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STRONG LAWS OF LARGE NUMBERS IN CERTAIN LINEAR SPACES

by **Wojbor A. Woyczyński.**

1. Introduction.

In the present paper we are concerned with the norm almost sure convergence of series of random vectors taking values in some linear metric spaces and strong laws of large numbers (SLLN) for sequences of such random vectors. All linear metric spaces appearing below are assumed to be complete, separable and real, and without any loss of generality we shall assume that the metric is generated by a (nonhomogeneous in general) F-norm (cf. [10]). Even in the case of sums of real random variables the research is still vigorously going on and the best account of up-to-date investigations we can recommend to the reader is V. V. Petrov's book [7] which also features the very complete bibliography. To be sure there is no such reference in the general case we are going to deal with. The exposition of some topics in the Banach space case can be found in J.-P. Kahane's book [6] and much is done in recent mimeographed notes by J. Hoffmann-Jørgensen [5] (cf. also references in [11]). We begin with a section on Banach-space-valued random vectors where we shall also give a survey of our earlier results. The second section is devoted to random vectors taking values in certain nonnecessarily locally convex spaces and the third one treats identically distributed random vectors. Some of our results seem to be new even when restricted to the real valued random variables.

2. Random vectors in Banach spaces.

In 1947 Kai-Lai Chung has proven that if $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ is a nondecreasing function such that $\varphi(t)/t$ and $t^2/\varphi(t)$ are nondecreasing and if $X_1, X_2, \ldots$ is a sequence of independent real random variables with $EX_n = 0$ and $\Sigma(E\varphi|X_n|/\varphi(n)) < \infty$ then the series $\Sigma(X_n/n)$ converges a.s. Now, Kronecker's Lemma yields the convergence of $(X_1 + \cdots + X_n)/n$ to zero a.s. The Kolmogorov's SLLN $(\varphi(t) = t^2)$ and « classical » SLLN saying that if $X_1, X_2, \ldots$ are independent, $EX_n = 0$, and $EX_n^2 \leqslant M < \infty$ then $(X_1 + \cdots + X_n)/n \to 0$ a.s. were obvious corollaries to Chung's theorem. It turned out that the analogues of the above mentioned SLLNs no longer hold in all Banach spaces. Roughly speaking this is due to the fact that the three series theorem is not available in this case. The validity of these SLLNs depends on some geometric properties of the unit sphere of the Banach space in question, as will be shown below.

Now, let $\mathscr{X}$ be a Banach space with the unit sphere denoted by $S_{\mathscr{X}}$ and the dual $\mathscr{X}^*$. A strongly measurable mapping X from a probability space $\langle \Omega, \mathscr{F}, P \rangle$ into $\mathscr{X}$ is said to be a random $\mathscr{X}$-variable. If $E\|X\| < \infty$ then the expectation EX is well defined by the Pettis or Bochner integral.

A. Beck [1] has given a complete characterization of all those Banach spaces in which the « classical » SLLN holds i.e. in which $(X_1 + \cdots + X_n)/n \to 0$ a.s. in norm for every sequence $X_1, X_2, \ldots$ of independent random $\mathscr{X}$-variables with $EX_n = 0$ and $E\|X_n\|^2 \leqslant M < \infty$. His theorem says that they are exactly the Banach spaces in which there are $k > 0, \varepsilon > 0$ such that for all k-tuples $a_1, \ldots, a_k \in S_{\mathscr{X}}$, $\| \pm a_1 \pm \cdots \pm a_k\| < k(1 - \varepsilon)$ for some combination of the $+$ and $-$ signs. The class of Banach spaces described above is now known as B — convex Banach spaces and includes, for instance, all uniformly convex spaces, and, in particular, L_p spaces with $1 < p < \infty$. As we mentioned above the Chung's theorem does not carry over even to all B — convex Banach spaces and in [11] we were able to show in every $\mathscr{X} = l^p, 1 < p < \alpha \leqslant 2$, a sequence

$X_1, X_2, \ldots$ of independent random l^p-variables such that $EX_n = 0$, $\Sigma(E\|X_n\|^\alpha/n^\alpha) < \infty$ and still $(X_1 + \cdots + X_n)/n$ diverges a.s. Recently A. Beck and D. P. Giesy showed in [2] that for an arbitrary Banach space $\mathfrak{X}$ the convergence of $\Sigma(E\|X_n\|^2/n^2)$ guarantees SLLN for a sequence $X_1, X_2, \ldots$ of independent random $\mathfrak{X}$-variables if additionally $(\sqrt{E\|X_1\|^2} + \cdots + \sqrt{E\|X_n\|^2})/n \to 0$ and that this result is strongest possible in the sense that if the mentioned above restrictions on $E\|X_n\|^2$ are weakened the resulting statement is no longer true for all Banach spaces.

In [11] we have shown, among others, that an analogue of Chung's SLLN is valid in Banach spaces satisfying the following $\mathcal{G}_\alpha$-condition.

DEFINITION. — *The Banach space $\mathfrak{X}$ is said to satisfy the condition $\mathcal{G}_\alpha$ for some $0 < \alpha \leqslant 1$ if there exists a map $G : \mathfrak{X} \to \mathfrak{X}^*$ such that $(\mathcal{G}_\alpha^{\mathrm{I}})\|G(x)\| = \|x\|^\alpha$, $(\mathcal{G}_\alpha^{\mathrm{II}})G(x)x = \|x\|^{1+\alpha}$ and $(\mathcal{G}_\alpha^{\mathrm{III}})\|G(x) - G(y)\| \leqslant A\|x - y\|^\alpha$ for all $x, y \in \mathfrak{X}$ and some positive* A.

The condition $\mathcal{G}_1$ was introduced by R. Fortet and E. Mourier in [4]. The importance of $\mathcal{G}_\alpha$ stems from the inequality

$$(1) \qquad E\|X_1 + \cdots + X_n\|^{1+\alpha} \leqslant A \sum_{j=1}^{n} E\|X_j\|^{1+\alpha}$$

which holds true for each integer n and for each n-tuple of independent random $\mathfrak{X}$-variables $X_1, \ldots, X_n$ with $\mathfrak{X} \in \mathcal{G}_\alpha$, $EX_j = 0$, and $E\|X_j\|^{1+\alpha} < \infty$, $j = 1, \ldots, n$ (cf. [11]). It is not difficult to see that if the norm of $\mathfrak{X}$ is Gateaux differentiable and $g : S_{\mathfrak{X}} \to S_{\mathfrak{X}^*}$ — the gradient of the norm is Lipschitzian with exponent α then $G(x) \overset{df}{=} \|x\|^\alpha g(x/\|x\|)$ satisfies $(\mathcal{G}_\alpha^{\mathrm{I}}) - (\mathcal{G}_\alpha^{\mathrm{III}})$. So L_p, l_p, $p \geqslant 2$ are in all $\mathcal{G}_\alpha$, $0 < \alpha \leqslant 1$, and $L^{1+\alpha}$, $l^{1+\alpha} \in \mathcal{G}_\beta$, $0 < \beta \leqslant \alpha \leqslant 1$ but $L^{1+\alpha}$, $l^{1+\alpha} \notin \mathcal{G}_\beta$ if $\beta > \alpha$ [11]. It would be interesting to check what other familiar Banach spaces are in $\mathcal{G}_\alpha$ (e.g. spaces of functions with finite p-variation, Orlicz spaces, spaces Λ_α of Lipschitz continuous functions and so on).

THEOREM 1 [11]. — *Let $X_1, X_2, \ldots,$ be a sequence of independent random $\mathfrak{X}$-variables with $EX_n = 0$, $\mathfrak{X}$ being a Banach space satisfying the condition $\mathcal{G}_\alpha$ for some $0 < \alpha \leqslant 1$.*

Then the convergence of $\Sigma E\varphi_0\|X_n\|$, *where* $\varphi_0(t) \overset{df}{=} \min(t, t^{1+\alpha})$, $t \geqslant 0$, *implies the strong a.s. convergence of* ΣX_n.

THEOREM 2 [11]. — *Let* $X_1, X_2, \ldots,$ *be a sequence of independent random* $\mathscr{X}$*-variables with* $EX_n = 0$, $\mathscr{X}$ *being a Banach space satisfying the condition* $\mathscr{G}_\alpha$ *for some* $0 < \alpha \leqslant 1$. *If* $\varphi_n : \mathbf{R}^+ \to \mathbf{R}^+$, $n = 1, 2, \ldots,$ *are continuous and such that* $\varphi_n(t)/t$ *and* $t^{1+\alpha}/\varphi_n(t)$ *are nondecreasing then for each sequence* $t_1, t_2, \ldots$ *of positive numbers the convergence of* $\Sigma(E\varphi_n\|X_n\|/\varphi_n(t_n))$ *implies the a.s. convergence in norm of* $\Sigma(X_n/t_n)$.

It is also possible to give a rather precise description of the rate of growth of partial sums of independent random vectors which is a corollary to the Theorem 2 and Kronecker's Lemma and, in a sense, is best possible.

THEOREM 3 [11]. — *Let* $X_1, X_2, \ldots,$ *and* $\varphi_n = \varphi$ *be as in Theorem* 2. *If* $E\varphi\|X_n\| < \infty$ *and* $A_n = \sum_{k=1}^{n} E\varphi\|X_k\| \uparrow \infty$ *then*

$$\|X_1 + \cdots + X_n\| = o\,(\varphi^{-1}(A_n\psi(A_n)))$$

a.s. for each function $\psi \in \Psi_c$.

Remind that φ^{-1} denotes the function inverse to φ and Ψ_c is the class of all functions $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ that do not decrease for $t > t_0$ for some $t_0 = t_0(\psi)$ and for which $\Sigma 1/(n\psi(n))$ converges. As a particular case of Theorem 3 and as a corollary to the example mentioned at the beginning of this section we get.

COROLLARY 1 (SLLN). — *If* $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ *is continuous and such that* $\varphi(t)/t$ *and* $t^{1+\alpha}/\varphi(t)$ *are nondecreasing and if* $X_1, X_2, \ldots$ *are independent random* $\mathscr{X}$*-variables,* $\mathscr{X} \in \mathscr{G}_\alpha$, *with* $EX_n = 0$ *then the convergence of* $\Sigma(E\varphi\|X_n\|/\varphi(n))$ *implies that* $(X_1 + \cdots + X_n)/n \to 0$ *a.s. in norm.*

COROLLARY 2. — *In* l^p, $1 < p < \infty$, *the convergence of* $\Sigma E\|X_n\|^{1+\alpha}/n^{1+\alpha}$, $0 < \alpha \leqslant 1$ *implies the SLLN for independent* $X_1, X_2, \ldots$ *with* $EX_n = 0$ *iff* $p \geqslant 1 + \alpha$ *or in other words iff* $l^p \in \mathscr{G}_\alpha$.

The Corollary 2 would suggest the following.

Conjecture: Does the existence in $\mathscr{X}$ of an equivalent (to $\|.\|$) norm satisfying $\mathscr{G}_\alpha$, characterize those Banach spaces $(\mathscr{X}, \|.\|)$ in which the convergence of $\Sigma E\|X_n\|^{1+\alpha}/n^{1+\alpha}$ implies the SLLN for a sequence $X_1, X_2, \ldots$ of independent random $\mathscr{X}$-variables with $EX_n = 0$?

In his recent mimeographed notes [5] J. Hoffmann-Jørgensen introduces the notion of p-type of a Banach space. Namely, a Banach space $\mathscr{X}$ is said to be of type p, $1 \leqslant p \leqslant 2$ iff for each sequence (X_j) in $\mathscr{X}$ with $\Sigma\|X_j\|^p < \infty$, the series $\Sigma\varepsilon_j X_j$, where (ε_j) is the Bernoulli (Rademacher) sequence of real random variables, converges a.s. in norm. Of course, thanks to Theorem 2, we know that if $\mathscr{X} \in \mathscr{G}_\alpha$ then it is of type $1 + \alpha$ and it is an open question whether the fact that $\mathscr{X}$ is of type $1 + \alpha$ implies the condition $\mathscr{G}_\alpha$ for $\mathscr{X}$. This, as we feel, is intimately related to our Conjecture quoted above. To support this statement we cite more results from [5] on Banach spaces of type p. First, it is not difficult to see that $\mathscr{X} \in$ (type p) iff the following implication holds: for each sequence (X_n) of independent random $\mathscr{X}$-variables of the form $X_n = \xi_n x_n$, where $x_n \in \mathscr{X}$ and ξ_n are identically distributed bounded real random variables with $E\xi_n = 0$, the convergence of $\Sigma E\|X_n\|^p$ implies that ΣX_n converges a.s. in norm. Moreover, one can prove that $\mathscr{X}$ is of type p iff $\forall$ (or $\exists$) $0 < r < \infty \ \exists K_r < \infty \ \forall n$

$$\left(E\Big\|\sum_{j=1}^{n} \xi_j x_j\Big\|^r\right)^{1/r} \leqslant K_r\left(\sum_{j=1}^{n} \|x_j\| p\right)^{1/p}$$

where $x_j \in \mathscr{X}$ and $\xi_1, \xi_2, \ldots$ are independent real r.v's with common distribution which is supposed to be symmetric, rapidly decreasing (for definition consult [5]) and nondegenerated. In particular, we see that for bounded nondegenerated and symmetric r.v's ξ_n, and independent $X_n = \xi_n x_n$ we have the implication $\Sigma E\|X_n\|^p < \infty \Longrightarrow \Sigma X_n$ converges a.s. in norm iff $\exists K \ \forall n$

$$E\|X_1 + \cdots + X_n\|^p \leqslant K \sum_{j=1}^{n} E\|X_j\|^p$$

and that inequality is clearly akin to (1).

Thus far we were concerned with the situation where some moments of X_n's were finite. Without these restrictions we are able to prove a theorem which even on the real line seems to be new in this generality (cf. [7], Ch. 9 for some special cases on $\mathbf{R}$).

THEOREM 4. — *Let* $X_1, X_2, \ldots$ *be independent random* $\mathscr{X}$*-variables,* $\mathscr{X} \in \mathscr{G}_\alpha$ *and* φ_n *be convex and such that* $\varphi_n(t)/t$ *and* $t^{1+\alpha}/\varphi_n(t)$ *do not decrease. If* $0 < t_n \uparrow \infty$ *then the convergence of the series*

$$(2) \qquad \sum_{n=1}^{\infty} \mathrm{E}\, \frac{\varphi_n\|X_n\|}{\varphi_n\|X_n\| + \varphi_n(t_n)}$$

implies that

$$(3) \qquad \frac{1}{t_n} \sum_{k=1}^{n} (X_k - \mathrm{E}Z_k) \to 0$$

a.s. in norm, where $Z_n \stackrel{df}{=} X_n \chi[\|X_n\| < t_n]$.

Proof: Because of the inequalities

$$(4) \quad \frac{\mathrm{E}\varphi_n\|Z_n\|}{2\varphi_n(t_n)} + \frac{1}{2}\,\mathrm{P}(\|X_n\| \geqslant t_n) \leqslant \mathrm{E}\, \frac{\varphi_n\|X_n\|}{\varphi_n\|X_n\| + \varphi_n(t_n)},$$

$n = 1, 2, \ldots, \ldots,$ we get that

$$(5) \qquad \sum_{n=1}^{\infty} \mathrm{P}(X_n \neq Z_n) < \infty,$$

and

$$(6) \qquad \sum_{n=1}^{\infty} \frac{\mathrm{E}\varphi_n\|Z_n\|}{\varphi_n(t_n)} < \infty.$$

Let $\varphi_0(t) \stackrel{df}{=} \min(t, t^{1+\alpha}),\ t \geqslant 0$. One can check that

$$(7) \qquad \varphi_0(t+s) \leqslant \mathrm{K}(\varphi_0(t) + \varphi_0(s)), \qquad t, s \geqslant 0,$$

for some constant K which may, of course, depend on α. Indeed, if $t + s \leqslant 1$ then (7) follows from the boundedness of the function $(1+t)^\alpha/(1+t^\alpha)$ on the real half-line, if $t + s \geqslant 1$ and $t, s \leqslant 1$ then (7) follows from the fact that $(t+s)/(t^{1+\alpha} + s^{1+\alpha}) \leqslant 2/(t^{1+\alpha} + (1-t)^{1+\alpha}) < 2$, and if $t + s \geqslant 1$ and (say) $t > 1,\ s < 1$ then (7) follows from the inequality $(t+s)/(t + s^{1+\alpha}) \leqslant (t+1)/t \leqslant 2$.

In view of (7), convexity of φ_n and the fact that

$\varphi_n(t_n t)/\varphi_n(t_n) \geqslant \varphi_0(t)$, $t \geqslant 0$ we get that

$$\mathrm{E}\varphi_0\left\|\frac{Z_n - EZ_n}{t_n}\right\| \leqslant K\left(\mathrm{E}\varphi_0\left(\frac{\|Z_n\|}{t_n}\right) + \varphi_0\left(\frac{E\|Z_n\|}{t_n}\right)\right)$$
$$\leqslant 2K\,\frac{\mathrm{E}\varphi_n\|Z_n\|}{\varphi_n(t_n)}$$

so that (6) implies the convergence of $\Sigma \mathrm{E}\varphi_0\|(Z_n - EZ_n)/t_n\|$. Clearly $Z_n - EZ_n$ are independent, zero mean and Theorem 1 yields that. $\Sigma(Z_n - EZ_n)/t_n$ converges a.s. in norm. Now, Kronecker's Lemma (cf. next section for Kronecker's Lemma in linear spaces) implies that $\sum_{k=1}^{n} (Z_k - EZ_k)/t_n \to 0$ a.s. in norm as $n \to \infty$. However, because of (5) and Borel-Cantelli Lemma $P(Z_k \neq X_k$ infinitely often$) = 0$ so that also $\sum_{k=1}^{n} (X_k - EZ_k)/t_n \to 0$ a.s. in norm. Q.E.D.

3. Random vectors in F-spaces.

We begin with the account of Toeplitz' and Kronecker's Lemmas which as we have seen earlier are essential in deriving laws of large numbers from theorems on the convergence of random series. Besides they are of independent interest and may be useful in possible further research. In the non-locally convex case their formulations involve very serious restrictions. However, fortunately enough, in numerous instances we shall be able to get around them and deduce theorems from theorems dealing with real-valued random variables.

In this section $\mathscr{X}$ is always an F-space i.e. a complete metric linear space equipped with the non-necessarily homogeneous F-norm $\|.\|$. In some places more stringent conditions will have to be imposed on $\mathscr{X}$.

As it can be easily guessed, the Toeplitz' Lemma, which in the simplest instance asserts that the consecutive arithmetic means of a sequence tend to zero whenever the sequence itself tends to zero, does not hold in non-locally convex spaces since if $\mathscr{X}$ is not locally convex then

$$\exists a > 0 \quad \forall n \exists X_1^n, \ldots, X_{k_n}^n \in \mathscr{X}$$

such that $\|X_i^n\| < 1/2^n$ $(i = 1, 2, \ldots, k_n)$ and still

$\|(X_1^n + \cdots + X_{k_n}^n)/k_n\| > a$. Here are two simple-minded counterexamples in two cases of spaces that will be of our interest in the sequel.

Example 1. — Let $\mathfrak{X} = \mathfrak{M}$-space of all Lebesgue measurable real functions on the unit interval and let

$$X_{2^k+l} = 2^k \chi[l2^{-k}, (l+1)2^{-k}]$$

$K = 0, 1, 2, \ldots, l = 0, \ldots, 2^k 1$. Obviously $X_n \to 0$ in the Lebesgue measure and still $(X_1 + \cdots + X_{2^k-1})/(2^k - 1) = 1$ for all $k \geq 1$ so that $(X_1 + \cdots + X_n)n \not\to 0$.

Example 2. — Let $\mathfrak{X} = l^p$, $0 < p < 1$ with the usual F-norm $\|X\| = \Sigma|X^i|^p$, $X = (X^i)$, let $0 < \alpha < (1/p) - 1$ and $X_n = n^{-\alpha}e^{(n)}$ where $e^{(n)}$ is the standard basis in l^p. Of course $\|X_n\| = n^{-p\alpha} \to 0$ as $n \to \infty$ and at the same time

$$\|(X_1 + \cdots + X_n)/n\| = n^{-p} \sum_{k=1}^{n} k^{-p\alpha} > n^{-p}n.n^{-p\alpha} \to \infty.$$

We are having the similar situation as far as Kronecker's Lemma $(\Sigma(X_n/n) < \infty \Rightarrow (X_1 + \cdots + X_n)/n \to 0)$ is concerned because each counter-example to the Toeplitz Lemma yields a counter-example to Kronecker's Lemma as follows: let $\mathfrak{X} \ni y_j \to 0$ and $(y_0 + \cdots + y_{n-1})/n \not\to 0$. Choose X_j so that $y_n = \sum_{j=1}^{n} (X_j/j)$. Clearly $\Sigma(X_j/j)$ is convergent but

$$\begin{aligned}(X_1 + \cdots + X_n)/n &= (1/n) \sum_{j=1}^{n} (y_j - y_{j-1})/j \\ &= y_n - (y_0 + \cdots + y_{n-1})/n \not\to 0.\end{aligned}$$

However, for some non-locally convex spaces we are able to prove (best possible) analogues of Toeplitz' and Kronecker's Lemmas.

Remind that a linear metric space $\mathfrak{X}$ is said to be locally p-convex for some $0 < p \leq 1$ if there is a sequence $(\|.\|_i)$ of p-homogeneous F-pseudonorms (i.e. such that

$$\|tX\|_i = |t|^p\|X\|_i, \quad t \in \mathbf{R}, \quad X \in \mathfrak{X})$$

determining a topology equivalent to the original one. Here are some criteria for local p-convexity:

a) if in $\mathfrak{X}$ there is a basis of neighbourhoods of zero $(\mathfrak{U}_n)$

with the modulus of concavity $c(\mathfrak{U}_n) \leqslant 2^{1/p}$ then $\mathscr{X}$ is locally p-convex. Here the modulus of concavity of a starlike ($t\mathrm{A} \subset \mathrm{A}$, $\forall t \in [0, 1]$) set $\mathrm{A} \subset \mathscr{X}$ is defined by $c(\mathrm{A}) = \inf (s > 0 : \mathrm{A} + \mathrm{A} \subset s\mathrm{A})$;

b) all locally bounded spaces are locally p-convex. Moreover, in this case, for a certain p, $0 < p \leqslant 1$, there is a single p-homogeneous F-norm in $\mathscr{X}$ which is equivalent to the original one. Recall that $\mathscr{X}$ is said to be locally bounded if it contains a bounded neighbourhood of zero and that $\mathrm{A} \subset \mathscr{X}$ is bounded if $\forall$ neighbourhood $\mathscr{U} \in 0$ $\exists a \in \mathbf{R}$ such that $\mathrm{A} \subset a\mathscr{U}$. If $p_0 = \log c(\mathscr{X})/\log 2$, where $c(\mathscr{X}) = \inf (c(\mathscr{U}) : \mathscr{X} \supset \mathscr{U}$, $\mathscr{U}$-open, bounded and balanced) is the modulus of concavity of the space $\mathscr{X}$ then for each p, $0 < p < p_0$, there is p-homogeneous F-norm equivalent to the original one (for proofs of the above stated facts and a variety of concrete examples of locally p-convex spaces consult [10], Ch. III).

TOEPLITZ' LEMMA. — *Let $\mathscr{X}$ be a locally p-convex space, $0 < p \leqslant 1$, and let (t_{nk}) be a double sequence of positive numbers for which*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} = 1, \tag{8}$$

$$\sup_{n} \sum_{k=k_0}^{\infty} t_{nk}^{p} = \mathrm{M} < \infty \tag{8'}$$

for some integer k_0, and

$$\lim_{n\to\infty} t_{kn} = 0, \qquad k = 1, 2, \ldots \tag{9}$$

Further, let $x_1, x_2, \ldots$ be a sequence of elements of $\mathscr{X}$ such that $x_n \to x \in \mathscr{X}$. Then

$$y_n = \sum_{k=1}^{\infty} t_{nk} x_k \to x. \tag{10}$$

Conversely, if (10) *holds for every sequence $x_n \to x$ then for the double sequence (t_{nk})* (8) *and* (9) *hold.*

Proof. — We have the inequality

$$\left\| \sum_{k=1}^{\infty} t_{nk} x_k - x \right\|_i \leqslant \left\| \sum_{k=1}^{\infty} t_{nk}(x_k - x) \right\|_i + \left\| x \sum_{k=1}^{\infty} t_{nk} - x \right\|_i$$

for each F-pseudonorm $\|\,.\,\|_i$. Because $x_k \to x$, $\forall\iota\ \forall\varepsilon > 0$ $\exists k_0\ \forall k \geqslant k_0\ \|x_k - x\|_i < \varepsilon/M$. Hence

$$\|x - \sum_{k=1}^{\infty} t_{nk}x_k\|_i \leqslant \|x \sum_{k=1}^{\infty} t_{nk} - x\|_i + \sum_{k=1}^{k_0} t_{nk}^p \|x_k - x\|_i + \frac{\varepsilon}{M} \sum_{k=k_0+1}^{\infty} t_{nk}^p,$$

and for n large enough we get $\left\|x - \sum_k t_{nk}x_k\right\| \leqslant 3\varepsilon$. Now, we prove the (partial) converse statement. To get (8) apply (10) to $x_n = x \neq 0$ and to get (9) put $x_k = x\,\delta_{kk_0}$, $x \neq 0$. Then $x = \lim_n y_n = \lim_n \left(\sum_k t_{nk} - t_{nk_0}\right)x$ so that

$$\lim_n \left(\sum_k t_{nk} - t_{nk_0}\right) = 1$$

what, in view of (8), gives (9).

Remarks 1. — If $p = 1$ then, of course, $\mathscr{X}$ is locally convex, (8′) is implied by (8) and we get the usual Toeplitz' Lemma as stated in [11].

2. If $\mathscr{X} = l^p$ (or

$$\mathscr{X} = l^{(pi)} = (x = (x^i) : \|x\| = \Sigma|x^i|^{pi} < \infty))$$

then we may also show that, not only (8) and (9) but also (8′) is indispensable in the formulation of Toeplitz' Lemma. Indeed, take $x_k = \beta_k e^{(k)}$. Then for each $\beta_k \to 0$, $x_k \to 0$ so that

$$(11) \qquad \left\|\sum_{k=1}^{\infty} t_{nk}\beta_k e^{(k)}\right\| = \sum_{k=1}^{\infty} t_{nk}^p \beta_k^p \to 0$$

as $n \to \infty$. Clearly then $\sum_k t_{nk}^p < \infty$ for each n. To prove that $\sup_n \sum_k t_{nk}^p = M < \infty$ it is sufficient to make an observation to the effect that $(t_{n1}^p, t_{n2}^p, \ldots) \in l_1$, $n = 1, 2, \ldots$, is a sequence in l^1 which is c_0-weakly convergent, so that it must be strongly bounded in l^1 (cf. [3], IV. 13.4) Q.E.D.

KRONECKER'S LEMMA. — *Let $x_1, x_2, \ldots \in \mathscr{X}$,— where $\mathscr{X}$ is locally p-convex and let $0 < t_k \uparrow \infty$ be such that*

$$(12) \qquad \sup_n \left(\sum_{k=0}^{n-1} (t_{k+1} - t_k)^p\right) \Big/ t_n^p < \infty$$

Then the convergence of $\sum_k (x_k/t_k)$ *implies that*

$$(x_1 + \cdots + x_k)/t_k \to 0.$$

Proof. — Having denoted $y_n = \sum_{k=1}^{n} (x_k/t_k)$ we have that

$$(x_1 + \cdots + x_n)/t_n = \left(\sum_{k=1}^{n} t_k(y_k - y_{k-1})\right)\Big/ t_n$$
$$= y_n - \left(\sum_{k=0}^{n-1} y_k(t_{k+1} - t_k)\right)\Big/ t_n \to 0$$

because of assumptions on t_k and above Toeplitz' Lemma Q.E.D.

Problem. — Let $\Lambda(\mathscr{X})$ be the set of all real sequences (λ_n) such that for any neighbourhood of zero $\mathscr{U} \subset \mathscr{X}$ there is a neighbourhood of zero V such that

$$\lambda_1 V + \lambda_2 V + \cdots \subset \mathscr{U}$$

Does the above Toeplitz' and thus Kronecker's Lemma hold true for all spaces $\mathscr{X}$ with $\Lambda(\mathscr{X}) \supset l^p$, $0 < p \leqslant 1$ (cf. [10], Ch. III)?

We are well aware that the above formulation of the Kronecker's Lemma may make it rather difficult to check its validity for a concrete sequence (t_n).

The direct computation shows that if $p < 1$ then $t_n = n$ is no good. One can also check that for any $\beta > 1$, $t_n = n^\beta$ does not satisfy (12) either, if only $p < 1$. On the other hand $t_n = q^n$, $q > 1$, is just fine for all $0 < p \leqslant 1$ because

$$\frac{1}{q^{np}} \sum_{k=0}^{n-1} (q^{k+1} - q^k)^p = (q-1)^p \frac{q^{pn} - 1}{(q^p - 1)q^{np}}$$

$n = 1, 2, \ldots,$ is a bounded sequence. More generally one can check that if $t_n \uparrow \infty$ is such that $\sum_{k=m}^{\infty} (1/t_k^p) = 0(1/t_m)$ and $t_{m+1}/t_m \leqslant M < \infty$ then also $\sum_{k=1}^{m} t_k^p = 0(t_m^p)$ and (12) is satisfied. Also all lacunary sequences would do here. Indeed

$$\frac{1}{t_n^p} \sum_{k=1}^{n-1} (t_{k+1} - t_k)^p = \frac{1}{t_n^p} \sum_{k=1}^{n-1} \frac{((t_{k+1}/t_k) - 1)^p}{(t_{k+1}/t_k)^p - 1} (t_{k+1}^p - t_k^p) \leqslant 1$$

whenever $t_{k+1}/t_k \geqslant \text{const} > 1$, because the function

$$(t-1)^p/(t^p-1)$$

is bounded on [const, ∞). Because

$$\lim\,(t-1)^p/(t^p-1) = +\,\infty$$

as $t \to 1+$ this also shows that the sequences with $t_{k+1}/t_k \to 1$ are not likely to satisfy (12).

Now, we turn to the investigation of random series in F-spaces. In comparison with the results of the previous section the theorems given below have the advantage of being valid without stochastic independence of summands X_n, and without the assumption that $EX_n = 0$ ($E\|X_n\|$ need not be even finite). It may be in order to mention here the fact that for functions with values in a non-locally convex space one can hardly define correct Lebesgue-type integral because always there are sequences of simple $\mathscr{X}$-valued functions tending to zero uniformly with the integrals that do not converge.

The next theorem could be also easily deduced from its special case when $\mathscr{X} = \mathbf{R}$ if only it existed in literature. In [7], however, we find it with the additional assumption that $X_n's$ are independent and $\mathscr{X} = \mathbf{R}$. So we give here an independent proof of it.

THEOREM 5. — *Let $X_1, X_2, \ldots$ be a sequence of random $\mathscr{X}$-variables, where $\mathscr{X}$ is an F-space. If $\varphi_n : \mathbf{R}^+ \to \mathbf{R}^+$, $n = 1, 2, \ldots, \varphi_n(t) > 0$ for $t > 0$, are continuous, nondecreasing and such that $t/\varphi_n(t)$ are nondecreasing then the convergence of $\Sigma(E\varphi_n\|X_n\|/\varphi_n(1))$ implies the strong (absolute) a.s. convergence of ΣX_n.*

Proof. — Let $\varphi_0(t) \overset{df}{=} \min\,(1, t)$, $t \geqslant 0$. Because

$$\varphi_n(t)/\varphi_n(1) \geqslant \varphi_0(t) \qquad \text{for} \qquad t \geqslant 0$$

it is sufficient to show that the convergence of $\Sigma E\varphi_0\|X_n\|$ implies that $\Sigma\|X_n\| < \infty$ a.s Indeed, for arbitrary integers N, M

$$E\varphi_0 \sum_{n=N}^{M} \|X_n\| \leqslant \sum_{n=N} E\varphi_0\|X_n\|$$

because of subadditivity of φ_0. Now, the Cauchy argument shows that $E\varphi_0 \sum_{n=N}^{M} \|X_n\|$ tends to zero as $N, M \to \infty$. Because $L\varphi_0 \overset{df}{=} (f\colon f\colon \Omega \to \mathbf{R},\ E\varphi_0|f| < \infty)$ with metric

$$d(f, g) = E\varphi_0|f - g|$$

is complete we have that $\Sigma\|X_n\|$ is convergent in L_{φ_0} and so $\Sigma\|X_n\| < \infty$ a.s. by Fatou Lemma. The completeness of $\mathscr{X}$ yields also the a.s. convergence of ΣX_n. Q.E.D.

Remarks 3. — Theorem 5 remains also valid (with the same proof) for complete separable metric group $\mathscr{X}$ and $\|.\| = d(e, X)$ where e is the unit element of $\mathscr{X}$ and d is (say) left invariant metric.

4. It is clear from the proof that as far as restrictions on φ_n are concerned it whould suffice to assume that $\varphi_n \geqslant C_n\varphi_0$ to get similar theorem.

COROLLARY 3. — *Let* $X_1, X_2, \ldots$ *be a sequence of random* $\mathscr{X}$*-variables, where* $\mathscr{X}$ *is a locally bounded space equipped with a p-homogeneous norm* $\|.\|$, $0 < p \leqslant 1$. *If* φ_n *are as in Theorem* 5 *then for each sequence* (t_n) *of positive numbers the convergence of* $\Sigma(E\varphi_n\|X_n\|/\varphi_n(t_n^p))$ *implies the strong (absolute) a.s. convergence of* $\Sigma(X_n/t_n)$.

Proof. — In view of Theorem 5 it is sufficient to show that $\Sigma(E\varphi_n\|t_nX_n\|/\varphi_n(t_n^p)) < \infty$ implies that $\Sigma E\varphi_0\|X_n\| < \infty$ and this is certainly true because $\varphi_n(t_n^pt)/\varphi_n(t_n^p) \geqslant \varphi_0(t)$, $t \in \mathbf{R}^+$.

Theorem 3 and Corollary 3 give us an immediate

COROLLARY 4. — (SLLN). *If* $\mathscr{X}$ *is a Banach space satisfying the condition* $\mathscr{G}_\alpha$ *for some* $0 < \alpha \leqslant 1$ *then for each* β, $0 < \beta \leqslant 1 + \alpha$ *the convergence of* $\Sigma E\|X_n\|^\beta/n^\beta$ *for independent and zero mean (these two restrictions are superfluous if* $\beta \leqslant 1$*) random* $\mathscr{X}$*-variables imply that* $(X_1 + \cdots + X_n)/n \to 0$ *a.s. in norm.*

We may ask whether in non-locally convex spaces the convergence of $\Sigma E\|X_n\|^\alpha$ for some $\alpha > 1$ does imply the a.s. convergence of ΣX_n. However, to this question we are able to supply the negative answer even in the presence of independence and centering assumptions.

Example 3. — Let $\mathscr{X} = l^p$, $0 < p < 1$, and let X_n be a sequence of independent $\mathscr{X}$-valued r.v's taking, with equal probabilities, the values $\pm n^\beta e^{(n)}$. Then

$$\Sigma E\|X_n\|^\alpha = \Sigma n^{\beta\alpha p} < \infty$$

if $\beta < -1/(\alpha p)$. On the other hand $\|\Sigma X_n\| = \Sigma n^{\beta p} = \infty$ if only $\beta \geqslant -1/p$ and one can obviously find β such that $-1/(\alpha p) > \beta \geqslant -1/p$ because $\alpha > 1$. In a like fashion one can get a counter-example to SLLN. Indeed, take again $\mathscr{X} = l^p$, $0 < p < 1$, as before and $X_n's$ as before. We have $\Sigma E\|X_n\|^\alpha/n^{\alpha p} = \Sigma n^{(\beta-1)\alpha p} < \infty$ if $\beta < 1 - 1/(\alpha p)$ and, on the other hand

$$\|(X_1 + \cdots + X_n)/n\| = n^{-\beta} \sum_{i=1}^{n} n^{p\beta} > \frac{n}{2} n^{-p} \left(\frac{n}{2}\right)^{p\beta} \not\to 0$$

if only $\beta \geqslant 1 - 1/p$ and it is certainly possible to find β such that $1 - 1/p \leqslant \beta < 1 - 1/(\alpha p)$ because $\alpha > 1$.

Next we shall give a description of the rate of growth of partial sums of random $\mathscr{X}$-variables. Fortunately, we are able to reduce the problem to real valued random variables and thus escape the use of Kronecker's Lemma as stated in this section. This fact also shows that the theorem is much less deep than Theorem 3 (cf. [8] for the case when $\mathscr{X} = \mathbf{R}$ and $\varphi(t) = t^\alpha$ $0 < \alpha \leqslant 1$).

THEOREM 6. — *Let* $\mathscr{X}$ *and* $\varphi_n = \varphi$ *be as in Theorem* 5, *and let additionally* φ *increases strictly to infinity. If* $E\varphi\|X_n\| < \infty$, $A_n = \sum_{k=1}^{n} E\varphi\|X_k\| \uparrow \infty$ then

$$\|X_1 + \cdots + X_n\| = o(\varphi^{-1}(A\text{-}\psi(A_n)))$$

a.s. for each $\psi \in \Psi_c$.

Proof. — See Theorem 3 for definition of Ψ_c. Denote $b_n = \varphi^{-1}(A_n\psi(A_n))$. Because for each sequence (a_n) of positive numbers with $A_n = \sum_{k=1}^{n} a_k \uparrow \infty$ the series $\Sigma a_n/(A_n\psi(A_n))$ is convergent whenever $\psi \in \Psi_c$ (cf. Lemma 15, [7], Ch. IX) we get that $\Sigma E\varphi\|X_n\|/(A_n\psi(A_n)) < \infty$ what in view of Theorem 5 implies that $\Sigma(\|X_n\|/b_n)$ converges a.s. Now, the (real) Kronecker's Lemma ($b_n \uparrow \infty$) and the triangle inequality yield the desired result.

In the case when the moments are infinite we have the following analogue of Theorem 4.

THEOREM 7. — *Let* $\mathscr{X}$, X_n, φ_n *be as in Theorem* 5. *Then the convergence of*

$$(13) \qquad \sum_{h=1}^{\infty} E \frac{\varphi_n \|X_n\|}{\varphi_n\|X_n\| + \varphi_n(t_n)}$$

where $t_n \uparrow \infty$, *implies that*

$$\|X_1 + \cdots + X_n\|/t_n \to 0$$

a.s. in norm.

Proof. — Put $Z_n = X_n \chi[\|X_n\| < t_n]$. Because of (13) and the inequality (4) we see that

$$(14) \qquad \sum_{n=1}^{\infty} P(X_n \neq Z_n) < \infty$$

$$(15) \qquad \sum_{n=1}^{\infty} \frac{E\varphi_n\|Z_n\|}{\varphi_n(t_n)} < \infty.$$

In view of (15), Corollary 3 (applied to the real sequence $\|Z_n\|$), (real) Kronecker's Lemma and the triangle inequality $\|Z_1 + \cdots + Z_n\|/t \to 0$ a.s. and so does $\|X_1 + \cdots + X_n\|/t_n$ in view of (14) and Borel-Cantelli Lemma (independence is not needed here) Q.E.D.

4. Identically distributed random vectors.

In this section we restrict our attention to the case of identically distributed random $\mathscr{X}$-variables. Some of the theorems given below are valid only for symmetric vectors and we recall that a random $\mathscr{X}$-variable X is called symmetric if there is a measure-preserving mapping $\tau : \Omega \to \Omega$ such that $X(\tau(\omega)) = - X(\omega)$ for almost all $\omega \in \Omega$. First, we deal with random vectors in Banach spaces.

THEOREM 8. — *Let* $X_1, X_2, \ldots$ *be symmetric, independent and identically distributed random* $\mathscr{X}$*-variables, where* $\mathscr{X} \in \mathscr{G}_\alpha$

for some $0 < \alpha \leqslant 1$, *let* φ *be convex and such that* $\varphi(t)/t$ *and* $t^{1+\alpha}/\varphi(t)$ *do not decrease, and let* $0 < t_n \uparrow \infty$ *be a sequence such that*

$$\sum_{k=n}^{\infty} (1/\varphi(t_k)) = 0(n/\varphi(_n)). \tag{16}$$

Then the condition

$$\sum_{n=1}^{\infty} \mathrm{P}(\|\mathrm{X}_1\| \geqslant t_n) < \infty \tag{17}$$

is necessary and sufficient in order that

$$(\mathrm{X}_1 + \cdots + \mathrm{X}_n)/t_n \to 0$$

a.s. in norm.

Proof. — *Sufficiency.* Let $\mathrm{Z}_n = \mathrm{X}_n \chi[\|\mathrm{X}_n\| < t_n]$ as in Theorem 4. Put $t_0 = 0$. Then, in view of (16) we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mathrm{E}\varphi\|\mathrm{Z}_n\|}{\varphi(t_n)} &= \sum_{n=1}^{\infty} \frac{1}{\varphi(t_n)} \sum_{k=1}^{n} \mathrm{E}\varphi(\|\mathrm{X}_1\|\chi[t_{k-1} \leqslant \|\mathrm{X}_1\| < t_k]) \\
&= \sum_{k=1}^{\infty} \mathrm{E}\varphi(\|\mathrm{X}_1\|\chi[t_{k-1} \leqslant \|\mathrm{X}_1\| < t_k]) \sum_{n=k}^{\infty} \frac{1}{\varphi(t_n)} \\
&\leqslant \text{const} \sum_{k=1}^{\infty} \frac{k}{\varphi(t_k)} \mathrm{E}\varphi(\|\mathrm{X}_1\|\chi[t_{k-1} \leqslant \|\mathrm{X}_1\| < t_k]) \\
&\leqslant \text{const} \sum_{k=1}^{\infty} k\mathrm{P}(t_{k-1} \leqslant \|\mathrm{X}_1\| < t_k) = \text{const} \sum_{k=0}^{\infty} \mathrm{P}(\|\mathrm{X}_1\| \geqslant t_k).
\end{aligned}$$

This, together with the proof of Theorem 4 implies that $\left(\sum_{k=1}^{n} (\mathrm{X}_k - \mathrm{EZ}_k)\right)\Big/ t_n \to 0$ a.s. in norm what completes the proof because $\mathrm{EZ}_k = 0$ in view of symmetry of $\mathrm{X}_n's$.

Necessity. — It follows from the fact that

$$\frac{\mathrm{X}_n}{t_n} = \frac{\mathrm{X}_1 + \cdots + \mathrm{X}_n}{t_n} - \frac{t_{n-1}}{t_n} \frac{\mathrm{X}_1 + \cdots + \mathrm{X}_{n-1}}{t_{n-1}} \to 0$$

a.s. in norm, so that were the series $\Sigma\mathrm{P}(\|\mathrm{X}_1\| \geqslant t_n)$ divergent, by Borel-Cantelli Lemma, $\mathrm{P}(\|\mathrm{X}_1\| \geqslant t_n$ infinitely often) would be 1 what would contradict (17). Q.E.D.

Remark 5. — It is easy to check that (16) is fulfilled whenever $\liminf \varphi(t_{2k})/\varphi(t_k) > 2$, what gives a handy criterion for a sequence (t_k) to satisfy (16). For instance we see that if

$\mathscr{X} \in \mathscr{G}_\alpha$, $0 < \alpha \leqslant 1$, $\varphi(t) = t^{1+\beta}$, $0 < \beta \leqslant \alpha$ then $t_k = k$ fulfills (16) and if $\beta = 0$ then it does not.

In the case of nonsymmetric random vectors we need more restrictions on (t_k).

THEOREM 9. — *Let* $X_1, X_2, \dots$ *be independent and identically distributed random* $\mathscr{X}$*-variables and let* $\mathscr{X}$, φ, (t_n) *be as in Theorem* 8. *If additionally* $EX_1 = 0$ *and*

$$t_k/t_n \leqslant Ck/n \tag{18}$$

$k \geqslant n$, *then* (17) *implies that*

$$(X_1 + \cdots + X_n)/t_n \to 0$$

a.s. in norm.

Proof. — The proof goes along the lines of the proof of Theorem 8 but to complete it making use of Theorem 4 we have to show that $\left(\sum_{k=1}^{n} EZ_k\right)\Big/t_n \to 0$, because here, in contrast with Theorem 8, EZ_k need not be zero. We shall do that, adapting the « real » idea of V. V. Petrov (Theorems 8 and 9 for $\mathscr{X} = \mathbf{R}$ and $\varphi(t) = t^2$ may be found in [7], Ch. IX) and utilizing (18) and $EX_1 = 0$ as follows :

$$\begin{aligned}
\frac{1}{t_n}\left\|\sum_{k=1}^{n} EZ_k\right\| &= \frac{1}{t_n}\left\|\sum_{k=1}^{n} E(X_k - Z_k)\right\| \\
&\leqslant \frac{1}{t_n}\sum_{k=1}^{n} E\|X_k - Z_k\| \\
&\leqslant \frac{1}{t_n}\sum_{k=1}^{n}\sum_{m=k}^{\infty} E(\|X_1\|\chi[t_m \leqslant \|X_1\| < t_{m+1}]) \\
&= \frac{1}{t_n}\left(\sum_{m=1}^{n}\sum_{k=1}^{m} + \sum_{m=n+1}^{\infty}\sum_{k=1}^{n}\right)E(\|X_1\|\chi[t_m \leqslant \|X_1\| < t_{m+1}]) \\
&\leqslant \frac{1}{t_n}\sum_{m=1}^{n} mt_{m+1}P(t_m \leqslant \|X_1\| < t_{m+1}) \\
&\quad + \frac{1}{t_n}\sum_{m=n+1}^{\infty} nt_{m+1}P(t_m \leqslant \|X_1\| < t_{m+1}).
\end{aligned}$$

Now, the first summand tends to zero because of Kronecker's Lemma and because of the fact that

$$\sum_{m=0}^{\infty} (m+1)P(t_m \leqslant \|X_1\| < t_{m+1}) = \sum_{m=0}^{\infty} P(\|X_1\| \geqslant t_m) < \infty,$$

and the second one tends to zero because of the above equality and (18) $(nt_{m+1}/t_n \leqslant C(m+1))$. Q.E.D.

THEOREM 10. — *Let* $X_1, X_2, \ldots$ *be a sequence of independent identically distributed random* $\mathfrak{X}$*-variables where* $\mathfrak{X}$ *is a Banach space. In order that there exists a vector* $x \in \mathfrak{X}$ *such that* $(X_1 + \cdots + X_n/n \to x$ *a.s. in norm it is necessary and sufficient that* $E\|X_1\| < \infty$. *Then also* $x = EX_1$.

Proof. — The sufficiency is proved in [9] p. 146 and the necessity may be shown as follows : if $(X_1 + \cdots + X_n)/n \to x$ then by Borel-Cantelli Lemma $\Sigma P(\|X_1 - x\| \geqslant n) < \infty$ (cf. proof of Theorem 8) and $E\|X_1\| = \infty$ would imply that

$$1 + \sum_{n=1}^{\infty} P(\|X_1 - x\| \geqslant n) = \sum_{m=1}^{\infty} mP(m - 1 \leqslant \|X_1 - x\| < m) \geqslant E\|X_1 - x\| = \infty$$

A contradiction.

Using the technique of Theorem 8 and applying Theorem 7 instead of Theorem 4 in the nonlocally convex case we get.

THEOREM 11. — *Let* $X_1, X_2, \ldots$ *be a sequence of identically distributed random* $\mathfrak{X}$*-variables, where* $\mathfrak{X}$ *is an* F*-space and let* $0 < t_n \uparrow \infty$ *be such that*

$$\sum_{k=n}^{\infty} \frac{1}{\varphi(t_k)} = 0\left(\frac{n}{\varphi(t_n)}\right)$$

for a continuous, nondecreasing $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ *with,* $\varphi(t) \neq 0$ *for* $t \neq 0$ *and nondecreasing* $t/\varphi(t)$. *Then the convergence of* $\Sigma P(\|X_1\| \geqslant t_n)$ *is sufficient (and also necessary when* X_n *are independent) in order that* $\|X_1 + \cdots + X_n\|/t_n \to 0$ *a.s. in* F*-norm.*

Remark 6. — Having investigated the SLLNs in the nonlocally convex case we were interested under which circumstances $\|X_1 + \cdots + X_n\|/t_n \to 0$. In the case of p-locally convex spaces our results permit also to give sufficient conditions for $\|(X_1 + \cdots + X_n)/t_n\| \to 0$. However to find such conditions in the case when $\|.\|$ is an arbitrary F-norm seems to be a more difficult problem.

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