$$

where  $*$  denotes convolution on  $K$  and  $\text{Tr}$  means trace.

For  $\tau \in \hat{K}$  let  $T_{\tau}(K)$  denote the linear span of all functions  $k \rightarrow (\tau(k)v, w)$ , where  $v, w \in H(\tau)$ . The map  $f \rightarrow d_{\tau} \chi_{\tau} * f$  is the  $L^2$ -orthogonal projection of  $L^2(K)$  onto  $T_{\tau}(K)$ . For  $\mu \in M(K)$ , the Fourier-Stieltjes series of  $\mu$  is defined as the formal series

$$\sum_{\tau \in \hat{K}} d_{\tau}(\chi_{\tau} * \mu)(k) = \sum d_{\tau} \text{Tr} \{ \hat{\mu}(\tau) \tau(k) \}.$$

**2.3. The spectrum of a measure.** — For  $\mu \in M(K)$ , let  $\text{spec } \mu$  be the support of  $\hat{\mu} : \text{spec } \mu = \{\tau \in \hat{K} : \hat{\mu}(\tau) \neq 0\}$ .

Clearly,  $\text{spec } \mu = \{\tau \in \hat{K} : \chi_\tau * \mu \neq 0\}$ .

Let  $X_\mu$  be the subspace of  $T(K)$  defined by

$$X_\mu = \{F * \mu : F \in T(K)\}.$$

$X_\mu$  determines  $\text{spec } \mu$  completely, since  $\text{spec } \mu = \{\tau \in \hat{K} : X_\mu \cap T_\tau(K) \neq \emptyset\}$ . Conversely,  $\text{spec } \mu$  does not determine  $X_\mu$  in general: note that  $X_\mu$  is spanned by the functions

$$k \rightarrow (\tau(k)v, w), \quad v \in \text{Range } \hat{\mu}(\tau), \quad w \in H(\tau):$$

a short computation shows that for  $w_1, w_2 \in H(\tau)$  one has

$$((\tau(\cdot)w_1, w_2) * \mu)(k) = (\tau(k)\hat{\mu}(\tau)w_1, w_2).$$

If  $K$  is abelian,  $\hat{K}$  can be identified with the character group of  $K$  and then  $X_\mu$  is the linear span of  $\text{spec } \mu$ .

It is expedient to use  $X_\mu$  instead of  $\text{spec } \mu$  when generalizing Shapiro's results to non-abelian  $K$ .

Recall that a space of functions  $Y$  on  $K$  is called invariant under left translation if  $f \in Y$  implies  ${}^k f \in Y$ , where  ${}^k f(x) := f(kx)$ . Note that  $X_\mu$  is invariant under left translation.

If  $Y$  is a subspace of  $T(K)$ , let  $Y^p$  denote the closure of  $Y$  in  $L^p(K)$ .

**2.4. THEOREM.** — *Let  $\mu$  in  $M(K)$  be singular with respect to  $dk$ . Then  $X_\mu^p$  has no nonzero continuous linear functionals if  $0 < p < 1$ .*

Compare [11], theorem 2.1. For the proof we need some lemmas.

**2.5. LEMMA.** — *There exists a sequence  $\{F_n\}$  of trigonometric polynomials, with  $\{\|F_n\|_1\}$  bounded such that*

- (i) *if  $\mu \in M(K)$ ,  $\mu \perp dk$ , then  $F_n * \mu \rightarrow 0$  in Haar measure as  $n \rightarrow \infty$ ;*
- (ii) *if  $f \in L^1(K)$ , then  $F_n * f \rightarrow f$  in  $L^1(K)$  as  $n \rightarrow \infty$ .*

*Proof.* — Let  $\{V_n : n \in \mathbf{N}\}$ ,  $V_{n+1} \subseteq V_n$ , be a countable basis of neighborhoods of  $e$  (the identity element) in  $K$ . Let  $h_n$  be the characteristic function of  $V_n$ , divided by the Haar measure of  $V_n$ . A rather straightforward argument shows that (i) and (ii) hold with  $F_n$  replaced by  $h_n$ , cf. [11], proof of lemma 1.1.

Since each  $h_n$  is in  $L^2(K)$ , there exists  $F_n \in T(K)$  such that  $\|F_n - h_n\|_1 < 2^{-n}$ . Hence (ii) holds. Furthermore,  $(F_n - h_n) * \mu \rightarrow 0$  in Haar measure as  $n \rightarrow \infty$  for every  $\mu \in M(K)$  since  $\|(F_n - h_n) * \mu\|_1 \rightarrow 0$ . Hence (i) follows. □

**2.6. LEMMA.** — *Let  $\{f_n\}$  be a sequence of functions in  $L^1(K)$  which converges to 0 in Haar measure. Suppose there exists a  $C > 0$  such that  $\|f_n\|_1 \leq C$  for all  $n$ . Then  $\|f_n\|_p \rightarrow 0$  as  $n \rightarrow \infty$  for all  $p \in (0,1)$ .*

*Proof.* — It is enough to observe the following: if  $E \subseteq K$  is Borel measurable then if  $|E|$  denotes the Haar measure of  $E$ ,

$$\begin{aligned} \int_E |f_n|^p dk &= \int |f_n|^p \chi_E dk \leq \|f_n\|_1^p |E|^{1-p} \\ &\leq C^p \cdot |E|^{1-p} \end{aligned}$$

by Hölder's inequality with exponent  $1/p$ . Now take for  $E = E_n$  the set where  $|f_n| > \varepsilon$ ; for large  $n$ , it will have small measure. □

*Proof of 2.4.* — Fix  $p, p \in (0,1)$  and write  $X$  for  $X_\mu^p$ . Let  $\{F_n\}$  be as in lemma 2.5.

Then  $f_n := F_n * \mu \rightarrow 0$  in  $L^p(K)$  by 2.5 (i), 2.6 and the fact that  $\|F_n * \mu\|_1 \leq \|F_n\|_1 \|\mu\| \leq C \|\mu\|$  for all  $n$ .

Suppose  $\Phi$  is an  $L^p$  continuous linear functional on  $X$ . Then  $\Phi$  is  $L^1$  continuous on  $L^1(K) \cap X$ , since  $\|\cdot\|_p \leq \|\cdot\|_1$ . By Hahn-Banach and the fact that the dual of  $L^1(K)$  is  $L^\infty(K)$ , there exists a  $\varphi$  in  $L^\infty(K)$  such that

$$\Phi(f) = \int_K f(x)\varphi(x^{-1}) dx, \quad f \in X \cap L^1(K).$$

By the left translation invariance of  $X_\mu$ ,  ${}^k f_n \in X_\mu$  for all  $k \in K$  and  ${}^k f_n \rightarrow 0$  in  $L^p(K)$  as  $n \rightarrow \infty$  since  $dk$  is left invariant. Therefore

$$\Phi({}^k f_n) = (f_n * \varphi)(k) = ((F_n * \mu) * \varphi)(k) \rightarrow 0, \quad n \rightarrow \infty.$$

Since  $\mu * \varphi \in L^1(K)$ ,  $F_n * (\mu * \varphi) \rightarrow (\mu * \varphi)$  in  $L^1(K)$  (2.5(ii)). Hence  $\mu * \varphi = 0$  a.e. . Since

$$\Phi(F * \mu) = ((F * \mu) * \varphi)(e), \quad F \in T(K),$$

$\Phi = 0$  on  $X$ . □

**2.7. THEOREM.** — *Let  $0 < p < 1$  and let  $Y \subseteq L^p(K)$  be a closed subspace, invariant under left translation. Suppose that  $Y \cap T(K)$  has sufficiently many  $L^p$  continuous linear functionals to separate points. Let  $\mu$  in  $M(K)$  be such that  $X_\mu \subseteq Y$ . Then  $\mu$  is absolutely continuous with respect to  $dk$ .*

Compare [11], corollary 5.2.

*Proof.* — Let  $\mu = fdk + \nu$  be the Lebesgue decomposition of  $\mu$ ,  $f \in L^1(K)$ ,  $\nu \perp dk$ .

Choose  $\{F_n\}$  as in lemma 2.5. Then by 2.5 (i) and (ii) and 2.6,

$$F_n * \mu \rightarrow f \quad \text{in } L^p \quad \text{as } n \rightarrow \infty \quad (0 < p < 1).$$

This implies that  $f \in Y$ , since  $X_\mu \subseteq Y$ .

Let  $V$  be the closed subspace of  $L^1(K)$  spanned by the left translates of  $f$ . Then  $V \subseteq Y$  since  $Y$  is closed under left translation. Hence  $F * f \in V \subseteq Y$  for all  $F \in T(K)$ . Also  $F * \mu \in Y$  for all  $F \in T(K)$ . Hence  $X_\nu \subseteq Y \cap T(K)$ . But this implies  $\nu = 0$ , by theorem 2.4.  $\square$

### 3. A general F. and M. Riesz theorem.

**3.1.** Our main theorem concerns compact groups  $K$  (with a countable neighborhood basis at  $e$ ) whose center  $Z(K)$  contains a circle group  $T$ . Throughout this section, let  $K$  be such a group, and fix an identification  $T \rightarrow Z(K)$ , so that  $e^{i\theta}$  denotes an element of  $K$  as well as of  $T$ . By Schur's lemma, there exists for each  $\tau \in \hat{K}$  an  $n(\tau) \in \mathbf{Z}$  such that

$$\tau(e^{i\theta}) = e^{in(\tau)\theta} \cdot \text{Id}, \quad \theta \in \mathbf{R}.$$

We can now formulate our main result.

**3.2. THEOREM.** — *Let  $\Delta \subseteq \hat{K}$  satisfy the following two conditions:*

- (i) *For each  $m \in \mathbf{Z}$  the set  $\{\tau \in \Delta : n(\tau) = m\}$  is finite.*
- (ii) *The set  $\{n(\tau) : \tau \in \Delta\}$  is bounded from below.*

*Let  $\mu \in M(K)$  be such that  $\text{spec } \mu \subseteq \Delta$ . Then  $\mu$  is absolutely continuous with respect to  $dk$ .*

In condition (ii) of 3.2 «from below» may be replaced by «from above»: just replace  $\mu$  by  $\bar{\mu}$ .

*Proof.* — Let  $Y$  be the linear span of the  $T_\tau(K)$  's with  $\tau \in \Delta$ . By theorem 2.7 it is sufficient to show that for  $p < 1$

$$(3.1a) \quad Y^p \cap T(K) = Y,$$

(3.1b)  $Y$  has sufficiently many  $L^p$  continuous linear functionals to separate points.

In the proof of (3.1a) and (3.1b) we will use the following lemma :

**3.3. LEMMA.** — For  $m \in \mathbf{Z}$  define the projection  $\Pi_m : T(K) \rightarrow T(K)$  by

$$\begin{aligned} \Pi_m f(k) &= \int_{-\pi}^{\pi} f(e^{i\theta}k) e^{-im\theta} d\theta/2\pi \\ &= \sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k). \end{aligned}$$

If  $Y$  is a subspace of  $T(K)$  such that the set  $\{n(\tau) : \exists f \in Y : \chi_\tau * f \neq 0\}$  is bounded from below, then  $\Pi_m$  is  $L^p$  continuous on  $Y$  for all  $p > 0$ . (The interesting case is of course  $0 < p < 1$ .)

*Proof.* — For  $k \in K$ ,  $f \in T(K)$  define the « slice function »  $f_k$  on  $T$  by  $f_k(e^{i\theta}) := f(e^{i\theta}k)$ . Obviously

$$\begin{aligned} f_k(e^{i\theta}) &= \sum_{m \in \mathbf{Z}} \left( \sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k) \right) e^{im\theta} \\ &= \sum_{m \in \mathbf{Z}} \Pi_m f(k) e^{im\theta}. \end{aligned}$$

Let  $N \in \mathbf{Z}$  be such that  $n(\tau) \geq N$  for all  $\tau$  for which  $d_\tau \chi_\tau * f \neq 0$  for some  $f \in Y$ . Suppose first that  $N \geq 0$ . Then  $f_k(e^{i\theta})$  is an analytic trigonometric polynomial for each  $f$  in  $Y$ . Hence, by a result from one variable  $H^p$  theory due to Hardy and Littlewood (cf. [3], theorem 6.4; cf. also [2], p. 68, for a short proof) for each  $p > 0$  and each  $m \in \mathbf{Z}$  there exists a constant  $C = C(p, m)$  such that

$$|\Pi_m f(k)|^p \leq C \int_{-\pi}^{\pi} |f(e^{i\theta}k)|^p d\theta/2\pi.$$

Integration over  $K$  yields the lemma when  $N \geq 0$ .

If  $N < 0$  then for each  $f \in Y$ ,  $f_k(e^{i\theta}) = e^{iN\theta} \cdot F(e^{i\theta})$  where  $F$  is again an analytic trigonometric polynomial on  $T$ . Apply the one variable result mentioned above to  $F$  and note that  $|F| = |f_k|$  on  $T$ . □



We now return to the proof of theorem 3.2. To prove (3.1a) suppose that  $f_n \in Y$ ,  $f \in T(\mathbf{K})$  such that  $f_n \rightarrow f$  in  $L^p(\mathbf{K})$  as  $n \rightarrow \infty$ . By lemma 3.3 applied to  $Y' = \text{span}\{Y, f\}$ ,  $\Pi_m(f_n) \rightarrow \Pi_m(f)$  in  $L^p(\mathbf{K})$  for all  $m \in \mathbf{Z}$ . Since  $Y$  is invariant under left translation,  $\Pi_m(f_n)$  belongs to  $Y \cap \bigoplus \{T_\tau(\mathbf{K}) : n(\tau) = m\}$ . The latter is a finite dimensional subspace of  $T(\mathbf{K})$  by condition 3.2(i). Since all vector space topologies on a finite dimensional vector space are complete,  $\Pi_m(f) \in Y$  for all  $m \in \mathbf{Z}$ , which implies that  $f \in Y$ .

For (3.1b) it is sufficient to show that for each  $\sigma \in \Delta$  and each  $k \in \mathbf{K}$  the linear functional

$$(3.2) \quad f \rightarrow d_\sigma(\chi_\sigma * f)(k)$$

is  $L^p$  continuous on  $Y$ . Take a  $\sigma$  in  $\Delta$ . Clearly, the linear functional (3.2) is equal to the composition of the projection  $\Pi_{n(\sigma)}$  with the restriction of (3.2) to  $\bigoplus \{T_\tau(\mathbf{K}) : \tau \in \Delta, n(\tau) = n(\sigma)\}$ . Since this subspace is finite dimensional, the  $L^p$  continuity of (3.2) follows from the  $L^p$  continuity of  $\Pi_{n(\sigma)}$ . This proves the theorem.  $\square$

**3.4. Remark.** — Recall that a subset  $\Sigma$  of  $\hat{\mathbf{K}}$  is called a  $\Lambda(1)$  set (Rudin [10]) if there exists a  $p < 1$  and a constant  $C$  such that for all  $f$  in  $\bigoplus \{T_\tau(\mathbf{K}) : \tau \in \Sigma\}$ ,

$$\|f\|_1 \leq C \|f\|_p,$$

i.e. if the  $L^1$  and  $L^p$  topologies coincide on  $\text{span}\{T_\tau(\mathbf{K}) : \tau \in \Sigma\}$ .

We can replace condition (i) of 3.2 by the following weaker condition :

(i)' For each  $m \in \mathbf{Z}$  the set  $\{\tau \in \Delta : n(\tau) = m\}$  is a  $\Lambda(1)$  subset of  $\hat{\mathbf{K}}$ .

The proof remains essentially the same: instead of the finite dimensionality of the subspaces  $\bigoplus \{T_\tau(\mathbf{K}) : \tau \in \Delta, n(\tau) = m\}$  we now use the equivalence, for some  $p < 1$ , of the  $L^1$  and  $L^p$  topologies on these subspaces, and the  $L^1$  continuity of the linear functionals (3.2) (for all  $\sigma \in \mathbf{K}$ ).

Similarly, we may also replace condition (ii) by

(ii)' The set  $\{n(\tau) : \tau \in \Delta\}$  is a  $\Lambda(1)$  subset of  $\mathbf{Z}$  (considered as the dual of  $\mathbf{T}$ ).

In this case the analogue of lemma 3.3 becomes trivial.

Note, by the way, that for arbitrary compact  $K$  the conclusion of the F. and M. Riesz theorem holds for all  $\Lambda(1)$  subsets of  $\hat{K}$ : this follows immediately from theorem 2.7 and the definition of a  $\Lambda(1)$  set.

**3.5. Example.** — Let  $K = \mathbf{T} \times \mathbf{T}$  and identify  $\mathbf{T}$  with a subgroup of  $K$  via the map  $e^{i\theta} \rightarrow (e^{i\theta}, 1)$ ,  $\theta \in (-\pi, \pi]$ . The irreducible unitary representations of  $K$  are the characters  $\chi_{p,q}: (e^{i\theta}, e^{i\psi}) \rightarrow e^{i(p\theta + q\psi)}$  and  $n(\chi_{p,q}) = p$ . In this case theorem 3.2 contains Bochner's theorem where the spectrum is required to lie in an angle to the right of opening less than  $\pi$  (cf. [9], theorem 8.2.5 for the precise formulation).

According to the remarks made in 3.4 it suffices to require in condition (i) that for each  $p$  the set  $\{q \in \mathbf{Z} : (p, q) \in \text{spec } \mu\}$  is a  $\Lambda(1)$  set. We refer to the appendix of [2] for another strengthening of Bochner's theorem which only requires that for each  $p$  these sets are either bounded from above or from below. This can easily be proved by the method of proof of theorem 3.2 if we note that  $H^p(\mathbf{T}) \cap \mathbf{T}(\mathbf{T})$  consists of analytic polynomials.

**3.6. F. and M. Riesz for homogeneous spaces.** — Let  $H$  be a closed subgroup of  $K$ . Functions and measures on  $K/H$  can be identified with functions and measures on  $K$  which are right  $H$ -invariant. If  $\mu \in M(K/H)$  is a right  $H$ -invariant measure on  $K$ , then  $\pi_\tau \mu := d_\tau \chi_\tau * \mu$  is again right  $H$ -invariant. Let  $\sigma$  be the  $K$ -invariant measure on  $K/H$ , normalized to 1. The map  $\pi_\tau: f \rightarrow d_\tau \chi_\tau * f$  ( $\tau \in \hat{K}$ ) is an  $L^2$  orthogonal projection of  $L^2(K/H, \sigma) = L^2(\sigma)$  which is different from zero iff  $\tau$  occurs in the left regular representation of  $K$  on  $L^2(\sigma)$ . As in the case of the unit sphere,  $\pi_\tau$  can be represented by an integral operator with continuous kernel.

Theorem 3.2 can now be formulated for measures on  $K/H$  entirely in terms of  $\pi_\tau$  and  $\sigma$ :

**3.7. THEOREM.** — Let  $\Delta \subseteq \hat{K}$  be such that all  $\tau \in \Delta$  occur in the left regular representation of  $K$  on  $L^2(\sigma)$  and suppose  $\Delta$  satisfies conditions (i) and (ii) of 3.2. Let  $\mu \in M(K/H)$  be such that  $\pi_\tau \mu = 0$  if  $\tau \notin \Delta$ . Then  $\mu$  is absolutely continuous with respect to  $\sigma$ .

If we take  $K = U(n)$ ,  $H = U(n-1)$  then  $K/H = S$  and we get theorem 1.1:  $Z(K)$  contains the multiplications by  $e^{i\theta}$ . If  $\tau_{pq}$  denotes the restriction of the left regular representation of  $U(n)$  on  $L^2(\sigma)$  to  $H(p, q)$ , i.e.  $\tau_{pq}(U)f(\zeta) = f(U^{-1}\zeta)$ ,  $f \in H(p, q)$ ,  $U \in U(n)$ ,  $\zeta \in S$ , then  $\tau_{pq}$  is irreducible,  $n(\tau_{pq}) = q - p$ , the  $\tau_{pq}$  are pairwise inequivalent and they represent all irreducible representations of  $U(n)$  which occur in  $L^2(\sigma)$  (cf. for example [7], chapter 12).

In case  $H = T$  theorem 3.7 becomes trivial:  $n(\tau) = 0$  for all  $\tau$  which occur in  $K/T$  and (i) then implies that  $\Delta$  is finite.

**3.8. Application to bounded symmetric domains.** — Let  $\Omega \subseteq \mathbb{C}^n$  be a bounded symmetric domain. (Cf. [6], [4] for the relevant facts.) We may assume that  $\Omega$  is convex and circular (i.e.  $z \in \Omega$  implies  $e^{i\theta} \cdot z \in \Omega$  for all  $\theta \in \mathbb{R}$ ). Let  $K$  be the stabilizer of 0 in the component of the identity of the group of holomorphic automorphisms of  $\Omega$ . The action of  $K$  on  $\Omega$  incorporates multiplication by  $e^{i\theta}$ ; in particular,  $T \subseteq Z(K)$ . Let  $S$  denote the Bergman-Shilov boundary of  $\Omega$ . Then  $K$  acts transitively on  $S$  and we can apply the principal theorems 3.2, 3.7 to  $S$ . As above, let  $\sigma$  be the normalized  $K$ -invariant measure on  $S$ .

Let  $H^2(S)$  be the closure in  $L^2(S, \sigma) = L^2(\sigma)$  of the holomorphic polynomials, restricted to  $S$ . Obviously  $H^2(S)$  is  $K$ -invariant under the left regular representation of  $K$  on  $L^2(\sigma)$ . Let  $\hat{K}_{\text{Hol}}$  be the set of irreducible representations of  $K$  which occur in  $H^2(S)$ ; for a description of  $\hat{K}_{\text{Hol}}$ , cf. [12].

We claim that  $\hat{K}_{\text{Hol}}$  satisfies conditions (i) and (ii) of 3.2. For let  $H(p)$  be the space of holomorphic polynomials which are homogeneous of degree  $p$ , restricted to  $S$ . By a well known theorem of H. Cartan (cf. [7], theorem 2.1.3)  $K$  acts on  $\Omega$  by complex linear transformations. Hence each  $H(p)$  is  $K$ -invariant and decomposes as a finite sum of representations in  $\hat{K}_{\text{Hol}}$ . Obviously,  $n(\tau) \leq 0$  if  $\tau$  is in  $\hat{K}_{\text{Hol}}$  and  $n(\tau) = -p$  if  $\tau$  occurs in  $H(p)$ . This proves the claim.

By theorem 3.7 a measure  $\mu$  in  $M(K/H)$  for which  $\text{spec } \mu \subseteq \hat{K}_{\text{Hol}}$  is absolutely continuous with respect to  $\sigma$ . By a familiar weak-\* compactness argument this implies the following result:

**3.9. COROLLARY.** — *If  $f$  is in the Hardy space  $H^1(\Omega)$  of  $\Omega$  then  $f$  can be written as the Poisson integral of a function in  $L^1(\sigma)$ . (Cf. [6], [14] for the definitions of Hardy space and Poisson kernel.)*  $\square$

The analogue of 3.9 for generalized Siegel half-planes is due to E. M. Stein [13]; 3.9 can be deduced from his result by using the generalized Cayley transform, cf. [6], p. 189. Corollary 3.9 can also be proved more directly, by using Bochner's method of slicing and the Hardy-Littlewood inequality for the radial maximal function from one dimensional  $H^p$  theory.

**4. A supplement to theorem 3.2.**

**4.1. THEOREM.** — *Let  $K$  be a compact Lie group whose center contains the circle group  $T$  and let  $H$  be a closed subgroup of  $K$  such that  $K/H$  is connected. Suppose  $f$  in  $L^1(K/H, \sigma)$  is such that  $\text{spec } f$  satisfies conditions (i) and (ii) of theorem 3.7. Then either  $f = 0$  or  $f(\xi) \neq 0$  a.e.  $[\sigma]$ .*

In particular, if  $\mu \in M(K/H)$  is as in theorem 3.7 and  $\mu \neq 0$  then  $\sigma \ll \mu$  as well as  $\mu \ll \sigma$ . The case  $K = T, H = \{1\}$  of this theorem is classical (cf. for example [3], theorem 2.2) and will be used in the proof of 4.1.

*Proof.* — Let  $f$  in  $L^1(K/H, \sigma)$  satisfy the conditions of the theorem and suppose that  $f = 0$  on a (Borel) set of nonzero measure. Identify  $f$  with a right  $H$ -invariant  $L^1$  function on  $K$ , also denoted by  $f$ .

For almost all  $k \in K$  the slice function  $f_k(e^{i\theta}) = f(e^{i\theta}k)$  is in  $L^1(T)$ . Set  $c_m(k) := \hat{f}_k(m)$ , the  $m$ -th Fourier coefficient of  $f_k$ . Then  $c_m \in L^1(K)$  and a calculation of  $d_\tau \chi_\tau * c_m$  shows that

$$(4.1) \quad c_m(k) = \sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k)$$

(note that the sum is finite). The case  $K = T$  of 4.1 now shows that for almost all  $k$  in  $K$

$$(4.2) \quad f_k = 0 \quad \text{or} \quad f_k(e^{i\theta}) \neq 0 \quad \text{a.e.} [d\theta].$$

Since  $f = 0$  on a set of nonzero Haar measure there exists an  $F \subseteq K$  of strictly positive Haar measure such that for all  $k$  in  $F$ ,  $f_k = 0$  on a subset of  $T$  of nonzero (one-dimensional) Lebesgue measure. By (4.2)  $f_k = 0$  for almost all  $k$  in  $F$  and hence, by (4.1)

$$\sum_{n(\tau)=m} d_\tau(\chi_\tau * f)(k) = 0 \quad \text{for all } m \in \mathbf{Z}, \quad \text{a.a. } k \in F.$$

Each  $\tau \in \hat{K}$  is an analytic function on  $K$  and therefore  $d_\tau \chi_\tau * f$  is an analytic function on the analytic manifold  $K/H$ . It is not difficult to prove that the zero set of a nonzero analytic function on a connected analytic manifold has Lebesgue measure zero. Hence

$$\sum_{n(\tau)=m} d_\tau \chi_\tau * f = 0$$

for all  $m$  and therefore  $d_\tau \chi_\tau * f = 0$  for all  $\tau \in \hat{K}$ , i.e.  $f = 0$ . □

It is easy to find a counterexample to 4.1 if  $K/H$  is not connected. Take  $K = T \times F$  and  $H = \{e\}$ , with  $F$  a finite non-commutative group. Let  $\tau$  be an irreducible representation of  $F$  and choose a matrix coefficient  $\tau_{mn}$  of  $\tau$  such that  $\tau_{mn}(e) = 0$ ,  $\tau_{mn}(x) \neq 0$  for some  $x \in F$ . Now take  $f = (1 \otimes \tau)_{mn}$ :  $\text{spec } f$  consists of one point but  $f = 0$  on the component of the unit element of  $K$ .

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Manuscrit reçu le 22 avril 1986.

R. G. M. BRUMMELHUIS,  
 Universiteit van Amsterdam  
 Mathematisch Instituut  
 Roetersstraat 15  
 1018 WB Amsterdam (The Netherlands).