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FRACTIONAL CARTESIAN PRODUCTS OF SETS

by Ron C. BLEI (*)

N-fold sums of « independent » sets serve in harmonic analysis as prototypical examples of $2N/(N+1)$-Sidon sets, and $\Lambda(q)$ sets whose $\Lambda(q)$ constants' growth is $\mathcal{O}(q^{N/2})$. Moreover, these features are exact : N-fold sums of independent sets are not $(2N/(N+1)-\varepsilon)$-Sidon and to not have $\Lambda(q)$ constants' growth asymptotic to $q^{(N/2-\varepsilon)}$, for any $\varepsilon > 0$ (see [4], [6] and [2]). In this paper, given any number $p \in (1, 2)$, we display a set that is p-Sidon but not $(p-\varepsilon)$-Sidon for any $\varepsilon > 0$. The same pool of examples contains, for any number $a \in [1/2, \infty)$, a set whose $\Lambda(q)$ constants' growth is $\mathcal{O}(q^{a})$ but not $\mathcal{O}(q^{a-\varepsilon})$ for any $\varepsilon > 0$. This answers questions raised in [4] and [6], and a question that is implicit in [2]. The type of sets displayed here exhibits « combinatorial » and « analytic » properties that one would expect « fractional » cartesian products (sums) of sets to possess, and hence the title of the paper. This class of sets naturally arises in the study of multidimensional extensions of Grothendieck's inequality ([1]); it is that study that led to the present work.

1. Definitions and main results.

We employ basic notation and facts of commutative harmonic analysis as presented and followed in [10]. Γ will be a countable discrete abelian group and $G = \hat{\Gamma}$ will denote its compact dual group. Throughout, group operations in Γ and G will be designated by multiplicative notation.

We now define the type of sets that is the object of the present study. Let $J \geqslant K > 0$ be arbitrary integers, and

$$\mathcal{F} = \{1, \ldots, J\}.$$

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For the sake of typographical convenience here and throughout the paper we let $N = \binom{J}{K}$. Let

$$\{S_1, \ldots, S_N\}$$

be the collection of all K-subsets of $\mathscr{F}$ (sets containing K elements of $\mathscr{F}$), where each $S_\alpha \subset \mathscr{F}$ is enumerated as

$$S_\alpha = (\alpha_1, \ldots, \alpha_K).$$

Let $\{P_1, \ldots, P_N\}$ be the collection of projections from $(\mathbf{Z}^+)^J$ into $(\mathbf{Z}^+)^K$ defined as follows : For $1 \leqslant \alpha \leqslant N$, and $j = (j_1, \ldots, j_J) \in (\mathbf{Z}^+)^J$

$$P_\alpha(j) = (j_{\alpha_1}, \ldots, j_{\alpha_K}).$$

Next, let $F \subset \Gamma$ and

$$F = \{\gamma_i\}_{i \in (\mathbf{Z}^+)^K}$$

be a K-fold enumeration of F. Finally, define

$$F_{J,K} = \{(\gamma_{P_1(j)}, \ldots, \gamma_{P_N(j)})\}_{j \in (\mathbf{Z}^+)^J} \subset F^N \subset \Gamma^N. \tag{1.1}$$

Throughout this paper, a set that is subscripted by J, K will denote the set defined by (1.1), for some fixed K-fold enumeration of F.

DEFINITION 1.1. — *Let* $M > 0$ *be a fixed integer.* $F = \{\gamma_j\}_{j=1}^{\infty} \subset \Gamma$ *is M-independent if for any* $L, L' > 0$ *the relation*

$$\prod_{j=1}^{L} \gamma_j^{\lambda_j} = \prod_{j=1}^{L'} \gamma_j^{\nu_j},$$

where the λ_j *'s and* ν_j *'s are integers in* $[-M, M]$, *implies that* $L = L'$ *and* $\lambda_j = \nu_j$ *for* $j = 1, \ldots, L$. *If* F *is* M*-independent for every* M, *then* F *is said to be independent.* 1*-independent sets are referred to as dissociate sets.*

DEFINITION 1.2 (1.6.2 and 1.6.3 in [9]). — *Let* $F = \{\gamma_j\}_{j=1}^{\infty} \subset \Gamma$, $s \in \mathbf{Z}^+$ *and* $\gamma \in \Gamma$. *Writing (formally) the Fourier series* $h \sim \left(\sum_{j=1}^{\infty} \gamma_j\right)^s$, *we define*

$$r_s(F, \gamma) = \hat{h}(\gamma).$$

Equivalently, $r_s(F, \gamma)$ is the number of ways to write γ in the form of

$$\gamma = \gamma_{i_1} \cdots \gamma_{i_s}, \tag{1.2}$$

where $\gamma_{i_1}, \ldots, \gamma_{i_s}$ are (not necessarily distinct) s elements in F, and where different permutations on the right hand side of (1.2) are counted as different representations.

For example, it is easy to see that if $F \subset \Gamma$ is independent then, for all $s > 0$, the J-fold cartesian product of F satisfies

$$\sup_{\gamma \in \Gamma^J} r_s(F_{J,1}, \gamma) = (s\,!)^J, \qquad \text{for all} \qquad s \in \mathbf{Z}^+. \tag{1.3}$$

The following is an extension to (1.3) and is evidence that $F_{J,K}$ could be viewed as a J/K-fold cartesian product of F.

THEOREM 1.3. – *Let* $F \subset \Gamma$ *be an independent set. For all* $J \geqslant K > 0$ *and* $s > 0$,

$$16^{-\left(\frac{J}{K}\right)s} s^{(J/K)s} \leqslant \sup_{\gamma \in \Gamma^N} r_s(F_{J,K}, \gamma) \leqslant s^{(J/K)s}. \tag{1.4}$$

We now list the « analytic » results that are based on the above « fractional cartesian products ». For $F \subset \Gamma$, $C_F(G)$ and $L^p_F(G)$, $1 \leqslant p \leqslant \infty$, will be the spaces of functions in $C(G)$ and $L^p(G)$, respectively, whose spectra lie in F. Recall that for $2 < q < \infty$ F is a $\Lambda(q)$ set if there is $A > 0$ so that for all $f \in L^2_F(G)$

$$A\|f\|_2 \geqslant \|f\|_q. \tag{1.5}$$

The « smallest » A for which (1.5) holds is the $\Lambda(q)$ constant of F and is denoted by $A(q,F)$. Of particular interest are sets $F \subset \Gamma$ for which $A(q,F)$ is $\mathcal{O}(q^a)$ for some $a \geqslant 1/2$. In fact, this growth condition can be neatly understood as follows : $A(q,F)$ is $\mathcal{O}(q^a)$ if and only if every $f \in L^2_F(G)$ also satisfies

$$\int_G \exp(\lambda |f|^{1/a}) < \infty, \qquad \text{for all} \qquad \lambda > 0$$

(see Remarque on p. 350 of [2]).

DEFINITION 1.4. – *Let* $\beta \in [1,\infty)$. $F \subset \Gamma$ *is a* Λ^β *set if* $A(q,F)$ *is* $\mathcal{O}(q^{\beta/2})$. F *is said to be exactly* Λ^β *when* F *is* Λ^a *if and only if* $a \in [\beta,\infty)$. F *is said to be exactly non-*Λ^β *when* F *is* Λ^a *if and only if* $a \in (\beta,\infty)$.

DEFINITION 1.5. – *Let* $p \in [1,2)$. $F \subset \Gamma$ *is a p-Sidon set if there is* $D > 0$ *so that for all* $f \in C_F(G)$

$$D\|f\|_\infty \geqslant \|\hat{f}\|_p. \tag{1.8}$$

The « smallest » D for which (1.8) holds is the p-Sidon constant of F and is denoted by $D(p,F)$. F is *exactly p-Sidon* when F is r-Sidon if and only if $r \in [p,2)$. F is exactly non-p-Sidon when F is r-Sidon if and only if $r \in (p,2)$.

J-fold cartesian products of dissociate sets are the classical examples of sets that are exactly Λ^J. These and other similar constructions of sets which are exactly Λ^J are studied extensively in [2]. The same J-fold products are also the simplest examples of sets which are exactly $2J/(J+1)$-Sidon ([7], [4] and [6]). The gaps left open between the J and $(J+1)$-fold products of dissociate sets are filled by the « fractional cartesian products » that were defined at the outset.

THEOREM 1.6. — *Let* $F \subset \Gamma$ *be an independent set.*

a) $F_{J,K} \subset \Gamma^N$ *is exactly* $\Lambda^{J/K}$. *Moreover, there is* $\eta_{J,K} > 0$ *so that for all* $q > 2$

$$\eta_{J,K} q^{J/2K} \leqslant A(q, F_{J,K}) \leqslant q^{J/2K}. \tag{1.9}$$

b) $F_{J,K}$ *is exactly* $2J/(K+J)$-*Sidon. Moreover,*

$$D\left(\frac{2J}{(K+J)}, F_{J,K}\right) \leqslant 2^{J/K}. \tag{1.10}$$

$\left(\textit{Recall that } N = \binom{J}{K}.\right)$

Constructions analogous to (1.1) can be carried out within Γ by replacing the cartesian product operation with the group operation in Γ. Given (an infinite set) $F \subset \Gamma$, let $\mathscr{P}$ be a N-partition of F :

$$\mathscr{P} = \{F_1, \ldots, F_N\}$$

(that is, $F_1, \ldots, F_N \subset \Gamma$ are mutually disjoint sets whose union is F), where each F_α is infinite. For each $1 \leqslant \alpha \leqslant N$, endow F_α with a K-fold enumeration

$$F_\alpha = \{\gamma_i^{(\alpha)}\}_{i \in (Z^+)^K}.$$

Define

$$F_{\mathscr{P}}^{J,K} = \{\gamma_{P_1(j)}^{(1)} \ldots \gamma_{P_N(j)}^{(N)}\}_{j \in (Z^+)^J} \subseteq \Gamma. \tag{1.11}$$

In this work, a set that is superscripted by J, K and subscripted by $\mathscr{P}$ will be the set defined in (1.11), where each member of $\mathscr{P}$ is understood to have a K-fold enumeration. When $\mathscr{P}$ is fixed and understood from the context, we write $F^{J,K}$ for $F_{\mathscr{P}}^{J,K}$. Observe that, for $F \subset \Gamma$, letting

$$F_\alpha = (1, \ldots, 1, F, 1, \ldots) \subset \Gamma^N$$
$$\uparrow$$
$$\alpha\text{-th coordinate}$$

and

$$\bar{F} = \bigcup_{\alpha=1}^{N} F_\alpha,$$

we have

$$F_{J,K} = \bar{F}^{J,K}_{\mathscr{P}} \subset \Gamma^N$$

where

$$\mathscr{P} = \{F_1,\ldots,F_N\}.$$

COROLLARY 1.7. — *Let* $F \subset \Gamma$ *be a dissociate set and* $\mathscr{P}$ *be any* N-*partition of* F.

a) $F^{J,K}_{\mathscr{P}} \subset \Gamma$ *is exactly* $\Lambda^{J/K}$. *Moreover, there are constants* $\eta_{J,K}$, $\zeta_{J,K} > 0$ *so that for all* $q > 2$

$$\eta_{J,K} q^{J/2K} \leqslant A(q,F^{J,K}) \leqslant \zeta_{J,K} q^{J/2K}. \tag{1.12}$$

b) $F^{J,K}$ *is exactly* $2J/(K+J)$-*Sidon.*

c) *Suppose that* Γ *contains element with arbitrarily large order. Then, for every* $\beta \in [1,\infty)$ *there are* F_1, $F_2 \subset \Gamma$ *so that* F_1 *is exactly* Λ^β *and* F_2 *is exactly non-*Λ^β.

d) *Let* Γ *be any discrete abelian group. Then, for every* $p \in [1,2)$ *there are* $F_1, F_2 \subset \Gamma$ *so that* F_1 *is exactly p-Sidon and* F_2 *is exactly non-p-Sidon.*

The organization of the paper is as follows. In section 2, we prove the right hand inequality of (1.4) in Theorem 1.3. In section 3, fitted for $F^{J,K}$, appropriate Riesz products are developed for use in later sections. The Λ^β property is treated in section 4 where Theorem 1.6 (*a*) and Corollary 1.7 (*a*), (*c*) are proved. The left hand inequality of (1.9) in Theorem 1.6 is then used to establish the left hand inequality of (1.4) in Theorem 1.3. *p*-Sidonicity is treated in section 5, where the remaining parts of Theorem 1.6 and Corollary 1.7 are proved. We conclude in section 6 with some problems.

2. A combinatorial property of $F_{J,K}$.

Let $F \subset \Gamma$ be an independent set. We prove here the right hand inequality of (1.4) : For all $s \in \mathbf{Z}^+$ and $\gamma \in \Gamma^N$

$$r_s(F_{J,K},\gamma) \leqslant s^{(J/K)s}. \tag{2.1}$$

We shall use an extension of Hölder's inequality which, to facilitate referencing here and in section 5, we state below.

2.1. *M-Hölder's inequality.* – Let X be a measure space, $M > 1$ be an arbitrary integer and $1 < p_1 < \cdots < p_M < \infty$ be so that

$$\sum_{i=1}^{M} \frac{1}{p_i} = 1 .$$

Then, for any $f_1, \ldots, f_M$, measurable functions on X,

$$\left| \int_X f_1, \ldots, f_M \right| \leqslant \|f_1\|_{p_1}, \ldots, \|f_M\|_{p_M} .$$

For typographical convenience, let

$$N = \binom{J}{K} \quad \text{and} \quad N_1 = \binom{J-1}{K-1} .$$

As usual, $l^{N_1}((\mathbf{Z}^+)^K)$ denotes all functions x on $(\mathbf{Z}^+)^K$ so that

$$\|x\|_{N_1} = \left(\sum_{i_1,\ldots,i_K} |x(i_1,\ldots,i_K)|^{N_1} \right)^{1/N_1} < \infty .$$

Lemma 2.2. – *Let* $x_1, \ldots, x_N \in l^{N_1}((\mathbf{Z}^+)^K)$. *Then,*

$$\left| \sum_{j\in(\mathbf{Z}^+)^J} x_1(P_1(j)) \ldots x_N(P_N(j)) \right| \leqslant \|x_1\|_{N_1}, \ldots, \|x_N\|_{N_1} .$$

Sketch of proof. – The key observation is that each $k \in \{1,\ldots,J\}$ appears in precisely $N_1 = \binom{J-1}{K-1}$ distinct K-subsets of $\{1,\ldots,J\}$. J successive applications of the N-Hölder inequality with $p_1 = \cdots = p_{N_1} = 1/N_1$ yield the desired inequality. □

Remark. – Another form of Lemma 2.2 was used in [1], where it served as a starting point for the study of the so-called « projectively bounded » multilinear forms on a Hilbert space (Lemma 1.2 of [1]).

Proof of (2.1). – First, by virtue of the canonical correspondence between $(\mathbf{Z}^+)^J$ and $F_{J,K}$, for notational simplicity we designate elements of $F_{J,K}$ as follows :

$$(\mathbf{Z}^+)^J \ni j \leftrightarrow \gamma(j) \in F_{J,K} ,$$

where

$$\gamma(j) = (\gamma_{P_1(j)}, \ldots, \gamma_{P_N(j)}).$$

Let $\gamma(j_1), \ldots, \gamma(j_s)$ be arbitrary elements of $F_{J,K}$ and write

$$\gamma = \gamma(j_1) \ \ldots \ \gamma(j_s), \tag{2.2}$$

that is,

$$\gamma = \left(\prod_{k=1}^{s} \gamma_{P_1(j_k)}, \ldots, \prod_{k=1}^{s} \gamma_{P_N(j_k)}\right) \tag{2.2'}$$

(Recall that $j_1, \ldots, j_s$ need not be distinct.)

Let

$$\begin{aligned} L_1 &= \{P_1(j_1), \ldots, P_1(j_s)\} \subset (\mathbf{Z}^+)^K \\ &\vdots \\ L_N &= \{P_N(j_1), \ldots, P_N(j_s)\} \subset (\mathbf{Z}^+)^K. \end{aligned}$$

Next, define

$$V = \{j \in (\mathbf{Z}^+)^J : P_\alpha(j) \in L_\alpha \text{ for } all \ 1 \leqslant \alpha \leqslant N\}.$$

By the independence of F, the only way that γ can be obtained as a product of s elements from $F_{J,K}$ is for these elements to have in their $1^{st}, \ldots, N^{th}$ coordinates the members of F that appear in the $1^{st}, \ldots, N^{th}$ coordinates of (2.2′), respectively. That is, if $j'_1, \ldots, j'_s \in (\mathbf{Z}^+)^J$ are so that

$$\gamma = \gamma(j'_1), \ldots, \gamma(j'_s),$$

then $j'_1, \ldots, j'_s \in V$. Therefore, the game plan is to estimate $|V|$, the « volume » of V, and exploit the fact that

$$r_s(F_{J,K}, \gamma) \leqslant |V|^s. \tag{2.3}$$

Let $\chi_1, \ldots, \chi_N$ be the characteristic functions of $L_1, \ldots, L_N$ in $(\mathbf{Z}^+)^K$. Clearly,

$$|V| = \sum_{j \in (\mathbf{Z}^+)^J} \chi_1(P_1(j)) \ \ldots \ \chi_N(P_N(j)).$$

By Lemma 2.2,

$$\begin{aligned} |V| &\leqslant \|\chi_1\|_{N_1} \ \cdots \ \|\chi_N\|_{N_1} \\ &\leqslant s^{N/N_1} = s^{J/K}. \end{aligned} \tag{2.4}$$

Combining (2.4) and (2.3), we obtain (2.1). □

3. Riesz products for $F_{J,K}$.

Let $F \subset \Gamma$ be a dissociate set. Let $F_{\mathscr{P}}^{J,K} = F^{J,K}$ be defined by (1.11) where $\mathscr{P}$ is an arbitrary N-partition of F (as usual, our convention is that $N = \binom{J}{K}$). For any $\gamma \in \Gamma$ and $\theta \in \mathbf{R}$, define

$$\cos(\gamma + \theta) = (e^{i\theta}\gamma + e^{-i\theta}\bar{\gamma})/2.$$

Next, let $\varphi_1, \ldots, \varphi_N \in l^\infty((\mathbf{Z}^+)^K)$ be so that

$$\|\varphi_1\|_\infty, \ldots, \|\varphi_N\|_\infty \leqslant 1,$$

and write, for $1 \leqslant \alpha \leqslant N$,

$$\varphi_\alpha(k) = |\varphi_\alpha(k)| e^{i\theta_\alpha(k)}, \ k \in (\mathbf{Z}^+)^K.$$

We now consider the following Riesz product :

$$(3.1) \quad \mu \sim \left[\prod_{i \in (\mathbf{Z}^+)^K} (1 + |\varphi_1(i)| \cos(\gamma_i^{(1)} + \theta_1(i))\right] \ldots \left[\prod_{i \in (\mathbf{Z}^+)^K} (1 + |\varphi_N(i)| \cos(\gamma_i^{(N)} + \theta_N(i))\right].$$

As usual, $\|\mu\| = 1$ and the spectral analysis of μ yields the following.

LEMMA 3.1. – *Let* $\varphi_1, \ldots, \varphi_N \in l^\infty((\mathbf{Z}^+)^K)$, $\|\varphi_1\|_\infty, \ldots, \|\varphi_N\|_\infty \leqslant 1$. *Then, there is* $\mu \in M(G)$ *so that*

$$(3.2) \qquad \|\mu\| \leqslant 2^N,$$

and

$$(3.3) \qquad \hat{\mu}(\gamma_{P_1(j)}^{(1)} \ldots \gamma_{P_N(j)}^{(N)}) = \varphi_1(P_1(j)) \ldots \varphi_N(P_N(j)),$$

for all $j \in (\mathbf{Z}^+)^J$.

If a higher degree of independence is assumed for $F \subset \Gamma$, then the norm estimate in (3.2) can be correspondingly improved. We illustrate this in $\Gamma = \mathbf{Z}$. Let $M > 1$ and $F = \bigcup_{\alpha=1}^{N} F_\alpha \subset Z$ be an M-independent set where

$$F_\alpha = \{\lambda_k^{(\alpha)}\}_{k \in (\mathbf{Z}^+)^K}$$

(as usual, the F_α's are infinite and mutually disjoint). Let $K_M \in L^1(T)$ be the M^{th} Fejer kernel :

$$K_M(t) = \sum_{l=-M}^{M} \left(1 - \frac{|l|}{M+1}\right) e^{ilt}.$$

Let $\varphi_1, \ldots, \varphi_N$ be unimodular functions in $l^\infty((\mathbf{Z}^+)^K)$ given by

$$\varphi_\alpha(k) = e^{i\theta_\alpha(k)} \tag{3.4}$$

for $k \in (\mathbf{Z}^+)^K$. Replacing (3.1) by

$$\mu \sim \left[\prod_{i \in (\mathbf{Z}^+)^K} K_M(\lambda_i^{(1)} t + \theta_1(i))\right] \ldots \left[\prod_{i \in (\mathbf{Z}^+)^K} K_M(\lambda_i^{(N)} t + \theta_N(i))\right],$$

we obtain

LEMMA 3.2. — *Let* $\varphi_1, \ldots, \varphi_N \in l^\infty((\mathbf{Z}^+)^K)$ *be given by* (3.4). *Then, there is* $\mu \in M(T)$ *so that*

$$\|\mu\| \leqslant (1 + 1/M)^N \tag{3.5}$$

and

$$\hat{\mu}(\lambda_{P_1(j)}^{(1)} + \cdots + \lambda_{P_N(j)}^{(N)}) = \varphi_1(P_1(j)) \ \ldots \ \varphi_N(P_N(j)), \tag{3.6}$$

for all $j \in (\mathbf{Z}^+)^J$.

DEFINITION 3.3. — $F_1, F_2 \subset \Gamma$ *are said to be harmonically separated if there is* $\mu \in M(G)$ *so that*

$$\hat{\mu} = \begin{cases} 1 \text{ on } F_1 \\ 0 \text{ on } F_2 \backslash F_1. \end{cases}$$

LEMMA 3.4. — *Let* $J > K > 0$, $J' > K' > 0$, *and* $N = \binom{J}{K}$, $N' = \binom{J'}{K'}$. *Let* $F \subset \Gamma$ *be dissociate, and* $\mathscr{P}$, $\mathscr{P}'$ *be* N,N'*-partitions of* F, *respectively. Then* $F_{\mathscr{P}}^{J,K}$ *and* $F_{\mathscr{P}'}^{J',K'}$ *are harmonically separated.*

Proof. — If $N \neq N'$, then an application (whose details are left to the reader) of a Riesz product such as the one given by (3.1) yields the desired conclusion. Assume that $N = N'$, and let

$$\mathscr{P} = \{F_1, \ldots, F_N\}$$

be an N-partition for F, where, as usual, F_α is enumerated

$$F_\alpha = \{\gamma_i^{(\alpha)}\}_{i\in(\mathbf{Z}^+)^K}.$$

Let $k = \binom{J-1}{K-1}$. $\otimes \mathbf{Z}_k = \Omega_{(k)}$ will denote the compact direct product of $\mathbf{Z}_k$, and $\oplus\, \mathbf{Z}_k = \hat{\Omega}_{(k)}$ will be its (discrete) dual group, the direct sum of $\mathbf{Z}_k$. $E^{(k)} = \{r_n^{(k)}\}_{n=1}^{\infty}$ will be the system of k-Rademacher functions realized as characters in $\hat{\Omega}_{(k)}$; The n^{th} k-Rademacher function $r_n^{(k)}$ is defined by

$$r_n^{(k)}(\omega) = \exp(2\pi i\omega(n)/k),$$

for all

$$\omega \in \Omega_{(k)} = \{(\omega(j))_{j=1}^{\infty} : \omega(j) \in \{0, 1, \ldots, k-1\}\}.$$

Observe that $E^{(k)}$ is $(k-1)$-independent and that for all $r \in E^{(k)}$, $r^m = 1$ iff $m \equiv 0 \pmod{k}$. In the sequel below, k is fixed and, for the sake of simplicity, will be omitted from all superscripts and subscripts. As usual, for $\gamma \in \Gamma$ and $\theta \in \mathbf{R}$, write

$$\cos(\gamma + \theta) = (e^{i\theta}\gamma + e^{-i\theta}\bar{\gamma})/2.$$

For each $\omega = (\omega_1,\ldots,\omega_J) \in \Omega^J$ define the Riesz product

$$\mu_\omega \sim \left\{\prod_{i\in(\mathbf{Z}^+)^K}\left[1 + \cos\left(\gamma_i^{(1)} + 2\pi\sum_{m=0}^{K}\omega_{1_m}(i(m))\right)\right]\right\}\cdots$$
$$\cdots\left\{\prod_{i\in(\mathbf{Z}^+)^K}\left[1 + \cos\left(\gamma_i^{(N)} + 2\pi\sum_{m=1}^{K}\omega_{N_m}(i(m))\right)\right]\right\},$$

where $i = (i(1), \ldots, i(K)) \in (\mathbf{Z}^+)^K$, and (see Section 1)

$$S_\alpha = (\alpha_1,\ldots,\alpha_K) \subset \{1,\ldots,J\},$$

for $1 \leqslant \alpha \leqslant N$. Next, we integrate over Ω^J the M(G)-valued function whose value at $\omega \in \Omega^J$ is μ_ω:

$$\mu = \int_{\Omega^J} \mu_\omega \, d\omega \in M(G). \tag{3.4.1}$$

The spectral analysis of μ yields the following :

$$\hat{\mu}(\gamma_{i_1}^{(1)} \cdots \gamma_{i_N}^{(N)}) = 2^{-N}\int_{\Omega^J}\left[\prod_{m=1}^{K} r_{i_1(m)}(\omega_{1_m})\right]\cdots$$
$$\cdots\left[\prod_{m=1}^{K} r_{i_N(m)}(\omega_{N_m})\right] d\omega_1 \ldots d\omega_J, \tag{3.4.2}$$

for all $i_1, \ldots, i_N \in (\mathbf{Z}^+)^K$.

Since every $l \in \{1,\ldots,J\}$ appears in precisely $k = \binom{J-1}{K-1}$ S_α's, and

$$\int_\Omega r^m = \begin{cases} 1 & \text{if } m \equiv 0 \pmod{k} \\ 0 & \text{otherwise} \end{cases}, \qquad r \in E^{(k)}$$

we obtain from (3.4.2) that

$$\hat{\mu}(\gamma^{(1)}_{i_1} \ldots \gamma^{(N)}_{i_N}) = \begin{cases} 2^{-N} & \text{if there is } j \in (\mathbf{Z}^+)^J \text{ so that} \\ & P_\alpha(j) = i_\alpha \text{ for all } 1 \leqslant \alpha \leqslant N; \\ 0 & \text{otherwise}. \end{cases}$$

□

4. The Λ^β property.

Proof of Theorem 1.6, *part* (*a*). – The right hand inequality in (1.9) follows immediately from (2.1) via the following.

LEMMA 4.1. (Théorème 3 in [2]). – *Let* s *be a positive integer. Then, for* $F \subset \Gamma$,

$$A(2s,F) \leqslant [\sup_{\gamma \in \Gamma} r_s(F,\gamma)]^{1/2s}.$$

We now prove that there is $\eta_{J,K} > 0$ so that

$$\eta_{J,K} q^{J/2K} \leqslant A(q,F_{J,K}),$$

where $F \subset \Gamma$ is independent (the idea for the argument that follows originates – as far as we can determine – in [5]). Let $n > 0$ be arbitrary, and denote

$$V_n = \{j = (j_1,\ldots,j_J) \in (\mathbf{Z}^+)^J : 1 \leqslant j_1,\ldots,j_J \leqslant n\}.$$

Let g be the trigonometric polynomial defined by

$$g = \sum_{j \in V_n} (\gamma_{P_1(j)}, \ldots, \gamma_{P_N(j)})$$

$\left(\text{as always, } N = \binom{J}{K}\right)$.

Clearly,

$$\|g\|_2 = n^{J/2}. \tag{4.1}$$

Next, let h be the Riesz product defined by

$$h(t_1,\ldots,t_N) = \left[\prod_{i\in U_n}(1+\cos\gamma_i(t_1))\right]\cdots\left[\prod_{i\in U_n}(1+\cos\gamma_i(t_N))\right],$$

where $\quad U_n = \{i = (i_1,\ldots,i_K) \in (\mathbf{Z}^+)^K : 1 \leqslant i_1, \ldots, i_K \leqslant n\}$,

$\cos\gamma = (\gamma+\bar{\gamma})/2$, and $t_1, \ldots, t_N \in G$. We observe that

$$\|h\|_1 = 1, \tag{4.2}$$

and

$$\|h\|_2 \leqslant \|h\|_\infty \leqslant 2^{Nn^K}. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain for any $1 < R < 2$

$$\|h\|_R \leqslant 2^{Nn^K/R'} \tag{4.4}$$

$(1/R + 1/R' = 1)$.

Also, note that

$$\hat{h} = 1/2^N \qquad \text{on} \qquad \{(\gamma_{P_1(j)}, \ldots, \gamma_{P_N(j)})\}_{j\in V_n}. \tag{4.5}$$

Letting $R' = n^K$ and applying (4.4) and (4.5), we deduce

$$\begin{aligned} 2^{-N}n^J = g * h(0) &\leqslant \|g\|_{n^K}\|h\|_{(n^K)'}, \\ &\leqslant \|g\|_{n^K}\cdot 2^N. \end{aligned}$$

Therefore (from (4.1)),

$$2^{-2N}\|g\|_2 n^{J/2} \leqslant \|g\|_{n^K}. \tag{4.6}$$

n was arbitrary, and the left hand inequality of (1.9) follows. □

Completion of the proof of Theorem 1.3. — The left hand inequality in (1.4) follows from Lemma 4.1 and (4.6). □

Exploiting (1.9) of Theorem 1.6 via an « averaging » procedure, we now prove part (*a*) of Corollary 1.7 :

Let $F \subset \Gamma$ be a dissociate set, and $F_{\mathscr{P}}^{J,K} = F^{J,K}$ be defined by (1.11). Let $f \in L^2_{F^{J,K}}(G)$ be given by

$$f \sim \sum_{j\in(\mathbf{Z}^+)^J} a_j \gamma^{(1)}_{P_1(j)} \cdots \gamma^{(N)}_{P_N(j)}.$$

We select $E = \{e_i\}_{i\in(\mathbf{Z}^+)^K}$, an infinite independent set in some Γ_0 $(= \hat{G}_0)$, and « randomize » the Fourier coefficients of f as follows. Let $t = (t_1,\ldots,t_N) \in G_0^N$, and

$$f_t \sim \sum_{j \in (\mathbf{Z}^+)^J} a_j e_{P_1(j)}(t_1) \ \ldots \ e_{P_N(j)}(t_N)\gamma^{(1)}_{P_1(j)} \ \ldots \ \gamma^{(N)}_{P_N(j)}.$$

For each $t = (t_1,\ldots,t_N) \in G_0^N$, let $\mu_t \in M(G)$ be so that

$$\hat{\mu}(\gamma^{(1)}_{P_1(j)} \cdots \gamma^{(N)}_{P_N(j)}) = \overline{e_{P_1(j)}(t_1) \ \ldots \ e_{P_N(j)}(t_N)},$$

and

$$\|\mu_t\| \leqslant 2^N \tag{4.7}$$

(Lemma 3.1).

Observe that

$$f = f_t * \mu_t.$$

Therefore, for all $t \in G_0^N$ and $q > 2$,

$$\|f\|_q^q \leqslant \|f_t\|_q^q \|\mu_t\|^q \leqslant 2^{Nq}\|f_t\|_q^q. \tag{4.8}$$

It follows from (4.8) that

$$\|f\|_q^q \leqslant 2^{Nq} \int_{G_0^N} \left(\int_G |f_t|^q \right) dt.$$

Interchanging the order of integration, and applying the right hand inequality of (1.9), we deduce that

$$\begin{aligned} \|f\|_q &\leqslant 2^N A(q,E_{J,K})\|f\|_2 \\ &\leqslant 2^N q^{J/2K}\|f\|_2. \end{aligned} \tag{4.9}$$

(4.9) proves the right hand inequality in (1.12). To show the left hand inequality, we follow a computation identical to the one used in showing the left hand inequality of (1.9). The proof of 1.7 (*a*) is complete. □

We assume now that Γ is such that for every $M \geqslant 1$, Γ contains an infinite M-independent set. In fact, for concreteness' sake, assume $\Gamma = \mathbf{Z}$.

LEMMA 4.3. – *Let* $J > K > 0$ *be arbitrary. There exists* $F \subset \mathbf{Z}$ *so that for any* $\mathcal{P}$, *an* N-*partition of* F, $\eta_{J,K} q^{J/2K} \leqslant A(q,F_{\mathcal{P}}^{J,K}) \leqslant 4q^{J/2K}$, *for some* $\eta_{J,K} > 0$ *and all* $q > 2$.

Proof. – Let $M > 0$ be so that

$$[(M+1)/M]^N \leqslant 4$$

$\left(N = \binom{J}{K}\right)$, and let $F \subset \mathbf{Z}$ be an M-independent set. Run through the argument leading to (4.9), where an application of Lemma 3.2 replaces that of 3.1. The result is an improved estimate (over (4.9)) :

$$A(q, F_{\mathscr{P}}^{J,K}) \leqslant 4q^{J/2K}. \qquad \square$$

Proof of Corollary 1.7, *part* (*c*). – Again, without loss of generality, we shall work in $\mathbf{Z}$. Let $\beta \in [1,\infty)$ be arbitrary, and $J_n > K_n > 0$, $n = 1, \ldots,$ be so that $(J_n/K_n)_{n=1}^{\infty}$ is a monotonically increasing sequence converging to β. Predictably, the strategy is to let $E_n \subset \mathbf{Z}$ be the « fractional » sum corresponding to $J_n > K_n > 0$ given by Lemma 4.3, and then select finite sets $F_n \subset E_n$ with the following properties :

(i) $$A\left(q, \bigcup_{k=1}^{\infty} l_k F_k\right) \leqslant 8q^{\beta/2},$$

where $l_{k+1} \gg l_k > 0$ are chosen appropriately

$$(l_n F_n = \{l_n \lambda_j^{(n)} : \lambda_j^{(n)} \in F_n\});$$

(ii) the F_n's are sufficiently « thick » so that $\bigcup_{k=1}^{\infty} l_k F_k$ is not $\Lambda^{\beta-\varepsilon}$ for any $\varepsilon > 0$.

We start with

$$E_n = \{\lambda_{P_1(j)}^{(1,n)} + \cdots + \lambda_{P_{N_n}(j)}^{(N_n,n)}\}_{j \in (\mathbf{Z}^+)^{J_n}},$$

a « fractional » sum corresponding to $J_n > K_n > 0$ as in Lemma 4.3 $\left(N_n = \binom{J_n}{K_n}\right)$. Next, fix a positive sequence $(\varepsilon_n)_{n=1}^{\infty}$ that is monotonically converging to 0. For each $n > 0$ select $L_n \in \mathbf{Z}^+$ so that

$$L_n^{\varepsilon_n K_n} \geqslant n 16^{N_n}. \tag{4.10}$$

Let

$$F_n = \{\lambda_{P_1(j)}^{(1,n)} + \cdots + \lambda_{P_{N_n}(j)}^{(N_n,n)}\}_{j \in V_{L_n}} \subset E_n,$$

where

$$V_{L_n} = \{j = (j_1, \ldots, j_{J_n}) \in (\mathbf{Z}^+)^{J_n} : 1 \leqslant j_\alpha \leqslant L_n\}.$$

Observe that by running through an argument identical to the one that led to (4.6), we obtain

$$g_n \in L_{F_n}^2(T)$$

so that

$$\|g_n\|_{(L_n)^{K_n}} \geqslant 4^{-N_n}(L_n)^{J_n/2}\|g_n\|_2 . \tag{4.11}$$

We now determine by induction that « dilation » factors of the F_n's (l_n in (i) above) : Let $l_1 = 1$, and suppose that $l_1, \ldots, l_k$ were determined so that

$$A\left(q, \bigcup_{n=1}^{k} l_n F_n\right) \leqslant 8q^{J_k/2K_k}, \tag{4.12}$$

for all $2 < q < \infty$. Observe that the cardinality of $\bigcup_{n=1}^{k} l_n F_n \cup F_{k+1}$ is $\sum_{n=1}^{k+1} (L_n)^{J_n}$, and, therefore, however we choose l_{k+1}, we will have

$$A\left(q, \bigcup_{n=1}^{k+1} l_n F_n\right) \leqslant \left[\sum_{n=1}^{k+1} (L_n)^{J_n}\right]^{1/2}, \tag{4.13}$$

for all $2 < q < \infty$. Guided by (4.13), we choose M_k so that

$$\left[\sum_{n=1}^{k+1} (L_n)^{J_n}\right]^{1/2} \leqslant 8M_k^{J_{k+1}/2K_{k+1}}. \tag{4.14}$$

Finally, select l_{k+1} so that

$$(\lambda_1 \pm \cdots \pm \lambda_r) + (\nu_1 \pm \cdots \pm \nu_{r'}) \neq 0 \tag{4.15}$$

for all $\{\lambda_i\}_{i=1}^{r} \subset \bigcup_{n=1}^{k} l_n F_n$ and $\{\nu_j\}_{j=1}^{r'} \subset l_{k+1}F_{k+1}$, where $r, r' \leqslant 2M_k$ (we allow repetitions in $\{\lambda_i\}_{i=1}^{r}$ and $\{\nu_j\}_{j=1}^{r'}$).

Claim. $A\left(q, \bigcup_{n=1}^{k+1} l_n F_n\right) \leqslant 8q^{J_{k+1}/2K_{k+1}}$, for all $2 < q < \infty$.

Let f be a trigonometric polynomial with spectrum in $\bigcup_{n=1}^{k+1} l_n F_n$ and write $f = f_1 + f_2$ where

$$\text{spectrum}(f_1) \subseteq \bigcup_{n=1}^{k} l_n F_n,$$

$$\text{spectrum}(f_2) \subseteq l_{k+1} F_{k+1}.$$

It follows from (4.15) that

$$\|f_1 + f_2\|_{2m}^{2m} = \|\,|f_1|^2 + |f_2|^2\|_m^m, \tag{4.16}$$

for all $m \leqslant M_k/2$. Therefore, it follows from (4.16) that

$$\|f_1 + f_2\|_{2m}^2 \leqslant \|f_1\|_{2m}^2 + \|f_2\|_{2m}^2 .$$

Applying the induction hypothesis, and the monotonicity of $(J_n/K_n)_{n=1}^{\infty}$, we obtain

$$(4.17) \qquad \|f_1 + f_2\|_{2m} \leqslant 8(2m^{J_{k+1}/2K_{k+1}})\|f_1 + f_2\|_2 .$$

Combining (4.14) and (4.17), we obtain the claim. Combining the « claim », (4.11) and (4.10), we obtain that $\bigcup_{n=1}^{\infty} l_n F_n$ is exactly Λ^{β}.

By choosing $J_n > K_n > 0$, where $(J_n/K_n)_{n=1}^{\infty}$ is a monotonically decreasing sequence converging to β, and following a procedure similar to the above, we obtain a set in $\mathbf{Z}$ which is exactly non-Λ^{β}. The details are left to the reader. □

5. The p-Sidonicity property.

Proof of Theorem 1.6, *part* (*b*). — First, for the sake of economy in notation, we adopt the following conventions : Let

$$U = \{i_1,\ldots,i_M\} \subset \{1,\ldots,J\} = F, \quad \text{and} \quad P_U : (\mathbf{Z}^+)^J \to (\mathbf{Z}^+)^M$$

be the projection defined by

$$P_U((j_1,\ldots,j_J)) = (j_{i_1},\ldots,j_{i_M}).$$

$\sum_U$ will denote summation over $P_U((\mathbf{Z}^+)^J$. For example,

$$\sum_U a_{j_1,\ldots,j_J} = \sum_{j_{i_1},\ldots,j_{i_M}}^{\infty} a_{j_1,\ldots,j_J}.$$

Also, in what follows γ_{S_α} will mean $\gamma_{P_\alpha(i)}$ whenever these occur in summands. For example,

$$\sum_{j=(j_1,\ldots,j_J)\in(\mathbf{Z}^+)^J} \gamma_{P_1(j)} \cdots \gamma_{P_N(j)} \equiv \sum_{j_1,\ldots,j_J} \gamma_{S_1} \cdots \gamma_{S_N}.$$

Let $E \subset \Gamma$ be an independent set, and $E_{J,K} \subset \Gamma^N$ be as in (1.1) (as usual, $N = \binom{J}{K}$).

LEMMA 5.1. – *Let f be a trigonometric polynomial in* $C_{E_{J,K}}(G^N)$ *given by*

$$f = \sum_{j_1,\dots,j_J} a_{j_1,\dots,j_J}(\gamma_{S_1},\dots,\gamma_{S_N}).$$

Then,

$$2^{J/K}\|f\|_\infty \geqslant \sum_{S_\alpha}\left(\sum_{\sim S_\alpha} |a_{j_1,\dots,j_J}|^2\right)^{1/2}$$

for all $\alpha = 1, \dots, N$.

Proof. – Let $1 \leqslant \alpha \leqslant N$ be arbitrary. Since E is a 1-Sidon set with Sidon constant $= 1$, it is easy to see that by a proper choice of $t_\alpha \in G$ we obtain

$$\text{(5.1.1)}\quad \|f\|_\infty \geqslant \sup_{t_1,\dots,t_{\alpha-1},t_{\alpha+1},\dots,t_N} \sum_{S_\alpha}\left|\sum_{\sim S_\alpha} a_{j_1,\dots,j_J}\gamma_{S_1}(t_1)\dots\gamma_{S_N}(t_N)\right|.$$

Since the sup-norm dominates the L^1-norm, it follows from (5.1.1) that

$$\|f\|_\infty \geqslant \sum_{S_\alpha}\int_{G^{N-1}}\left|\sum_{\sim S_\alpha} a_{j_1,\dots,j_J}(\gamma_{S_1},\dots,\gamma_{S_{\alpha-1}},\gamma_{S_{\alpha+1}},\dots,\gamma_{S_N})\right|$$

(we make the obvious modification for $\alpha = 1$ and $\alpha = N$). Since $E_{J,K}$ is a $\Lambda(2)$ set whose $\Lambda(2)$ constant is bounded by $2^{J/K}$ (this follows from part (*a*) of Theorem 1.6), we obtain

$$2^{J/K}\|f\|_\infty \geqslant \sum_{S_\alpha}\left(\sum_{\sim S_\alpha}|a_{j_1,\dots,j_J}|^2\right)^{1/2}. \qquad \square$$

In a previous version of this manuscript, by following a multidimensional version of Littlewood's rearrangement argument (see p. 168 of [7] and Lemma 3 of [6]) we proved that

$$\text{(5.1)}\quad \sum_{\alpha=1}^{N}\sum_{S_\alpha}\left(\sum_{\sim S_\alpha}|a_{j_1,\dots,j_J}|^2\right)^{1/2} \geqslant \frac{1}{K!}\left(\sum_{j_1,\dots,j_J}|a_{j_1,\dots,j_J}|^{\frac{2J}{(K+J)}}\right)^{\frac{(K+J)}{2J}}.$$

Then, combining Lemma 5.1 and (5.1), we deduce that $E_{J,K}$ is a $2J/(K+J)$-Sidon and that

$$\text{(5.2)}\qquad D\left(\frac{2J}{(K+J)}, E_{J,K}\right) \leqslant 2^{J/K}\binom{J}{K}K!.$$

The estimate in (5.2), however, is not as tight as we would like it to be. In order to obtain the existence of sets that are exactly p-Sidon, for *any* $p \in (1,2)$, we require the sharper Lemma 5.3 below. We are grateful to S. Kaijser at Uppsala University for pointing out to us that Littlewood's classical inequality (the case $J = 2$ and $K = 1$ in (5.1)) can be proved without Littlewood's rearrangement argument. Indeed, this is a key observation in the demonstration below. In the course of the proof of 5.3 we greatly benefited also from stimulating conversations with M. Benedicks at the Institut Mittag-Leffler.

We require two basic inequalities : Minkowski's inequality which, to facilitate referencing, we state below, and the M-Hölder inequality that is given in section 2 ((2.1)).

(5.2) *Minkowski's inequality.* – Let X and Y be measure spaces and g be a measurable function on $X \times Y$. For any $1 < r < \infty$,

$$\left(\int_X \left(\int_Y |g(x,y)|\right)^r\right)^{1/r} \leqslant \int_Y \left(\int_X |g(x,y)|^r\right)^{1/r}. \tag{5.3}$$

LEMMA 5.3. – *Let* $J > K > 0$.

$$\left(\sum_{j_1,\ldots,j_J} |a_{j_1,\ldots,j_J}|^{\frac{2J}{(K+J)}}\right)^{\frac{(K+J)}{2J}} \leqslant \prod_{\alpha=1}^{N}\left[\sum_{S_\alpha}\left(\sum_{\sim S_\alpha} |a_{j_1,\ldots,j_J}|^2\right)^{1/2}\right]^{1/N} \tag{5.4}$$

$$\left(N \equiv \binom{J}{K}\right).$$

Proof. – We prove (5.4) by induction on J. We start with the case $J = 2$ and $K = 1$. Write

$$\sum_{i,j} |a_{ij}|^{4/3} = \sum_{i,j} |a_{ij}|^{2/3}\, |a_{ij}|^{2/3}. \tag{5.3.1}$$

By applying 2-Hölder's inequality to the sum over j with $p_1 = 3/2$ and $p_2 = 3$ for the first and second factors, respectively, in the summand of the right side of (5.3.1), we obtain

$$\sum_{i,j} |a_{ij}|^{4/3} \leqslant \sum_i \left(\sum_j |a_{ij}|\right)^{2/3} \left(\sum_j |a_{ij}|^2\right)^{1/3}. \tag{5.3.2}$$

Next, applying 2-Hölder's inequality to the sum over i with p_1 and p_2 for the second and first factors, respectively, in the summand of the right side of

(5.3.2.), we obtain

$$(5.3.3)\qquad \sum_{i,j}|a_{ij}|^{4/3} \leqslant \left[\sum_i\left(\sum_j |a_{ij}|\right)^2\right]^{1/3}\left[\sum_i\left(\sum_j |a_{ij}|^2\right)^{1/2}\right]^{2/3}.$$

From Minkowski's inequality, we have

$$\left(\sum_i\left(\sum_j |a_{ij}|\right)^2\right)^{1/2} \leqslant \sum_j\left(\sum_i |a_{ij}|^2\right)^{1/2},$$

and, therefore (from (5.3.3)), we deduce

$$(5.3.4)\qquad \left(\sum_{i,j}|a_{ij}|^{4/3}\right)^{3/4} \leqslant \left[\sum_i\left(\sum_i |a_{ij}|^2\right)^{1/2}\right]^{1/2}\left[\sum_i\left(\sum_j |a_{ij}|^2\right)^{1/2}\right]^{1/2}.$$

(5.3.4) starts the inductive proof of (5.4), and we now assume that $J > 2$ and that (1.4) holds for all $J_1 < J$. We write

$$(5.3.5)\qquad \sum_{j_1,\ldots,j_J} |a_{j_1,\ldots,j_J}|^{\frac{2J}{K+J}} = \sum_{j_1,\ldots,j_J} |a_{j_1,\ldots,j_J}|^{\frac{2}{K+J}}\ldots|a_{j_1,\ldots,j_J}|^{\frac{2}{K+J}}.$$

For notational reasons, we shall write

$$f_n(j_1,\ldots,j_J) = a_{j_1,\ldots,j_J}$$

for the n-th factor in the summand of the right hand side of (5.3.5) (even though $f_1 \equiv \cdots \equiv f_J$), and whenever there is no confusion we shall write merely f_n :

$$(5.3.5')\qquad \sum_{j_1,\ldots,j_J} |a_{j_1,\ldots,j_J}|^{\frac{2J}{K+J}} = \sum_{j_1,\ldots,j_J} |f_1|^{\frac{2}{(K+J)}}\ldots|f_J|^{\frac{2}{(K+J)}}.$$

Now, apply J-Hölder's inequality to the sum over j_J with

$$p_1 \equiv p = (K+J)/2$$

for f_1 and $p_2 = \cdots = p_J \equiv q = \dfrac{(J-1)(K+J)}{(K+J-2)}$ for $f_2, \ldots, f_J$ and obtain that the right side of (5.3.5′) is majorized by

$$(5.3.6)\qquad \sum_{j_1,\ldots,j_{J-1}}\left(\sum_{j_J}|f_1|\right)^{\frac{2}{(K+J)}}\prod_{n=2}^{J}\left(\sum_{j_J}|f_n|^{\frac{2(J-1)}{(K+J-2)}}\right)^{\frac{K+J-2}{(J-1)(K+J)}}.$$

Next, apply J-Hölder's inequality to the sum over j_{J-1} with p for the factor containing f_2 and q for the remaining factors in the summand of (5.3.6), and obtain that (5.3.6) is majorized by

(5.3.7)

$$\sum_{j_1,\ldots,J-2}\left\{\left[\sum_{j_{J-1}}\left(\sum_{j_J}|f_1|\right)^{\frac{2(J-1)}{(K+J-2)}}\right]^{\frac{(K+J-2)}{(J-1)(K+J)}}\cdot\left[\sum_{j_{J-1}}\left(\sum_{j_J}|f_2|^{\frac{2(J-1)}{(K+J-2)}}\right)^{\frac{(K+J-2)}{2(J-1)}}\right]^{\frac{2}{(K+J)}}\right.$$
$$\left.\cdot\prod_{n=3}^{J}\left[\sum_{j_{J-1},j_J}|f_n|^{\frac{2(J-1)}{(K+J-2)}}\right]^{\frac{(K+J-2)}{(J-1)(K+J)}}\right\}.$$

We continue in this fashion : At then n-th step, $n > 2$, we apply the J-Hölder inequality to the sum over j_{J-n+1} with p for the factor that contains f_n and with q for the remaining factors. After J such operations we obtain that (5.3.5′) is majorized by

(5.3.8)

$$\left[\sum_{j_1,\ldots,j_{J-1}}\left(\sum_{j_J}|f_1|\right)^{\frac{2(J-1)}{(K+J-2)}}\right]^{\frac{(K+J-2)}{(J-1)(K+1)}}\cdots$$
$$\left[\sum_{j_1,\ldots,j_{J-n}}\left(\sum_{j_{J-n+1}}\left(\sum_{j_{J-n+2},\ldots,j_J}|f_n|^{\frac{2(J-1)}{(K+J-2)}}\right)^{\frac{(K+J-2)}{2(J-1)}}\right)^{\frac{2(J-1)}{(K+J-2)}}\right]^{\frac{(K+J-2)}{(J-1)(K+1)}}\cdots$$
$$\cdots\left[\sum_{j_1}\left(\sum_{j_2,\ldots,j_J}|f_J|^{\frac{2(J-1)}{(K+J-2)}}\right)^{\frac{(K+J-2)}{2(J-1)}}\right]^{\frac{2}{(K+J)}}.$$

Now, apply Minkowki's inequality to each of the first $(J-1)$ factors in (5.3.8) as follows : To the first factor, apply Minkowski's inequality (as stated above) with $|g| = |f_1|$,

$$\int_X = \sum_{j_1,\ldots,j_{J-1}}, \qquad \int_Y = \sum_{j_J}, \qquad \text{and} \qquad r = \frac{2(J-1)}{(K+J-2)}\cdot$$

To the n-th factor, $1 < n < J$, apply Minkowski's inequality with

$$|g| = \sum_{j_{J-n+2},\ldots,j_J}|f_n|^{\frac{2(J-1)}{(K+J-2)}},$$
$$\int_X = \sum_{j_1,\ldots,j_{J-n}}, \qquad \int_Y = \sum_{j_{J-n+1}},$$

and again with $r = \dfrac{2(J-1)}{(K+J-2)}\cdot$

We therefore obtain

$$(5.3.9)\quad \left(\sum_{j_1,\ldots,j_J} |a_{j_1,\ldots,j_J}|^{\frac{2J}{(K+J)}}\right)^{\frac{(K+J)}{2J}} \leqslant \prod_{n=1}^{J}\left(\sum_{j_n}\left(\sum_{\sim j_n} |a_{j_1,\ldots,j_J}|^{\frac{2(J-1)}{(K+J-2)}}\right)^{\frac{(K+J-2)}{2(J-1)}}\right)^{1/J}$$

$\left(\sum_{\sim j_n}\right.$ denotes summation with respect to indices in $\left.\{j_1,\ldots,j_J\}\backslash\{j_n\}\right)$. If $K = 1$, (5.3.9) reduces to (5.4) and the lemma is proved. We now assume $K > 1$ and apply the induction hypothesis that (5.4) holds for $J-1$ and $K-1$ for each of the J factors on the right hand side of (5.3.9). Namely, for each $1 \leqslant n \leqslant J$ we have

$$(5.3.10)\quad \sum_{j_n}\left(\sum_{\sim j_n} |a_{j_1,\ldots,j_J}|^{\frac{2(J-1)}{(K+J-2)}}\right)^{\frac{(K+J-2)}{2(J-1)}} \leqslant \sum_{j_n}\prod_{\alpha=1}^{N_1}\left(\sum_{S_\alpha^n}\left(\sum_{\sim S_\alpha^n} |a_{j_1,\ldots,j_J}|^2\right)^{1/2}\right)^{1/N_1}$$

[Recall that $N_1 \equiv \binom{J-1}{K-1}$. $\{S_1^n,\ldots S_{N_1}^n\}$ denotes the collection of all $(K-1)$-subsets of $\{1,\ldots,J\}\backslash\{n\}$; $\sum_{S_\alpha^n}$ denotes summation with respect to the indices $j_{\alpha_1},\ldots,j_{\alpha_{K-1}}$, where $S_\alpha^n = (\alpha_1,\ldots,\alpha_{K-1})$; $\sum_{\sim S_\alpha^n}$ denotes summation with respect to the remaining indices (except j_n).] We now apply N_1-Hölder inequality, on the right hand side of (5.3.10), to the sum over j_n with $p_1 = \cdots = p_{N_1} = 1/N_1$ for each of the $N_1 \equiv \binom{J-1}{K-1}$ factors in the summand. Combining this application with (5.3.9), we obtain that the left hand side of (5.3.9) is majorized by

$$(5.3.11)\quad \prod_{n=1}^{J}\prod_{\alpha=1}^{N_1}\left[\sum_{j_n}\sum_{S_\alpha^n}\left(\sum_{\sim S_\alpha^n} |a_{j_1,\ldots,j_J}|^2\right)^{1/2}\right]^{\frac{1}{J\cdot N_1}}.$$

Finally, observe that

$$(5.3.12)\quad \{\{1\}\cup S_1^1,\{1\}\cup S_2^1,\ldots,\{1\}\cup S_{N_1}^1,\ldots,\{n\}\cup S_1^n,\ldots,\{n\}\cup S_{N_1},\ldots,$$
$$\{J\}\cup S_1^J,\ldots,\{J\}\cup S_{N_1}^J\} = \{S_1,\ldots,S_N\}.$$

Also, notice that each set in the enumeration of the collection on the left hand side of (5.3.12) occurs precisely K times. Therefore, the expression in (5.3.11) equals

$$\prod_{\alpha=1}^{N_1 \cdot J/K} \left(\sum_{S_\alpha} \left(\sum_{\sim S_\alpha} |a_{j_1,\ldots,j_J}|^2 \right)^{1/2} \right)^{\frac{K}{N_1 \cdot J}} = \prod_{\alpha=1}^{N} \left(\sum_{S_\alpha} \left(\sum_{\sim S_\alpha} |a_{j_1,\ldots,j_J}|^2 \right)^{1/2} \right)^{1/N},$$

and the proof of the Lemma is complete. □

Combining Lemmas 5.1 and 5.3, we deduce that

$$D(2J/(K+J), E_{J,K}) \leqslant 2^{J/K}.$$

We now proceed to show that

$$D([2J/(K+J)] - \varepsilon, E_{J,K}) = \infty$$

for all $\varepsilon > 0$. The argument that follows is similar to the one used in showing that $E_{J,K}$ is not $\Lambda^{J/K-\varepsilon}$ for any $\varepsilon > 0$ (and is an adaptation of a proof used in 2.7 of [4]). We use the fact that if $F \subset \Gamma$ is p-Sidon, then there is $B > 0$ so that for all $f \in L^1_F(G)$

(5.5) $$\|f\|_b \leqslant B\sqrt{b}\,\|\hat{f}\|_a$$

for all $1 < b < \infty$, where

$$a = 2p/(3p-2)$$

(see (9) in [3]). Let $n > 0$ be arbitrary, and let

$$V_n = \{j = (j_1,\ldots,j_J) \in (\mathbf{Z}^+)^J : 1 \leqslant j_1,\ldots,j_J \leqslant n\}.$$

Let

$$g = \sum_{j \in V_n} (\gamma_{P_1(j)},\ldots,\gamma_{P_N(j)}).$$

Clearly,

(5.6) $$\|\hat{g}\|_a = n^{J/a}.$$

Next, let $U_n = \{i = (i_1,\ldots,i_K) \in (\mathbf{Z}^+)^K : 1 \leqslant i_1,\ldots,i_K \leqslant n\}$ and define

$$h(t_1,\ldots,t_N) = \left[\prod_{i \in U_n} (1 + \cos \gamma_i(t_1))\right] \cdots \left[\prod_{i \in U_n} (1 + \cos \gamma_i(t_N))\right].$$

As in section 4, we conclude (see (4.2)-(4.4)) that for any $1 < R < 2$

$$\|h\|_R \leqslant 2^{\frac{Nn^K}{R'}},$$

and

$$\hat{h} = 1/2^N \quad \text{on} \quad \{(\gamma_{P_1(j)},\ldots,\gamma_{P_N(j)})\}_{j \in V_n}.$$

Therefore, as in section 4,

$$(5.7)\qquad 2^{-N}n^{J} = |g * h(0)| \leqslant \|g\|_{n^{K}}\|h\|_{(n^{K})}, \leqslant \|g\|_{n^{K}}\cdot 2^{N}.$$

If $E_{J,K}$ is p-Sidon, it follows from (5.5), (5.6) and (5.7) that

$$(5.8)\qquad 2^{-2N}n^{J} \leqslant Bn^{\left(\frac{K}{2}\right)+\left(\frac{J}{a}\right)}.$$

But, (5.8) holds for all $n > 0$ only if $p \geqslant \frac{2J}{(K+J)}\cdot$ The proof of part b) of Theorem 1.6 is complete. □

Proof of Corollary 1.7, *part* (*b*). – Let $F \subset \Gamma$ be dissociate and $F^{J,K}_{\mathscr{P}} = F^{J,K}$ be given by (1.11), for an arbitrary $\mathscr{P}$. Let f be a trigonometric polynomial in $C_{F^{J,K}}(G)$ given by

$$f = \sum_{j\in(\mathbf{Z}^{+})^{J}} a_j\gamma^{(1)}_{P_1(j)} \cdots \gamma^{(N)}_{P_N(j)}.$$

Select $E = \{e_i\}_{i\in(\mathbf{Z}^{+})^{K}}$, an infinite independent set in some $\Gamma_0(=\hat{G}_0)$. Let $t = (t_1,\ldots,t_N) \in G_0^N$ be arbitrary, and $\mu_t \in M(G)$ be so that

$$\hat{\mu}_t(\gamma^{(1)}_{P_1(j)} \cdots \gamma^{(N)}_{P_N(j)}) = e_{P_1(j)}(t_1)\ldots e_{P_N(j)}(t_N)$$

for all $j \in (\mathbf{Z}^{+})^{J}$, and

$$\|\mu_t\| \leqslant 2^{N}$$

(Lemma 3.1).

Therefore,

$$\sup_{t\in G_o^N} \Big|\sum_{j\in(\mathbf{Z}^{+})^{J}} a_j e_{P_1(j)}(t_1)\ldots e_{P_N(j)}(t_N)\Big| = \sup_{t\in G_o^N} |\langle f,\mu_t\rangle| \leqslant 2^{N}\|f\|_{\infty}.$$

By Theorem 1.6, part (b), we have

$$(5.9)\qquad \Bigg(\sum_{j\in(\mathbf{Z}^{+})^{J}} |a_j|^{\frac{2J}{(K+J)}}\Bigg)^{\frac{(K+J)}{2J}} \leqslant 2^{J/K}\cdot 2^{N}\|f\|_{\infty}.$$

This proves that $F^{J,K}$ is $\frac{2J}{(K+J)}$ – Sidon. To show that $F^{J,K}$ is exactly $\frac{2J}{(K+J)}$ – Sidon we follow the same route that was used to prove the corresponding fact for $E_{J,K}$. □

COROLLARY 5.4. — *Let* $J \geqslant K > 0$ *and* $J' \geqslant K' > 0$ *be arbitrary. Let* $F \subset \Gamma$ *be dissociate and* $\mathscr{P}$, $\mathscr{P}'$ *be* $\binom{J}{K}$, $\binom{J'}{K'}$ — *partitions of* F *respectively. Then* $F^{J,K}_{\mathscr{P}} \cup F^{J',K'}_{\mathscr{P}'}$ *is exactly a p-Sidon set, where*

$$p = \max\left\{\frac{2J}{(K+J)}, \frac{2J'}{(K'+J')}\right\}.$$

Proof. — Apply Lemma 3.4. □

Proof of Corollary 1.7, *part* (*d*). — We consider two cases.

I. Γ *contains elements with arbitrarily large order.* As usual, we shall assume that $\Gamma = \mathbf{Z}$. First, observe that for any $J > K > 0$ the (M-independent) set $F \subset \mathbf{Z}$ given in Lemma 4.3 has the property

$$D(2J/(K+J), F^{J,K}) \leqslant 16 \cdot 2^{J/K}. \tag{5.10}$$

(5.10) is achieved by applying, en route to (5.9), Lemmas 3.2 and 4.3 in place of Lemma 3.1 and Theorem 1.6 (*b*), respectively, whence 2^N is replaced by 4 and $2^{J/K}$ by $4 \cdot 2^{J/K}$ in (5.9). Let $p \in [1, \infty)$, and $J_n > K_n > 0$ be so that $(J_n/K_n)_{n=1}^{\infty}$ is a monotonically increasing sequence converging to $p/(2\text{-}p)$. For each $n > 0$, let $F_n \subset \mathbf{Z}$ be the set given in Lemma 4.3 corresponding to $J_n > K_n > 0$. Select finite sets $E_n \subset F_n$ so that

$$D([2J_n/(J_n+K_n)] - 1/n, E_n) > n. \tag{5.11}$$

Next, determine $d_n \in \mathbf{Z}^+$ with the following property : Whenever f is a trigonometric polynomial with spectrum in $\bigcup_{n=1}^{\infty} d_n E_n$, then

$$6\,\|f\|_{\infty} \geqslant \sum_{n=1}^{\infty} \|f_n\|_{\infty}, \tag{5.12}$$

where $f = \sum_{n=1}^{\infty} f_n$ and spectrum$(f_n) \subset d_n E_n$. Combining (5.10), (5.11), and (5.12), we conclude that $\bigcup_{n=1}^{\infty} d_n E_n$ is exactly p-Sidon.

To obtain a set in $\mathbf{Z}$ which is exactly non p-Sidon, we choose $J_n > K_n > 0$ so that $(J_n/K_n)_{n=1}^{\infty}$ is a monotonically decreasing sequence to $p/(2\text{-}p)$, and carry out a construction similar to the one above. Details are left to the reader.

II. Γ *contains* $\oplus\, \mathbf{Z}_a$ *for some* $a \in \mathbf{Z}^+$. Clearly, we may assume that for any given $J > K > 0$, there is a dissociate set

$$E = \{\gamma_i\}_{i \in (\mathbf{Z}^+)^K} \subset \oplus\, \mathbf{Z}_a$$

so that

$$(5.13) \quad E_{J,K} = \{(\gamma_{P_1(j)}, \ldots, \gamma_{P_N(j)})\}_{j \in (\mathbf{Z}^+)^J} \subset \oplus\, \mathbf{Z}_a = (\oplus \mathbf{Z}_a)^N .$$

SUBLEMMA. — *Let* $E_{J,K} \subset \oplus\, \mathbf{Z}_a$ *be given by* (5.13). *Then,*

$$A(q, E_{J,K}) \leqslant (Bq)^{[J/K]/2},$$

where $[J/K]$ = *least integer equal to or greater than* J/K, *and* B *is the* 1-*Sidon constant of* E.

Proof. — Let $g \in L^2_{E_{J,K}}(\otimes \mathbf{Z}_a)$, $\|g\|_2 = 1$, be given by

$$g \sim \sum_{j \in (\mathbf{Z}^+)^J} a_j (\gamma_{P_1(j)}, \ldots, \gamma_{P_N}(j)) .$$

Observe that there are $[J/K]$ subsets of $\{1, \ldots, J\}$, $S_1, \ldots, S_{[J/K]}$, so that $\bigcup_{n=1}^{[J/K]} S_n = \{1, \ldots, J\}$. Therefore, we may estimate the q^{th}-norm of g as follows :

$$\|g\|_q^q = \int_{(\omega_1, \ldots, \omega_N) \in (\otimes \mathbf{Z}_2)^N} \Big| \sum_{j \in (\mathbf{Z}^+)^J} a_j \gamma_{P_1(j)}(\omega_1) \ldots \gamma_{P(j)}(\omega_N) \Big|^q \, d\omega_1 \ldots d\omega_N$$

$$= \int_{(\otimes \mathbf{Z}_a)^{N-[J/K]}} \int_{(\otimes \mathbf{Z}_a)^{[J/K]}} \Big| \sum_j a_j \gamma_{P_1(j)} \cdots \gamma_{P_N(j)} \Big|^q \leqslant \int_{(\otimes \mathbf{Z}_a)^{N-[J/K]}} (Bq)^{[J/K]q/2}$$

$$\leqslant (Bq)^{[J/K]q/2}$$

(the appearance of B above is explained by (2) of 5.7.7 in [10]). □

To prove Corollary 1.7 (*d*), in the present context, we follow a route identical to the one followed in Case I where the use of (5.10) is replaced by a use of the Sublemma. □

6. Problems.

The « J/K-fractional » product of a dissociate set $E \subset \Gamma$ was defined in this work as a subset of the $\binom{J}{K}$-fold product of E. Subsequently, through the use of Riesz products, $\binom{J}{K}$ appeared in some of the estimates. The first four problems focus on replacing $\binom{J}{K}$ by J/K in the various computations.

Problem 6.1. — Improve the lower bound on $\sup_{\gamma \in \Gamma^N} r_s(F_{J,K}, \gamma)$ in (1.4) of Theorem 1.3. In particular, is there $C > 0$ so that for all $s \in \mathbf{Z}^+$

$$C^{-J/K} s^{(J/K)s} \leqslant \sup_{\gamma \in \Gamma^N} r_s(F_{J,K}, \gamma),$$

where $F \subset \Gamma$ is independent ?

Problem 6.2. — Improve the norm estimate in (3.2) of Lemma 3.1. In particular, is there $\mu \in M(G)$ fulfilling (3.3) and so that

$$\|\mu\| \leqslant C^{J/K},$$

for some (universal) $C > 0$?

Problem 6.3. — Given any rational $\beta \in [1, \infty]$, we constructed in this paper $E \subset \Gamma$ for which there are η_β, $\zeta_\beta > 0$ so that for all $q > 2$

$$\eta_\beta q^{\beta/2} \leqslant A(q, E) \leqslant \zeta_\beta q^{\beta/2}. \tag{6.4.1}$$

While (6.4.1) is a stronger statement than « E is exactly Λ^β », in Corollary 1.7 (*c*) we deduced for *all* $\beta \in [1, \infty)$ no more than the existence of sets that are exactly Λ^β. Given any $\beta \in [1, \infty)$, can we find sets (say in $\mathbf{Z}$) for which (6.4.1) holds ?

Problem 6.4. — Prove Corollary 1.7 (*c*) for $\Gamma = \oplus \mathbf{Z}_a$, $a \in \mathbf{Z}^+$.

Problem 6.5. — A classical theorem due to Rudin ([10]) states that every 1-Sidon set is Λ^1. The converse was recently established by Pisier ([9]). Could it be that $E \subset \Gamma$ is a Λ^β set if and only if E is $2\beta/(1+\beta)$-Sidon ?

Problem 6.6. — In Section 5, an essentially probabilistic method was employed to show that $E_{J,K}$ is not $(2J/(K+J)-\varepsilon)$-Sidon for any $\varepsilon > 0$. In the case $J = 2$ and $K = 1$, an explicit construction of trigonometric polynomials (bounded bilinear forms) displaying this fact is given on p. 172 of [8]. Replace the probabilistic procedure in Section 5 by a constructive one.

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