Annales de l'institut Fourier

Gérard L. G. Sleijpen

The order structure of the space of measures with continuous translation

Annales de l'institut Fourier, tome 32, nº 2 (1982), p. 67-110
<http://www.numdam.org/item?id=AIF_1982__32_2_67_0>

Ann. Inst. Fourier, Grenoble
32, 2 (1982), 67-110.

THE ORDER STRUCTURE OF THE SPACE OF MEASURES WITH CONTINUOUS TRANSLATION

by Gerard L. G. SLEIJPEN

Introduction.

Let S be a stip; this is a locally compact semigroup with identity element 1 of which the topology is induced by a neighbourhood base of 1 [cf. (2.1)]. In view of the results in e.g. [1], [3], [15] one may state that the algebra L(S) of all bounded Radon measures on S with continuous translations [see definition (2.3)] is the natural analogue of the group algebra $L^1(G)$ of a locally compact group G. Therefore, it is tempting, now, to look for an analogue on S of the $L^1(G)$-module $L^\infty(G)$. For this purpose, since L(S) is essentially a measure algebra and not a function space, we look among the measure spaces for a candidate.

If, for instance, S is compact, the space of all bounded Radon measures μ for which the collection of all translates $|\mu| * \bar{x}$ [where $\bar{x}$ is the point mass at x] ($x \in S$) has an upper bound in L(S) seems to be suitable; if, moreover, S is a group this space « coincides » with $L^\infty(S)$. However, simplicity of a definition only is not a sufficient justification for a study; many other generalizations of $L^\infty(G)$ are conceivable [see for instance § 7 of [19]]. Therefore, in order to deepen our understanding in the structure of $L^\infty(G)$, we listed a number of properties that the least a proper analogue of $L^\infty(G)$ sould have. Thus, we came to the notion of « *pseudo* L^∞*-space* » [these are Riesz ideals of $L(S)_{loc}$ with a Banach lattice structure that has certain completeness properties [cf. (2.5.1-2)]]; furthermore the unit ball is vaguely bounded [cf. (2.5.3)], and it « contains » all its translates [cf. (2.5.4)]. In the case that S is a group, these spaces are [or, to be more precise, can be identified via the Haar measure with] invariant solid Banach function spaces as have been studied in e.g. [6] and [7]. By studying the properties of the pseudo L^∞-spaces, and to observe how they work in the induced spaces, we hope to establish those that are essential for the $L^\infty(G)$.

In [19], we paid some attention to the L^p-spaces, induced by a pseudo L^∞-space in a way as described by J.-P. Bertrandias in [2]. In the present paper, we concentrate on the subspace of a pseudo L^∞-space consisting of all measures of which the translation is [uniformly] continuous with respect to the norm of the pseudo L^∞-space. To be more precise : let $L^\infty(S,B)$ be a pseudo L^∞-space with norm $\| \ \|_\infty^B$. The collection of all $\mu \in L^\infty(S,B)$ for which $r_\mu[r_\mu(x) := \mu * \bar{x} \ (x \in S)]$ is a continuous map from S into $L^\infty(S,B)$ is denoted by $L_{RUC}(S,B)$. The closure of $\{\mu \in L_{RUC}(S,B) \,|\, \text{support of } \mu \text{ is compact}\}$ is denoted by $L_{RUC}(S,B)_\infty$. If S is a group with right Haar-measure m and $L^\infty(S,B) \cong L^\infty(S,m)$ [i.e. $L^\infty(S,B) = \{fm \,|\, f \in L^\infty(S,m)\}$], then $L_{RUC}(S,B) \cong \{f \,|\, f : S \to \mathbf{C}$ uniformly continuous$\}$ and $L_{RUC}(S,B)_\infty \cong C_\infty(S)$.

The problems we solve here, mainly have to do with the order structure of the spaces in question. We show how certain order-continuity properties of $\| \ \|_\infty^B$ are related to the conditions « $L_{RUC}(S,B)$ [or $L_{RUC}(S,B)_\infty$] is a Riesz ideal of $L^\infty(S,B)$ » and « $L_{RUC}(S,B)$ [or $L_{RUC}(S,B)_\infty$] is a Riesz subspace of $L^\infty(S,B)$ ». The main result we obtain is new and of interest also in the case that S is a group. If S is a non-discrete group with right Haar measure m, this result runs as follows :

1) $L_{RUC}(S,B)_\infty$ *is a Riesz ideal if and only if* $\| \ \|_\infty^B$ *is order continuous on* $\{fm \,|\, f \in L^\infty(S,m)$, *the support of* f *is compact*$\}$ [cf. (4.14)].

2) $L_{RUC}(S,B)$ *is a Riesz ideal if and only if* $\| \ \|_\infty^B$ *is order continuous on* $\{fm | (\sup\{f_x | x \in U\})m \in L^\infty(S,B)\}$, *where* U *is a compact neighbourhood of* 1; [see (5.10) and (5.11.2)].

In § 2, we explain our notations and conventions. Further, we give the definitions and properties that are basic to the theory of stips, and we introduce the pseudo L^∞-spaces.

We consider the Banach-module structure of $L_{RUC}(S,B)$ in the next section. In § 4, we discuss the case that $L_{RUC}(S,B)_\infty$ is a Riesz ideal. Next, in §5, we generalize the obtained results to $L_{RUC}(S,B)$. In the last section, we study the conditions under which $L_{RUC}(S,B)$ is a Riesz subspace of $L^\infty(S,B)$.

I wish to express my gratitude to dr. G. Groenewegen for stimulating discussions on the subject of this paper.

2. Notations, definitions and elementary properties.

In this section, we explain notations and conventions. Furthermore, we collect some elementary properties. Conventions that are not explained in the text are the same as the ones in [15]. Related properties can be found in [15], [19] and [20]. For some background information concerning Riesz spaces we refer the interested reader to [9] and [14].

S is a locally compact semigroup [the topology is locally compact Hausdorff and the multiplication is jointly continuous] with an identity element 1.

$\mathscr{K}$ denotes the collection of all compact subsets of S. For any subset A of S, ξ_A denotes the characteristic function of A. The collection of all locally Borel measurable functions f from S into **C** [i.e. $f\xi_F$ is Borel measurable for all $F \in \mathscr{K}$] is denoted by $m(S)$. For each $f \in m(S)$, $\|f\|_\infty := \sup\{|f(x)| | x \in S\}$. The subspace of the bounded continuous functions in $m(S)$ is denoted by $C(S)$. $C_{00}(S) := \{f \in C(S) | \text{there is an } F \in \mathscr{K} \text{ such that } f(x) = 0\ (x \in S\backslash F)\}$ and $C_\infty(S)$ is the closure of $C_{00}(S)$ with respect to the $\|\cdot\|_\infty$-norm.

The space of the [not necessarily bounded] Radon measures on S is denoted by $\bar{M}(S)$. We will identify $\bar{M}(S)$ with $C_{00}(S)^*$, the topological dual space of $C_{00}(S)$ [the topology on $C_{00}(S)$ is given by the seminorms $f \to \|fh\|_\infty$ $(f \in C_{00}(S))$, where h is any continuous function on S]. $\bar{M}(S)$ is a [complex] Riesz space under the obvious ordering. $\bar{M}_\sigma(S) := \{\mu \in \bar{M}(S) | \mu \text{ is } \sigma\text{-finite}\}$, while $M(S) := \{\mu \in \bar{M}(S) | \mu \text{ is bounded}\}$.

For a $\mu \in \bar{M}(S)^+$ and a $\nu \in \bar{M}_\sigma(S)^+$ we will write $\mu * \nu \in \bar{M}(S)$ if for each $f \in C_{00}(S)^+$ and each $x \in \operatorname{supp}(\nu)$ the bounded continuous function $f_x : y \to f(yx)\ (y \in S)$ is μ-integrable and the function $\mu \circ f : x \to \mu(f_x)$ $(x \in S)$ is ν-integrable : in this case $\mu * \nu$ is given by

$$\mu * \nu(f) := \int \mu \circ f \, d\nu \quad \text{for all} \quad f \in C_{00}(S)^+.$$

By splitting the measures into their Jordan components it will be clear what we mean by $\mu * \nu \in \bar{M}(S)$ for a $\mu \in \bar{M}(S)$ and a $\nu \in \bar{M}_\sigma(S)$. If both μ,

$\nu \in \bar{M}_\sigma(S)$ and $\mu * \nu \in \bar{M}(S)$ then

$$\mu * \nu(f) = \iint f(xy)\, d\mu(x)\, d\nu(y) = \iint f(xy)\, d\nu(y)\, d\mu(x)$$

for every $f \in m(S)$ that is $|\mu| * |\nu|$-integrable.

If $B \subseteq \bar{M}(S)$ is a Banach space under a certain norm ρ, then $B_{\mathscr{K}}$ is the collection of all $\mu \in B$ for which supp $(\mu) \in \mathscr{K}$ and $B_\infty := \rho\text{-clo}(B_{\mathscr{K}})$.

2.1. Definition [cf. [15], (2.1), (2.3)]. — *A stip* S *is a locally compact semi group with identity element* 1 *for which for each neighbourhood* U *of* 1 :

(1) $\quad x \in \text{int}[U^{-1}(Ux) \cap (xU)U^{-1}] \quad$ *for all* $\quad x \in S$

[*where* $A^{-1}B = \{y \mid Ay \cap B \neq \emptyset\}$ $(A, B \subseteq S)$];

(2) $\quad 1 \in \text{int}\,[U^{-1}v \cap wU^{-1}] \quad$ *for some* $v, w \in U$.

Put $\mathring{S} := \bigcap\{J \mid J \subseteq S,\ \bar{J} = S,\ JS \cap SJ \subseteq J\}$, [where $\bar{J}$ is the closure clo J of J].

2.2. Proposition [cf. [15], (2.4), (2.7)]. — *Let* S *be a stip.*

Then $\text{clo}(\mathring{S}) = S$, $S\mathring{S}S = S = \mathring{S}\mathring{S}$.

For each $x \in S$, *for each open set* U *and* V *of* S *and each* $u \in U \cap \mathring{S}$ *we have that the sets* $U^{-1}(Vx)$, $(xV)U^{-1}$, $(U \cap \mathring{S})^{-1}x$ *and* $x(U \cap \mathring{S})^{-1}$ *are open and*

$$x \in \text{int}\,[u^{-1}((U \cap \mathring{S})x) \cap (x(U \cap \mathring{S}))u^{-1}].$$

2.3. Définition. — *Let* S *be a stip.*

L(S) *is the collection of all* $\mu \in M(S)$ *for which one of the maps* r_μ *or* $l_\mu[r_\mu(x) := \mu * \bar{x}, l_\mu(x) := \bar{x} * \mu \ (x \in S)]$ *from* S *into* M(S) *is weakly continuous at* 1. $L(S)_{\text{loc}} := \{\mu \in \bar{M}(S) \mid \mu|_K \in L(S)$ *for all* $K \in \mathscr{K}\}$. *The collection of all Borel subsets* A *of* S *for which* $\mu(A) = 0$ *for all* $\mu \in L(S)$ *is denoted by* $\mathscr{N}$.

2.4. Proposition [cf. [15], (3.13) and [20], (12.7), (6.9)]. — *Let* S *be a stip.* $L(S)_{\text{loc}}$ *is a Riesz ideal of* $\bar{M}(S)$. L(S) *is an* L-*ideal in* M(S). *If* $\mu \in L(S)$ *then both* r_μ *and* l_μ *are norm-continuous. A* $\mu \in \bar{M}(S)$ *belongs to*

$L(S)_{loc}$ *as soon as* $\mu(F) = 0$ *for all* $F \in \mathcal{N} \cap \mathcal{K}$. *If* $Z \subseteq S$ *such that* $ZS \subseteq Z$ *or* $SZ \subseteq Z$ *then* $\bar{Z}\backslash Z$ *is* μ*-negligible for all* $\mu \in L(S)$.

Throughout this paper S *is a stip with the additional properties :*

1) $\mathrm{clo}\{\mathrm{supp}(\mu) \mid \mu \in L(S)\} = S$;

2) *the identity element has a countable neighbourhood base.*

A stip S with property (1) belongs to the class of the *foundation semigroups* [cf. [15], (2.2)]. In [18], the reader can find a discussion whether each stip has property (1).

We require S to have property (2), only in order to avoid a number of rather technical complications. Most of the results in this paper can also be proved without this topological restriction, by exploiting the δ-isolated idempotents e [i.e. $e^2 = e$, and $\{e\}$ is a G_δ-subset of $\{f \in S | f^2 = f, ef = fe = f\}$] and the compact subgroups of S that are G_δ-sets [cf. [20], ch. XI].

Furthermore, *throughout this paper :*

2.5. Definition [cf. [19], (5.3)]. – $L^\infty(S,B)$ *is a pseudo* L^∞*-space under the norm* $\| \ \|_\infty^B$: *i.e.* $L^\infty(S,B)$ *is a Riesz ideal of* $L(S)_{loc}$ *and the norm* $\| \ \|_\infty^B$ *on* $L^\infty(S,B)$ *has the following properties :*

1) $L^\infty(S,B)$ *is a Banach lattice under* $\| \ \|_\infty^B$;

2) $\| \ \|_\infty^B$ *has the [extended] Fatou-Levi property* [*i.e.* if $V \subseteq L^\infty(S,B)$ such that (i) for each $v', v'' \in V$ there is a $v \in V$ for which $v' \leqslant v$, $v'' \leqslant v$ [we write $V\uparrow$] and (ii) $\|v\|_\infty^B \leqslant 1$ for all $v \in V$, then V has a least upper bound $\mu \in L^\infty(S,B)$ [we write $V\uparrow\mu$] and $\|\mu\|_\infty^B \leqslant 1$];

3) $B := \{\rho \in L^\infty(S,B) | \ \|\rho\|_\infty^B \leqslant 1\}$ *is vaguely bounded* [*i.e.* $\sup\{|\rho(F)| \,|\rho \in B\} < \infty$ for all $F \in \mathcal{K}$];

4) *The modular function* Δ *from* S *into* $[0,\infty]$ *defined by* $\Delta(x) := \sup\{\|\rho * \bar{x}\|_\infty^B \mid |\rho \in B \cap L(S)\}$ $(x \in S)$ *is locally bounded* [*i.e.* $\|\Delta\xi_F\|_\infty < \infty$ for all $F \in \mathcal{K}$].

In case S is a group the pseudo L^∞-spaces can be identified, via the Haar measure, with invariant solid BF-spaces having property L.4 as defined in [6].

2.6. *Examples* [see also (3.3) and (5.4) of [19] and in this paper (3.7), (4.1), (4.16), (4.18), (5.7)].

1) Let S be a group with right Haar measure m_r and left Haar measure m_l. For each $p \in [1,\infty]$, the space $L^p(S,m_r)$ is a pseudo L^∞-space with modular function equal to 1. The space $L^p(S,m_l)$ is also a pseudo L^∞-space. In this case the modular function is $\delta^{1/q}$, where $q \in [1,\infty]$ such that $1/q + 1/p = 1$ and δ is given by $\delta(x) := m_l(Kx^{-1})/m_l(K)\ (x \in S)$ for some $K \in \mathcal{K}$ with $m_l(K) \neq 0$.

2) L(S) is a pseudo L^∞-space with modular function identically 1.

3) Let U be a compact neighbourhood of 1.

For each $\mu \in L(S)$, let $\|\mu\|_U^v := \|\mu_U^v\|$, whenever $\{|\mu| * \check{x} | x \in U\}$ has a least upper bound μ_U^v in L(S), otherwise $\|\mu\|_U^v := \infty$.

The space $L_U^v(S) := \{\mu \in L(S) | \|\mu\|_U^v < \infty\}$ is a pseudo L^∞-space under the norm $\|\ \|_U^v$ [cf. § 7 of [19]].

In case S is a group and $U^{-1} = U$, $m(U) = 1$ for a right Haar measure m, we have that $m(UxU)/m(UU) \leqslant \Delta(x) \leqslant m(UxU)(x \in S)$.

The space $\{\mu \in L(S) | \|\sup\{\check{x} * |\mu|\,| x \in U\}\| < \infty\}$ is a pseudo L^∞-space as well. The modular function is equal to 1. For the case where S is a group, this space has been studied in [12], [5], [8].

2.7. Proposition. — *a) For each* $K \in \mathcal{K}$, *there is an* $M_K \in (0,\infty)$ *such that*

$$\|\mu|_K\| \leqslant M_K \|\mu\|_\infty^B \quad \textit{for all} \quad \mu \in L^\infty(S,B).$$

b) For each $f \in m(S)$, *put*

$$|f|_1^B := \sup\{|\mu(f)|\,|\,\mu \in B\}.$$

A $\mu \in L(S)_{loc}$ *belongs to* $L^\infty(S,B)$ *as soon as* $c := \sup\{|\mu(f)|\,|\,f \in m(S), |f|_1^B \leqslant 1\} < \infty$, *in which case* $\|\mu\|_\infty^B = c$.

c) The modular function Δ *is lower semicontinuous [i.e.* $\Delta^{-1}([0,\alpha])$ *is closed* $(\alpha > 0)$] *and* $\Delta(xy) \leqslant \Delta(x)\Delta(y)$ *for all* $x,y \in S$.

d) With $\delta = 1/\Delta$, *for each* $\mu \in L^\infty(S,B)$, $\nu \in M(S)$ *we have that*

$$\mu * (\delta\nu) \in L^\infty(S,B) \quad \textit{and} \quad \|\mu * (\delta\nu)\|_\infty^B \leqslant \|\mu\|_\infty^B \|\nu\|.$$

(e) Put $Q := \mathrm{clo} \bigcup \{\mathrm{supp}(\rho) | \rho \in L^\infty(S,B)\}$. *If* $\mu \in L(S)$ *such that* $|\mu|(S \backslash Q) = 0$ *then* $\mu \ll \rho$ *for some* $\rho \in B$. *For each* $F \in \mathcal{K}$, *there is a* $\rho \in B$ *such that* $\mu|_F \ll \rho$ *for all* $\mu \in L^\infty(S,B)$.

Proof. — (*a*) Is a trivial consequence of the vague boundedness of B. (*b*) By an adaptation of the proof of theorem (13.5) in [20] [see also theorem (4.8) in [16]], for each compact subset F of S, we can find an $m \in L(S)^+$ such that

$$\mu|_F \ll m \qquad \text{for all} \qquad \mu \in L^\infty(S,B).$$

Therefore, locally, $L^\infty(S,B)$ can be viewed as a Köthe function space. Since $\| \ \|_\infty^B$ has the Fatou property, we locally have (*b*).

Finally, the [extended] Fatou property now implies (*b*) [see also prop. VII and theorem IV of [2]].

The proof of (*c*), (*d*) and (*e*) can be found in [19], (5.5), (5.9), (5.8), respectively.

2.8. *Remarks.* — (1) The proof of (*b*), as suggested above, depends on the fact that $\{1\}$ is a G_δ-subset of S. However, by an adaptation of the arguments in § 4 of [19], one can also prove (*b*) without this countability restriction for $\{1\}$.

(2) Let $(L(S),\circledast)$ be the Banach space endowed with the product $\circledast$ given by

$$\mu \circledast \nu = \Delta\left[\frac{1}{\Delta}\mu * \frac{1}{\Delta}\nu\right] \quad (\mu,\nu \in L(S)).$$

Then $(L(S),\circledast)$ is a Banach algebra, a so-called *Beurling algebra* [cf. e.g. [6], p. 142] and $L^\infty(S,B)$ is a right $(L(S),\circledast)$-module under the module operation suggested in (*d*) [cf. [6], lemma 1.5].

3. B-uniformly continuous measures.

In this section, we introduce the B-uniformly continuous measures and we prove some elementary properties.

The notion of « B-uniformly continuous measure » can be viewed as a generalization of the notion of « uniformly continuous function » on a group; in case S is a group with right Haar measure m, the measure fm ($f \in L^\infty(S,m)$) of which the right translation r_{fm} from S into $\bar{M}(S)$ is continuous with respect to $\| \ \|_\infty [\|fm\|_\infty := \text{ess sup}\,\{|f(x)|\,|x \in S\}]$ can be identified with a uniformly continuous function [cf. [4]].

3.1. DÉFINITION. — *A* $\mu \in L^\infty(S,B)$ *is said to be* B-*uniformly continuous if the map* r_μ *from* S *into* $L^\infty(S,B)$ *is continuous with respect to the* $\| \ \|_\infty^B$-*norm.* The collection of all B-uniformly continuous measures is denoted by $L_{RUC}(S,B)$.

Recall that $L_{RUC}(S,B)_{\mathscr{K}} = \{\mu \in L_{RUC}(S,B) \mid \text{supp}(\mu) \in \mathscr{K}\}$ [not to be confused with

$$\{\mu|_F | \mu \in L_{RUC}(S,B), F \in \mathscr{K}\}]$$

and

$$L_{RUC}(S,B)_\infty = \| \ \|_\infty^B\text{-clo}(L_{RUC}(S,B)_{\mathscr{K}}).$$

The spaces $L_{RUC}(S,B)$ and $L_{RUC}(S,B)_\infty$ obviously are closed subspaces of $L^\infty(S,B)$. However, it is far from clear whether these spaces are Riesz subspaces or Riesz ideals. Before we concentrate on these problems in § 4, 5 and § 6 we give some « properties of Banach module type ».

If the space $L_{RUC}(S,B)$ is considered as a generalization of RUC, the space of uniformly continuous functions on a group, then $L_{RUC}(S,B)_{\mathscr{K}}$ and $L_{RUC}(S,B)_\infty$ are generalizations of $C_{00}(S)$, respectively of $C_\infty(S)$. The correctness of the view, suggested here, is emphasized by the following property, for whose proof we refer to [19], (5.12).

As in [2] has been explained, $L^\infty(S,B)$ *introduces* L^p-*spaces* [see also [19], § 3]. As in the group case, *these* B-*uniformly continuous measures with compact support form a dense subset in any of these* L^p-*spaces.*

3.2. LEMMA. — *Put* $\delta(x) := \Delta(x)^{-1}$ *for all* $x \in S$.

Then δ *is locally bounded.*

Put $\gamma := \sup\{1/\|\Delta \xi_U\|_\infty | U \subseteq S, 1 \in \text{int}(U)\}$, *and let* V *be a compact neighbourhood of* 1.

Then for each $\nu \in L(S)$, $\varepsilon > 0$ *there is a* $\rho \in L(S)^+$ [*or if*

$$S = \text{clo} \cup \{\text{supp}(\mu) | \mu \in L^\infty(S,B)\}$$

there is a $\rho \in L(S)^+ \cap L^\infty(S,B)$] *such that*

$$\text{supp}(\rho) \subseteq V, \qquad \|\rho\| \leqslant 2/\gamma \qquad \textit{and} \qquad \|\nu \circledast \rho - \nu\| < \varepsilon$$

[*where* $\nu \circledast \rho = \Delta(\delta\nu * \delta\rho)$]. *In particular, we have that the Beurling algebra* $(L(S),\circledast)$ *has an approximate identity with bound* $2/\gamma$.

Proof. – The local boundedness of δ follows easily from the fact that the sets $\{x \in S | \delta(x) < N\}$ are open [use (2.7.*c*)].

Let $\nu \in L(S)$, $\varepsilon > 0$. Put

$$\varepsilon' := \varepsilon/7, \qquad \varepsilon'' := \min(\gamma, \varepsilon'\gamma/||\nu||)$$

and

$$V' := \operatorname{int}\{x \in V | \delta(x) \in (\gamma - \varepsilon'', \gamma + \varepsilon'')\}.$$

From the definition of γ and the upper semicontinuity of δ, it follows that $1 \in \operatorname{clo}(V')$. There is an $F \in \mathscr{K}$ such that $|\nu|(S \backslash F) < \varepsilon'$. Consider $\mu := \nu|_F$. Since Δ and δ are locally bounded and μ belongs to $L(S)$ we have that

$$W := \{x \in S | \, ||\Delta(\delta\mu * \bar{x}) - \mu|| < \varepsilon'\}$$

is an open neighbourhood of 1. Take a $\rho' \in L(S)^+$ such that

$$||\rho'|| = 1/\gamma \qquad \text{and} \qquad \operatorname{supp}(\rho') \subseteq V' \cap W.$$

Then for each $f \in C_\infty(S)$ with $||f||_\infty \leqslant 1$ we find that

$$\begin{aligned} |\mu \circledast \rho'(f) - \mu(f)| &= \left| \int \delta\mu * \bar{x}(\Delta f) \, d\,\delta\rho'(x) - \int \gamma\mu(f) \, d\rho'(x) \right| \\ &\leqslant \left| \int [\delta\mu * \bar{x}(\Delta f) - \mu(f)]\delta(x) \, d\rho'(x) \right| + \left| \int \mu(f)(\delta(x) - \gamma) \, d\rho'(x) \right| \\ &\leqslant \varepsilon' \int \delta \, d\rho' + ||\mu|| \int |\delta(x) - \gamma| \, d\rho'(x) \leqslant 4\varepsilon'. \end{aligned}$$

Hence

$$||\mu \circledast \rho' - \mu|| \leqslant 4\varepsilon'.$$

If $S = \operatorname{clo} \bigcup \{\operatorname{supp}(\pi) \, | \, \pi \in L^\infty(S,B)\}$ then, by (2.7.*e*), there is a $\rho \in B$ such that

$$||\rho - \rho'|| < \min(\varepsilon'/||\nu||, \gamma)$$

[actually, $\rho = (f \wedge n)\sigma$, where $\sigma \in B^+$ and $f \in L^1(S,\sigma)$ such that $\rho' = f\sigma$, and n is a suitable natural number]. Otherwise, $\rho := \rho'$. Then

$$\begin{aligned} ||\nu \circledast \rho - \nu|| &\leqslant ||\nu \circledast \rho - \mu \circledast \rho|| + ||\mu \circledast \rho - \mu \circledast \rho'|| \\ &\quad + ||\mu \circledast \rho' - \mu|| + ||\mu - \nu|| \leqslant 7\varepsilon' = \varepsilon. \quad \square \end{aligned}$$

3.3. THEOREM. – $L_{RUC}(S,B) = \{\mu * \delta\nu \mid \mu \in L^{\infty}(S,B),\ \nu \in L(S)\}$ *and*

$$L_{RUC}(S,B)_{\infty} = \{\mu * \delta\nu \mid \mu \in L^{\infty}(S,B)_{\infty}, \nu \in L(S)\}.$$

Proof. – Let $\mu \in L^{\infty}(S,B)$ and $\nu \in L(S)_{\mathscr{K}}$. Put $F := \mathrm{supp}(\nu)$. Let $x \in S$ with compact neighbourhood X.

Then

$$\|\mu * \delta\nu * \bar{x} - \mu * \delta\nu * \bar{y}\|_{\infty}^{B} = \|\mu * \delta(\Delta(\delta\nu * \bar{x} - \delta\nu * \bar{y}))\|_{\infty}^{B}$$
$$\leqslant \|\mu\|_{\infty}^{B}\|\Delta(\delta\nu * \bar{x} - \delta\nu * \bar{y})\| \leqslant \|\mu\|_{\infty}^{B}\|\Delta\xi_{FX}\|_{\infty}\|\delta\nu * \bar{x} - \delta\nu * \bar{y}\|.$$

Since Δ and δ are locally bounded and $\delta\nu \in L(S)$, the continuity of r_{μ} at x follows. Furthermore for a $\rho \in L(S)$ we have

$$\|\mu * \delta\rho - \mu * \delta\rho|_{K}\|_{\infty}^{B} \leqslant \|\mu\|_{\infty}^{B}\|\rho|_{S\setminus K}\| \quad (K \in \mathscr{K})$$

and, consequently,

$$\mu * \delta\rho \in \|\ \|_{\infty}^{B}\text{-clo}\,\{\mu * \delta\rho|_{K} \mid K \in \mathscr{K}\}.$$

Apparently, $\{\mu * \delta\nu \mid \mu \in L^{\infty}(S,B), \nu \in L(S)\} \subseteq L_{RUC}(S,B)$.

Take a $\mu \in L_{RUC}(S,B)$, and $\varepsilon > 0$.
Then $V := \{x \in S \mid \|\mu * \bar{x} - \mu\|_{\infty}^{B} < \varepsilon\}$ is a neighbourhood of 1. There is a $\nu \in L(S)$ such that $\|\nu\| = \nu(V)$, $\|\delta\nu\| = 1$. By a combination [for details see (2.1) of [11]] of the Eberlein-Šmulian and the Banach-Grothendieck theorem, for any $f \in m(S)$, with $|f|_{1}^{B} \leqslant 1$ we have that

$$|(\mu * \delta\nu - \mu)(f)| = \left|\int \mu * \bar{x}(f) - \mu(f)\, d\delta\nu(x)\right|$$
$$\leqslant \int \|\mu * \bar{x} - \mu\|_{\infty}^{B}\, d\delta\nu(x) < \varepsilon.$$

The factorization theorem of Cohen leads now to the result in the theorem. □

Several characterizations of measures $\mu \in L_{RUC}(S,B)$ can be given. A basic one is formulated in the next theorem; the proof as presented is an adaptation of the arguments in (3.2) of [15].

Another characterization can be found by generalizing the results in [13], in the following way.

If $\mu \in L^\infty(S,B)$ such that $\{\mu * \bar{x} | x \in A\}$ is separable in $L^\infty(S,B)$ for some σ-compact subset A of S of which 1 is an L(S)-density point [i.e. for each open V with $1 \in V$, there is a $\nu \in L(S)$ for which $\nu(A \cap V) \neq 0$] then $\mu * \bar{x} \in L_{RUC}(S,B)$ for any $x \in \mathring{S}$. [Take an $x \in \mathring{S}$. By a reasoning similar to the one in [13], find a compact K contained in $A \cap \mathring{S}^{-1} x$ that is not L(S)-negligible and on which r_μ is continuous. Next, look for a $v \in S$ and a compact neighbourhood V of 1 such that $KKv \subseteq xV$ and prove that $r_{\mu * \bar{x}}$ is continuous on V. Finally, apply the next theorem in order to obtain the announced result.] In particular, if $\mu \in L^\infty(S,B)$ then $\mu * \bar{x} \in L_{RUC}(S,B)$ $(x \in \mathring{S})$ as soon as r_μ is L(S)-measurable.

3.4. THEOREM. – *Let* $\mu \in L^\infty(S,B)$.

Then $\mu \in L_{RUC}(S,B)$ *if and only if* r_μ *is weakly continuous at* 1 *[i.e. continuous with respect to the weak topology of* $L^\infty(S,B)$*].*

If $\mu \in L_{RUC}(S,B)$ *and* $f \in C(S)$ *is uniformly continuous [i.e.* $x \to f_x$ *is a continuous map from* S *into* C(S)*] then* $f\mu \in L_{RUC}(S,B)$. *In particular, we have that* $f\mu \in L_{RUC}(S,B)$ *for all* $\mu \in L_{RUC}(S,B)$ *and* $f \in C_\infty(S)$.

Proof. – Note that $h_x \in L^\infty(S,B)^*$ for each $h \in L^\infty(S,B)^*$, $x \in S$ if h_x is defined by $h_x(\nu) := h(\nu * \bar{x})$ $(\nu \in L^\infty(S,B))$.

Let $\mu \in L^\infty(S,B)$ for which r_μ is weakly continuous at 1. In order to prove that r_μ is norm-continuous on S, we may suppose that μ is real.

First, we shall show that r_μ is weakly continuous on S. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a set in S that converges to $x \in S$. Suppose $h \in L^\infty(S,B)^*$ is real and such that $(h(\mu * \bar{x}_\lambda))_{\lambda \in \Lambda}$ converges to a $C \in \mathbf{R}$. We shall prove that $C = h(\mu * \bar{x})$; then we may conclude that r_μ is weakly continuous at x. According to the Hahn-Banach theorem there is an $\tilde{h} \in L^\infty(S,B)^*$ such that for each real $\nu \in L^\infty(S,B)$

$$\liminf_\lambda h(\nu * \bar{x}_\lambda) \leqslant \tilde{h}(\nu) \leqslant \limsup_\lambda h(\nu * \bar{x}_\lambda).$$

Let $\varepsilon > 0$ and let U be a compact neighbourhood of 1. V is the collection of all $v \in U$ for which both

$$|\tilde{h}(\mu * \bar{v}) - \tilde{h}(\mu)| < \varepsilon \qquad \text{and} \qquad |h_x(\mu * \bar{v}) - h_x(\mu)| < \varepsilon.$$

Then $1 \in \text{int}(V)$. Take a $v \in \text{int}(V) \cap \mathring{S}$ and note that $x \in \text{int}[v^{-1}(Vx)]$ [cf. (2.2)]. Therefore, there are a $\lambda_0 \in \Lambda$ and a family $(v_\lambda)_{\lambda \in \Lambda}$ in V for which $v_\lambda x = v x_\lambda$ for all $\lambda \in \Lambda$, $\lambda \geqslant \lambda_0$. We find that

$$\begin{aligned} C - 2\varepsilon &= \lim_\lambda h(\mu * \bar{x}_\lambda) - 2\varepsilon = \tilde{h}(\mu) - 2\varepsilon \leqslant \tilde{h}(\mu * \bar{v}) - \varepsilon \\ &\leqslant \limsup_\lambda h(\mu * \bar{v} * \bar{x}_\lambda) - \varepsilon = \limsup_\lambda h_x(\mu * \bar{v}_\lambda) - \varepsilon \\ &\leqslant h_x(\mu) \leqslant \liminf_\lambda h_x(\mu * \bar{v}_\lambda) + \varepsilon = \liminf_\lambda h(\mu * \bar{v} * \bar{x}_\lambda) + \varepsilon \\ &\leqslant \tilde{h}(\mu * \bar{v}) + \varepsilon \leqslant \tilde{h}(\mu) + 2\varepsilon = C + 2\varepsilon . \end{aligned}$$

Apparently, $C = h(\mu * \bar{x})$.

Now, by a combination of the Eberlein-Smulian and the Banach-Grothendieck theorem [cf. (2.11) of [11]] we find that

$$h(\mu * \nu) = \int h(\mu * \bar{x})\, d\nu(x) \qquad \text{for all} \qquad \nu \in M(S)_{\mathscr{K}}, \qquad h \in L^\infty(S,B)^* .$$

$L_{RUC}(S,B)$ is norm-closed and hence weakly closed. Therefore, since $\{\mu * \nu | \nu \in L(S)_{\mathscr{K}}\} \subseteq L_{RUC}(S,B)$ [by (3.3)], it easily follows now that $\mu \in L_{RUC}(S,B)$.

The proofs of the other assertions in the theorem are left to the reader.

□

3.5. *Note.* – In [15], we proved that a $\mu \in M(S)$ belongs to $L(S)$ as soon as $x \to \mu * \bar{x}(f)$ from S into **C** is continuous at 1 for all $f \in m(S)$. In view of this result one could hope that a $\mu \in L^\infty(S,B)$ belongs to $L_{RUC}(S,B)$ as soon as $x \to \mu * \bar{x}(f)$ from S into **C** is continuous at 1 for all $f \in m(S)$, for which $|f|_1^B \leqslant 1$. However, on **R** with Lebesgue measure λ, the function $x \to \sin x^2$ induces a measure in $L^\infty(\mathbf{R},\lambda)$ that does not belong to $L_{RUC}(\mathbf{R},\lambda)$ but for which $x \to \int \sin (x+y)^2 f(y)\, dy$ is continuous for all $f \in L^1(\mathbf{R},\lambda)$.

For measures $\mu \in L_{RUC}(S,B)$ we can approximate $\|\mu\|_\infty^B$ with the aid of the B-uniformly continuous measures in B^+.

3.6. PROPOSITION. – *There is an $\alpha > 0$ such that for each $\mu \in L_{RUC}(S,B)$*

$$\|\mu\|_\infty^B \leqslant \inf \{c \in \mathbf{R} |\ |\mu| \leqslant cm \text{ for some } m \in L_{RUC}(S,B) \cap B^+\} \leqslant \alpha \|\mu\|_\infty^B .$$

[If $L_{RUC}(S,B)$ is a Riesz space one obviously can take α to be 1.]

Proof. – For each $\mu \in L_{RUC}(S,B)$, put

$$||\mu||_\infty^U := \inf\{c \in \mathbf{R} |\, |\mu| < cm \text{ for some } m \in L_{RUC}(S,B) \cap B^+\}.$$

Obviously, we have that $||\mu||_\infty^B \leqslant ||\mu||_\infty^U$.

Let $\mu \in L_{RUC}(S,B)$. By (3.3), there are a $\nu \in L_{RUC}(S,B)$ and a $\rho \in L(S)^+$ such that $\mu = \nu * \delta\rho$. Note that

$$|\nu| * \delta\rho \in L_{RUC}(S,B) \quad \text{and} \quad |\mu| \leqslant |\nu| * \delta\rho .$$

Therefore $||\mu||_\infty^U < \infty$.

It is not hard to prove that $L_{RUC}(S,B)$ is also a Banach space under $||\cdot||_\infty^U$. The proposition follows now as a corollary of the open mapping theorem. □

The following example shows that α may happen to be unequal to 1.

3.7. *Example.* – Let $S := \{(x,y) \in [0,\infty) \times \mathbf{R} | y = 1$ or $x \in [0,1]$ and $x = y\}$ be endowed with the restriction topology. The multiplication is given by

$$(x,y)(p,q) := \begin{cases} (x+p,\, y+q) & \text{if} \quad (x+p,\, y+q) \in S \\ (x+p,\, 1) & \text{otherwise.} \end{cases}$$

Let λ be the Lebesgue measure on $[0,\infty) \times \{1\}$ and let λ' be the Lebesgue measure on $S \backslash [0,\infty) \times \{1\}$, normalized such that

$$\lambda([0,1] \times \{1\}) = 1 = ||\lambda'||.$$

For a $\mu \in L(S)_{\text{loc}} = \{f(\lambda+\lambda') | f \in L^1(S,\lambda+\lambda')_{\text{loc}}\}$ put

$$||\mu||_\infty^B := \inf\{c \in \mathbf{R} |\, |\mu| \leqslant c(\lambda+\lambda')\}.$$

Let $f: S \to \mathbf{R}$ be given by

$$f(x,y) := \begin{cases} x & \text{if} \quad y = 1 \quad \text{and} \quad x \leqslant 1 \\ -x & \text{if} \quad y \neq 1 \quad \text{and} \quad x < 1 \\ 0 & \text{if} \quad x > 1. \end{cases}$$

Consider $\mu := f(\lambda+\lambda')$. Note that $\mu \in L_{RUC}(S,B)$, but $|\mu| \notin L_{RUC}(S,B)$.

In this case we have that

$$||\mu||_\infty^B = 1, \quad \text{while} \quad ||\mu||_\infty^U = 2.$$

Clearly, $\alpha \geqslant 2$. However, one can show that $\alpha = 2$.

4. The case where $L_{RUC}(S,B)_\infty$ is a Riesz ideal.

Consider a linear subspace L of $L^\infty(S,B)$. If V is a downward directed subset of L with infimum 0 in L [i.e. if $\mu \in L$ such that $0 \leqslant \mu \leqslant \nu$ for all $\nu \in V$ then $\mu = 0$], we put $V \downarrow 0(L)$.

We say that $|| \ ||_\infty^B$ is *absolutely continuous on* L if for each countable subset V of L for which $V \downarrow 0(L)$ we have that

$$\text{(1)} \qquad \inf\{||\nu||_\infty^B | \nu \in V\} = 0.$$

In case (1) holds for all subsets V of L for which $V \downarrow 0(L)$ we say that $|| \ ||_\infty^B$ is *order continuous on* L.

Note that we do not require L to be a Riesz subspace of $L^\infty(S,B)$ [see (3.6) and (3.7)]. Furthermore, we have that $|| \ ||_\infty^B$ is order continuous on L whenever $|| \ ||_\infty^B$ is absolutely continuous on L and $L \subseteq L^\infty(S,B)_\infty$; to see the correctness of this statement it is sufficient to note that $L^\infty(S,B)_\infty$ is a subspace of $\bar{M}_\sigma(S)$ and that for each $\mu \in \bar{M}_\sigma(S)^+$, $\{\nu \in \bar{M}_\sigma(S) | \nu \ll \mu\}$ is super Dedekind complete [see also (5.13)].

In this section, we obtain characterizations for the case where $L_{RUC}(S,B)_\infty$ is a Riesz ideal of $L^\infty(S,B)$ in terms of the order continuity of $|| \ ||_\infty^B$ on certain subspaces of $L^\infty(S,B)$ [see (4.14)].

Note that $L_{RUC}(S,B)_\infty$ is a Riesz ideal of $L^\infty(S,B)$ as soon as $L_{RUC}(S,B)$ is one. However, the converse need not be true [$L_{RUC}(S,B)_\infty$ can be a Riesz ideal while $L_{RUC}(S,B)$ is not] as the following example (4.1) shows. The results concerning $L_{RUC}(S,B)$ depend on those for $L_{RUC}(S,B)_\infty$. Therefore we study the situation for $L_{RUC}(S,B)_\infty$ firstly.

4.1. *Example.* – Let S be the additive group of the real numbers, λ denotes the Lebesgue measure on S. For an $f \in L^1(S,\lambda)_{loc}$, define

$$||f\lambda||_\infty^B := \inf\{||h||_1 + ||g||_\infty | h,g \in L^1(S,\lambda)_{loc}, f = h+g\}.$$

Then $L^\infty(S,B) := \{f\lambda \in L(S)_{loc} | \, ||f\lambda||_\infty^B < \infty\}$ is a pseudo L^∞-space; the so-called *Gould space* [cf. [10]]. It is not hard to check that $L^\infty(S,B)_\infty \cong L^1(S,\lambda)$ and, consequently, that $\|\ \|^B_\infty$ is order continuous on $L^\infty(S,B)_\infty$. However, $\|\ \|^B_\infty$ is not order continuous on $L^\infty(S,B)$. We have that $L_{RUC}(S,B)_\infty \cong L^1(S,\lambda)$ is a Riesz ideal of $L^\infty(S,B)$, while $L_{RUC}(S,B)$ is not : $\lambda \in L_{RUC}(S,B)$, but $\sum_{n=1}^{\infty} \xi_{[n,n+1/n]}\lambda \notin L_{RUC}(S,B)$.

4.2. Proposition. – *Let* $L^\infty(S,B)$ *be such that* $L_{RUC}(S,B)_\infty$ *is a Riesz ideal. Then* $\|\ \|^B_\infty$ *is order continuous on* $L_{RUC}(S,B)_\infty$.

The proof of the proposition is based on the following three lemmas. In these lemmas $L_{RUC}(S,B)_\infty$ is supposed to be a Riesz ideal of $L^\infty(S,B)$.

4.3. Lemma A.– *If* $\mu \in L_{RUC}(S,B)^+$ *and* $(F_n)_{n\in N}$ *is a decreasing sequence of compact subsets of* S *such that with* $F := \bigcap\{F_n | n \in N\}$, $F^{-1}F$ *is not a neighbourhood of* 1 [F *is emaciated in the terminology of* [16]], *then*

$$\lim_n \| |\mu|_{F_n} \|^B_\infty = 0.$$

4.4. Lemma B.– *If* $\mu \in L_{RUC}(S,B)^+_{\mathscr{K}}$ *and* $(F_n)_{n\in N}$ *is a decreasing sequence of compact subsets of* S *such that* $\bigcap\{F_n | n \in N\}$ *is* L(S)-*negligible, then*

$$\lim_n \| |\mu|_{F_n} \|^B_\infty = 0.$$

4.5. Lemma C. – *If* $\mu \in L_{RUC}(S,B)^+_{\mathscr{K}}$ *and* O *is an open subset of* S, *then*

$$\inf\{\| |\mu|_{O\setminus F} \|^B_\infty | F \in \mathscr{K}, F \subseteq O\} = 0.$$

In order to prove lemma B we need lemma A. A combination of lemma B and lemma C leads to the order continuity of $\|\ \|^B_\infty$; we shall first prove this last implication.

4.6. *Proof of* (4.2). – It is left to the reader to verify that the order continuity of $\|\ \|^B_\infty$ follows from the following property (∗).

(∗) For each $\mu \in L_{RUC}(S,B)^+_{\mathscr{K}}$ and each decreasing sequence $(U_n)_{n\in N}$ of open subsets of S for which $\bigcap\{U_n | n \in N\}$ is L(S)-negligible we have that $\lim_n \| |\mu|_{U_n} \|^B_\infty = 0$.

In order to prove $(*)$, suppose that for some $\mu \in L_{RUC}(S,B)^+$ and for some decreasing sequence $(O_n)_{n \in N}$ of open sets whose intersection is L(S)-negligible we have that

$$\lim_{n \to \infty} \|\mu|_{O_n}\|_\infty^B = \alpha > 0 .$$

In view of lemma C for each $n \in N$ we can find a compact subset F_n of O_n such that

$$\|\mu|_{O_n \setminus F_n}\|_\infty^B \leqslant \frac{1}{2} 2^{-n} \alpha .$$

Put $K_n := \bigcap_{i=1}^{n} F_i$. Then $(K_n)_{n \in N}$ is a decreasing sequence of compact sets for which $\bigcap \{K_n \mid n \in N\}$ is L(S)-negligible, while

$$\begin{aligned} \|\mu|_{K_n}\|_\infty^B &\geqslant \|\mu|_{O_n}\|_\infty^B - \|\mu|_{O_n \setminus K_n}\|_\infty^B \\ &\geqslant \|\mu|_{O_n}\|_\infty^B - \sum_{i=1}^{n} \|\mu|_{O_i \setminus F_i}\|_\infty^B > \frac{1}{2}\alpha \qquad \text{for all } n . \end{aligned}$$

Clearly this violates the result in lemma B.

Apparently $\| \ \|_\infty^B$ has property $(*)$ and consequently is order continuous. □

4.7. *Proof of Lemma* A. – Suppose there are a $\mu \in L_{RUC}(S,B)_{\mathscr{K}}^+$ and a decreasing sequence $(F_n)_{n \in N}$ of compact subsets of S such that with $F := \bigcap \{F_n | n \in N\}$, $F^{-1} F$ is not a neighbourhood of 1 while $\lim_{n \to \infty} \|\mu|_{F_n}\|_\infty^B = \alpha > 0$.

Put $M := \text{supp}(\mu)$. Note that $M \in \mathscr{K}$.

Without loss of generality we may assume that

$$F_{n+1} \subseteq \text{int}(F_n) \qquad \text{for all} \qquad n \in N .$$

Let $(V_n)_{n \in N}$ be a decreasing sequence of relatively compact open neighbourhoods of 1 such that $\{1\} = \bigcap \{V_n \mid n \in N\}$.

By induction we construct sequences $(x_n)_{n \in N}$, $(y_n)_{n \in N}$ in S and $(K_n)_{n \in N}$ of compact subsets as follows.

For $n \in \mathbf{N}$, assume that $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in S$ and compact sets $K_1, \ldots, K_n$ are such that for all $i = 1, \ldots, n$,

$$x_i, y_i \in V_i, \qquad K_i = F_j \quad \text{for a certain } j,$$

$$K_n \subseteq K_{n-1} \subseteq \cdots \subseteq K_1,$$

$$M \cap K_i x_i^{-1} \subseteq K_{i-1}, \qquad K_j x_j^{-1} \cap K_i = \varnothing \qquad \text{for all } j, \quad j \leqslant i.$$

$$M \cap A_i y_i \cap K_i x_i^{-1} = \varnothing, \qquad \text{where } A_i := K_1 x_1^{-1} \cup \ldots \cup K_i x_i^{-1},$$

and

$$K_i y_i \cap K_i x_i^{-1} = \varnothing.$$

Then choose $y_{n+1} \in V_{n+1}$ such that

$$A_n y_{n+1} \cap F = \varnothing, \qquad F y_{n+1} \cap F = \varnothing,$$

which is possible since $1 \notin \operatorname{int}(F^{-1} F \cap V_{n+1})$ and by assumption $A_n \cap F \subseteq A_n \cap K_n = \varnothing$. Next choose an $x_{n+1} \in V_{n+1}$ such that

$$(*) \qquad \begin{aligned} & x_{n+1} \notin (A_n y_{n+1})^{-1} F \cup (F y_{n+1})^{-1} F, \\ & F x_{n+1}^{-1} \cap M \subseteq K_n, \ F x_{n+1}^{-1} \cap F = \varnothing, \\ & (F x_{n+1}^{-1}) y_{n+1} \cap F x_{n+1}^{-1} \cap M = \varnothing. \end{aligned}$$

Finally, take $K_{n+1} \in \{F_m \mid m \in \mathbf{N}\}$ such that $K_{n+1} \subseteq K_n$ and the properties $(*)$ hold with K_{n+1} instead of F.

Put $A = M \cap \bigcup \{K_n x_n^{-1} \mid n \in \mathbf{N}\}$. Then we have that

$$M \cap K_n x_n^{-1} \cap A y_n = \varnothing \qquad \text{for all} \qquad n \in \mathbf{N};$$

since $K_n x_n^{-1} \cap A_n y_n \cap M = \varnothing$ and for all $m \in \mathbf{N}$, $m > n$

$$M \cap K_n x_n^{-1} \cap (K_m x_m^{-1} \cap M) y_n \subseteq K_n x_n^{-1} \cap K_n y_n = \varnothing.$$

Apparently,

$$\mu|_{K_n x_n^{-1}} \leqslant |\mu|_A - \mu|_A * \bar{y}_n| \qquad \text{for all} \qquad n \in \mathbf{N}.$$

Since $(y)_{n \in \mathbf{N}}$ converges to 1 and μ belongs to the Riesz ideal $L_{RUC}(S,B)_\infty$ we have that

$$0 \leqslant \lim_{n \to \infty} \|\mu|_{K_n x_n^{-1}}\|_\infty^B \leqslant \lim_{n \to \infty} \|\mu|_A - \mu|_A * \bar{y}_n\|_\infty^B = 0.$$

Therefore, we can find a subsequence $(C_n)_{n \in \mathbf{N}}$ of $(K_n)_{n \in \mathbf{N}}$ such that, with

$z_n := x_l$ whenever $C_n = K_l$,

$$\|\mu|_{C_n z_n^{-1}}\|_\infty^B \leqslant \frac{1}{2} 2^{-n} \qquad \text{for all} \qquad n \in \mathbf{N}.$$

Put $D := \bigcup\{C_n z_n^{-1} \mid n \in \mathbf{N}\}$. Note that,

$$\mu|_{C_n \setminus D} \leqslant |\mu|_D * \bar{z}_n - \mu|_D + \mu|_{C_n} - \mu * \bar{z}_n|_{C_n}| \qquad \text{for all} \qquad n \in \mathbf{N},$$

whence

$$\|\mu|_{C_n \setminus D}\|_\infty^B \leqslant \|\mu|_D * \bar{z}_n - \mu|_D\|_\infty^B + \|\mu - \mu * \bar{z}_n\|_\infty^B.$$

Since $\mu|_D \in L_{RUC}(S,B)$, we see that

$$\lim_{n \to \infty} \|\mu|_{C_n \setminus D}\|_\infty^B = 0.$$

Now, note that by our choice of the x_n,

$$C_n \cap D \subseteq \bigcup_{m=n+1}^{\infty} C_m z_m^{-1}$$

and we find that

$$\|\mu|_{C_n}\|_\infty^B \leqslant \|\mu|_{C_n \setminus D}\|_\infty^B + \|\mu|_{C_n \cap D}\|_\infty^B$$

$$\leqslant \|\mu|_{C_n \setminus D}\|_\infty^B + \sum_{m=n+1}^{\infty} \|\mu|_{C_m z_m^{-1}}\|_\infty^B \leqslant \|\mu|_{C_n \setminus D}\|_\infty^B + 2^{-n}.$$

We have to conclude that $\lim_{n \to \infty} \|\mu|_{C_n}\|_\infty^B = 0$ which, however, violates the fact that $\|\mu|_{F_k}\|_\infty^B \geqslant \alpha > 0$ for all $k \in \mathbf{N}$ [recall that for each $n \in \mathbf{N}$, $C_n \in \{F_k \mid k \in \mathbf{N}\}$]. □

4.8. *Proof of lemma* B. – Let $\mu \in L_{RUC}(S,B)_{\mathscr{X}}^+$ and let $(F_n)_{n \in \mathbf{N}}$ be a decreasing sequence of compact sets such that $F := \bigcap\{F_n \mid n \in \mathbf{N}\}$ is L(S)-negligible. Take an $x \in \mathring{S}$. For each $f \in C_\infty(S)$, put

$$p(f) = \limsup_{n \to \infty} \|f\mu|_{F_n x^{-1}}\|_\infty^B.$$

Then p is a seminorm on $C_\infty(S)$,

$$p(1) = \lim_{n \to \infty} \|\mu|_{F_n x^{-1}}\|_\infty^B, \qquad \text{and} \qquad p(f) \leqslant \|f\|_\infty \|\mu\|_\infty^B.$$

According to the Hahn-Banach theorem, there exists a measure $\nu \in M(S)$ such that

$$\nu(S) = p(1) \quad \text{and} \quad |\nu(f)| \leqslant p(f) \quad \text{for all} \quad f \in C_\infty(S).$$

Obviously, $p(f) = 0$ whenever $f \in C_\infty(S)$ and $f = 0$ on Fx^{-1}. Apparently, $\operatorname{supp}(\nu) \subseteq Fx^{-1}$.

Let $K \in \mathscr{K}$ such that $K^{-1}K$ is not a neighbourhood of 1. Take an $\varepsilon > 0$. Then, by lemma A there exists an $f \in C_\infty(S)$ such that

$$0 \leqslant \xi_K \leqslant f \leqslant 1 \quad \text{and} \quad \|f\mu\|_\infty^B < \varepsilon.$$

This shows that $\nu(K) = 0$.
By theorem (3.4) of [16], we now have that $\nu * \bar{x} \in L(S)$.

Since $\operatorname{supp}(\nu * \bar{x}) \subseteq (Fx^{-1})x \subseteq F$ is $L(S)$-negligible, we find that $\nu * \bar{x} = 0$. Therefore,

$$0 = \nu * \bar{x}(S) = \nu(S) = p(1) = \lim_{n\to\infty} \|\mu|_{F_n x^{-1}}\|_\infty^B.$$

To complete the proof, note that

$$\begin{aligned}\|\mu|_{F_n}\|_\infty^B &\leqslant \|\mu|_{F_n} - \mu * \bar{x}|_{F_n}\|_\infty^B + \|\mu|_{F_n x^{-1}} * \bar{x}\|_\infty^B \\ &\leqslant \|\mu - \mu * \bar{x}\|_\infty^B + \Delta(x)\|\mu|_{F_n x^{-1}}\|_\infty^B.\end{aligned}$$

The facts that $\mu \in L_{RUC}(S,B)$ and Δ is bounded on a neighbourhood of 1 show that

$$\lim_{n\to\infty} \|\mu|_{F_n}\|_\infty^B = 0. \qquad \square$$

Before we proceed to the proof of lemma C we separate two steps in the proof in the form of the following lemmas (4.9) and (4.10).

4.9. Lemma. — *Assume there are a* $\mu \in L_{RUC}(S,B)_{\mathscr{K}}^+$, *an* α, $\varepsilon \in \mathbf{R}^+$ *and a sequence* $(E_n)_{n\in\mathbf{N}}$ *of Borel measurable subsets such that*

(i) $\bigcap_m \bigcup_{n\geqslant m} E_n = \varnothing$

(ii) $\|\mu|_{E_n}\|_\infty^B \geqslant \alpha + \varepsilon$ *for all* $n \in \mathbf{N}$.

Let V *be an open subset of* S.

Then for each $m \in \mathbf{N}$ *there are an* $x \in \mathrm{V}$ *and an* $n \in \mathbf{N}$, $n \geqslant m$ *such that*

$$\| \mu|_{\mathrm{E}_n \setminus \mathrm{E}_n x^{-1}} \|_\infty^{\mathrm{B}} > \alpha .$$

Proof. – Let $\nu \in \mathrm{L(S)} \cap \mathrm{B}$ such that $\nu(\mathrm{V}) = \|\nu\| = 1$. Take an $m \in \mathbf{N}$. Consider an $f \in m(\mathrm{S})$, $|f|_1^{\mathrm{B}} < \infty$, an $n \in \mathbf{N}$ and $\mathrm{E} := \mathrm{E}_n \cap \operatorname{supp}(\mu)$. Then

$$\begin{aligned}\int |\mu|_{\mathrm{E} \cap \mathrm{E}x^{-1}}(f)| \, d\nu(x) &\leqslant \int |f\mu|_{\mathrm{E}}| * \bar{x}(\mathrm{E}) \, d\nu(x) \\ &= \int \bar{y} * \nu(\mathrm{E}) \, d|f\mu|_{\mathrm{E}}|(y) \leqslant \sup_{y \in \mathrm{E}} \bar{y} * \nu(\mathrm{E}) \|f\mu|_{\mathrm{E}}\| \\ &\leqslant \sup_{y \in \mathrm{E}} \bar{y} * \nu(\mathrm{E}) |\mu|_{\mathrm{E}}|(|f|) \leqslant \sup_{y \in \mathrm{E}} \bar{y} * \nu(\mathrm{E}) |f|_1^{\mathrm{B}} \|\mu\|_\infty^{\mathrm{B}} .\end{aligned}$$

Since $\mathrm{E} \subseteq \operatorname{supp}(\mu) \in \mathscr{K}$ and $\nu \in \mathrm{L(S)}$, we can find an $n \in \mathbf{N}$, $n \geqslant m$ such that

$$\sup_{y \in \mathrm{E}} \bar{y} * \nu(\mathrm{E}) \leqslant \varepsilon (2 \|\mu\|_\infty^{\mathrm{B}})^{-1} .$$

Therefore, for each $f \in m(\mathrm{S})$ with $|f|_1^{\mathrm{B}} \leqslant 1$ we have that

$$\begin{aligned}\int |\mu|_{\mathrm{E}_n \setminus \mathrm{E}_n x^{-1}}(f)| \, d\nu(x) &= \int |\mu|_{\mathrm{E}_n}(f) - \mu|_{\mathrm{E}_n \cap \mathrm{E}_n x^{-1}}(f)| \, d\nu(x) \\ &\geqslant \int |\mu|_{\mathrm{E}_n}(f)| \, d\nu(x) - \int |\mu|_{\mathrm{E}_n \cap \mathrm{E}_n x^{-1}}(f)| \, d\nu(x) \\ &\geqslant |\mu|_{\mathrm{E}_n}(f)| - \varepsilon/2 .\end{aligned}$$

Since $\|\mu|_{\mathrm{E}_n}\|_\infty^{\mathrm{B}} \geqslant \alpha + \varepsilon$, there is some $f \in m(\mathrm{S})$ with $|f|_1^{\mathrm{B}} \leqslant 1$ such that $|\mu|_{\mathrm{E}_n}(f)| > \alpha + \varepsilon/2$ [see (2.7.*b*)].
Apparently, we have that

$$\int \|\mu|_{\mathrm{E}_n \setminus \mathrm{E}_n x^{-1}}\|_\infty^{\mathrm{B}} \, d\nu(x) \geqslant \int |\mu|_{\mathrm{E}_n \setminus \mathrm{E}_n x^{-1}}(f)| \, d\nu(x) > \alpha .$$

The existence of an $x \in \mathrm{V}$ with the required property follows. □

In the proof of lemma C, we will have to choose compact sets F with an additional property : *viz.* $\mathrm{F}xx^{-1} = \mathrm{F}$ for some $x \in \mathring{\mathrm{S}}$. Unlike the group case, for semigroups, this is not a trivial property. The following lemma overcomes this problem. The proof of this lemma may be based on an observation as in the last few sentences of the proof of lemma B; we omit the details.

4.10. LEMMA. – *Assume there are a* $\mu \in L_{RUC}(S,B)^+_{\mathscr{K}}$, *a* $\beta \in \mathbf{R}^+$ *and an open subset* O *of* S *such that*

$$\|\mu|_{O\setminus F}\|^B_\infty \geqslant \beta \quad \textit{for all} \quad F \in \mathscr{K} \quad \textit{for which} \quad F \subseteq O.$$

Then there exists an $x_0 \in \mathring{S}$ *and an* $\alpha \in (0,\beta)$ *such that*

$$\|\mu|_{Ox_0^{-1}\setminus F}\|^B_\infty > \alpha \quad \textit{for all} \quad F \in \mathscr{K} \quad \textit{for which} \quad F \subseteq Ox_0^{-1}. \ \square$$

[Note that, whenever $F \in \mathscr{K}$, $F \subseteq Ox_0^{-1}$ and $F' := Fx_0x_0^{-1} \cap \operatorname{supp}(\mu)$ we have that $F' \in \mathscr{K}$, $F' \subseteq Ox_0^{-1}$, $F \subseteq F'$, $F'x_0x_0^{-1} \cap \operatorname{supp}(\mu) = F'$.]

4.11. *Proof of lemma* C. – Suppose there exist a $\mu \in L_{RUC}(S,B)^+_{\mathscr{K}}$ and an open set O of S such that for some $x_0 \in \mathring{S}$, α, $\varepsilon \in \mathbf{R}^+$ we have that

$$\|\mu|_{Ox_0^{-1}\setminus F}\|^B_\infty > \alpha + \varepsilon \quad \text{for all} \quad F \in \mathscr{K}, \quad F \subseteq Ox_0^{-1};$$

if we can deduce a contradiction then, in view of the above lemma, we may conclude that lemma C holds. Put $M := \operatorname{supp}(\mu)$. Without loss of generality, we may assume that there exists a sequence $(G_n)_{n\in N}$ of compact subsets of S such that

$$\varnothing = G_0 \subseteq G_1 \subseteq \operatorname{int}(G_2) \subseteq G_2 \subseteq \operatorname{int}(G_3) \subseteq \ldots\ldots \subseteq O$$

and

$$O = \bigcup\{G_n | n \in \mathbf{N}\}.$$

Let $(n(k))_{k\in N}$ be a sequence in $\mathbf{N}$ such that for all $k \in \mathbf{N}$

$$(1) \qquad n(k) > n(k-1) + 2 \quad [\text{where } n(0) = 0].$$

Put $K_1 := G_{n(1)}x_0^{-1} \cap M$, $U_1 := (\operatorname{int} G_{n(1)+1})x_0^{-1}$ and for each $k \in \mathbf{N}$, $k > 1$ put

$$(2) \qquad K_k := (G_{n(k)}x_0^{-1} \cap M)\setminus(\operatorname{int} G_{n(k-1)+2})x_0^{-1}$$

and

$$U_k := (\operatorname{int} G_{n(k)+1})x_0^{-1}\setminus G_{n(k-1)+1}x_0^{-1}.$$

Note that

(3) $\quad K_k$ is compact, U_k is open $\quad (k \in \mathbf{N})$

(4) $\quad K_k \subseteq U_k \subseteq Ox_0^{-1}, \quad U_n \cap U_k = \varnothing \quad (k \in \mathbf{N}, n \in \mathbf{N}, n \neq k)$

(5) $$(K_k x_0)x_0^{-1} \cap M = K_k \quad (k \in \mathbf{N}).$$

By induction, we shall show that, in addition to (1), the seauence $(n(k))_{k \in \mathbf{N}}$ can be chosen such that for any $k \in \mathbf{N}$ $\quad x_0^{-1}$

(6) $$\|\mu|_{K_k}\|_\infty^B > \alpha + \varepsilon .$$

By the Fatou-Levi property of $\| \ \|_\infty^B$, we can find an $n(1) \in \mathbf{N}$ such that (6) holds with $k = 1$.

Now consider a $p \in \mathbf{N}$ and suppose that $n(1), \ldots, n(p)$ have been choosen such that (1) and (6) hold for $k \leqslant p$. Since

$$\|\mu|_{Ox_0^{-1}\backslash G_{n(p)+2}x_0^{-1}}\|_\infty^B > \alpha + \varepsilon$$

and

$$(O\backslash G_{n(p)+2})x_0^{-1} \subseteq \bigcup_{m=n(p)+2}^{\infty} (G_m x_0^{-1} \backslash G_{n(p)+2} x_0^{-1}),$$

again by the Fatou-Levi property, we can find an $n(p+1) \in \mathbf{N}$ such that $n(p+1) > n(p) + 2$ and (6) holds with $k = p+1$.

For each $m \in \mathbf{N}$, put

$$\tilde{K}_m := \bigcup_{n=m}^{\infty} K_n \quad \text{and note that} \quad \bigcap_m \tilde{K}_m = \varnothing .$$

There exists a sequence $(V_n)_{n \in \mathbf{N}}$ of open relatively compact neighbourhoods of 1 such that

(7) $$x_0 \mathring{S}^{-1} \supseteq V_1 \supseteq V_2 \supseteq \cdots, \quad \bigcap_{n=1}^{\infty} V_n = \{1\}$$

and

(8) $$(K_n V_n)V_n^{-1} \cap M \subseteq U_n \quad \text{for all} \quad n \in \mathbf{N};$$

since M is compact and (4), (5) hold.

Let $(\gamma(n))_{n\in\mathbf{N}}$ and $(\lambda(n))_{n\in\mathbf{N}}$ be sequences in $\mathbf{N}$ such that for all $n \in \mathbf{N}$, $n > 1$,

(9) $$\lambda(n+1) \geqslant \gamma(n) > \lambda(n) > \gamma(1) = \lambda(1) = 1.$$

Let $(x_n)_{n\in\mathbf{N}}$ be a sequence in S such that $x_n \in V_{\lambda(n)}$ for all $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, $n > 1$, put

$$F_n := K_{\lambda(n)} \quad \text{and} \quad C_n := F_n \backslash (F_n x_{n-1}^{-1} \cup \tilde{K}_{\gamma(n)} x_{n-1}).$$

Then

(10) $$(x_n)_{n\in\mathbf{N}} \text{ converges to } 1.$$

In order to prove that

(11) $$C_j \cap C_{i+1} x_i^{-1} = \varnothing \quad \text{for all} \quad i, j \in \mathbf{N},$$

we distinguish three cases.

If $j \leqslant i$ then

$$\begin{aligned} C_j \cap C_{i+1} x_i^{-1} &\subseteq (C_j x_i \cap C_{i+1}) x_i^{-1} \subseteq (K_{\lambda(j)} V_{\lambda(j)} \cap K_{\lambda(i+1)}) x_i^{-1} \\ &\subseteq (U_{\lambda(j)} \cap K_{\lambda(i+1)}) x_i^{-1} = \varnothing \quad \text{[by (7) and (4)]}. \end{aligned}$$

If $j > i + 1$ then

$$\begin{aligned} C_j \cap C_{i+1} x_i^{-1} &\subseteq K_{\lambda(j)} \cap C_{i+1} x_i^{-1} \subseteq \tilde{K}_{\gamma(j-1)} \cap C_{i+1} x_i^{-1} \\ &\subseteq \tilde{K}_{\gamma(i+1)} \cap C_{i+1} x_i^{-1} = \varnothing \quad \text{[by (9)]}. \end{aligned}$$

If $j = i + 1$ then $C_j \cap C_{i+1} x_i^{-1} = C_{i+1} \cap C_{i+1} x_i^{-1} = \varnothing$.

Hence (11) holds.

Finally we shall show that the sequences $(\gamma(n))$, $(\lambda(n))$ and (x_n) can be chosen such that, in addition to (10) and (11), for all $n \in \mathbf{N}$, $n > 1$, also

(12) $$\| \mu|_{C_n} \|_\infty^B > \alpha.$$

By lemma (4.9), we can find an $x_1 \in V_1$ and a $\lambda(2) \in \mathbf{N}$, $\lambda(2) > 1$ such that

$$\| \mu|_{F_2 \backslash F_2 x_1^{-1}} \|_\infty^B > \alpha.$$

Now, note that $(F_2 \backslash (F_2 x_1^{-1} \cup \tilde{K}_m x_1))_{m\in\mathbf{N}} \uparrow F_2 \backslash F_2 x_1^{-1}$ [here $(A_n)_{n\in\mathbf{N}} \uparrow A$ means $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \cup_n A_n$]; since, by (5) and (7),

$$(K_m y) y^{-1} \cap M = K_m \quad \text{for all} \quad m \in \mathbf{N}, \quad y \in x_0 \mathring{S}^{-1}$$

we have

$$\bigcap_m F_2 \cap \tilde{K}_m x_1 \subseteq (F_2 x_1^{-1} \cap \bigcap_m \tilde{K}_m) x_1 = \emptyset .$$

Since $\| \ \|_\infty^B$ has the Fatou-Levi property, there exists a $\gamma(2) \in \mathbf{N}$, $\gamma(2) > \lambda(2)$ such that $\|\mu|_{C_2}\|_\infty^B > \alpha$.

Now, consider a $p \in \mathbf{N}$ and suppose that $\gamma(1), \ldots, \gamma(p)$, $\lambda(1), \ldots, \lambda(p)$ in $\mathbf{N}$ and $x_1, \ldots, x_{p-1}$ in S are as required.

By (4.9) there are a $\lambda(p+1) \in \mathbf{N}$ and an $x_p \in V_{\lambda(p)}$ such that

$$\lambda(p+1) \geqslant \gamma(p) \quad \text{and} \quad \|\mu|_{F_{p+1} \setminus F_{p+1} x_p^{-1}}\|_\infty^B > \alpha .$$

As above, by the Fatou-Levi property, there is a $\gamma(p+1) \in \mathbf{N}$ such that $\gamma(p+1) > \lambda(p+1)$ and (12) holds with $n = p+1$.

Finally, put $A := \bigcup_{j=1}^{\infty} C_{j+1} x_j^{-1}$. Then, by (11),

$$C_j \cap A = \emptyset \quad \text{for all} \quad j \in \mathbf{N} .$$

Since

$$|\mu|_{C_j \setminus A}| \leqslant |\mu|_{C_j} - \mu * \bar{x}_{j-1}|_{C_j} + \mu|_A * \bar{x}_{j-1} - \mu|_A| \quad \text{for all} \quad j \in \mathbf{N}$$

and by (12), we find that

$$\begin{aligned} \alpha < \|\mu|_{C_j}\|_\infty^B = \|\mu|_{C_j \setminus A}\|_\infty^B &\leqslant \|\mu|_{C_j} - \mu * \bar{x}_{j-1}|_{C_j}\|_\infty^B + \|\mu|_A * \bar{x}_{j-1} - \mu|_A\|_\infty^B \\ &\leqslant \|\mu - \mu * \bar{x}_{j-1}\|_\infty^B + \|\mu|_A * \bar{x}_{j-1} - \mu|_A\|_\infty^B \quad \text{for all} \quad j \in \mathbf{N}. \end{aligned}$$

This inequality cannot hold; because $(x_{j-1})_{j \in \mathbf{N}}$ converges to 1 [see (10)], while both μ and $\mu|_A$ belong to $L_{RUC}(S,B)$. □

It is not hard to see how the case where $L_{RUC}(S,B)_\infty$ is a Riesz ideal is related to the order continuity of $\| \ \|_\infty^B$ on $L_{RUC}(S,B)_\infty$. However, we can also link this situation to the order continuity of $\| \ \|_\infty^B$ on a subspace of $L^\infty(S,B)_\infty$ that does not explicitely depend on $L_{RUC}(S,B)_\infty$ [cf. (4.14)], and even on a subspace of L(S) of which the definition is intrinsically based on S itself and has nothing to do with $\| \ \|_\infty^B$ [cf. (4.15.2)]. In (4.12), we introduce these spaces and in the subsequent proposition we show that these spaces [as Riesz ideals of $L^\infty(S,B)$] are natural objects.

4.12. *Notation.* – Let U be a compact neighbourhood of 1. The Riesz ideal of $L^\infty(S,B)$ consisting of all $\rho \in L^\infty(S,B)$ for which the collection $\{|\rho| * \bar{x} | x \in U\}$ has an upper bound in $L^\infty(S,B)$ is denoted by $L_U^\nu(S,B)$. $L_U^\nu(S)$ denotes the space of all $\mu \in L(S)$ for which $\{|\mu| * \bar{x} | x \in L(S)\}$ has an upper bound in $L(S)$ [$L_U^\nu(S) = L_U^\nu(S,B)$ if $L^\infty(S,B) = L(S)$, $\| \|_\infty^B = \| \|$].

If A is a subspace of $\bar{M}(S)$ then we put

$$A^o := \{\mu * \bar{x} \in \bar{M}(S) \mid \mu \in A, x \in \mathring{S}\}$$

and

$$\mathrm{Supp}\,(A) := \mathrm{clo} \bigcup \{\mathrm{supp}(\mu) \mid \mu \in A\}.$$

4.13. Proposition. – *Let* U *and* V *be compact neighbourhoods of* 1. *Then :*

(1) $L_U^\nu(S,B)^o_{\mathscr{K}} = L_V^\nu(S,B)^o_{\mathscr{K}} [:= (L_V^\nu(S,B)_{\mathscr{K}})^o]$;

(2) *If* $\mu \in L^\infty(S,B)$ *and* $\rho \in L_U^\nu(S)_{\mathscr{K}}$ *then* $\mu * \rho \in L_U^\nu(S,B)$;

(3) $L_U^\nu(S,B)^o_{\mathscr{K}} \subseteq L_U^\nu(S)$ [see also (4.15.2)]. □

One can prove (1) by adapting the arguments in (2.7) of [18]. The proof of (2) and (3) is easy.

4.14. Theorem. – *Let* U *be a compact neighbourhood of* 1. *Consider the following properties :*

(1) $L_{RUC}(S,B)_\infty$ *is a Riesz ideal of* $L^\infty(S,B)_\infty$;

(2) $\| \|_\infty^B$ *is order continuous on* $L_{RUC}(S,B)_\infty$;

(3) $\| \|_\infty^B$ *is order continuous on* $L_U^\nu(S,B)^o_{\mathscr{K}}$.

Then, (1) *and* (2) *are equivalent and both imply* (3).

If, in addition, S = Supp $L_U^\nu(S)$) *then all the properties* (1), (2) *and* (3) *are equivalent* [see also (4.15.1) and (4.15.2)].

Proof. – « (1) ⇒ (2) » is the content of (4.2).

Before we prove « (2) ⇒ (1) », we make some observations concerning the order denseness of $L_{RUC}(S,B)_\infty$ in $\tilde{L}_{RUC} := \{\nu \in L^\infty(S,B) \mid |\nu| \leqslant |\mu|$ for some $\mu \in L_{RUC}(S,B)_\infty\}$. A linear subspace L′ [not necessarily a Riesz subspace] of a Riesz space L is said to be *order dense*, if for each $\mu \in L$ there are nets $(\nu_\lambda)_{\lambda \in \Lambda}$ in L′ and $(\mu_\lambda)_{\lambda \in \Lambda}$ in L such that $|\mu - \nu_\lambda| \leqslant \mu_\lambda$ for all $\lambda \in \Lambda$, while $(\mu_\lambda) \downarrow 0(L)$.

Note that

(4) for each $m \in \bar{M}(S)^+$, $C_{00}(S)$ is order dense in $L^\infty(S,m)$ [cf. [14], ch. III, ex. 13; here $f \leqslant g$ if $f \leqslant g$ *m*-a.e. and any function in C(S) is

identified with its equivalence class]. In particular, by (3.4) and (3.6) we have that

(5) $L_{RUC}(S,B)_\infty$ is order dense in $\tilde{L}_{RUC}$.

Suppose that (2) holds. In case, in addition, $L_{RUC}(S,B)_\infty$ is a Riesz subspace of $L^\infty(S,B)$ we may apply theorem 5.10 of [14] in order to see that $L_{RUC}(S,B)_\infty$ is Dedekind σ-complete. Then (5) implies that $L_{RUC}(S,B)_\infty = \tilde{L}_{RUC}$. Unfortunately, $L_{RUC}(S,B)_\infty$ need not be a Riesz subspace [see (3.7)]. However, we can adapt the proof of 5.10 of [14] as follows.

Let $D \subseteq L_{RUC}(S,B)_\infty$ such that $D\downarrow$ and $\mu \geqslant 0\,(\mu \in D)$. Consider the subcollections A of $L_{RUC}(S,B)_\infty^+$ for which

(6) $\Sigma E \leqslant \mu$ for all $\mu \in D$ and for every finite subset E of A.

By Zorn's lemma, there exists a subset A_0 of $L_{RUC}(S,B)_\infty^+$ that is maximal with property (6). Then

$$\{\mu - \Sigma E \,|\, \mu \in D, E \subseteq A_0, E \text{ finite}\} \downarrow 0\,(L_{RUC}(S,B)_\infty).$$

And now the order continuity of $\|\ \|_\infty^B$ on $L_{RUC}(S,B)_\infty$ shows that D is a Cauchy net. Consequently, D has an infimum in $L_{RUC}(S,B)_\infty$ and, moreover, this infimum is precisely the infimum of D in $L^\infty(S,B)$. Therefore, by (4) and the fact that $\{fm \,|\, f \in C_{00}(S)\}$ is a Riesz space, we have that $\{fm \,|\, f \in L^\infty(S,m)_{\mathscr{X}}\} \subseteq L_{RUC}(S,B)_\infty$ for all $m \in L_{RUC}(S,B)^+$. By (3.6) and the norm closedness of $L_{RUC}(S,B)_\infty$, we obtain that $L_{RUC}(S,B)_\infty = \tilde{L}_{RUC}$.

« (1) ⇒ (3) ». Suppose that $L_{RUC}(S,B)_\infty$ is a Riesz ideal. We shall show that $L_U^v(S,B)_{\mathscr{X}}^o \subseteq L_{RUC}(S,B)_\infty$; then, since $L_U^v(S,B)_{\mathscr{X}}^o$ is a Riesz ideal, (3) follows from (2).

Let $\rho \in L_U^v(S,B)_{\mathscr{X}}^+$. Take a $\nu \in L(S)^+$ for which $1 \in \operatorname{supp}(\nu) \in \mathscr{K}$ and $\|\nu\| = 1$. There is a $\rho^v \in L^\infty(S,B)_{\mathscr{X}}^+$ such that

$$\rho * \bar{x} \leqslant \rho^v \qquad \text{for all} \qquad x \in U.$$

Note that $\rho^v * \nu \in L_{RUC}(S,B)_{\mathscr{X}}$. Furthermore, for any $x_0 \in \mathring{S} \cap \operatorname{int}(U)$, with $d := \nu(U^{-1}x_0)$ we have that $d > 0$ and

$$\begin{aligned}\rho^v * \nu(f) = \int \rho^v * \bar{y}(f)\,d\nu(y) &\geqslant \int_{U^{-1}x_0} \rho^v * \bar{y}(f)\,d\nu(y)\\ &\geqslant \int_{U^{-1}x_0} \rho * \bar{x}_0(f)\,d\nu(y) = d\rho * \bar{x}_0(f) \quad \text{for all} \quad f \in C_\infty(S)^+.\end{aligned}$$

Hence

$$0 \leqslant d\rho * \bar{x}_0 \leqslant \rho^{v} * \nu .$$

Consequently, $\rho * \bar{x}_0 \in L_{RUC}(S,B)_{\mathscr{X}}$, which shows that

$$L_U^{v}(S,B)^{o}_{\mathscr{X}} \subseteq L_{RUC}(S,B)_{\infty} .$$

« (3) $\Rightarrow$ (1) ». Assume that $S = \operatorname{Supp} L_U^{v}(S)$ and that $\| \|_{\infty}^{B}$ is order continuous on $L_U^{v}(S,B)^{o}_{\mathscr{X}}$. Let $\mu \in L_{RUC}(S,B)_{\infty}$ and let f be a Borel measurable function from S into $\{z \in \mathbf{C} |\ |z| \leqslant 1\}$. Put $\nu := f\mu$. In order to show that $\nu \in L_{RUC}(S,B)_{\infty}$, let $\varepsilon > 0$. Then $W := \{x \in S |\ \|\mu * \bar{x} - \mu\|_{\infty}^{B} < \varepsilon\}$ is a neighbourhood of 1. There is a $\rho \in L_U^{v}(S)^{o}_{\mathscr{X}}$ such that $\rho(W) = \|\rho\| = 1$. Then

$$\mu * \rho \in L_U^{v}(S,B)^{o}_{\mathscr{X}} \cap L_{RUC}(S,B)_{\infty} ,$$
$$\|\mu * \rho - \mu\|_{\infty} < \varepsilon .$$

The order continuity of $\| \|_{\infty}^{B}$ on $L_U^{B}(S,B)^{o}_{\mathscr{X}}$ and the order denseness of the subcollection $\{g(\mu * \rho) | g \in C_{00}(S)\}$ of $L_{RUC}(S,B)_{\infty}$ in $\{h(\mu * \rho) | h \in m(S), \|h\|_{\infty} \leqslant 1\}$ imply that

$$f(\mu * \rho) \in L_{RUC}(S,B)_{\infty} .$$

Finally, the inequalities

$$\begin{aligned} \|\nu * \bar{x} - \nu * \bar{y}\|_{\infty}^{B} &\leqslant \|\nu * \bar{x} - f(\mu * \rho) * \bar{x}\|_{\infty}^{B} \\ &\quad + \|f(\mu * \rho) * \bar{x} - f(\mu * \rho) * \bar{y}\|_{\infty}^{B} + \|f(\mu * \rho) * \bar{y} - \nu * \bar{y}\|_{\infty}^{B} \\ &\leqslant \varepsilon(\Delta(x) + \Delta(y)) + \|f(\mu * \rho) * \bar{x} - f(\mu * \rho) * \bar{y}\|_{\infty}^{B} \end{aligned}$$

clear that $\nu \in L_{RUC}(S,B)_{\infty}$. □

4.15. *Remarks.* – Let U be a compact neighbourhood of 1.

(1) In [18], we gave sufficient topological conditions on S [e.g. $\mathring{S}$ is a G_δ-subset of S] under which $S = \operatorname{Supp} L_U^{v}(S)$. However, it is still an unsolved problem whether $S = \operatorname{Supp} L_U^{v}(S)$ for all [foundation] stips S.

(2) Let Λ be the collection of all measures in $L(S)$ of the form $\bar{x} * \mu * \bar{y}$, where $x,y \in \mathring{S}$ and $\mu \in L(S)_{\mathscr{X}}$ such that $\{|\mu| * \bar{z} | z \in U\} \cup \{\bar{z} * |\mu| \,|\, z \in U\}$ has an upper bound in $L(S)$. [As a linear space Λ does not depend on the choice of the compact neighbourhood U of 1.] Suppose that $S = \operatorname{Supp} \Lambda = \operatorname{Supp} L^{\infty}(S,B)$. Then $L_U^{v}(S,B)^{o}_{\mathscr{X}} = \Lambda$. Therefore, concerning this case, we may state that $L_U^{v}(S,B)^{o}_{\mathscr{X}}$ does not depend on B. [One can show, by techniques as used in the proofs in § 3 of [15], that $\operatorname{Supp}(\Lambda) = \operatorname{Supp} L_U^{v}(S)$.]

(3) In case S is a group with right Haar measure m, we obviously have that $S = \mathrm{Supp}(\Lambda) = \mathrm{Supp}\, L^{\infty}(S,B)$ [unless $L^{\infty}(S,B)=\{0\}$], whence

$$L^{v}_{U}(S,B)^{o}_{\mathscr{X}} = \Lambda = \{fm \mid f \in L^{\infty}(S,m)\}_{\mathscr{X}}.$$

(4) The following example shows that in (4.14.3) one may not replace $L^{v}_{U}(S,B)^{o}_{\mathscr{X}}$ by $L^{v}_{U}(S,B)_{\mathscr{X}}$.

4.16. *Example.* – Let S be the additive subsemigroup $[0,\infty)$ of the real numbers. λ is the Lebesgue measure on S.

For each $f \in L^1(S,\lambda)_{\mathrm{loc}}$ we define

$$\|f\lambda\|^{B}_{\infty} := \sup\left\{\frac{1}{\varepsilon}\int_0^{\varepsilon} |f(t)|\,dt + \int_{\varepsilon}^{\infty} |f(t)|\,dt \quad \Big|\, \varepsilon > 0\right\}$$

$L^{\infty}(S,B) := \{f\lambda \mid f \in L^1(S,\lambda)_{\mathrm{loc}}, \|f\lambda\|^{B}_{\infty} < \infty\}$. Then $\|\ \|^{B}_{\infty}$ is order continuous on $L^{v}_{U}(S,B)^{o}_{\mathscr{X}}$, while for each n, $\mu_n := \lambda|_{[0,1/n)}$ belongs to $L^{v}_{U}(S,B)_{\mathscr{X}}$, $(\mu_n)\downarrow 0$, but $\|\mu\|^{B}_{\infty} = 1$ for all $n \in \mathbf{N}$.

4.17. Corollary. – $L_{RUC}(S,B)_{\infty} = L^{\infty}(S,B)_{\infty}$ *if and only if* $\|\ \|^{B}_{\infty}$ *is order continuous on* $L^{\infty}(S,B)_{\infty}$.

Proof. – Assume that $\|\ \|^{B}_{\infty}$ is order continuous on $L^{\infty}(S,B)_{\infty}$. Then, since $L_{RUC}(S,B)_{\infty}$ is a norm closed Riesz ideal, $L_{RUC}(S,B)_{\infty}$ is a band in $L^{\infty}(S,B)_{\infty}$. Therefore, in order to show that $L_{RUC}(S,B)_{\infty} = L^{\infty}(S,B)_{\infty}$, we only have to prove that

$$\left.\begin{array}{l}\text{for each } \mu \in L^{\infty}(S,B)^{+}_{\mathscr{X}} \text{ there is an } m \in L_{RUC}(S,B)_{\infty}\\ \text{such that } \mu \ll m.\end{array}\right\} \quad (1)$$

Let $\mu \in L^{\infty}(S,B)^{+}_{\mathscr{X}}$. Take a $\nu \in L(S)^{+}_{\mathscr{X}}$ for which $1 \in \mathrm{supp}(\nu)$. Then

$$\mu * \nu \in L_{RUC}(S,B)_{\infty}$$

and, moreover,

$$\mu \ll \mu * \nu;$$

because, if $F \in K$ and $\mu * \nu(F) = 0$ then

$$1 \in \mathrm{clo}\{x \in S \mid \mu * \bar{x}(F) = 0\},$$

whence $\mu(F) = 0$. □

Maybe needless to note that $L_{RUC}(S,B)_{\infty}$ can be a Riesz-ideal while $\|\ \|^{B}_{\infty}$ is not order continuous on $L^{\infty}(S,B)_{\infty}$ [see the following example].

4.18. *Example.* – Let $S := \{z \in \mathbf{C} \mid |z| = 1\}$, endowed with the usual topology and multiplication. λ denotes the Lebesgue measure on S. For each $f \in L^1(S,\lambda)_{loc} = L^1(S,\lambda)$ define

$$\|f\lambda\|_\infty^B := \sup \left\{ \frac{1}{\varepsilon} \int_0^\varepsilon |f(\exp(it+is))| \sqrt{s}\, ds| \quad t, \varepsilon \in (0,2\pi] \right\}.$$

Then $L^\infty(S,B) := \{f\lambda \mid f \in L^1(S,\lambda), \|f\lambda\|_\infty^B < \infty\}$.

Now, $L_U^v(S,B) = \{f\lambda \mid f \in L^\infty(S,\lambda)\}$. In order to show that $\| \ \|_\infty^B$ is order continuous on $L_U^v(S,B)$, let $(f_n)_{n \in \mathbf{N}}$ be a decreasing sequence such that $1 \geqslant (f_n)_{n \in \mathbf{N}} \downarrow 0$.

Suppose that $\inf \|f_n\lambda\|_\infty^B = \alpha > 0$.

Then for each $n \in \mathbf{N}$ there are some $\varepsilon > 0$ and some $t \in (0,2\pi]$ such that

$$\alpha/2 \leqslant \frac{1}{\varepsilon} \int_0^\varepsilon |f_n(\exp(it + is))| \sqrt{s}\, ds \leqslant \frac{1}{\varepsilon} \int_0^\varepsilon \sqrt{s}\, ds = \frac{2}{3} \sqrt{\varepsilon}.$$

Clearly $\varepsilon \geqslant \varepsilon_0 := (3\alpha/4)^2$ and

$$\begin{aligned} \alpha/2 &\leqslant \frac{1}{\varepsilon} \int_0^\varepsilon |f_n(\exp(it+is))| \sqrt{s}\, ds \\ &\leqslant \frac{2\pi}{\varepsilon_0} \int_0^{2\pi} |f_n(\exp(it+is))|\, ds = \frac{2\pi}{\varepsilon_0} \|f_n\|_1, \end{aligned}$$

which is impossible by the Fatou lemma. However, with

$$f_n(\exp(is)) := s^{-\frac{1}{2}} \xi_{[0,1/n]}(s) \quad \text{for all} \quad s \in [0,2\pi),$$

we have a sequence $(f_n)_{n \in \mathbf{N}}$ for which

$$(f_n) \downarrow 0, \qquad f_n\lambda \in L^\infty(S,B)$$

and

$$\|f_n\lambda\|_\infty^B = 1 \qquad \text{for all} \qquad n \in \mathbf{N}.$$

5. The case where $L_{RUC}(S,B)$ is a Riesz ideal.

In view of the results in the previous section one might hope that ($*$) $L_{RUC}(S,B)$ *is a Riesz ideal if and only if* $\| \ \|_\infty^B$ *is order continuous on* $L_{RUC}(S,B)$.

A short reflection with $l^\infty(\mathbf{Z})$ in mind clears that this hope is vain. However, in case S is a group, $l^\infty(\mathbf{Z})$ is essentially the only counter example : if S is discrete then $L^\infty(S,B) = L_{RUC}(S,B)$ and otherwise the above conjecture $(*)$ is correct [cf. (5.10) and (5.11.2)]. In general, the situation is more complicated. We have to split the semigroup into two disjoint sets, one of which consists of the elements t that have a kind of « fix-point property » $[1 \in \operatorname{int}\{x \in S \mid tx = t\}]$.

In (5.1)-(5.3), we introduce and discuss the mentioned partition of S. Next, in (5.4)-(5.5) we obtain results on the « non-disastrously collapsing » part of S. The complementary part is discussed in (5.6)-(5.9). Finally, the main result can be found in (5.10).

5.1. Lemma. – *For each* $t \in S$, *put* $H(t) := \{x \in S \mid tx = t\}$. *Then* $H(t)$ *is a closed subsemigroup of* S *and*

$$1 \in H(t)^{-1}\, H(t) \subseteq H(t) \quad (t \in S).$$

For each $t \in S$, $H(t)$ *is either meagre or open.*

Proof. – The proof of the first claim is left to the reader.
Suppose that $\operatorname{int}[H(t)] \neq \varnothing$. Then $1 \in \operatorname{int}[H(t)^{-1}\, H(t)]$, by (2.2),

$$y \in \operatorname{int}[H(t)^{-1}\, H(t)y] \subseteq H(t) \qquad \text{for all} \qquad y \in H(t).$$

Therefore $H(t)$ is open. □

5.2. *Notation.* – Let Z be the left ideal $\{t \in S \mid H(t) \text{ is open}\}$.

5.3. *Remarks.* – (*a*) Since Z is an ideal by (2.4), we have that $\bar{Z}\backslash Z \in \mathscr{N}$.

(*b*) If S is connected then $\bar{Z}$ is the collection of the left zeros of S.

(*c*) If $\bar{Z} = S$ then $\{1\} = \bigcap\{H(t) \mid H(t) \text{ open and closed}\}$ [because $x = 1$ if $tx = t$ for all $t \in S$] and, consequently, S has a zero dimensional topology.

(*d*) If $\{1\}$ is open [or, equivalently, if S discrete] then $\bar{Z} = S$.

(*e*) In case S is a group, we have that either $\bar{Z} = S$ [if S is discrete] or $\bar{Z} = \varnothing$ [if S is not discrete].

5.4. Lemma. – *Let* V *be an open subset of* S *and let* $F \in \mathscr{K}$ *be such that*

$$F \cap \bar{Z} = \varnothing.$$

There exists an $x \in V$ [even $x \in \mathring{S}$] *such that* $tx \neq t$ *for all* $t \in F$. *Then*

$$\bigcap_{n=1}^{\infty} Fx^{-n} = \varnothing .$$

Put $\Pi(F,x) := \bigcup_{j=0}^{\infty} \left(A_{2j+1} \backslash \bigcup_{i=0}^{2j} A_i \right)$, *where* $A_i := (F \backslash Fx^{-1})x^{-i} (i \in \mathbf{N})$ *and* $A_0 = F \backslash Fx^{-1}$. *Then*

$$\Pi(F,x)x^{-1} \cap \Pi(F,x) = \varnothing$$

and

$$F \cap Fx^{-1} \subseteq \Pi(F,x)x^{-1} \cup \Pi(F,x).$$

Proof. – Take an $r \in \mathring{S}$ such that $rF \cap \bar{Z} = \varnothing$.

For each $t \in F$, $S^{-1}rt$ is a neighbourhood of t. Therefore, there are $t_1, \ldots, t_m \in F$ such that

$$F \subseteq S^{-1}rt_1 \cup \ldots \cup S^{-1}rt_m .$$

Since $H(rt_i)$ is meagre $(i=1,\ldots,m)$, there is an $x \in V \backslash \bigcup_{i=1}^{m} H(rt_i)$. One easily checks that $tx \neq t$ for all $t \in F$.

Suppose that $t \in \bigcap_{n=1}^{\infty} Fx^{-n}$. Then $T := \bigcap_{m} \operatorname{clo}\{tx^n \mid n \geqslant m\}$ is a non-empty compact subset of F for which $Tx \subseteq T$. Hence there is a « fix-point » q in T; i.e. $qx = q$. But this violates our choice of x.

The proof of the last claim is straightforward : we omit this. □

5.5. Proposition. – *Assume that* $L_{RUC}(S,B)$ *is a Riesz ideal. Then*

$$\{\mu \in L_{RUC}(S,B) \mid \mu|_Z \in L_{RUC}(S,B)_\infty\} \subseteq L_{RUC}(S,B)_\infty .$$

Proof. – Let $\mu \in L_{RUC}(S,B)^+$ such that $\mu|_Z = 0$.

Let V_0 be a compact neighbourhood of 1 and $\rho \in \mathbf{R}$ such that $\rho > \|\Delta\xi_{V_0}\|_\infty$.

Firstly, we shall show that for any $K \in \mathscr{K}$

(1) $$1 \in \operatorname{int}\{x \in S \mid \|\mu|_{K \cap Kx^{-1}}\|_\infty^B > \|\mu|_K\|_\infty^B/\rho\}$$

and for any $K \in \mathscr{K}$ and any countable subset A of $\mathring{S}$ with $1 \in \bar{A}$

(2) $$1 \in \operatorname{clo}\{x \in A \mid \|\mu|_{Kx^{-1}\backslash K}\|_\infty^B \leqslant \varepsilon\} \quad \text{for all} \quad \varepsilon > 0 .$$

Next, with the aid of (1) and (2), we shall show that

$$(3)\,\beta = 0, \quad \text{where} \quad \beta := \frac{1}{2}\inf\{\|\mu|_{S\backslash F}\|_\infty^B \mid F \in \mathscr{K},\ \ F \cap \bar{Z} = \emptyset\}.$$

The proposition follows easily from (3).

Property (1) follows from the observation that $\mu|_K \in L_{RUC}(S,B)$ and

$$\|\mu|_K\|_\infty^B \leqslant \|\mu|_K * \bar{x}|_K\|_\infty^B + \|\mu|_K * \bar{x}|_K - \mu|_K\|_\infty^B$$
$$\leqslant \Delta(x)\|\mu|_{K \cap Kx^{-1}}\|_\infty^B + \|\mu|_K * \bar{x} - \mu|_K\|_\infty^B.$$

In order to prove (2), consider a subset A' of $\mathring{S}$ with $1 \in \operatorname{clo} A'$. By induction, one can construct sequences $(U_n)_{n\in N}$ and $(V_n)_{n\in N}$ of open subsets of S and $(x_n)_{n\in N}$ of elements of A' such that

$$\bar{U}_1 \in \mathscr{K}, \qquad \{1\} = \bigcap V_n, \qquad K = \bigcap U_n,$$
$$x_{n+1} \in V_{n+1} \subseteq V_n \cap x_n V_n^{-1}, \qquad \bar{U}_{n+1}V_{n+1} \subseteq U_n \quad \text{for all} \quad n \in N.$$

Put $E_n := \bar{U}_n x_n^{-1}\backslash U_1$. Note that $E_{n+1} \subseteq E_n$ for all $n \in N$. Since $\bar{U}_1$ is compact, by (4.2), it is sufficient to show that

$$\alpha = 0, \qquad \text{where} \qquad \alpha := \frac{1}{2}\lim \|\mu|_{E_n}\|_\infty^B.$$

By exploiting the Fatou-Levi property and the fact that all the sets E_n are closed, one can show by induction that there exist sequences $(\alpha(n))_{n\in N}$, $(\beta(n))_{n\geqslant 0}$ of natural numbers and $(F_n)_{n\in N}$ of compact sets such that

$$\beta(0) = \alpha(1) = 1, \quad \alpha(n+1) > \beta(n) > \alpha(n) \qquad \text{for all} \qquad n \in N$$
$$F_n \subseteq E_{\beta(n-1)}\backslash E_{\alpha(n)}, \qquad F_n V_{\beta(n)} \cap E_{\alpha(n)} = \emptyset$$

and

$$\|\mu|_{F_n}\|_\infty^B \geqslant \alpha.$$

Put $y_n := x_{\beta(n-1)} (n \in N)$. Note that $F_m y_n \subseteq U_1$ for all $m, n \in N$, $m \geqslant n$, because

$$F_m y_n \subseteq E_{\beta(m-1)} y_n \subseteq (U_{\beta(m-1)} x_{\beta(m-1)}^{-1}) x_{\beta(n-1)} \subseteq U_1.$$

Put $A := \bigcup_{n=1}^{\infty} F_n$. Then

$$Ay_n\backslash U_1 \subseteq F_1 y_n \cup \ldots \cup F_{n-1}y_n \subseteq F_1 V_{\beta(1)} \cup \ldots \cup F_{n-1}V_{\beta(n-1)},$$

whence

$$A\backslash Ay_n \supseteq F_n \qquad \text{for all} \qquad n \in N.$$

Apparently,

$$\alpha \leqslant \|\mu|_{F_n}\|_\infty^B \leqslant \|\mu|_A * \bar{y}_n - \mu|_A\|_\infty^B \qquad \text{for all} \qquad n \in \mathbf{N}.$$

Since $\mu|_A \in L_{RUC}(S,B)$ we have that $\alpha = 0$.

To prove (3), note that $S\backslash\bar{Z}$ is the union of the open relatively compact subsets U of S for which $\bar{U} \cap \bar{Z} = \varnothing$. Therefore, in view of the Fatou-Levi property, there is a sequence $(U_n)_{n \geqslant 0}$ of open relatively compact subsets of S such that

$$\bar{U}_n \cap \bar{Z} = \varnothing, \qquad \varnothing = U_0 \subseteq \bar{U}_n \subseteq U_{n+1} \qquad \text{for all} \qquad n \in \mathbf{N}$$

and

$$\|\mu|_{U_{n+1}\backslash\bar{U}_n}\|_\infty^B \geqslant \beta \qquad \text{for all} \qquad n \geqslant 0.$$

Via an inductive construction, based again on the Fatou-Levi property and, furthermore on (1), (2) and lemma (5.4), one can find sequences $(F_n)_{n \in N}$ of compact subsets of S and $(x_n)_{n \in N}$ in $\mathring{S}$ such that $(x_n)_{n \in N}$ converge to 1,

$$tx_n \neq t \qquad \text{for all} \qquad t \in F_n \qquad \text{for all} \qquad n \in \mathbf{N},$$

$$\left.\begin{array}{l} F_n \cup F_n x_m \subseteq U_n\backslash\bar{U}_{n-1} \\ F_m \cap F_n x_n^{-1} = \varnothing \end{array}\right\} \qquad \text{for all} \qquad n,\ m \in \mathbf{N},\ \ m > n$$

and

$$\|\mu|_{F_n \cap F_n x_n^{-1}}\|_\infty^B \geqslant \frac{1}{2}\,\beta/\rho \qquad \text{for all} \qquad n \in \mathbf{N}.$$

[Actually,

$$\|\mu|_{F_n x_n^{-1}\backslash F_n}\|_\infty^B \leqslant 2^{-n-2}\,\beta \qquad \text{and} \qquad F_n \subseteq U_n\backslash\left(\bar{U}_{n-1} \cup \bigcup_{j=1}^{n-1} F_j x_j^{-1}\right).\Bigg]$$

Put $C_n := \Pi(F_n,x_n) \cap F_n$ and $C'_n := \Pi(F_n,x_n)x_n^{-1} \cap F_n$; the notation here is as in (5.4). Then, by (5.4),

$$F_n \cap F_n x_n^{-1} \subseteq C_n \cup C'_n \qquad \text{and} \qquad C_n \cap C'_n = \varnothing.$$

Hence, we have that

$$\text{(i)}\ \ \|\mu|_{C_n}\|_\infty^B \geqslant \frac{1}{4}\,\beta/\rho \qquad \text{or} \qquad \text{(ii)}\ \ \|\mu|_{C'_n}\|_\infty^B \geqslant \frac{1}{4}\,\beta/\rho.$$

Now, take A_n to be C_n in case (i) or else $A_n = C'_n$. Note that

$$A_n x_n \cap A_n = \varnothing$$

if $m > n$, $A_m \cap A_n x_n^{-1} \subseteq F_m \cap F_n x_n^{-1} = \varnothing$,
if $m < n$, $A_m x_n \cap A_n \subseteq F_m x_n \cap F_n \subseteq U_m \cap F_n = \varnothing$.

Put $A := \bigcup_{n=1}^{\infty} A_n$. Then $A_n \subseteq A \backslash A x_n$ for all $n \in \mathbf{N}$, which leads to

$$\frac{1}{4}\ \beta/\rho \leqslant \|\mu|_{A_n}\|_\infty^B \leqslant \|\mu|_A * \bar{x}_n - \mu|_A\|_\infty^B .$$

Since $\mu|_A \in L_{RUC}(S,B)$, this shows that $\beta = 0$. □

By a combination of (4.14) and (5.6) we obtain a description of the case where $\{\mu \in L_{RUC}(S,B) | \mu|_Z = 0\}$ is a Riesz ideal [see (5.10)].

We now consider the measures that vanish outside $\bar{Z}$. If the identity element has a connected neighbourhood V, then there are no problems : because in this case $tV = t$ for all $t \in Z$, whence $\mu|_Z * \bar{v} = \mu|_Z$ for every $v \in V$ and $\mu \in L^\infty(S,B)$. Consequently, here $\mu|_Z \in L_{RUC}(S,B)$ for every $\mu \in L^\infty(S,B)$.

In general, however, the situation is more complicated as the following example may show.

5.6. *Example.* – Let G be the product space $\{-1,+1\}^{\mathbf{N}}$. $\mathbf{N}_\infty := \mathbf{N} \cup \{\infty\}$ is endowed with the discrete topology. S is the subspace

$$\{(n,\bar{t}) \in \mathbf{N}_\infty \times G | \bar{t}(m) = 1 \quad \text{for all} \quad m > n\}$$

of the topological product $\mathbf{N}_\infty \times G$. The multiplication on S is given by

$$(n,\bar{t})(m,\bar{s}) := (\min(n,m), \overline{ts}),$$

where

$$\overline{ts}\,(i) := \begin{cases} 1 & \text{if } i > \min(n,m) \\ t(i).s(i) & \text{if } i \leqslant \min(n,m) \end{cases} \qquad ((n,\bar{t}),(m\bar{s}) \in S).$$

Then S is a foundation stip. Put $S_\infty := \{\infty\} \times G$ and for each $n \in \mathbf{N}$ $S_n = \{(n,t) | \text{where } t \in G \text{ such that } (n,t) \in S\}$. Then $\bar{Z} = S \backslash S_\infty$.

For each $n \in \mathbf{N}_\infty$ let π_n be the Haar measure on the subgroups S_n normalized such that $\|\pi_n\| := 2^{-n}$ if $n \in \mathbf{N}$ and $\|\pi_\infty\| = 1$. Put

$m := \sum_{n=1}^{\infty} \pi_n + \pi_\infty$. For a $\mu \in L(S)_{loc}$,

$$\|\mu\|_\infty^B := \inf \{c \in \mathbf{R}^+ \mid |\mu| \leqslant cm\}.$$

Then $m \in L_{RUC}(S,B)$, while $m|_A \notin L_{RUC}(S,B)$ where

$$A := \{(n,\bar{t}) \in S \mid n \in \mathbf{N}, \text{ if } \bar{t} = (t_m) \text{ then } t_n = 1\}.$$

In order to describe the case where $\{\mu \in L_{RUC}(S,B) \mid \mu|_Z = \mu\}$ is a Riesz ideal, we use the sets $Z(x) := \{t \in \bar{Z} \mid tx = t\}$ $(x \in S)$ [see (5.9)]. In the proof of (5.9), we need a partition of the sets $Z \backslash Z(x)$. This partition is introduced in (5.7). Its measurability properties are discussed in (5.8).

5.7. LEMMA. – *Let* $x \in S$. *Put* $Z(x) := \{t \in \bar{Z} \mid tx = t\}$ *and* $Q(x) := Z \backslash Z(x)$. *There exists a set* $\tilde{Q}(x) \subseteq Q(x)$ *such that*

$$\tilde{Q}(x)x \cap \tilde{Q}(x) = \tilde{Q}(x)x^2 \cap \tilde{Q}(x)x = \varnothing,$$
$$Q(x) = \tilde{Q}(x)x \cup \tilde{Q}(x) \cup \tilde{Q}(x)x^{-1}.$$

Proof. – Consider the following sets.

$A := \{t \in \bar{Z} \mid tx^n \in Z(x) \text{ for some } n \in \mathbf{N}\}$,
$B := \{t \in \bar{Z} \mid tx^m = t \text{ for some } m \in \mathbf{N},\ m \geqslant 2\}$,
$C := \{t \in \bar{Z} \mid tx^n \in B \text{ for some } n = 0,1,2,\ldots\}$ [where $x^0 := 1$] and
$D := \{t \in \bar{Z} \mid tx^n \neq tx^m \text{ for all } n,\ m \in \mathbf{N},\ n \neq m\}$.

Note that all these sets are fixed under multiplication by x [i.e. $Ax \subseteq A$, etc.].

Put $A' := \{t \in A \mid tx^{2n} \in Z(x) \text{ and } tx^{2n-1} \notin Z(x) \text{ for some } n \in \mathbf{N}\}$.

Then $A'x \cap A' = A'x^2 \cap A'x = \varnothing$, $A'x \cup A' = A \backslash Z(x)$.

With the aid of Zorn's lemma, one can find a subset B' of B such that

$$B'x \cap B' = \varnothing, \qquad B'x^2 \cap B'x = \varnothing$$

and

$$B \subseteq B'x \cup B' \cup B'x^{-1}.$$

Put $B_0 := B'x$ and $B_e := B' \cup (B \backslash B'x)$. Now, let

$C' := \{t \in C \mid \text{there are an } n \geqslant 1,\ b \in B_e \text{ for which } tx^{2n} = b, \text{ while } tx^{2n-1} \notin B\} \cup \{t \in C \mid \text{for certain } n \geqslant 0,\ b \in B_0,\ tx^{2n+1} = b, \text{ while } tx^{2n} \notin B\} \cup B'$.

Then $C'x \cap C' = \varnothing$, $C'x^2 \cap C'x = \varnothing$, $C'x \cup C' \cup C'x^{-1} = C$.

Choose a subset E of D such that for each $t \in D$ the set $E \cap \{s \in D | sx^n = tx^m$ for some $n, m \in \mathbf{N}\}$ contains exactly one element.

Put $D' := \bigcup \{(tx^{2n})x^{-2m} | t \in E,\ n,\ m \in \mathbf{N}\}$.

Then $D'x \cap D' = \varnothing$, $D'x^2 \subseteq D'$ and $D'x \cup D' = D$.

Now, the set $\tilde{Q}(x) := A' \cup C' \cup D'$ fulfills the required conditions.

□

5.8. Lemma. – *For each* $U \subseteq S$, *put* $Z(U) := \{t \in \bar{Z} | tU = t\}$.

Let U *be a neighbourhood of* 1. *Then* Z(U) *is a closed, discrete subset of* S; since on Z(U) the neighbourhood $(tU)U^{-1}$ of $t \in Z(U)$ coincides with $\{t\}$. In particular, we have that *each subset of* Z(U) *is a Borel set. If* $x \in U \subseteq V \subseteq S$ *then* $Z(V) \subseteq Z(U) \subseteq Z(x)$. *If* $(V_n)_{n \in \mathbf{N}}$ *is a sequence of neighbourhoods of* 1 *such that* $\bigcap_{n=1}^{\infty} V_n = \{1\}$ *then* $Z = \bigcup_{n=1}^{\infty} Z(V_n)$. □

5.9. Proposition. – $\{\mu \in L_{RUC}(S,B) | \mu|_Z = \mu\}$ *is a Riesz ideal if and only if*

$$1 \in \text{int}\,\{x \in S |\ \|\mu|_{Z \setminus Z(x)}\|_\infty^B < \varepsilon\} \quad \textit{for each} \quad \varepsilon > 0,$$
$$\mu \in L_{RUC}(S,B) \quad \textit{with} \quad \mu|_Z = \mu.$$

Proof. – Suppose there is a $\mu \in L_{RUC}(S,B)$ with $\mu|_Z = \mu$ and an $\alpha > 0$ such that with $W := \{x \in S |\ \|\mu|_{Z \setminus Z(x)}\|_\infty^B > \alpha\}$ we have that $1 \in \text{clo}\,(W)$. We shall show that under this assumption $\{\nu \in L^\infty(S,B) \,|\, 0 \leqslant |\nu| \leqslant \mu\} \nsubseteq L_{RUC}(S,B)$; then the « only if »part of the proposition follows.

Let $V_0 := \left\{x \in S |\ \|\mu * \bar{x} - \mu\|_\infty^B < \frac{1}{4}\alpha\right\}$. Then V_0 is a neighbourhood of 1 and for each $x \in V_0 \cap W$ we have that

$$\begin{aligned}\|\mu|_{Z \setminus Z(x)} * \bar{x}\|_\infty^B &\geqslant \|\mu|_{Z \setminus Z(x)}\|_\infty^B - \|\mu|_{Z \setminus Z(x)} * \bar{x} - \mu|_{Z \setminus Z(x)}\|_\infty^B \\ &> \alpha - \|\mu * \bar{x} - \mu\|_\infty^B \geqslant \frac{3}{4}\alpha;\end{aligned}$$

because $\mu|_{Z(x)} * \bar{x} = \mu|_{Z(x)}$.

Using the Fatou-Levi property, by induction, we can find sequences

$(V_n)_{n\in N}$ of neighbourhoods of 1 and $(x_n)_{n\in N}$ in S such that

$$x_n \in V_{n-1} \qquad \text{for all} \qquad n,$$

$$V_n \subseteq \bigcap_{i=1}^{n-1} [x_i^{-1}(V_i^{-1}(V_i x_i)) \cap V_i] \cap V_0, \qquad \{1\} = \bigcap_{n=1}^{\infty} V_n$$

and with $X_n := Z(V_n)\backslash Z(x_n)$ we have that

$$\| \mu|_{X_n} * \bar{x}_n \|_\infty^B > \frac{3}{4}\alpha \qquad \text{for all} \qquad n \in N$$

[take $x_n \in V_{n-1} \cap W$ and find a V_n with the required properties]. Note that

if $m < n$ then $X_m \subseteq Z(V_m) \subseteq Z(V_{n-1}) \subseteq Z(x_n)$,
if $m > n$ then $X_n x_n \subseteq Z(V_n)x_n \subseteq Z(x_m)$; because $x_n x_m \in V_n^{-1}(V_n x_n)$.

Hence

$$(X_m \cap X_n x_n)\backslash Z(x_n) = \varnothing \qquad \text{for all} \qquad m,\ n \in N,\ m \neq n.$$

For each $n \in N$, choose Y_n to be either

$$\tilde{Q}(x_n)x_n^{-1} \cap X_n, \qquad \text{or} \qquad \tilde{Q}(x_n) \cap X_n \qquad \text{or} \qquad \tilde{Q}(x_n)x_n \cap X_n$$

such that

$$\| \mu|_{Y_n} * \bar{x}_n \|_\infty^B > \frac{1}{4}\alpha \quad \text{[use the lemmas (5.7) and (5.8)].}$$

Put $Y := \bigcup_{n=1}^{\infty} Y_n$. Then Y is measurable and

$$\begin{aligned} |\mu|_Y * \bar{x}_n - \mu|_Y| &= |\mu|_{Y\backslash Z(x_n)} * \bar{x}_n - \mu|_{Y\backslash Z(x_n)}| \\ &\geqslant |\mu|_{Y_n\backslash Z(x_n)} * \bar{x}_n - \mu|_{Y_n x_n \cap Y\backslash Z(x_n)}| = |\mu|_{Y_n} * \bar{x}_n| . \end{aligned}$$

So we find that

$$\frac{1}{4}\alpha \leqslant \| \mu|_{Y_n} * \bar{x}_n \|_\infty^B \leqslant \| \mu|_Y * \bar{x}_n - \mu|_Y \|_\infty^B \qquad \text{for all} \qquad n \in N.$$

Since $(x_n)_{n\in N}$ converges to 1, this shows that $\mu|_Y \notin L_{RUC}(S,B)$.

The « if » part follows easily from the observation that if

$\nu, \mu \in L^\infty(S,B)$, $|\nu| \leqslant |\mu|$ and $\mu|_Z = \mu$ then

$$\|\nu * \bar{x} - \nu\|_\infty^B = \|\nu|_{Z\setminus Z(x)} * \bar{x} - \nu|_{Z\setminus Z(x)}\|_\infty^B$$
$$\leqslant \|\mu|_{Z\setminus Z(x)} * \bar{x}\|_\infty^B + \|\mu|_{Z\setminus Z(x)}\|_\infty^B \leqslant \|\mu|_{\bar{Z}\setminus Z(x)}\|_\infty^B (1 + \Delta(x)). \qquad \square$$

Tying the results of the propositions (5.5) and (5.9) together, we come to the following theorem.

5.10. THEOREM. – *Let* U *be a compact neighbourhood of* 1. *Consider the following properties :*

(1) $L_{RUC}(S,B)$ *is a Riesz ideal of* $L^\infty(S,B)$.

(2) $\begin{cases} (a)\ \|\ \|_\infty^B \textit{ is order continuous on } \{\mu|_{S\setminus \bar{Z}} | \mu \in L_{RUC}(S,B)\} \\ (b)\ 1 \in \text{int}\,\{x \in S \mid \|\mu|_{\bar{Z}\setminus Z(x)}\|_\infty^B < \varepsilon\} \textit{ for all } \varepsilon > 0,\ \mu \in L_{RUC}(S,B). \end{cases}$

(3) $\begin{cases} (a)\ \|\ \|_\infty^B \textit{ is order continuous on } \{\rho|_{S\setminus \bar{Z}} | \rho \in L_U^v(S,B)^o\} \textit{ and} \\ (b)\ 1 \in \text{int}\,\{x \in S \mid \|\rho|_{\bar{Z}\setminus Z(x)}\|_\infty^B < \varepsilon\} \textit{ for all } \varepsilon > 0,\ \rho \in L_U^v(S,B)^o. \end{cases}$

(4) $\{\mu|_{S\setminus \bar{Z}} | \mu \in L_{RUC}(S,B)| \subseteq L_{RUC}(S,B)_\infty$.

Then (1) *and* (2) *are equivalent and they both imply* (3) *and* (4). *If, in addition,* $S = \text{Supp}\, L_U^v(S)$, *then* (1), (2) *and* (3) *are equivalent.*

Proof. – « (4) $\Leftarrow$ (1) $\Rightarrow$ (2) » is a combination of (5.6), (5.9) and (4.2). Now, suppose that (2) holds. We shall show that (2*a*) implies (4); then (1) follows from (2), (4.2) and (5.9).

Let $\mu \in L_{RUC}(S,B)$. Let

$$V := \{f \in C(X) \mid \xi_Z \leqslant f \leqslant 1 \text{ and } 1 - f \in C_{00}(S)\}.$$

Then $V \downarrow \xi_Z$ and $f\mu - \mu|_Z = f\mu|_{S\setminus Z}$. Since $f\mu \in L_{RUC}(S,B)$ $(f \in V)$ and $L_{RUC}(S,B)$ is norm closed, (2*a*) implies that $\mu|_Z \in L_{RUC}(S,B)$. In particular, we have that $(1-f)\mu|_{S\setminus Z} \in L_{RUC}(S,B)_\infty$ $(f \in V)$ and since

$$\mu|_{S\setminus Z} - (1 - f)\mu|_{S\setminus Z} = f\mu|_{S\setminus Z} \qquad (f \in V)$$

we see that $\mu|_{S\setminus Z} \in L_{RUC}(S,B)_\infty$.

By an adaptation of the arguments in the proof of « (1) $\Rightarrow$ (3) » and « (3) $\Rightarrow$ (1) » of (4.14), one can complete the proof of this theorem. $\square$

5.11. *Remarks.* – (1) If 1 has a connected neighbourhood V [for instance if {1} is open] then $\bar{Z}\setminus Z(x) = \varnothing$ for all $x \in V$; because

$V \subseteq H(t)$ for all $t \in Z$ and hence for all $t \in \bar{Z}$. Therefore, in this situation both the conditions (*b*) in (2) and (3) are redundant.

(2) This is also the case if S is a group [if $\{1\}$ is not open then $\bar{Z} = \emptyset$].

(3) Example (5.6) shows that, in general, these conditions (*b*) are meaningful.

5.12. COROLLARY. – $L_{RUC}(S,B) = L^\infty(S,B)$ *if and only if* (*a*) $\|\ \|_\infty^B$ *is order continuous on* $\{\mu|_{S\setminus Z} | \mu \in L_{RUC}(S,B)\}$ *and* (*b*) $1 \in \operatorname{int}\{x \in S \mid \|\mu|_{Z\setminus Z(x)}\|_\infty^B < \varepsilon\}$ *for all* $\varepsilon > 0$, $\mu \in L^\infty(S,B)$.

Proof. – For a $\mu \in L^\infty(S,B)$, note that

$$\begin{aligned}\|\mu|_Z * \bar{x} - \mu|_Z\|_\infty^B &\leqslant \|\mu|_{Z\setminus Z(x)} * \bar{x} - \mu|_{Z\setminus Z(x)}\|_\infty^B \\ &\leqslant \|\mu|_{Z\setminus Z(x)}\|_\infty^B(\Delta(x)+1).\end{aligned}$$

Now, by making some observations simular to those in the proof of (4.17), the corollary follows. □

We conclude this section with the following observation (5.13). In this one, we prove that under certain restrictions on the size of S [discrete subsets have to be of measurable cardinality] $\|\ \|_\infty^B$ is order continuous on $L_{RUC}(S,B)|_{S\setminus Z}$ as soon as $\|\ \|_\infty^B$ is absolutely continuous on this space $L_{RUC}(S,B)|_{S\setminus Z}$. As a consequence, under the mentioned restriction, in (2) and (3) of (5.10), one may replace « order continuous » by « absolutely continuous » [in order to see the correctness of this statement as far as (3) concerns one may for instance inspect the arguments in the proof of « (3) ⇒ (1) » in (4.14)].

In the proof of this observation (5.13), we use a result from [17]. A discussion of this restriction of the size and references concerning the notion of measurable cardinality and the other notion [σ-smooth, τ-smooth] used in the proof of (5.13) can also be found in [17].

5.13. PROPOSITION. – *Let* S *be such that*

(i) *each discrete subset [i.e. discrete if endowed with the restriction topology] is of measurable cardinality,*

(ii) *for each* $F \in \mathcal{K}$ *there is a neighbourhood* V *of* 1 *such that* $V^{-1}F$ *is* σ*-compact.*

Let $g \in m(S)$ *be such that* $0 \leqslant g \leqslant 1$ *and* $\|\ \|_\infty^B$ *is absolutely continuous on* $M := \{g\mu | \mu \in L_{RUC}(S,B)\}$.

Then $\mathrm{M} \subseteq \mathrm{L}^\infty(\mathrm{S,B})_\infty$ *and, in particular,* $\| \|_\infty^{\mathrm{B}}$ *is order continuous on* M.

Proof. – For an $h \in \mathrm{C(S)}$ and a $\rho \in \mathrm{L(S)}$, put

$$\rho \circ h(t) := \rho * \bar{t}(\mathrm{h})(t \in \mathrm{S}).$$

Note that $\rho \circ h$ is uniformly continuous and that $\rho \circ (h_x) = (\rho \circ h)_x$. Furthermore, in view of (ii), for each $h \in \mathrm{C_{00}(S)}$ we can find a $\rho \in \mathrm{L(S)}^+$ such that $\|\rho\| = 1$ and $\rho \circ h$ vanishes outside a σ-compact subset of S.

Let $\mu \in \mathrm{L_{RUC}(S,B)}$. Take an $x \in \mathring{\mathrm{S}}$ and put

$$\mathrm{V} := \{f \in \mathrm{C(S)} \,|\, 0 \leqslant f \leqslant 1,\ \ 1 - f \in \mathrm{C_{00}(S)}\}.$$

Then $\mathrm{V} \downarrow 0$.

Let $p:\ \mathrm{C(S)} \to [0, \infty)$ be defined by

$$p(h) := \inf\{\|g((\rho \circ hf_x\mu) * \bar{x})\|_\infty^{\mathrm{B}} | f \in \mathrm{V}, \rho \in \mathrm{L(S)}^+, \|\rho\| = 1\} \quad (h \in \mathrm{C(S)}).$$

Then p is a seminorm on C(S) for which

$$p(h) \leqslant \|h\|_\infty \|\mu\|_\infty\, \Delta(x) \qquad (h \in \mathrm{C(S)}).$$

Consider an $h \in \mathrm{C_{00}(S)}$. There is a $\rho \in \mathrm{L(S)}^+$, $\|\rho\| = 1$ for which $\rho \circ h(\mu * \bar{x}) \in \bar{\mathrm{M}}_\sigma(\mathrm{S})$. Moreover, $f(\rho \circ h)(\mu * \bar{x}) \in \mathrm{L_{RUC}(S,B)}$ [see (3.4)].

Since $\mathrm{L}^1(\mathrm{S},\nu)$ is super Dedekind complete for any $\nu \in \bar{\mathrm{M}}_\sigma(\mathrm{S})$, we have that

$$(1)\quad p(h_x) = \inf\{\|(g\rho \circ hf)(\mu * \bar{x})\|_\infty^{\mathrm{B}} | f \in \mathrm{V}, \rho \in \mathrm{L(S)}^+,\ \ \|\rho\| = 1\} = 0$$

$$\text{for all} \quad h \in \mathrm{C_{00}(S)}.$$

According to the Hahn-Banach theorem there is a $\varphi \in \mathrm{C(S)}^*$ such that

$$\varphi(1) = p(1) \quad \text{and} \quad |\varphi(h)| \leqslant p(h) \leqslant \|h\|_\infty \|\mu\|_\infty^{\mathrm{B}}\, \Delta(x) \quad \text{for all} \quad h \in \mathrm{C(S)}.$$

Since $\rho \circ hf_x$ is uniformly continuous we have that $(\rho \circ \mathrm{h}f_x)(\mu * \bar{x})$ belongs to $\mathrm{L_{RUC}(S,B)}$ and therefore, by assumption, we see that φ is σ-smooth. Consequently, by (5.4) of [17], $\varphi * \bar{x} :\ h \to \varphi(h_x)\ (h \in \mathrm{C(S)})$ is a τ-smooth functional on C(S). However, by (1), $\varphi * \bar{x}(h) = 0$ for all $h \in \mathrm{C_{00}(S)}$ and the τ-smoothness implies that

$$0 = \varphi * \bar{x}(1) = \varphi(1) = p(1).$$

Apparently,

$$\inf\{\|gf(\mu * \bar{x})\|_\infty^B | f \in V\} = 0.$$

Consider

$$\|fg\mu\|_\infty^B \leqslant \|fg(\mu * \bar{x})\|_\infty^B + \|fg(\mu * \bar{x}) - fg\mu\|_\infty^B$$
$$\leqslant \|fg(\mu * \bar{x})\|_\infty^B + \|\mu * \bar{x} - \mu\|_\infty^B .$$

Recall that $\mu \in L_{RUC}(S,B)$ in order to see that

$$\inf\{\|fg\mu\|_\infty^B | f \in V\} = 0.$$

Since $1 - f \in C_{00}(S)$ $(f \in V)$, this shows that

$$g\mu \in \text{clo}\, L^\infty(S,B)_\infty \subseteq L^\infty(S,B)_\infty .$$ □

6. The case where $L_{RUC}(S,B)$ is a Riesz subspace.

In case S is a group, we have that

$$|\mu * \bar{x}| = |\mu| * \bar{x} \qquad \text{for all} \qquad \mu \in L^\infty(S,B), \qquad x \in S$$

and, since $\| \, |\mu| - |\mu * \bar{x}| \, \|_\infty^B \leqslant \|\mu - \mu * \bar{x}\|_\infty^B$, we see that $|\mu| \in L^\infty(S,B)$ whenever $\mu \in L^\infty(S,B)$. However, example (3.7) shows that, in general, $L_{RUC}(S,B)$ need not to be a Riesz subspace..

6.1. *Notation.* – Let U be a compact neighbourhood of 1. $L_U^v(S,B)$ is a pseudo L^∞-space under the norm $\| \; \|_U^v$ given by

$$\|\mu\|_U^v := \inf\{c \in R^+ | \sup\{|\mu| * \bar{x} | x \in U\} \leqslant cm \text{ for some } m \in B^+\}$$

for each $\mu \in L_U^v(S,B)$ [see also § 7 of [19]].

The collection of all $\mu \in L_U^v(S,B)$ for which r_μ is continuous with respect to $\| \; \|_U^v$ is denoted by $L_{RUC}^v(S,B)$.

Note that $L^\infty(S,B) * L_U^v(S)_{\mathscr{K}} * L(S)_{\mathscr{K}} \subseteq L_{RUC}^v(S,B) \subseteq L_{RUC}(S,B)$.

By exercising with some triangle inequalities, theorem (3.4), and techniques as presented in the proof of (4.14), one can prove the following result : we omit the details.

6.2. THEOREM. – *Let* U *be a compact neighbourhood of* 1. *Consider the following properties :*

(1) $L_{RUC}(S,B)$ *is a Riesz subspace of* $L^\infty(S,B)$.

(2) *For each* $\mu \in L_{RUC}(S,B)$, $\varepsilon > 0$, $x \in \mathring{S}$, $X \in \mathscr{K}$ *with* $x \in \operatorname{int}(X)$

$1 \in \operatorname{int}\{x \in S \mid \| |\mu * \bar{y}| * \bar{z} - |\mu * \bar{x}| \|_\infty^B < \varepsilon,$ *for some* $y \in X \cap xz^{-1}\}$.

(3) *For each* $\mu \in L^v_{RUC}(S,B)$, $\varepsilon > 0$, $x \in \mathring{S}$, $X \in \mathscr{K}$ *with* $x \in \operatorname{int}(X)$,

$1 \in \operatorname{int}\{x \in S \mid \| |\mu * \bar{y}| * \bar{z} - |\mu * \bar{x}| \|_\infty^B < \varepsilon,$ *for some* $y \in X \cap xz^{-1}\}$.

Then (1) *and* (2) *are equivalent and both imply* (3). *If, in addition,* $S = \operatorname{Supp}(L^v_U(S))$ *then all the properties* (1), (2) *and* (3) *are equivalent.*

A similar statement holds if one replaces

$$L_{RUC}(S,B) \quad \textit{by} \quad L_{RUC}(S,B)_\infty$$

and simultaneously $L^v_{RUC}(S,B)$ *by* $L^v_{RUC}(S,B)^o_{\mathscr{K}}$. □

6.3. *Remark.* – (1) In view of (4.12) and (6.1), it will be clear what we mean by $L^v_{RUC}(S)$.

One can show that $L^v_{RUC}(S,B)_{\mathscr{K}} \subseteq L^v_{RUC}(S)$ if $S = \operatorname{Supp} L^v_U(S)$ and if, in addition, $S = \operatorname{Supp} L^\infty(S,B)$ then we even have that

$$L^v_{RUC}(S,B)^o_{\mathscr{K}} = \Lambda \cap L^v_{RUC}(S)_{\mathscr{K}}.$$

In case S is a group,

$$L^v_{RUC}(S,B)_{\mathscr{K}} = L^v_{RUC}(S,B)^o_{\mathscr{K}} = \{fm \mid f \in C_{00}(S)\},$$

where m is a right Haar measure.

(2) The property in (6.2.2-3) can be viewed as a weak kind of order continuity. To be more precise : let $\mu \in L^\infty(S,B)$.

If $x \in \mathring{S}$, $X \in \mathscr{K}$, $x \in \operatorname{int}(X)$ and $(z_\lambda)_{\lambda \in \Lambda}$ is a net in $\mathring{S}$ that converges to 1 such that $z_\lambda \in Sz_\gamma$ for all λ, $\gamma \in \Lambda$ with $\lambda \leqslant \gamma$ then

$$0 \leqslant \{|\mu * \bar{y}_\lambda| * \bar{z}_\lambda - |\mu * \bar{x}| \mid \lambda \in \Lambda\} \downarrow 0,$$

where $y_\lambda \in X$ such that $y_\lambda z_\lambda = x$.

Now, we have that

$$0 = \inf \{\| |\mu * \bar{y}_\lambda| * \bar{z}_\lambda - |\mu * \bar{x}| \|_\infty^B \mid \lambda \in \Lambda\}$$

for all these x, X and $(z_\lambda)_{\lambda \in \Lambda}$ if and only if a property as in (6.2.2) holds.

(3) If $L^{v}_{RUC}(S,B)$ is a Riesz subspace of $L^{\infty}(S,B)$ then so is $L_{RUC}(S,B)$. The converse, however, need not be true [consider once more the semigroup from example (3.7) where the « pseudo L^{∞}-norm » now is given by

$$\|f\lambda' + g\lambda\|^{B}_{\infty} := \|f\|_{\infty} + \|g\|_{1}].$$

6.4. Corollary. – *Assume that* S *has a zero-dimensional topology. Then* $L_{RUC}(S,B)$ *is a Riesz subspace.*

Proof. – Take an $x \in \mathring{S}$ with compact neighbourhood X of x.

Let V be an open relatively compact neighbourhood of 1 such that $\bar{V}x \subseteq \operatorname{int}(X)$. Since $x \in \operatorname{clo}(\mathring{S}^{-1}x)$ there are $x_1, x_2 \in \mathring{S}$ such that $x = x_2x_1$, $x_2 \in V$, $Vx_1 \subseteq X$. Then [cf. (2.2)],

$$1 \in \operatorname{int}[x_1^{-1}((V \cap \mathring{S})^{-1}x)] \subseteq (Vx_1)^{-1}x.$$

Hence, as in (4.5) of [18], there is an open compact subsemigroup H of S such that

$$1 \in H \subseteq (Vx_1)^{-1}x \cap \mathring{S}^{-1}x_2.$$

Take an idempotent e in the kernel of H. Then $e \in \mathring{S}$ [cf. (4.5) of [18]] and eHe is a group. Furthermore $x_1e = x_1$ and $xe = x$.

Consider a $\mu \in L^{\infty}(S,B)$. If $z \in H$ then $yz = x$ for some $y \in Vx_1$. Since

$$\operatorname{supp}(|\mu * \bar{y}| * \bar{z}) \subseteq \operatorname{clo} Syz \subseteq \operatorname{clo} Sx \subseteq Se \quad \text{and} \quad ye = y,$$

we have that $|\mu * \bar{y}| * z = |\mu * \bar{y}| * \overline{eze}$. Finally, the fact that eze belongs to the group eHe, while $\operatorname{supp}(|\mu * \bar{y}|) \subseteq Se$, implies that

$$|\mu * \bar{y}| * \overline{eze} = |\mu * \bar{y} * \overline{eze}| = |\mu * \bar{x}|.$$

Apparently, for each $z \in H$, there is a $y \in X \cap xz^{-1}$ such that

$$|\mu * \bar{y}| * \bar{z} - |\mu * \bar{x}| = 0. \qquad \square$$

BIBLIOGRAPHY

[1] A. C. Baker and J. W. Baker, Algebras of measures on a locally compact semigroup II, *J. London Math. Soc.* (2), 4 (1972), 685-695.

[2] J.-P. Bertrandias, Unions et intersections d'espaces L^p sur un espace localement compact, *Bull. Sc. Math.*, (2), 101 (1977), 209-247.

[3] H. A. M. Dzinotyiweyi and G. L. G. Sleijpen, A note on measures on foundation semigroups with weakly compact orbits, *Pac. Journal Math.* (1), 81 (1979), 61-69.

[4] D. A. Edwards, On translates of L^∞-functions, *J. London Math. Soc.*, 36 (1961), 431-432.

[5] R. E. Edwards, E. Hewitt and G. Ritter, Fourier multipliers for certain spaces of functions with compact support, *Inventiones Math.*, 40 (1977), 37-57.

[6] H. G. Feichtinger, On a class of convolution algebras of functions, *Ann. Inst. Fourier*, Grenoble, 27, 3 (1977), 135-162.

[7] H. G. Feichtinger, Multipliers of Banach spaces of functions on groups, *Math. Z.*, 152 (1976), 47-58.

[8] H. G. Feichtinger, A characterization of Wiener's algebra on locally compact groups, *Arch. Math.*, 39 (1977), 136-140.

[9] D. H. Fremlin, *Topological Riesz-spaces and measure theory*, Cambridge Univ. Press, (1974).

[10] G. G. Gould, On a class of integration spaces, *J. London Math. Soc.*, 34 (1959), 161-172.

[11] H. Kharaghani, The weaky continuous left translations of measures with applications to invariant means, preprint.

[12] T.-S. Liu, A. C. M. van Rooij and J.-K. Wang, On some group modules related to Wiener's algebra M_1, *Pac. Journal Math.* (2), 55 (1974), 507-520.

[13] H. Porta, L. A. Rubel and A. L. Shields, Separability of orbits of functions on locally compact groups, *Studia Math.*, 48 (1973), 89-94.

[14] H. H. Schaeffer, *Banach-Lattices and Positive Operators*, Springer-Verlag, Berlin Heidelberg New-York, (1974).

[15] G. L. G. Sleijpen, Locally compact semigroups and continuous translations of measures, *Proc. London Math. Soc.*, (3), 37 (1978), 75-97.

[16] G. L. G. Sleijpen, Emaciated sets and measures with continuous translations, *Proc. London Math. Soc.*, (3), 37 (1978), 98-119.

[17] G. L. G. Sleijpen, L-multipliers for foundation semigroups with identity element, *Proc. London Math. Soc.*, (3), 39 (1979), 299-330.

[18] G. L. G. Sleijpen, The support of the Wiener algebra on stips, *Indag. Math.*, 42 (1980), 61-82.

[19] G. L. G. Sleijpen, L^p-spaces on foundation semigroups with identity element, Report 7906, Mathematical Institute, Catholic University, Nijmegen (1979).

[20] G. L. G. Sleijpen, Convolution measure algebras on semigroups, Thesis, Catholic University, Nijmegen (1976).

Manuscrit reçu le 11 mai 1981.

Gérard L. G. Sleijpen,
Mathematical Institute
Catholic University
Toernooiveld
6525 ED Nijmegen (The Netherlands).