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THE TRACE INEQUALITY AND EIGENVALUE ESTIMATES FOR SCHRODINGER OPERATORS

by R. KERMAN (1) and E. SAWYER (2)

1. Introduction.

This paper deals with potential operators T_{Φ} given at Lebesgue measurable f on \mathbb{R}^n by a convolution integral

$$(\mathbf{T}_{\Phi}f)(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y) \, dy,$$

provided this integral exists for almost all $x \in \mathbf{R}^n$. The kernels $\Phi(y)$ are radially decreasing (r.d.) functions; that is, they are nonnegative, locally integrable radial functions on \mathbf{R}^n , which are nonincreasing in |y|. These T_{Φ} include the Riesz potential operator I_{α} whose kernel K_{α} is defined directly as

$$\mathbf{K}_{\alpha}(y) = |y|^{\alpha - n}, \quad 0 < \alpha < n$$

and the Bessel potential operator J_{α} with kernel G_{α} defined in terms of its Fourier transform \hat{G}_{α} by

$$\widehat{\mathbf{G}}_{\alpha}(\zeta) = \int_{\mathbb{R}^n} \mathbf{G}_{\alpha}(x) e^{-i\zeta \cdot x} \, dx = (1+|\zeta|^2)^{-\frac{\alpha}{2}}, \qquad 0 < \alpha < n.$$

Given an r.d. kernel Φ and $1 , we wish to characterize the (possibly singular) positive Borel measures <math>\mu$ on \mathbf{R}^n for which there exists C > 0 such that

(1.1)
$$\int_{\mathbb{R}^n} (T_{\Phi} f)(x)^p d\mu(x) \leqslant C \int_{\mathbb{R}^n} f(x)^p dx$$

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for all nonnegative measurable f. Clearly this will be true if and only if T_{Φ} is a bounded linear operator between the Lebesgue spaces $L^{p}(\mathbf{R}^{n})$ and $L^{p}(\mathbf{R}^{n},\mu)$. An important special case, with p=2 and $\Phi=G_{1}$, arises in estimating the spectrum of Schrödinger operators and will be considered in detail below. Another special case is treated in Stein [19], where it is

shown that (1.1) holds for J_{α} when $\mu = \mu_k$, $\alpha > \frac{n-k}{p}$, where

$$\mu_k(\mathbf{E}) \equiv m_k(\mathbf{E} \cap \mathbf{R}^k),$$

 m_k being k-dimensional Lebesgue measure on \mathbb{R}^k considered as a subset of \mathbb{R}^n . The inequality of [19] can be stated in the equivalent form

$$\int_{\mathbb{R}^n} (J_{\alpha}f)(x_1, \dots, x_k, 0, \dots, 0)^p dx_1, \dots, dx_k$$

$$\leq C \int_{\mathbb{R}^n} f(x_1, \dots, x_n)^p dx_1, \dots, dx_n.$$

It is thus a statement about the restriction, or trace, of $J_{\alpha}f$. For this reason we follow other authors in referring to (1.1) as « the trace inequality ».

Generalizing results of Adams [1] and Maz'ya [14], K. Hansson in [12] has characterized the μ satisfying (1.1) in terms of capacities (see also B. Dahlberg [8]). He shows the trace inequality holds if and only if K>0 exists for which

$$\mu(E) \leqslant K \operatorname{cap}(E)$$

whenever E is a compact subset of \mathbb{R}^n . Here cap (E) denotes the L^p capacity associated with the kernel Φ ,

$$\operatorname{cap}(E) = \inf \left\{ \int_{\mathbb{R}^n} f(x)^p \, dx : f \geqslant 0 \text{ and } T_{\Phi} f \geqslant 1 \text{ on } E \right\}.$$

A criterion such as (1.2) can be difficult to verify for all compact sets E. On the other hand if one only requires (1.2) to hold for a class of simple sets such as all cubes Q with sides parallel to the coordinate axes, the resulting condition is no longer sufficient (D. Adams [2]). For example, when n=p=2, $I_{\frac{1}{2}}$ doesn't satisfy (1.1) with μ_1 , yet inequality (1.2) for cubes, which amounts to $\mu_1(Q) \leq K|Q|^{\frac{1}{2}}$, holds. In fact, with $f(x) = x_2^{-\frac{1}{2}} |\ln x_2|^{-1} \chi_{\left[0,\frac{1}{2}\right]} \times \left[0,\frac{1}{2}\right] (x_1,x_2)$, $I_{\frac{1}{2}}$ f is infinite on

 $\{(x_1,0): 0 \le x_1 \le \frac{1}{2}\}$ and thus the left side of (1.1) is infinite while the right side is finite. Examples of this nature were first pointed out in [2].

Theorem 2.3 below gives a necessary and sufficient condition for (1.1) that involves testing an inequality over dyadic cubes Q, namely

(1.3)
$$\int_{Q} (M_{\Phi} X_{Q} \mu)(x)^{p'} dx \leq K \int_{Q} d\mu < \infty$$

where $p' = \frac{p}{p-1}$, the constant K > 0 is independent of Q, and

$$(\mathbf{M}_{\Phi}f\mu)(x) = \sup_{\mathbf{x} \in \mathbf{Q}} \left[\frac{1}{|\mathbf{Q}|} \int_{|\mathbf{y}| \leq |\mathbf{Q}|^{\frac{1}{n}}} \Phi(\mathbf{y}) \, d\mathbf{y} \right] \int_{\mathbf{Q}} f(\mathbf{y}) \, d\mu(\mathbf{y}).$$

Alternatively, (1.1) is equivalent to

(1.4)
$$\int_{\mathbb{R}^n} (T_{\Phi} \chi_{Q} \mu)(x)^{p'} dx \leq K \int_{Q} d\mu < \infty \text{ for all dyadic cubes } Q.$$

To compare (1.2) and (1.4), we note that (1.2) is equivalent by an elementary argument (see Theorem 4 in [2]) to testing the inequality in (1.4) over all compact sets Q. The reduction in (1.4) to testing over dyadic cubes Q is essential in obtaining sharp estimates for the higher eigenvalues of Schrödinger operators in § 3. For a different characterization involving test functions see Stromberg and Wheeden [21].

In the special case where $T_{\Phi} = I_{\alpha}$, the equivalence of (1.1) and (1.3) can be established by dualizing inequality (1.1), using the « good λ inequality » of B. Muckenhoupt and R. L. Wheeden [15] in order to replace I_{α} by its associated maximal operator M_{α} , and then using the characterization of the weighted inequality for M_{α} in [18]. The general case of the theorem is proved along similar lines, the crucial new estimate being an extension (Theorem 2.2) of the « good λ inequality » in [15].

As an application of Theorem 2.3 we obtain a sharpened form of recent results of C. L. Fefferman and D. H. Phong on the distribution of eigenvalues of Schrödinger operators, $H = -\Delta - v$, $v \ge 0$ ([10]; Theorem 5, 6 and 6' in Chapter II). Roughly speaking, their results show that for many $v \ge 0$, the negative eigenvalues of $H = -\Delta - v$ are approximately given by $-|Q|^{-\frac{2}{n}}$ as Q varies over the minimal dyadic

cubes satisfying $|Q|^{\frac{2}{n}-1}\int_{Q}v\geqslant C$. Theorem 3.3 below shows, as suggested by condition (1.3), that this picture extends to arbitrary $v\geqslant 0$ if the fractional average, $|Q|^{\frac{2}{n}-1}\int_{Q}v$, is replaced by

$$\frac{1}{|Q|_{v}}\int [I_{1}(\chi_{Q}v)(x)]^{2} dx = \frac{1}{|Q|_{v}}\int_{Q}I_{2}(\chi_{Q}v)(x)v(x) dx,$$

the *v*-average over Q of the Newtonian potential of $\chi_Q v$. Certain of the results in [10] have been generalized by S. Y. A. Chang, J. M. Wilson and T. H. Wolff ([5]) and by S. Chanillo and R. L. Wheeden ([6]). This is discussed in more detail in § 3. Further applications of Theorem 2.3 have been announced in [13].

2. The trace inequality.

We begin by deriving the basic properties of r.d. kernels Φ and Borel measures μ for which the trace inequality holds. For the sake of completeness, we consider here and in § 3 the more general trace inequality

(2.1)
$$\left[\int_{\mathbb{R}^n} (\mathsf{T}_{\Phi} f)(x)^q \, d\mu(x)\right]^{\frac{1}{q}} \leqslant \mathsf{C} \left[\int_{\mathbb{R}^n} f(x)^p \, dx\right]^{\frac{1}{p}}$$

for all nonnegative measurable f, where 1 . For <math>p < q and many r.d. kernels Φ , the trace inequality (2.1) can be characterized in terms of very simple conditions — see e.g. [12]. However, many applications, such as that in the next section, require the case p = q.

PROPOSITION 2.1. — If (2.1) holds for a non-trivial r.d. kernel Φ and a non-trivial Borel measure μ , then (i) μ is locally finite, that is, $\int_Q d\mu < \infty$ for all cubes Q, and (ii) Φ satisfies

(2.2)
$$\int_{|y| \geqslant r} \Phi(y)^{p'} dy < \infty \quad \text{for all } r > 0.$$

Proof. – Choose $\varepsilon > 0$ so that $\Phi(2\varepsilon) > 0$. If B is any ball of radius ε , and if γ_n denotes the measure of the surface of the unit ball in

 \mathbb{R}^n , then

$$\begin{split} \gamma_{n} \varepsilon^{n} \Phi(2\varepsilon) \bigg(\int_{\mathbb{B}} d\mu \bigg)^{\frac{1}{q}} & \leqslant \left[\int_{\mathbb{B}} (T_{\Phi} \chi_{\mathbb{B}})^{q} d\mu \right]^{\frac{1}{q}} \\ & \leqslant [\gamma_{n} \varepsilon^{n}]^{\frac{1}{p}} ||T_{\Phi}||_{0p} < \infty \; . \end{split}$$

Hence $\int_{B} d\mu < \infty$ and this proves that μ is locally finite.

To obtain (2.2), fix R > 0 so that $\int_B d\mu > 0$ where B is the ball of radius R centred at the origin. Momentarily fix S > 2R and let $f(x) = \Phi(x)^{p'-1}\chi_{\{2R \leqslant |y| \leqslant S\}}(x)$. For $|x| \leqslant R$, we have $T_{\Phi}f(x) = \int_{2R \leqslant |y| \leqslant S} \Phi(x-y)\Phi(y)^{p'-1} dy \geqslant C \int_{2R \leqslant |y| \leqslant S} \Phi(y)^{p'} dy$. Indeed, $\Phi(x-y) \geqslant \Phi(y)$ for all y satisfying $|x-y| \leqslant |y|$ and this in turn holds provided $|x| \leqslant R$, $|y| \geqslant 2R$ and the distance between $\frac{x}{|x|}$ and $\frac{y}{|y|}$ is sufficiently small. With this estimate, (2.1) yields

$$C \int_{2R \leqslant |y| \leqslant S} \Phi(y)^{p'} dy \left(\int_{B} d\mu \right)^{\frac{1}{q}} \leqslant \left[\int (T_{\Phi} f)^{q} d\mu \right]^{\frac{1}{q}}$$
$$\leqslant C \left[\int_{2R \leqslant |y| \leqslant S} \Phi(y)^{p'} dy \right]^{\frac{1}{p}}.$$

Letting $S \to \infty$ yields $\int_{|y| \ge 2R} \Phi(y)^{py} dy < \infty$ and this proves (2.2).

To obtain a criterion for (2.1) to hold, we look at the inequality dual to it. A standard argument shows this dual is, with the same C > 0,

$$(2.3) \qquad \left[\int_{\mathbb{R}^n} (\mathsf{T}_{\Phi} f \mu)(x)^{p'} \, dx\right]^{\frac{1}{p'}} \leqslant \mathsf{C} \left[\int_{\mathbb{R}^n} f(x)^{q'} \, d\mu(x)\right]^{\frac{1}{q'}},$$

where
$$p' = \frac{p}{p-1}$$
, $q' = \frac{q}{q-1}$, and

$$(\mathrm{T}_{\Phi}f\mu)(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y) \, d\mu(y) \, .$$

The behaviour of T_{Φ} in (2.3) is determined by that of the maximal operator M_{Φ} given at a positive Borel measure ν by

$$(\mathbf{M}_{\Phi}\mathbf{v})(x) = \sup_{x \in \mathbf{Q}} \left[\frac{1}{|\mathbf{Q}|} \int_{|y| \leq |\mathbf{Q}|^{\frac{1}{n}}} \Phi(y) \, dy \right] \int_{\mathbf{Q}} d\mathbf{v} \, .$$

Note that the first factor on the right side is the average of Φ over the ball of radius $|Q|^{\frac{1}{n}}$ centred at the origin. In the case when Φ is the kernel K_{α} for the Riesz potential operator, then M_{Φ} is the usual fractional maximal operator M_{α} (see e.g. [3] or [15]).

THEOREM 2.2. — Let Φ be an r.d. kernel and ν a positive locally finite Borel measure on \mathbb{R}^n . Then

(a)
$$(\mathbf{M}_{\Phi} \mathbf{v})(x) \leqslant \mathbf{C}_n \mathbf{M}(\mathbf{T}_{\Phi} \mathbf{v})(x), \quad x \in \mathbf{R}^n$$

where M denotes the usual Hardy-Littlewood maximal operator and the constant $C_n > 0$ depends only on the dimension n.

(b) There exists $\gamma > 1$ and a positive constant C_n depending only on n so that for all $\lambda > 0$ and all $\beta \in (0,1]$,

$$|\{T_{\Phi}v > \gamma\lambda \text{ and } M_{\Phi}v \leqslant \beta\gamma\}| \leqslant C_n \frac{\beta}{\gamma} |\{M(T_{\Phi}v) > \lambda\}|.$$

Proof. — To a given cube Q in \mathbb{R}^n associate the cube Q* having the same centre as Q but edges $7\sqrt{n}$ times as long as those of Q.

To prove (a) fix $x \in \mathbb{R}^n$ and a cube Q containing x. Then

$$\int_{Q^*} (T_{\Phi} v)(y) \, dy \geqslant \int_{Q^*} dy \, \int_{Q} \Phi(y-z) \, dv \, (z)$$

$$\geqslant \int_{Q} dv(z) \, \int_{Q^*} \Phi(y-z) \, dy$$

$$\geqslant \int_{|y| \leq |Q|^{\frac{1}{p}}} \Phi(y) \, dy \, \int_{Q} dv \, (y)$$

since $\{y; |y-z| \le |Q|^{\frac{1}{n}} \subset Q^*$, whenever $z \in Q$. Hence,

$$M(T_{\Phi}v)(x) \geqslant \frac{7^{-n}n^{-\frac{n}{2}}}{|Q|} \int_{|y| \leq |Q|^{\frac{1}{n}}} \Phi(y) \, dy \int_{Q} dv \, (y)$$

and so

$$M_{\Phi}v)(x) \geqslant 7^n n^{\frac{n}{2}} M(T_{\Phi}v)(x), x \in \mathbb{R}^n.$$

We now show (b). Given $\lambda > 0$, let

$$\Omega_{\lambda} = \{M(T_{\Phi}v) > \lambda\}.$$

Decompose Ω_{λ} into disjoint Whitney cubes Q with $Q^* \cap \Phi_{\lambda}^c \neq \emptyset$. See De Guzman [11]. Let $\{Q_k\}$ be those Whitney cubes for which there is an $x_k \in Q_k$ satisfying $(M_{\Phi}v)(x_k) \leq \beta \lambda$. Fixing attention on such a Q_k , which we'll denote simply by Q, we define v_1 and v_2 to be restrictions of the measure v; the first to Q^* , the second to $\mathbb{R}^n - Q^*$. We claim it is enough to obtain a dimensional constant $C_n > 0$ such that

$$(2.4) T_{\Phi} v_2 \leqslant C_n \lambda$$

on Q. Suppose for the moment that (2.4) has been proved and take $\gamma > 2C_n$. Then

$$\{x \in Q; (T_{\Phi}v)(x) > \gamma\lambda\} \subset \left\{x \in Q; (T_{\Phi}v_1)(x) > \frac{\gamma\lambda}{2}\right\}.$$

Now.

(2.5)
$$\int_{Q} \Phi(x-z) dx \leqslant \int_{|y| \leqslant |\frac{\sqrt{n}}{2}|Q|^{\frac{1}{n}}} \Phi(y) dy.$$

This means

$$\begin{split} \int_{Q} (T_{\Phi} v_{1})(x) \, dx &= \int_{Q} dx \int_{Q^{*}} \Phi(x-y) \, dv(y) \\ &= \int_{Q^{*}} dv(y) \int_{Q} \Phi(x-y) \, dx \leqslant \int_{|y| \leqslant |\frac{\sqrt{n}}{2}|Q|^{\frac{1}{n}}} \Phi(y) \, dy \int_{Q^{*}} dv(y) \end{split}$$

$$\leq (7\sqrt{n})^n |Q| (M_{\Phi} v)(x_k) \leq (7\sqrt{n})^n \beta \lambda |Q|.$$

Thus with $C = 2(7\sqrt{n})^n$,

$$\left|\left\{x \in \mathbb{Q}; (T_{\Phi} \mathsf{v}_1)(x) > \frac{\gamma \lambda}{2}\right\}\right| \leq \frac{2}{\gamma \lambda} \int_{\mathbb{Q}} (T_{\Phi} \mathsf{v}_1)(x) \, dx > C \frac{\beta}{\gamma} |\mathbb{Q}|.$$

Therefore,

$$\begin{split} |\{T_{\Phi}v > \gamma\lambda \ \text{ and } \ M_{\Phi}v \leqslant \beta\lambda\}| &= \sum_{k} |\{x \in Q_{k}; (T_{\Phi}v)(x) > \gamma\lambda\}| \\ &\leqslant \frac{C\beta}{\gamma} \sum_{k} |Q_{k}| \leqslant C\frac{\beta}{\gamma} \ |\{M(T_{\Phi}v) > \lambda\}| \,. \end{split}$$

1

To prove (2.4) we'll require the fact that $C'_n > 0$ exists with

(2.6)
$$\Phi(y) \leqslant \frac{C'_n}{r^n} \int_{|y-z| \leqslant r} \Phi(z) \, dz, \ 0 < r \leqslant |y|.$$

As Φ is nonincreasing, this would be true if it were known to hold whenever Φ is the characteristic function of a ball centred at the origin. For this it suffices to know that the set of z in the ball $|y-z| \le r$ satisfying $|z| \le |y|$ occupies at least a fixed fraction of the ball. The change of variable z = |y|v, followed by the rotation that sends $\frac{y}{|y|}$ to $e_1 = (1,0,\ldots 0)$, reduces the problem to the relative size of the intersection of the balls $|v| \le 1$ and $|v-e_1| \le s$, 0 < s < 1, to the size of the ball $|v-e_1| \le s$ itself. But for these sets the result in clear.

If $x \in Q$ (where Q denotes some fixed Q_k) and $y \in \mathbb{R}^n - Q^*$, then $|x-y| \ge |Q|^{\frac{1}{n}}$. Thus taking $r = |Q|^{\frac{1}{n}}$ in (2.6), we get

$$(\operatorname{Tv}_{2})(x) = \int_{\mathbb{R}^{n} - \mathbb{Q}^{*}} \Phi(x - y) \, d\nu(y)$$

$$\leq \frac{C'_{n}}{r^{n}} \int_{\mathbb{R}^{n} - \mathbb{Q}^{*}} d\nu(y) \int_{|z| \leq r} \Phi(x - y - z) \, dz.$$

Making the substitution v = x - z, the last expression becomes

$$\frac{C_n'}{r^n} \int_{|x-v| \le r} (T_{\Phi} v_2)(v) \, dv \leqslant \frac{C_n'}{r^n} \int_{\Omega^*} (T_{\Phi} v)(x) \, dx \leqslant \frac{C_n'}{r^n} \, \lambda |Q^*| = C_n \lambda$$

with $C_n = (7\sqrt{n})^n C_n'$, since Q^* intersects $R^n - \Omega_{\lambda} = \{M(T_{\Phi} v) \leq \lambda\}$ by the Whitney condition. This completes the proof.

Theorem 2.3. — Suppose Φ is a nonnegative, locally integrable radially decreasing function satisfying (2.2). Then for $1 and <math>\mu$ a positive locally finite Borel measure on \mathbf{R}^n , the following statements are equivalent:

1. There exists C > 0 so that whenever f is a nonnegative measurable function on \mathbf{R}^n

$$\left[\int_{\mathbb{R}^n} (\mathsf{T}_{\Phi} f)(x)^q \, d\mu(x)\right]^{\frac{1}{q}} \leqslant \mathsf{C} \left[\int_{\mathbb{R}^n} f(x)^p \, dx\right]^{\frac{1}{p}}.$$

2. There exists C' > 0 so that for all dyadic cubes Q

$$\left[\int_{\mathbb{R}^n} T_{\Phi}(\chi_{\mathbb{Q}}\mu)(x)^{p'} dx\right]^{\frac{1}{p'}} \leqslant C'[\mu(\mathbb{Q})]^{\frac{1}{q'}} < \infty$$

where
$$p' = \frac{p}{p-1}$$
, $q' = \frac{q}{q-1}$.

3. There exists K > 0 so that for all dyadic cubes Q

$$\left[\int_{Q} (M_{\Phi} \chi_{Q} \mu)(x)^{p'} dx\right]^{\frac{1}{p'}} \leqslant K[\mu(Q)]^{\frac{1}{q'}} < \infty.$$

Moreover, the least possible C, C' and K in the above are all within constant multiples of one another, the constants being independent of Φ and μ .

Proof. – Let M_{Φ}^{dy} denote the dyadic analogue of M_{Φ} given by

$$M_{\Phi}^{dy}v(x) = \sup_{x \in Q \text{ dyadic}} \left[\frac{1}{|Q|} \int \int_{Q} \Phi(y) dy \right] \int_{Q} dv$$

for $x \in \mathbb{R}^n$ and v a locally finite positive measure. We claim that for all such v,

(2.7)
$$\int_{\mathbb{R}^n} |M_{\Phi}^{dy} v|^{p'} \leqslant \int_{\mathbb{R}^n} |M_{\Phi} v|^{p'} \leqslant C_1 \int_{\mathbb{R}^n} |T_{\Phi} v|^{p'},$$

(2.8)
$$\int_{\mathbb{R}^n} |T_{\Phi} \nu|^{p'} \leqslant C_2 \int_{\mathbb{R}^n} |M_{\Phi} \nu|^{p'} \leqslant C_3 \int_{\mathbb{R}^n} |M_{\Phi}^{dy} \nu|^{p'},$$

where the constants C_1 , C_2 , C_3 depend only on n and $p(1 . The first inequality in (2.7) is trivial and the second inequality follows from part (a) of Theorem 2.2 and the classical <math>L^{p'}$ inequality for M ([18]). The first inequality in (2.8) follows from part (b) of Theorem 2.2 as in [6]. Finally, to prove the second inequality in (2.8), we apply a standard covering argument to $\{M_{\Phi}v > \lambda\}$ (where $\lambda > 0$) to obtain the existence of cubes $(Q_k)_k$ with disjoint triples satisfying

(i)
$$\left[\frac{1}{|Q_k|} \int_{|y| \le |Q_k|^{\frac{1}{n}}} \Phi(y) \, dy \right] \int_{Q_k} dv > \lambda for all k$$

$$|\{M_{\Phi} v > \lambda\}| \leqslant C \sum_k |Q_k|.$$

Now each Q_k is covered by at most 2^n dyadic cubes $(I_k^j)_{1 \le j \le 2^n}$ with

 $2^{-n}|Q_k|\leqslant |I_k^j|\leqslant |Q_k|$. There is at least one of these dyadic cubes, say $I_k=I_k^j$, with $\int_{I_k}d\nu\geqslant 2^{-n}\int_{Q_k}d\nu$. Then, since Φ is r.d. and $|I_k|\leqslant |Q_k|$,

$$\left[\frac{1}{|I_k|}\int_{|y|\leqslant |I_k|^{\frac{1}{n}}}\Phi(y)\,dy\right]\int_{I_k}d\nu > 2^{-n}\lambda \quad \text{for all } k$$

and so $\bigcup_k I_k \subset \{M_\Phi^{dy} v > 2^{-n} \lambda\}$. Since the I_k 's are pairwise disjoint, we have

$$\begin{aligned} |\{M_{\Phi}v > \lambda\}| &\leqslant C \sum_{k} |Q_{k}| \leqslant C \sum_{k} |I_{k}| \\ &\leqslant C |\{M_{\Phi}^{dy}v > 2^{-n}\lambda\}| \end{aligned}$$

and (2.8) follows upon multiplying this inequality by $\lambda^{p'-1}$ and then integrating over $(0,\infty)$.

From (2.3), (2.7) and (2.8) we obtain that the trace inequality in 1. holds if and only if there is C > 0, comparable to the one in (2.1), for which

$$(2.9) \quad \left[\int_{\mathbb{R}^n} (\mathsf{M}_{\Phi}^{dy} f \mu)(x)^{p'} \, dx \right]^{\frac{1}{p'}} \leqslant \mathsf{C} \left[\int_{\mathbb{R}^n} f(x)^{q'} \, d\mu(x) \right]^{\frac{1}{q'}}, \quad \text{for all } f.$$

Theorem A of [16] (with M_{Φ}^{dy} in place of $M_{\mu,\alpha}$, the proof is unchanged) shows that (2.9) holds if and only if there is C > 0, comparable to that in (2.9), for which

$$(2.10) \qquad \left[\int_{\mathbb{R}^n} \left[M_{\Phi}^{dy}(\chi_{\mathcal{Q}} d\mu)\right]^{p'}\right]^{\frac{1}{p'}} \leqslant C\mu(\mathcal{Q})^{\frac{1}{q'}} < \infty$$

for all dyadic cubes Q. Theorem 2.3 now follows easily. The trace inequality 1. implies its dual (2.3) which in turn implies 2. upon taking $f = \chi_Q$. Inequality 2. implies 3. by (2.7) and finally, $3. \Rightarrow (2.10) \Rightarrow (2.9) \Rightarrow 1$.

3. Schrödinger operators.

In this section, Theorem 2.3 is used to refine the estimates for eigenvalues of a Schrödinger operator $H = -\Delta - v$ given in Theorem 5, Chapter II, of [10]. By eigenvalues, we mean the numbers

 $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N \ldots$ where λ_N is the maximum over all N-1 tuples $\Phi_1, \ldots \Phi_{N-1}$ of the quantity $\inf \frac{\langle Hu,u \rangle}{\langle u,u \rangle}$, the infimum being over all $u \in Q(H), \langle u,\Phi_j \rangle = 0$, $j=1,\ldots N-1$. Here Q(H) denotes the form domain of H (see [16]) and $\langle Hu,u \rangle = \int_{\mathbb{R}^n} (|\nabla u|^2 - v|u|^2)$ for $u \in Q(H)$. Recall that $I_2 f(x) = \int_{\mathbb{R}^n} |x-y|^{2-n} f(y) \, dy$ denotes the Newtonian potential of f.

Theorem 3.1. — Let $H=-\Delta-v$, where $v(x)\geqslant 0$ is locally integrable on \mathbf{R}^n and $n\geqslant 3$. Denote the v measure of Q, $\int_Q v(x)\,dx$, by $|Q|_v$. There are positive constants C, c depending only on the dimension n such that the least eigenvalue λ_1 of H satisfies $E_{sm}\leqslant -\lambda_1\leqslant E_{big}$ where

$$\begin{split} &E_{sm}=\sup\left\{|Q|^{-2/n};\,|Q|_v^{-1}\int_QI_2(\chi_Qv)v\!\geqslant\!C\right\}\\ &E_{big}=\sup\left\{|Q|^{-2/n};\,|Q|_v^{-1}\int_QI_2(\chi_Qv)v\!\geqslant\!c\right\}. \end{split}$$

Example 3.2. — Consider Example V in [10]: a particle in a rectangular box $B = B_1 \times B_2 \times \cdots B_n$ with side lengths $\delta_1 \leq \delta_2 \leq \cdots \delta_n$. Let $v = \chi_B$ and let x_B denote the centre of B. Since

$$\sup_{\mathbf{Q}} |\mathbf{Q}_{v}|^{-1} \int_{\mathbf{Q}} \mathbf{I}_{2}(\chi_{\mathbf{Q}}v)v \approx \mathbf{I}_{2}v(\chi_{\mathbf{B}}) \approx \delta_{1}^{2} + \delta_{1} \delta_{2} + \delta_{1} \delta_{2} \log (\delta_{3}/\delta_{2})$$
$$\approx \delta_{1} \delta_{2} \log (1 + \delta_{3}/\delta_{2}),$$

Theorem 3.1 yields the correct order of magnitude for the energy, $E_{critical}$, needed to trap a particle in B, namely

$$\mathbf{E}_{\text{critical}} = \sup \left\{ 1 - 0; -\Delta - \mathbf{E}v \geqslant 0 \right\} = 1/\delta_1 \, \delta_2 \log \left(1 + \delta_3/\delta_2 \right).$$

A refinement of Theorems 6 and 6' in Chapter II of [10], similar to the one above, is given in

THEOREM 3.3. — Let $H = -\Delta - v$ where $v(x) \ge 0$ is locally integrable on \mathbb{R}^n and $n \ge 3$. There are positive constants C, c depending only on the dimension n such that:

(A) Suppose $\lambda \ge 0$ and let Q_1, \ldots, Q_N be a collection of cubes of side length at most $\lambda^{-\frac{1}{2}}$ whose doubles are pairwise disjoint. Suppose further that

 $|Q_j|_v^{-1}\int_{Q_j}I_2(\chi_{Q_j}v)v\geqslant C\,,\quad 1\leqslant j\leqslant N\,.\quad \text{Then}\quad H\quad \text{has}\quad \text{at least}\quad N$ eigenvalues $\leqslant -\lambda\,.$

(B) Conversely, suppose $\lambda \geqslant 0$ and that H has at least CN eigenvalues $\leqslant -\lambda$. Then there is a collection of pairwise disjoint (dyadic) cubes Q_1,\ldots,Q_N of side lengths at most $\lambda^{-\frac{1}{2}}$ that satisfy $|Q_j|_v^{-1}\int_{Q_j}I_2(\chi_{Q_j}v)v\geqslant c$, $1\leqslant j\leqslant N$.

Roughly speaking, Theorem 3.3 says that the negative eigenvalues of H are approximately given by $-|Q|^{-2/n}$ as Q ranges over the minimal dyadic cubes satisfying $|Q|_v^{-1} \int_O I_2(\chi_Q v) v \ge C$.

In [10], results corresponding to Theorems 3.1 and 3.3 were obtained with the quantity $|Q|_v^{-1}\int_Q I_2(\chi_Q v)v$ replaced by the simpler average $C|Q|^{\frac{2}{n}-1}\int_Q v$ in part (A) of Theorem 3.3 and by $C_p|Q|^{\frac{2}{n}-\frac{1}{p}}\bigg(\int_Q v^p\bigg)^{\frac{1}{p}}$ in part (B). A comparison of these quantities is made in Remark 3.5 at the end of this section. Chang, Wilson, and Wolff [5] show part (B) of Theorem 3.3 holds for v if $\sup_Q |Q|^{\frac{2}{n}-1}\int_Q v(x)\Phi(|Q|^{\frac{2}{n}}v(x))\,dx<\infty$, where $\Phi:[0,\infty]\to[1,\infty]$ is increasing and $\int_1^\infty \frac{dx}{x\Phi(x)}<\infty$. See also Chanillo and Wheeden [6].

Proof of Theorem 3.1. — The Schwartz class S is dense in Q(H) and thus we have

$$-\lambda_{1} = -\inf_{u \in Q(H)} \frac{\langle Hu, u \rangle}{\langle u, u \rangle} = \sup_{u \in S} \frac{\int |u|^{2}v - \int |\nabla u|^{2}}{\int |u|^{2}}$$

$$= \inf \{\alpha > 0; \int |u|^{2}v \leqslant \int |\nabla u|^{2} + \alpha |u|^{2}$$

$$= \int (|\xi|^{2} + \alpha)|\hat{u}(\xi)|^{2} d\xi, u \in S\}$$

$$= \inf \{\alpha > 0; \int (I_{1}^{\alpha}f)^{2}v \leqslant \int f^{2}, f \geqslant 0\}$$

where I_1^{α} is the operator with r.d. kernel K_1^{α} defined by $(K_1^{\alpha})^{\wedge}(\xi)=(|\xi|^2+\alpha)^{-\frac{1}{2}}.$ Thus $K_1^1(x)=G_1(x)$ and

$$\mathbf{K}_{1}^{\alpha}(x) = \alpha^{\frac{n-1}{2}} \mathbf{G}_{1}(\alpha^{\frac{1}{2}}x).$$

If we let C_{α} denote the least constant such that

$$\int (I_1^{\alpha} f)^2 v \leqslant C_{\alpha} \int f^2 \quad \text{for all } f \geqslant 0,$$

then $-\lambda_1 = \inf \{\alpha; C_{\alpha} \leq 1\}$. By Theorem 2.3,

(3.1)
$$C_{\alpha} \approx \sup_{Q} \frac{1}{|Q|_{\nu}} \int [I_{1}^{\alpha}(\chi_{Q}\nu)]^{2}$$

in the sense that the ratio of the left and right sides is bounded between two constants independent of α and v. We now show that, in fact, the supremum in (3.1) need only be taken over those cubes Q with $|Q|^{\frac{1}{n}} \leqslant \alpha^{-\frac{1}{2}}$. To this end, set $M = \sup_{\substack{Q \\ |Q|^{1/n} \leqslant \alpha^{-1/2}}} \frac{1}{|Q|_v} \int [I_1^\alpha(\chi_Q v)]^2$ and

suppose Q is a cube with $|Q|^{\frac{1}{n}} > \alpha^{-\frac{1}{2}}$. Express Q as a union of congruent cubes, Q_j , having pairwise disjoint interiors and common sidelengths, $|Q_j|^{\frac{1}{n}}$, satisfying $\frac{1}{2}\alpha^{-\frac{1}{2}} \le |Q_j|^{\frac{1}{n}} \le \alpha^{-\frac{1}{2}}$. Then, we claim

(3.2)
$$\int [I_1^{\alpha}(\chi_{Q}v)]^2 = \sum_{i,j} \int I_1^{\alpha}(\chi_{Q_i}v) I_1^{\alpha}(\chi_{Q_j}v)$$
$$\leq C \sum_i \int [I_1^{\alpha}(\chi_{Q_i}v)]^2$$
$$\leq CM \sum_i |Q_i|_v = CM|Q|_v.$$

The second inequality holds by definition of M and since $|Q_i|^{\frac{1}{n}} \le \alpha^{-\frac{1}{2}}$. To prove the first inequality, we consider two cases. First, when Q_i and Q_j are adjacent, we simply use

$$\int I_1^\alpha(\chi_{{\bf Q}_i}v)I_1^\alpha(\chi_{{\bf Q}_j}v) \leqslant \frac{1}{2}\int [I_1^\alpha(\chi_{{\bf Q}_i}v)]^2 \, + \, \frac{1}{2}\int [I_1^\alpha(\chi_{{\bf Q}_j}v)]^2 \, .$$

To treat the case when Q_i and Q_j have a distance of roughly k

sidelengths between them, $k \ge 1$, we require the facts that $K_2^{\alpha}(x) \approx |x|^{2-n}$ if $|x| \le \alpha^{-\frac{1}{2}}$ and $K_2^{\alpha}(x) \le C\alpha^{\frac{n-2}{2}}e^{-\sqrt{\alpha}|x|}$ if $|x| > \alpha^{-\frac{1}{2}}$, for which see [4]. We then have

$$\int I_1^{\alpha}(\chi_{Q_i}v)I_1^{\alpha}(\chi_{Q_j}v) = \int_{Q_i} I_2^{\alpha}(\chi_{Q_j}v)(x)v(x) dx \leqslant C\alpha^{\frac{n-2}{2}}e^{-k}|Q_i|_v|Q_j|_v.$$

However, $I_1^{\alpha}(\chi_{Q_i})(x) \ge C\alpha^{-\frac{1}{2}}$ for $x \in Q_i$ and so

$$|Q_i|_v \leqslant \frac{\alpha^{\frac{1}{2}}}{C} \int_{Q_i} I_1^{\alpha}(\chi_{Q_i}) v = \frac{\alpha^{\frac{1}{2}}}{C} \int_{Q_i} I_1^{\alpha}(\chi_{Q_i} v)(x) dx.$$

Thus

$$\begin{split} 2|Q_i|_v|Q_j|_v &\leqslant |Q_i|_v^2 + |Q_j|_v^2 \\ &\leqslant C\alpha \bigg(\bigg[\int_{Q_i} I_1^\alpha(\chi_{Q_i}v)\bigg]^2 + \bigg[\int_{Q_j} I_1^\alpha(\chi_{Q_j}v)\bigg]^2\bigg) \\ &\leqslant C\alpha^{1-\frac{n}{2}} \bigg(\int_{Q_i} [I_1^\alpha(\chi_{Q_i}v)]^2 + \int_{Q_i} [I_1^\alpha(\chi_{Q_j}v)]^2\bigg). \end{split}$$

Now, for a fixed cube Q_i , there are at most Ck^{n-1} cubes Q_j at a distance of roughly k sidelengths from Q_i . Combining all of the above, we obtain

$$\sum_{\substack{i,j\\i\neq i}} \int \mathbf{I}_1^{\alpha}(\chi_{\mathbf{Q}_i}v) \mathbf{I}_1^{\alpha}(\chi_{\mathbf{Q}_j}v) \leqslant \mathbf{C} \left[1 + \sum_{k=1}^{\infty} k^{n-1} e^{-k}\right] \sum_i \int [\mathbf{I}_1^{\alpha}(\chi_{\mathbf{Q}_i}v)]^2$$

which yields the first inequality in (3.2). From (3.1) and (3.2), we have $C_{\alpha} \approx M$ and since $\int [I_1^{\alpha}(\chi_Q v)]^2 = \int I_2^{\alpha}(\chi_Q v)v \approx \int I_2(\chi_Q v)v$ when $|Q|^{\frac{1}{n}} \leqslant \alpha^{-\frac{1}{2}}$, we finally have

$$C_{\alpha} \approx \sup_{\substack{Q \ |Q|^{1/n} \leq \alpha^{-1/2}}} \frac{1}{|Q|_{v}} \int_{Q} I_{2}(\chi_{Q}v)v$$

and Theorem 3.1 follows readily.

Proof of Theorem 3.3, part (A). — As in [10], it suffices by elementary functional analysis to construct an N-dimensional subspace $\Omega \subset Q(H)$ so

that $\langle Hu,u\rangle\leqslant -\lambda\int |u|^2$ for u in Ω . Our hypothesis implies

$$\frac{1}{|Q_j|_v}\int_{Q_j}I_2^{\lambda}(\chi_{Q_j}v)v\geqslant C \qquad \text{for } j=1,\ldots N.$$

Since $\int_Q I_2^{\lambda}(\chi_Q v)v \leqslant \left(\int_Q [I_2^{\lambda}(\chi_Q v)]^2 v\right)^{\frac{1}{2}} |Q|_v^{\frac{1}{2}}$ by Holder's inequality, we actually have

$$\int_{Q_j} [I_2^{\lambda}(\chi_{Q_j} v)]^2 v \, \geqslant \, C \int_{Q_j} I_2^{\lambda}(\chi_{Q_j} v) v \,, \quad 1 \, \leqslant j \, \leqslant \, N \,.$$

This suggests we let Ω be the linear span of $\{f_j\}_{j=1}^N$ where $f_j = \Phi_j I_2^{\lambda}(\chi_{Q_j} v)$ and $\Phi_j = 1$ on $\frac{3}{2} Q_j$ with supp Φ_j contained in $2Q_j$. Here the Φ_j are dilates and translates of a fixed $\Phi \in C_c^{\infty}(\mathbb{R}^n)$. We have immediately that

(3.3)
$$\int f_j^2 v \geqslant C \int_{Q_j} I_2^{\lambda}(\chi_{Q_j} v) v \quad \text{for} \quad 1 \leqslant j \leqslant N.$$

By hypothesis, the supports of the f_j are pairwise disjoint and so we need only establish

(3.4)
$$\langle (-\Delta + \lambda)f_j, f_j \rangle \leqslant \int (f_j)^2 v \quad \text{for } 1 \leqslant j \leqslant N$$

in order to conclude $\langle Hu,u\rangle \leqslant -\lambda \int |u|^2$ for u in Ω , as required. To prove (3.4), we let $G_j = 2Q_j - \frac{3}{2}Q_j$ and compute that

$$(-\Delta + \lambda)f_j = (-\Delta + \lambda)[\Phi_j I_2^{\lambda}(\chi_{Q_j} v)]$$

$$= \chi_{Q_j} v + \chi_{G_j} (-\Delta + \lambda)[\Phi_j I_2^{\lambda}(\chi_{Q_j} v)]$$

$$= A_j + B_j$$

since $I_2^{\lambda} = (-\Delta + \lambda)^{-1}$. Now

$$\langle \mathbf{A}_j, f_j \rangle = \int_{\mathbf{Q}_j} \mathbf{I}_2^{\lambda}(\chi_{\mathbf{Q}_j} v) v \leqslant \frac{1}{\mathbf{C}} \int f_j^2 v \text{ (by 4.3)}) \leqslant \frac{1}{2} \int f_j^2 v$$

provided C is chosen $\geqslant 2$. It remains to verify $\langle \mathbf{B}_j, f_j \rangle \leqslant C' \int_{\Omega_i} \mathbf{I}_2^{\lambda}(\chi_{\mathbf{Q}_j} v) v$ for all j since then (3.4) will follow from (3.3)

and the previous estimate provided $C \ge 2C'$. Now

(3.5)
$$|\mathbf{B}_{j}| \leq \chi_{G_{j}}[\Phi_{j}|\Delta I_{2}^{\lambda}(\chi_{Q_{j}}v)| + 2|\nabla\Phi_{j}||\nabla I_{2}^{\lambda}(\chi_{Q_{j}}v)| + (\lambda + |\Delta\Phi_{j}|)[I_{2}^{\lambda}(\chi_{Q_{j}}v)]$$

= $\mathbf{D}_{i} + \mathbf{E}_{i} + \mathbf{F}_{i}$.

Using the estimates $|D^sK_2^{\lambda}(x)| \le C|x|^{2-n-s}$, for $s \ge 0$ and $|x| \le C\lambda^{-\frac{1}{2}}$ (see [4]) we obtain that on G_i ,

$$\begin{split} & I_2^{\lambda}(\chi_{Q_j}v)(x) \leqslant C|Q_j|^{\frac{2}{n}-1} \int_{Q_j} v \\ & |\nabla I_2^{\lambda}(\chi_{Q_j}v)(x)| \leqslant C|Q_j|^{\frac{1}{n}-1} \int_{Q_j} v \\ & |\Delta I_2^{\lambda}(\chi_{Q_j}v)(x)| \leqslant C|Q_j|^{-1} \int_{Q_j} v \,. \end{split}$$

These inequalities, together with $|\Phi_j|\leqslant 1$, $|\nabla\Phi_j|\leqslant C|Q_j|^{-\frac{1}{n}}$, $|\Delta\Phi_j|\leqslant C|Q_j|^{-\frac{2}{n}}$ and the hypothesis $\lambda\leqslant |Q_j|^{-\frac{2}{n}}$, yields

(3.6)
$$D_{j}, E_{j}, F_{j} \leq C|Q_{j}|^{-1}|Q_{j}|_{v}.$$

Since $f_j(x) \le C|Q_j|^{\frac{2}{n}-1} \int_{Q_j} v$ on G_j , (3.5) and (3.6) imply

$$\langle \mathbf{B}_{i}, f_{i} \rangle \leqslant \mathbf{C} |\mathbf{Q}_{i}|^{\frac{2}{n} - 1} |\mathbf{Q}_{i}|^{2}_{v}.$$

Finally,

$$\begin{split} |Q_j|^{\frac{2}{n}-1} \bigg(\int_{Q_j} v \bigg)^2 & \leqslant C(\min_{x \in Q_j} I_2^{\lambda}(\chi_{Q_j} v)) \bigg(\int_{Q_j} v \bigg) \\ & \leqslant C \int_{Q_j} I_2^{\lambda}(\chi_{Q_j} v) v \end{split}$$

and this, combined with (3.7), shows that $\langle B_j, f_j \rangle \leq C' \int_{Q_j} I_2^{\lambda}(\chi_{Q_j} v) v$ and completes the proof of part (A) of Theorem 3.3.

Proof of Theorem 3.3, part (B). — We follow closely the argument of C. L. Fefferman and D. H. Phong in ([10]; proof of Theorem 6 in Chapter II), but with certain modifications designed to avoid the use of a square function. As in [10], it suffices to suppose v bounded and to show that if Q_1, \ldots, Q_N are the minimal dyadic cubes satisfying

 $\frac{1}{|Q_j|_v}\int_{Q_j}I_2(\chi_{Q_j}v)v\geqslant c \quad \text{and} \quad |Q_j|^{\frac{1}{n}}\leqslant \lambda^{-\frac{1}{2}}, \quad \text{then} \quad H=-\Delta-v \quad \text{has at most } CN \quad \text{eigenvalues} \quad \leqslant -\lambda \quad \text{(where the constant } C \quad \text{is of course independent of the bound on } v). \quad \text{As usual, this will be accomplished by exhibiting a subspace} \quad \Omega \subset L^2 \quad \text{of codimension} \quad \leqslant CN \quad \text{such that}$

(3.8)
$$\langle Hu,u\rangle \geqslant -\lambda \int |u|^2$$
 for all u in Ω .

We consider only the case $\lambda=0$, the case $\lambda>0$ requiring easy modifications. We begin by defining additional cubes Q_{N+1},\ldots,Q_M as in [10]; i.e. let B be the collection of all dyadic cubes Q with $\frac{1}{|Q|_v}\int_Q I_2(\chi_Q v)v\geqslant c$ and define the additional cubes Q_{N+1},\ldots,Q_M to consist of (i) the maximal cubes in B, (ii) the branching cubes in B and (iii) the descendents of branching cubes in B. The descendents of a cube Q in B are those $Q'\in B$ which are maximal with respect to the property of being properly contained in Q. A cube in B « branches » if it has at least two descendents. As shown in [10], $M\leqslant CN$. Still following [10] we define $E_0=\mathbb{R}^n-\bigcup_{j=1}^M Q_j$ and $E_j=Q_j$ minus its descendents for $j\geqslant 1$. In analogy with estimates (i) and (ii) of [10], we shall prove that the weights $v_j=\chi_{E_i}v$ satisfy

(3.9)
$$\frac{1}{|Q|_{v_j}} \int_Q I_2(\chi_Q v_j) v_j \leqslant Cc \quad \text{for all } 0 \leqslant j \leqslant M, Q \quad \text{dyadic cube.}$$

In order to make use of (3.9) and the trace inequalities it implies we shall have to define the subspace Ω so that

$$(3.10) \quad |u(x)| \leqslant \operatorname{CI}_1(\chi_{\operatorname{E}_j}|\nabla u|)(x) \qquad \text{for} \quad x \in \operatorname{E}_j, \, 0 \leqslant j \leqslant \operatorname{M}, \, u \in \Omega \, .$$
 Indeed, if both (3.9) and (3.10) hold, then for $u \in \Omega$,

$$\int |u|^2 v = \sum_{j=0}^{M} \int_{E_j} |u|^2 v_j$$

$$\leq C \sum_{j=0}^{M} \int_{E_j} [I_1(\chi_{E_j} |\nabla u|]^2 v_j \quad \text{by (3.10)}$$

$$\leq Cc \sum_{j=0}^{M} \int_{E_j} |\nabla u|^2 \quad \text{by (3.9) and Theorem 2.3}$$

$$\leq \int |\nabla u|^2 \quad \text{if } c \text{ small enough,}$$

and this is (3.8) for $\lambda = 0$. Thus it remains to construct Ω of codimension \leq CN such that (3.10) holds. In the case $1 \leq j \leq N$, E_j is a cube and (3.10) holds whenever $\int_{E_j} u = 0$ by the following inequality of E. Fabes, C. Kenig and R. Serapioni ([9]; Lemma 1.4)

$$(3.11) \quad \left| u(x) - \frac{1}{|Q|} \int_{Q} u \right| \leq CI_1(\chi_Q |\nabla u|)(x) \quad \text{for } x \in Q, Q \text{ a cube.}$$

For the case when E_j is not a cube we will need the following lemma.

Lemma 3.4. — Suppose Q_1, \ldots, Q_k are pairwise disjoint dyadic subcubes of a dyadic cube Q in \mathbf{R}^n . Then there are (not necessarily dyadic or disjoint) cubes I_1, \ldots, I_m such that $Q - \bigcup_{j=1}^k Q_j = \bigcup_{i=1}^m I_i$ and $m \leq Ck$ where C is a constant depending only on the dimension n. The above holds also for $Q = \mathbf{R}^n$ if we allow the cubes I_i to be infinite, i.e. of the form $J_1 \times J_2 \times \cdots J_n$ where each J_i is a semi-infinite interval.

This lemma has been obtained independently by S. Chanillo and R. L. Wheeden [6], with a proof much simpler than that appearing in a previous version of this paper. As a result, we refer the reader to [6] for a proof of the lemma.

We can now define the subspace Ω . For each j with j=0 or $N+1\leqslant j\leqslant M$, apply Lemma 3.4 with $Q=Q_j$ and Q_1,\ldots,Q_k the descendents of Q_j (for j=0, take $Q=\mathbb{R}^n$ and Q_1,\ldots,Q_k to be the maximal cubes in B), to obtain cubes $I_1^{(j)},\ldots,I_{m_j}^{(j)}$ with $E_j=\bigcup_{i=1}^{m_j}I_i^{(j)}$ and $m_j\leqslant C$ (# of descendents of Q_j). Note that $E_j=Q_j$ for $1\leqslant j\leqslant N$. Now define

$$\Omega = \{u; \int_{Q_j} u = 0 \text{ for } 1 \le j \le N \text{ and } \int_{I_i^{(j)}} u = 0$$

$$\text{for } N + 1 \le j \le M, \ j = 0 \text{ and } 1 \le i \le m_j \}.$$

If $x \in E_j$, $N+1 \le j \le M$ or j=0, then $x \in \text{some } I_i^{(j)}$ and thus for $u \in \Omega$, $|u(x)| \le \operatorname{CI}_1(\chi_{I_i^{(j)}}|\nabla u|)(x) \le \operatorname{CI}_1(\chi_{E_j}|\nabla u|)(x)$ by (3.11). Thus (3.10) holds. Finally, the codimension of Ω is at most

$$N + \sum_{\substack{j=0 \\ N+1 \leq j \leq M}} m_j \leq N + C \sum_{\substack{j=0 \\ N+1 \leq j \leq M}} (\# \text{ of descendents of } Q_j)$$

$$\leq N + C(M+1) \leq CM.$$

It remains now to establish (3.9). We begin with the case $j \neq 0$ of (3.9), and follow the corresponding argument in [10]. Since supp $v_j \subset Q_j$, we need only check (3.9) for dyadic cubes $Q \in B$ with $Q \subset Q_j$ and in fact, only for proper dyadic subcubes of Q_j (since if $Q = \bigcup_{i=1}^{2^n} Q_i$, then

$$\begin{split} \int_{Q} I_{2}(\chi_{Q}v) &= \int [I_{1}(\chi_{Q}v)]^{2} \\ &= \sum_{i,j} \int I_{1}(\chi_{Q_{i}}v)I_{1}(\chi_{Q_{j}}v) \leqslant \frac{1}{2} \sum_{i,j} \int [I_{1}(\chi_{Q_{j}}v)]^{2} \\ &\leqslant C_{n} \sum_{i=1}^{2^{n}} \int [I_{1}(\chi_{Q_{i}}v)]^{2} \\ &= C_{n} \sum_{i=1}^{2^{n}} \int_{Q_{i}} I_{2}(\chi_{Q_{i}}v)v). \end{split}$$

As in [10], the only «non-trivial» case occurs when $Q_j \in B$ is neither minimal nor branching and Q contains $Q^{\#_j}$, the unique maximal Q_i , $1 \le i \le M$, that is properly contained in Q_j (see the argument on p. 157-158 of [10]). To obtain (3.9) in this case we use a Whitney decomposition in place of the Calderon-Zygmund decomposition used in [10]. There is a dimensional constant C so large that we can choose pairwise disjoint dyadic subcubes \hat{Q}_{α} of $Q - Q^{\#} (= E_j \cap Q)$ such that each \hat{Q}_{α} satisfies

(3.12) either
$$|\hat{Q}_{\alpha}| = |Q_{j}^{\#}|$$
 and dist $(\hat{Q}_{\alpha}, Q_{j}^{\#}) \leq C$ or $2 \leq \frac{\operatorname{dist}(\hat{Q}_{\alpha}, Q_{j}^{\#})}{\operatorname{diam}\hat{Q}_{\alpha}} \leq 2C$.

Then

$$\begin{split} \int_{Q} I_{2}(\chi_{Q}v_{j})v_{j} &= \sum_{\alpha,\beta} \int_{Q_{\alpha}} I_{2}(\chi_{Q_{\beta}}v)v \\ &\leqslant C \sum_{\{\alpha,\beta;Q_{\alpha} \text{ touches } Q_{\beta}\}} \int I_{1}(\chi_{Q_{\alpha}}v)I_{1}(\chi_{Q_{\beta}}v) \\ &+ C \sum_{\substack{\{\alpha,\beta;|Q_{\beta}| \leqslant |Q_{\alpha}| \\ \text{and } Q_{\alpha},Q_{\beta} \text{ do not touch}\}}} \int_{Q_{\alpha}} I_{2}(\chi_{Q_{\beta}}v)v = D + E \,. \end{split}$$

Now (3.12) shows that the number of \hat{Q}_{β} touching a given \hat{Q}_{α} doesn't

exceed a dimensional constant and so

$$\mathbf{D} \leqslant \mathbf{C} \sum_{\alpha} \int [\mathbf{I}_{1}(\chi_{\bar{\mathbb{Q}}_{\alpha}} v)]^{2} = \mathbf{C} \sum_{\alpha} \int_{\bar{\mathbb{Q}}_{\alpha}} \mathbf{I}_{2}(\chi_{\bar{\mathbb{Q}}_{\alpha}} v) v \leqslant \mathbf{C} c \sum_{\alpha} \int_{\bar{\mathbb{Q}}_{\alpha}} v_{j} = \mathbf{C} c \int_{\mathbb{Q}} v_{j}$$

since the \hat{Q}_{α} are not in B. Condition (3.12) also shows that if $|\hat{Q}_{\beta}| \leqslant |\hat{Q}_{\alpha}|$ and \hat{Q}_{β} , \hat{Q}_{α} do not touch, then dist $(\hat{Q}_{\beta},\hat{Q}_{\alpha}) \geqslant c |\hat{Q}_{\alpha}|^{\frac{1}{n}}$. Thus

$$\mathbf{E} \leqslant \mathbf{C} \sum_{\alpha} \left(\int_{\mathbf{Q}_{\alpha}} v \right) |\hat{\mathbf{Q}}_{\alpha}|^{\frac{2}{n} - 1} \sum_{\beta: |\hat{\mathbf{Q}}_{\beta}| \leq |\hat{\mathbf{Q}}_{\alpha}|} \left[\int_{\mathbf{Q}_{\beta}} v \right].$$

But $|\hat{Q}_{\beta}|^{\frac{2}{n}-1} \int_{Q_{\beta}} v \leq \frac{1}{|\hat{Q}_{\beta}|_{v}} \int_{Q_{\beta}} I_{2}(\chi_{Q_{\beta}}v)v \leq c$ since $\hat{Q}_{\beta} \notin B$ and, by (3.12), the number of \hat{Q}_{β} of a given size does not exceed a dimensional constant. Thus

$$E \leqslant Cc \sum_{\alpha} \left(\int_{Q} v \right) |\hat{Q}_{\alpha}|^{\frac{2}{n} - 1} \sum_{\{k; 2^{kn} \leqslant |\hat{Q}_{\alpha}|\}} \left[\sum_{|\hat{Q}_{\beta}| = 2^{kn}} |\hat{Q}_{\beta}|^{1 - \frac{2}{n}} \right]$$

$$\leqslant Cc \sum_{\alpha} \int_{Q_{\alpha}} v = Cc \int_{Q} v_{j} \quad \text{(since } n \geqslant 3\text{)}$$

and this completes the verification of (3.9) for $j \neq 0$. For j = 0, we again suppose Q dyadic in B. If $Q \subset \text{some } Q_1, \ldots, Q_M$, then supp $v_0 \cap Q = \emptyset$ and (3.9) holds trivially. Otherwise, Q contains a unique maximal $Q_i(1 \leq i \leq M)$, say $Q^{\#}$, and we may argue as above to obtain (3.9). This completes the proof of Theorem 3.3.

Remark 3.5. — In [10] it is shown that $\sup_{Q} |Q|^{\frac{2}{n}-1} \int_{Q} v \leqslant C$ is necessary and $\sup_{Q} |Q|^{\frac{2}{n}-\frac{1}{p}} \bigg(\int_{Q} v^p \bigg)^{1/p} \leqslant C_p, p > 1$, sufficient for the L^2 trace inequality (1.1) with $T_{\Phi} = I_1$. We give here a direct proof that

(3.20)
$$\sup_{Q} |Q|^{\frac{2}{n}-1} \int_{Q} v \leq C \sup_{Q} |Q|^{-1} \int_{Q} I_{2}(\chi_{Q}v)v$$

$$\leq C_{p} \sup_{Q} |Q|^{\frac{2}{n}-\frac{1}{p}} \left(\int_{Q} v^{p} \right)^{1/p}, \qquad p > 1.$$

The first inequality in (3.20) follows from the observation that $I_2(\chi_Q v)(x) \geqslant C|Q|^{\frac{2}{n}-1} \int_Q v$ for x in a cube Q.

Let $B_p = \sup_Q |Q|^{\frac{2}{n}-\frac{1}{p}} \bigg(\int_Q v^p \bigg)^{1/p}$. Suppose first that v satisfies the A_∞ condition of B. Muckenhoupt. Choose p so close to 1 that the reverse Hölder condition $\bigg(|Q|^{-1} \int_Q v^p \bigg)^{1/p} \leqslant C_p |Q|^{-1} \int_Q v$ holds for all cubes Q. Let $M_\alpha f(x) = \sup_{x \in Q} |Q|^{\frac{\alpha}{n}-1} \int_Q |f|$. Since $M_2(\chi_Q v) \leqslant B_p$ on Q,

$$(3.21) \quad \int_{Q} I_{2}(\chi_{Q}v)v \leqslant \left(\int_{Q} I_{2}(\chi_{Q}v)^{p'}\right)^{\frac{1}{p'}} \left(\int_{Q} v^{p}\right)^{1/p}$$

$$\leqslant C_{p} \left(\int_{Q} M_{2}(\chi_{Q}v)^{p'}\right)^{1/p'} \left(\int_{Q} v^{p}\right)^{1/p} \quad (\text{see [15]})$$

$$\leqslant C_{p}B_{p}|Q|^{1/p'} \left(\int_{Q} v^{p}\right)^{1/p} \leqslant C_{p}B_{p} \int_{Q} v.$$

For the general case, we use the observations in [10] that $v^+(x) = \sup_{x \in Q} \left(|Q|^{-1} \int_Q v^p \right)^{1/p}$ satisfies the A_∞ condition and $M_2 v^+ \leqslant C_p B_p$ ([10]; p. 153). The above argument then yields (3.21) with v^+ in place v. Since $v \leqslant v^+$, (3.20) follows. This is of course obvious from Theorem 2.3, but can also be proved directly. Finally, we point out that the condition $M_{2p}(v^p) \leqslant C_p$ is equivalent to the boundedness of M_p from L^2 to $L^2(v^p)$ ([17]). Together with the inequality $|I_1f(x)| \leqslant C_p M_p |f|(x)^{1/p} Mf(x)^{1/p'}$ of D. R. Adams, this yields another proof that $M_{2p}(v^p) \leqslant C_p$ is sufficient for the L^2 trace inequality (1.1) with $T_{\Phi} = I_1$. J. M. Wilson has recently communicated to us yet another proof.

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