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THE MEASURE EXTENSION PROBLEM FOR VECTOR LATTICES

by J. D. Maitland WRIGHT

Introduction.

This paper is mainly concerned with measures which take their values in a boundedly σ-complete vector lattice but some applications are made to Boolean algebras and classical measure theory. In particular a problem of Sikorski and Matthes is solved by showing that all Boolean σ-algebras with the weak σ-extension property are weakly σ-distributive. A generalisation of a result of Horn and Tarski [5] follows from Theorem T.

In classical measure theory Hopf's extension theorem states that a finitely additive measure on a field of sets can be extended to a σ-additive measure on the generated σ-field provided a simple necessary condition is satisfied. This result is usually obtained by constructing inner and outer measures but it is shown in a corollary to Theorem E that the Hopf -Kolmogorof extension theorem can easily be deduced from the Riesz representation theorem. This observation may not be new but seems of mild interest.

Let V be a boundedly σ-complete vector lattice, that is, a vector lattice such that each sequence $\{a_n\}$ $(n = 1, 2, \ldots)$ in V which is bounded above has a least upper bound $\bigvee_{n=1}^{\infty} a_n$. Let $(X, \mathcal{B})$ be a measurable space and $m : \mathcal{B} \to V$ a positive finitely additive measure then m is a σ-measure if, whenever $\{E_n\}$ $(n = 1, 2, \ldots)$ is an increasing sequence in $\mathcal{B}$, $m \bigcup_{1}^{\infty} E_n = \bigvee_{1}^{\infty} mE_n$. In the special case where V is the dual of a Banach space and is a Banach lattice this is the same as requiring that $m \bigcup_1^\infty E_n = \lim mE_n$ in the weak*-topology. But Floyd [1] gives an example of a boundedly complete Banach lattice for which there does not exist any Hausdorff vector topology such that each bounded monotone increasing sequence converges to its least upper bound.

It is not difficult to construct a V-valued integral with respect to a σ-additive V-valued measure and obtain analogues of all the Lebesgue convergence theorems. In § 3 of [15] this construction is performed when V is a Stone algebra but, since each boundedly σ-complete vector lattice is necessarily Archimedean, both the results and proofs remain valid when V is an arbitary boundedly σ-complete vector lattice.

Let $\varphi : C(Z) \to V$ be a positive linear map where Z is compact Hausdorff and V is a boundedly σ-complete vector lattice. It follows from Theorem 4.1 of [15] that there exists a unique σ-additive V-valued measure m on the Baire sets of Z such that for all $f \in C(Z)$,

$$\varphi(f) = \int_Z f\, dm.$$

The natural way to attempt a proof of this result is by a straightforward adaption of the Daniell method but this fails in general and a different approach has to be adopted. In view of a previous misunderstanding it is perhaps worth mentioning that Theorem 4.1 [15] is not a consequence of McShane's far reaching generalisation of the Daniell extension method. The Daniell construction breaks down because the Baire measure m may fail to be regular. These points are clarified at the end of § 2.

The existence of a successful analogue of the Riesz representation theorem makes it reasonable to ask the following question. Let $\mathcal{F}$ be a field of subsets of a set X, $\mathcal{F}^\infty$ the σ-field generated by $\mathcal{F}$ and $m : \mathcal{F} \to V$ a positive finitely additive measure on $(X, \mathcal{F})$ such that $m \bigcap_1^\infty E_n = 0$ whenever $\{E_n\}$ $(n = 1, 2, \ldots)$ is a monotone decreasing sequence in $\mathcal{F}$ with $\bigcap_1^\infty E_n = \emptyset$. Does there always exist a σ-additive measure

$m^* : \mathcal{F}^\infty \to V$ which is an extension of m? We shall see that the answer is no, unless V has special properties.

Various conditions sufficient to ensure that V has the measure extension property are given, or implicit, in the work of several mathematicians, notably Kantorovich [6], McShane [7] and Matthes [9]. The main point of this paper is that we completely characterise boundedly σ-complete vector lattices with the measure extension property, that is, find necessary and sufficient conditions. The first and perhaps most useful characterisation is that *the measure extension property for* V *is equivalent to each* V-*valued Baire measure on a compact Hausdorff space being* « *near regular* ». This leads to an intrinsic algebraic characterisation; V *has the measure extension property if, and only if,* V *is weakly* σ-*distributive*. The sufficiency of the latter condition follows from Matthes elegant and delicate lattice theoretic methods [9] but an independent proof of this result is given here in Corollary P of Theorem N. It is further shown that when B has the measure extension property each V-valued Baire measure on a compact Hausdorff space is regular.

Summary.

The structure of this paper is as follows. The first key result is Theorem E where the problem of extending a finitely additive V-valued measure m on a field of sets $\mathcal{F}$ to a σ-measure on the generated σ-field is shown to be equivalent to the near regularity of a corresponding measure $\hat{m}$ on a (totally disconnected) compact Hausdorff space, the Boolean structure space of the field.

Next, in Theorem N, it is shown that when V is weakly σ-distributive then each V-valued Baire measure on a compact Hausdorff space is regular and hence near regular. So it follows, in Corollary P, that when V is weakly σ-distributive then V has the measure extension property.

Then, in Theorem Q, it is shown that when each V-valued Baire measure on a totally disconnected compact Hausdorff space is near regular then V is weakly σ-distributive.

In Theorem T, the main theorem, the above results are used to characterize the measure extension property. Some applications are then made of this result.

1. Preliminary definitions and results.

Throughout this paper V is a boundedly σ-complete vector lattice, $\mathcal{F}$ is a field of subsets of a set X and $\mathcal{F}^\infty$ is the σ-field generated by $\mathcal{F}$. A *measure* on $(X, \mathcal{F})$ is a map $m : \mathcal{F} \to V$ such that

1) $mA \geqslant 0$ for each $A \in \mathcal{F}$.

2) $mA \cup B + mA \cap B = mA + mB$ for all $A, B \in \mathcal{F}$.

3) $\bigwedge_{n=1}^{\infty} mA_n = 0$ whenever $\{A_n\}$ $(n = 1, 2, \ldots)$ is a monotone decreasing sequence in $\mathcal{F}$ with $\bigcap_1^\infty A_n = \varnothing$. A σ-measure is a V-valued measure on a σ-field of subsets of a set X. A measure m on $(X, \mathcal{F})$ with values in V is *extendable* if there exists a σ-measure m^* on $(X, \mathcal{F}^\infty)$ with values in V such that $m^*F = mF$ for each $F \in \mathcal{F}$. The vector lattice V has the *measure extension property* when, for each set X and each field $\mathcal{F}$ of subsets of X, each measure $m : \mathcal{F} \to V$ is extendable. The definition of V-valued measure adopted here differs slightly from that given in [15] since infinite values have been excluded. This avoids obscuring the main ideas with irrelevant difficulties. The modifications needed when non-finite measures are considered will be dealt with in part of a later work.

The field $\mathcal{F}$ is said to be *reduced*, § 7 Sikorski [12], if it separates the points of X.

LEMMA A. — *If, whenever $\mathcal{F}'$ is a reduced field of subsets of a set X', each V-valued measure on $(X', \mathcal{F}')$ is extendable then V has the measure extension property.*

Let m be a V-valued measure on $(X, \mathcal{F})$ and suppose that $\mathcal{F}$ is not a reduced field. We form a new set X' by identifying points of X which are not separated by $\mathcal{F}$. Then, in the notation of § 7 Sikorski [12], $\mathcal{F}$ is isomorphic to $\mathcal{F}'$ which is a reduced field of subsets of X'. So we can transform the measure m to $\mathcal{F}'$ by defining $lA' = mA$ for all $A \in \mathcal{F}$. Since $\mathcal{F}'$ is a reduced field of subsets of X' there exists a V-valued σ-measure l^* on the σ-field generated by

$\mathcal{F}'$. Let $m^*A = l^*A'$ for each $A \in \mathcal{F}^\infty$ then m^* is a σ-measure on $(X, \mathcal{F}^\infty)$ which extends m.

Since the above argument shows that it is enough to consider only reduced fields of sets we shall, for the rest of this paper, always suppose that $\mathcal{F}$ is a reduced field of subsets of X.

When b is a positive element of a boundedly σ-complete vector lattice V,

$$1/2 \bigwedge_{n=1}^{\infty} 1/n \ b = \bigwedge_{n=1}^{\infty} 1/(2\ nb) \geqslant \bigwedge_{k=1}^{\infty} 1/k \ b$$

and hence $\bigwedge_{n=1}^{\infty} 1/nb = 0$. So V is Archimedean. For each positive element e of V let $V[e] = \{b \in V : -re \leqslant b \leqslant re$ for some positive real number $r\}$. Whenever $\{a_n\}$ $(n = 1,2,...)$ is an increasing sequence in $V[e]$ with an upper bound $b \in V[e]$ then $\bigvee_1^{\infty} a_n \in V[e]$. Thus $V[e]$ is a boundedly σ-complete vector lattice with order unit e. So when $V[e]$ is given the order unit norm it is a Banach space. By the fundamental Stone-Krein-Kakutani-Yosida vector lattice representation theorem, see Theorem 4.1 of Kadison [16], there exists a compact Hausdorff space S such that $V[e]$ is isometrically and lattice isomorphic to C(S). Since C(S) is boundedly σ-complete it follows from the work of Stone [14] that S is totally disconnected and the closure of a countable union of clopen subsets of S is clopen so that the clopen subsets of S form a Boolean σ-algebra. Following [15] we call C(S) a σ-Stone algebra

Proposition B. — *Let* Z *be a compact Hausdorff space and* $\varphi : C(Z) \to V$ *a positive linear map. Then there exists a unique* V*-valued* σ*-measure* m *on the Baire sets of* Z *such that, for each* $f \in C(Z)$,

$$\varphi(f) = \int_Z f\, dm$$

The existence of m follows from Theorem 4.1 of [15] after making the observation that φ maps C(Z) into $V[\varphi(1)]$

and $V[\varphi(1)]$ is isomorphic to a σ-Stone algebra. Since m must take its values in $V[\varphi(1)]$ it also follows that m is unique.

The following definitions are used throughout this paper. Let $\hat{X}$ be the Boolean space of $\mathcal{F}$ so that $\hat{X}$ may be identified with the set of Boolean homomorphisms of $\mathcal{F}$ into the two element Boolean algebra $\{0, 1\}$ and $\hat{X}$ is equipped with a compact Hausdorff totally disconnected topology, (see § 18 Halmos [2] or § 8 Sikorski [12]). Since $\mathcal{F}$ is a reduced field of sets each $x \in X$ corresponds to a unique homomorphism $A \to \chi_A(x)$. So X may be identified with a subset of $\hat{X}$. Let $\hat{\mathcal{F}}$ be the Boolean algebra of clopen subsets of $\hat{X}$; $\mathcal{B}^\infty$ the σ-field of Baire subsets of $\hat{X}$, that is, the smallest σ-field of subsets of $\hat{X}$ which contains $\hat{\mathcal{F}}$; and let $\rho B = B \cap X$ for each $B \in \mathcal{B}^\infty$. It is clear from the proof of the Stone representation theorem given in § 18 Halmos [2] that the restriction of ρ to $\hat{\mathcal{F}}$ is an isomorphism of $\hat{\mathcal{F}}$ onto $\mathcal{F}$. So there is no non-empty clopen subset of $\hat{X} - X$ and hence $K = \overline{\rho K}$ for each $K \in \hat{\mathcal{F}}$.

LEMMA C. — *ρ is a Boolean σ-homomorphism of $\mathcal{B}^\infty$ onto $\mathcal{F}^\infty$ whose kernel is the σ-ideal of Baire sets disjoint from* X.

Clearly ρ is a Boolean σ-homomorphism of $\mathcal{B}^\infty$ onto a σ-field of sets containing $\mathcal{F}$. So $\rho[\mathcal{B}^\infty] \supset \mathcal{F}^\infty$. Then $\rho^{-1}[\mathcal{F}^\infty]$ is a σ-field of subsets of $\mathcal{B}^\infty$ and $\hat{\mathcal{F}} \subset \rho^{-1}[\mathcal{F}^\infty]$ so that $\rho^{-1}[\mathcal{F}^\infty] = \mathcal{B}^\infty$.

2. The measure extension property.

LEMMA D. — *Let m be a V-valued measure on $(X, \mathcal{F})$. Then there exists a unique V-valued σ-measure $\hat{m}$ on the Baire sets of $\hat{X}$ such that $\hat{m}K = mK \cap X$ for each $K \in \hat{\mathcal{F}}$ or equivalently, $mF = \hat{m}\overline{F}$ for each $F \in \mathcal{F}$.*

Let $V[mX] \cong C(S)$. Let $\mathcal{A}$ be the subalgebra of $C(\hat{X})$ consisting of all functions which assume only a finite number of values. Each such function is of the form $\sum_{i=1}^{n} a_i \chi_{K_i}$ where

$\{K_1, K_2, \ldots, K_n\}$ are pairwise disjoint clopen sets. Because m is finitely additive a positive linear operator $\varphi_0 : \mathcal{B} \to C(S)$ is defined by

$$\varphi_0\left(\sum_{i=1}^{n} a_i \chi_{K_i}\right) = \sum_{1}^{n} a_i m(K_i \cap X).$$

Since φ_0 is a positive linear operator it is a bounded linear operator with $\|\varphi_0\| = \|\varphi_0(1)\| = \|mX\| = 1$.

By the Stone-Weierstrass theorem $\mathcal{A}$ is uniformly dense in $C(\hat{X})$ and, since $C(S)$ is a Banach space, φ_0 has a unique extension φ which maps $C(X)$ into $C(S) \cong V[mX] \subset V$. By Proposition B of § 1 there exists a Baire measure $\hat{m}$ on $\hat{X}$ such that for each $f \in C(X)$,

$$\varphi(f) = \int_{\hat{X}} f \, d\hat{m}.$$

So, for each clopen set $K \in \hat{\mathcal{F}}$,

$$\hat{m}K = \varphi(\chi_K) = mK \cap X.$$

For each $F \in \mathcal{F}$, $\overline{F}$ is a clopen subset of $\hat{X}$ with $F = \overline{F} \cap X$ so that $\hat{m}\overline{F} = mF$.

The uniqueness of $\hat{m}$ is trivial.

Theorem E. — *Let m be a V-valued measure on $(X, \mathcal{F})$. There exists a σ-measure m^* on $\mathcal{F}^\infty$ which extends m if, and only if, $\hat{m}B = 0$ for each Baire set $B \subset \hat{X} - X$. When m^* exists then $\hat{m}B = m^*B \cap X$ for all $B \in \mathcal{B}^\infty$.*

When m^* exists we define for each $B \in \mathcal{B}^\infty$

$$lB = m^*(\rho B) = m^*(B \cap X).$$

Since, by Lemma C of § 1, ρ is a Boolean σ-homomorphism it follows that l is a V-valued σ-measure on the Baire sets of $\hat{X}$. In particular, for each $K \in \hat{\mathcal{F}}$,

$$lK = m^*K \cap X = mK \cap X$$

since m^* is an extension of m. So, by Lemma D, $l = \hat{m}$. Therefore when m^* exists

$$\hat{m}B = m^*B \cap X \qquad \text{for all} \qquad B \in \mathcal{B}^\infty.$$

Hence, if B is a Baire subset of $\hat{X} - X$ then

$$\hat{m}B = m^*B \cap X = m^*\varnothing = 0.$$

Conversely, suppose $\hat{m}B = 0$ for each Baire set $B \subset \hat{X} - X$. So, if B_1, B_2 are Baire sets such that $B_1 \cap X = B_2 \cap X$ then $B_1 \Delta B_2 \subset \hat{X} - X$ and hence $\hat{m}\{(B_1 - B_2) \cup (B_2 - B_1)\} = 0$. Thus $\hat{m}B_1 = \hat{m}B_2$ and we may properly define a function

$$m^* : \mathfrak{F}^\infty \to V \qquad \text{by} \qquad m^*B \cap X = \hat{m}B.$$

Then m^* is a σ-measure on $\mathfrak{F}^\infty$ and for each $F \in \mathfrak{F}$,

$$m^*F = m^*\overline{F} \cap X = \hat{m}\overline{F} = mF.$$

A V-valued σ-measure on the Baire sets of a compact Hausdorff space Z is *near regular* if, whenever E is a Baire subset of Z such that $lF = 0$ for each closed Baire set F contained in E then $lE = 0$. When $V = \mathbf{R}$, the real numbers, then each $\mathbf{R}$-valued Baire measure is regular and hence near regular. In [15] an example is given of a V-valued Baire measure on the unit interval which is not near regular.

Corollary F. — *When every V-valued Baire measure on each totally disconnected compact Hausdorff space is near regular then V has the measure extension property.*

Let X, $\hat{X}$, $\mathfrak{F}$, $\hat{\mathfrak{F}}$, m, $\hat{m}$ be as in Theorem E. Since $\hat{X}$ is the Boolean space of $\mathfrak{F}$ it is totally disconnected and so, by hypothesis, $\hat{m}$ is near regular. If each closed Baire set $F \subset \hat{X} - X$ has zero $\hat{m}$-measure then $\hat{m}E = 0$ for each Baire set $E \subset \hat{X} - X$ and so, by Theorem E, m is extendable to a σ-measure on $\mathfrak{F}^\infty$.

Let F be a closed Baire subset of $\hat{X} - X$. In any compact Hausdorff space a closed Baire set is the intersection of a sequence of open sets. When O is an open set containing F then O is the union of a collection of clopen sets because $\hat{X}$ is totally disconnected. But F is compact so that it is covered by a finite collection of clopen subsets of O and so there exists a clopen set K such that $F \subset K \subset O$. So there exists a monotone decreasing sequence of clopen sets

$\{K_n\}$ $(n = 1, 2, \ldots)$ such that $F = \bigcap_{n=1}^{\infty} K_n$. Then

$$\hat{m}F = \bigwedge_{n=1}^{\infty} \hat{m}K_n = \bigwedge_{n=1}^{\infty} mK_n \cap X = m\bigcap_{n=1}^{\infty} K_n \cap X$$
$$= mF \cap X = m\varnothing = 0.$$

Corollary G. — (Hopf-Kolmogorof extension theorem) *Let μ be a finite finitely additive positive real valued measure on $(X, \mathscr{F})$ such that* $\lim \mu E_n = 0$ *whenever* $\{E_n\}$ $(n = 1, 2, \ldots)$ *is a monotone decreasing sequence in $\mathscr{F}$ with empty intersection. Then there exists a σ-additive measure μ^* on $(X, \mathscr{F}^{\infty})$ which extends μ.*

Each real Baire measure on a compact Hausdorff space is regular and thus near regular. So it follows from Corollary F that **R** has the measure extension property. To obtain the corresponding result when μ is not a finite measure we may slightly modify Theorem E by taking $\mathscr{F}$ to be the Boolean ring without identity of sets of finite measure so that $\hat{X}$ is locally compact. This point will be covered in a later paper.

A linear functional $\psi : V \to \mathbf{R}$ is *σ-normal* if, when $\{b_n\}$ $(n = 1, 2, \ldots)$ is a monotone increasing sequence with least upper bound b then $\psi(b) = \lim \psi(b_n)$. The functional ψ is not required to be positive.

Corollary H. — *If V has a separating family of σ-normal functionals then V has the measure extension property.*

Let Z be a compact Hausdorff space and l a V-valued Baire measure on Z. Let E be any Baire subset of Z such that $lE \neq 0$. Then there exists a σ-normal functional ψ on V such that $\psi(lE) \neq 0$. For each Baire set B let $\mu B = \psi(lB)$ so that μ is a finite signed real valued Baire measure on Z.

The classical Hahn decomposition theorem ensures the existence of a Baire set $D \subset Z$ such that if, for each Baire set B, $\mu^+B = \mu B \cap D$ and $\mu^-B = -\mu B \cap D$ then μ^+, μ^- are positive finite real valued Baire measures on Z. Since $\mu E \neq 0$ we may suppose, without loss of generality, that $\mu^+E > 0$ and so $\mu^+E \cap D > 0$. Since each real positive

Baire measure on a compact Hausdorff space is regular there exists a closed Baire set $F \subset E \cap D$ with $\mu^+F > 0$. So $\mu F = \mu F \cap D = \mu^+F > 0$ and hence $lF \neq 0$. It follows that l is near regular and so, by Corollary F, V has the measure extension property.

A topology $\mathfrak{J}$ for V is σ-compatible with the partial ordering if, whenever $\{b_n\}$ $(n = 1, 2, \ldots)$ is a monotone increasing sequence with least upper bound b then $b_n \to b$ in the $\mathfrak{J}$-topology. There are many vector lattices for which a locally convex Hausdorff σ-compatible vector topology can be found. Indeed, if V is any partially ordered Banach dual space such that $V^+ \cap (b - V^+)$ is norm bounded and closed for each b in V^+ then the weack* — topology is σ — compatible. However Floyd [1] gives a simple example of a boundedly complete vector lattice for whichf no such topology exists.

Corollary J. — *If there exists a locally convex Hausdorff σ-compatible vector topology $\mathfrak{J}$ for V then V has the measure extension property.*

By the Hahn-Banach theorem the $\mathfrak{J}$-continuous linear functionals on V separate the points of V. Since $\mathfrak{J}$ is a σ-compatible topology each $\mathfrak{J}$-continuous linear functional is σ-normal. The result now follows from Corollary H.

We see from the preceding paragraphs that there are many vector lattices with the measure extension property. The following example shows that this property can fail even when V has an order unit, is boundedly complete and satisfies the countable chain condition. Further, the V-valued measure which has no σ-extension is defined on a field of subsets of a countable set, the set of rationals between 0 and 1.

The complete Boolean algebra of regular open subsets of [0, 1] is isomorphic to the algebra of idempotents in C(S), where S is its Boolean space. By a theorem of Birkoff and Ulam, see § 21 Sikorski [12], there is a Boolean σ-homomorphism k of the Borel subsets of [0,1]onto the algebra of idempotents in C(S) whose kernel is the σ-ideal of meagre Borel sets. C(S) is a boundedly complete vector lattice which can be shown to satisfy the countable chain condition.

Let X be the set of all rationals in $(0, 1)$. Let $\mathscr{B}$ be the field of subsets of $[0, 1]$ consisting of all finite unions of intervals (open or closed at either end) whose end points are elements of $[0, 1] - X$.

Let $\mathscr{F}$ be the field of all sets of the form $B \cap X$ where $B \in \mathscr{B}$. If $A \in \mathscr{B}$ and $A \cap X = \varnothing$ then A is a finite set so $m: \mathscr{F} \to C(S)$ is properly defined by $mB \cap X = kB$ for each $B \in \mathscr{B}$. When $\{B_n : n = 1, 2, \ldots\}$ is a monotone decreasing sequence in $\mathscr{B}$ such that $\bigcap_1^\infty B_n$ is disjoint from X then $\bigcap_1^\infty \overline{B}_n$ is a closed set disjoint from X and hence nowhere dense. So

$$\bigwedge_1^\infty mB_n \cap X = \bigwedge_1^\infty kB_n \leqslant \bigwedge_1^\infty k\overline{B}_n = k\bigcap_1^\infty \overline{B}_n = 0.$$

So m is a C(S)-valued measure on $(X, \mathscr{F})$.

Suppose there exists a σ-measure m^* on $\mathscr{F}^\infty$ which extends m. Let $\mathscr{G} = \{B \in \mathscr{B}^\infty : kB = m^*B \cap X\}$ then $\mathscr{G}$ contains $\mathscr{B}$ also $\mathscr{G}$ is closed under the unions of monotone increasing sequences and the intersections of monotone decreasing sequences. So, by Theorem 21.6 [4] or Theorem B § 6 Chapter 1 [3], $\mathscr{G}$ contains the σ-field generated by $\mathscr{B}$ which is the field of Borel subsets of $[0, 1]$. Then

$$1 = k([0, 1] - X) = m^*\varnothing = 0$$

which is impossible.

In the following definition N is the set of positive integers and N^N is the set of all maps from N into N, that is, all sequences of positive integers. The boundedly σ-complete vector lattice V is defined to be *weakly σ-distributive* if, whenever $\{b_{n,r} : n = 1, 2, \ldots, r = 1, 2, \ldots\}$ is an order bounded subset of V with $b_{n,r+1} \leqslant b_{n,r}$ for each n and r then

$$\bigvee_{n=1}^\infty \bigwedge_{r=1}^\infty b_{n,r} = \bigwedge_{\varphi \in N^N} \bigvee_{n=1}^\infty b_{n,\varphi(n)}:$$

As an immediate consequence of this definition, V is weakly

σ-distributive if, and only if, $V[e]$ is weakly σ-distributive for each $e \geqslant 0$. ($V[e]$ is defined at the end of § 1).

In any topological space a *σ-meagre* set is a subset of the union of a countable family of closed nowhere dense Baire sets. A proof of the following Lemma is included for the convenience of the reader.

LEMMA K. — (Stone [14]). *When* $C(S)$ *is a* σ*-Stone algebra and* $\{f_n\}$ $(n = 1, 2, \ldots)$ *is a sequence in* $C(S)$ *which is bounded below then*

$$\left\{ s \in S : \inf f_n(S) > \left(\bigwedge_{r=1}^{\infty} f_r\right)(s) \right\}$$

is a σ*-meagre set.*

Let F_r be the closed Baire set

$$\bigcap_{n=1}^{\infty} \left\{ s \in S : \left(\bigwedge_{j=1}^{\infty} f_j\right)(s) \leqslant f_n(s) - 1/r \right\}$$
$$= \left\{ s \in S : \left(\bigwedge_{j=1}^{\infty} f_j\right)(s) \leqslant \inf f_n(s) - 1/r \right\}.$$

If K is a clopen subset of F_r then $\bigwedge_{j=1}^{\infty} f_j + 1/r\, \chi_K \leqslant f_n$ for all n and so K is empty. So F_r is nowhere dense.

We need the following simple Lemma. See Kelley [17] for a related result and also his beautiful characterization of measure algebras

LEMMA L. — *A* σ*-Stone algebra* $C(S)$ *is weakly* σ*-distributive if, and only if, each* σ*-meagre subset of* S *is nowhere dense.*

First suppose $C(S)$ to be weakly σ-distributive. Let $\{F_n\}$ $(n = 1, 2, \ldots)$ be a sequence of closed nowhere dense Baire subsets of S. Then for each n there exists a decreasing sequence of clopen sets $\{K_{n,r}\}$ $(r = 1, 2, \ldots)$ such that $F_n = \bigcap_{r=1}^{\infty} K_{n,r}$. So

$$0 = \bigwedge_{r=1}^{\infty} \chi_{K_{n,r}} = \bigvee_{n=1}^{\infty} \bigwedge_{r=1}^{\infty} \chi_{K_{n,r}} = \bigwedge \left\{ \bigvee_{n=1}^{\infty} \chi_{K_{n,\varphi(n)}} : \varphi \in N^N \right\}.$$

Thus the closed set $\bigcap \left\{ \bigvee_{n=1}^{\infty} \mathrm{K}_{n, \varphi(n)} : \varphi \in \mathrm{N}^{\mathrm{N}} \right\}$ is nowhere dense and this set contains $\bigcup_{1}^{\infty} \mathrm{F}_n$.

Now suppose that each σ-meagre subset of S is nowhere dense. Let $\{b_{n,r}\}$ $(n = 1, 2, \ldots)$ $(r = 1, 2, \ldots)$ be an order bounded double sequence in C(S) with $b_{n,r+1} \leqslant b_{n,r}$ for all n, r. Then, for each $\varphi \in \mathrm{N}^{\mathrm{N}}$,

$$\bigvee_{n=1}^{\infty} \bigwedge_{r=1}^{\infty} b_{n,r} \leqslant \bigvee_{n=1}^{\infty} b_{n, \varphi(n)}.$$

Assume that there exists a greater lower bound g in C(S) for the set $\left\{ \bigvee_{1}^{\infty} b_{n, \varphi(n)} : \varphi \in \mathrm{N}^{\mathrm{N}} \right\}$. There exists a σ-meagre set M such that for all n and each $s \notin \mathrm{M}$,

$$\left(\bigwedge_{r=1}^{\infty} b_{n,r} \right)(s) = \inf_{r} b_{n,r}(s)$$

and

$$\left(\bigvee_{n=1}^{\infty} \bigwedge_{r=1}^{\infty} b_{n,r} \right)(s) = \sup_{n} \inf_{r} b_{n,r}(s).$$

By assumption, the open set $\mathrm{O} = \left\{ s \in \mathrm{S} : \left(\bigvee_{n=1}^{\infty} \bigwedge_{r=1}^{\infty} b_{n,r} \right)(s) < g(s) \right\}$ is non-empty. By hypothesis $\overline{\mathrm{M}}$ is a closed nowhere dense set so that $\mathrm{O} - \overline{\mathrm{M}}$ is a non-empty open subset of the totally disconnected space S. Thus there exists a non-empty clopen set $\mathrm{K} \subset \mathrm{O} - \overline{\mathrm{M}}$ and a positive ε such that

$$\chi_{\mathrm{K}} \left(\bigvee_{n=1}^{\infty} \bigwedge_{r=1}^{\infty} b_{n,r} \right) + \varepsilon \chi_{\mathrm{K}} < g \chi_{\mathrm{K}}$$

For each n, $\{\chi_{\mathrm{K}} b_{n,r}\}$ $(r = 1, 2, \ldots)$ is a monotone decreasing sequence converging pointwise to the continuous function $\chi_{\mathrm{K}} \bigwedge_{r=1}^{\infty} b_{n,r}$. So, by Dini's theorem, the convergence is uniform and thus we can choose $\varphi(n)$ such that

$$\chi_K b_{n,\varphi(n)} \leqslant \varepsilon/2\, \chi_K + \chi_K \bigwedge_{r=1}^{\infty} b_{n,r}.$$

Hence

$$\chi_K \bigvee_{n=1}^{\infty} b_{n,\varphi(n)} \leqslant \varepsilon/2\, \chi_K + \chi_K \bigvee_{n=1}^{\infty} \bigwedge_{r=1}^{\infty} b_{n,r},$$

So

$$g\chi_K \leqslant (g - \varepsilon/2)\chi_K.$$

This contradiction shows that $\bigvee_{n=1}^{\infty} \bigwedge_{r=1}^{\infty} b_{n,r}$ is the greatest lower bound.

Corollary M. — C(S) *is weakly σ-distributive if, and only if, the Boolean σ-algebra of clopen subsets of* S *is weakly σ-distributive.*

This follows at once from the Lemma and Theorem 30.1 of Sikorski [12].

A V-valued Baire measure m on a compact space Z is said to be *regular* when, for each Baire set $E \subset Z$,

$mE = \bigvee\{mF : F \subset E$ and F is a closed Baire set$\}$.

Theorem N. — *Let* V *be weakly σ-distributive,* Z *a compact Hausdorff space and* m *a* V*-valued Baire measure on* Z. *Then* m *is a regular Baire measure.*

The measure m takes its values in $V[mZ]$ which may be identified with C(S) where C(S) is a weakly σ-distributive σ-Stone algebra.

For each Baire set $E \subset Z$ let $\mathcal{U}(E)$ be the family of all open Baire sets containing E and let $\mathcal{L}(E)$ be the family of all closed Baire subsets of E. For each $s \in S$ let

$$(\underline{m}E)(s) = \sup\{(mF)(s) : F \in \mathcal{L}(E)\}$$
$$(\overline{m}E)(s) = \inf\{(mO)(s) : O \in \mathcal{U}(E)\}.$$

Let a subset of S be called *negligible* if it is contained in the closure of a σ-meagre set. Then a countable union of negligible sets is negligible and each negligible set is nowhere dense. Let $\mathcal{R}$ be the collection of all Baire sets E for which

$\{s \in S : (\underline{m}E)(s) < (\overline{m}E)(s)\}$ is a negligible set. When O is any open Baire set then it is the union of an increasing sequence of closed Baire sets so, by appealing to Lemma K, it follows that $mO = \underline{m}O$ except on a σ-meagre set. Hence all open Baire sets of Z are in $\mathfrak{R}$.

Let $E_j \in \mathfrak{R}$ and $U_j \in \mathfrak{U}(E_j)$ for $j = 1, 2$. Then

$$\begin{aligned} m(U_1 \cap U_2) - m(E_1 \cap E_2) \\ &= m[(U_1 \cap U_2 - E_1) \cup (U_1 \cap U_2 - E_2)] \\ &\leqslant m(U_1 - E_1) + m(U_2 - E_2). \end{aligned}$$

So $\overline{m}E_1 \cap E_2 = mE_1 \cap E_2$ except on a negligible set. By a similar argument $\underline{m}E_1 \cap E_2 = mE_1 \cap E_2$ except on a negligible set and thus $E_1 \cap E_2 \in \mathfrak{R}$. It is clear that if $E \in \mathfrak{R}$ then $Z - E \in \mathfrak{R}$ so that $\mathfrak{R}$ is a field of subsets of Z which contains all the open Baire sets.

Let $\{E_n\}$ $(n = 1, 2, ...)$ be a sequence of pairwise disjoint elements of $\mathfrak{R}$ and let $E = \bigcup_1^\infty E_n$. Then there exists a negligible set $M \subset S$ such that, for all $s \in S - M$,

$$(mE)(s) = \sum_1^\infty (mE_n)(s)$$

and for each n

$$(\underline{m}E_n)(s) = (mE_n)(s) = (\overline{m}E_n)(s).$$

Since C(S) is weakly σ-distributive it follows from Lemma L that each negligible subset of S is nowhere dense and so $S - \overline{M}$ is a dense open subset of S. Choose $s_0 \in S - \overline{M}$ then there exists a clopen neighbourhood K of s_0 such that $K \subset S - \overline{M}$.

The family of continuous functions $\{mO : O \in \mathfrak{U}(E_n)\}$ is filtering downwards with pointwise limit $\overline{m}E_n$. But $\chi_K\overline{m}E_n = \chi_K mE_n$ so, by Dini's theorem, $\chi_K mE_n$ is the uniform limit of $\{\chi_K mO : O \in \mathfrak{U}(E_n)\}$. Fix $\varepsilon > 0$ and then for each n choose $O_n \in \mathfrak{U}(E_n)$ such that

$$\chi_K mE_n + \varepsilon/2^n \chi_K \geqslant \chi_K mO_n.$$

So

$$\bigvee_{n=1}^\infty \sum_1^n \chi_K mO_r \leqslant \varepsilon\chi_K + \chi_K \bigvee_{n=1}^\infty \sum_1^n mE_r.$$

So

$$\chi_K m \bigcup_1^\infty O_n \leqslant \varepsilon\chi_K + \chi_K m E.$$

Thus

$$\chi_K \overline{m} E \leqslant \varepsilon\chi_K + \chi_K m E.$$

Since ε was arbitrary we have

$$(\overline{m}E)(s_0) \leqslant (mE)(s_0) \leqslant (\overline{m}E)(s_0).$$

But this is true for any $s_0 \in S - \overline{M}$ and hence $\overline{m}E = mE$ except on a negligible set.

Clearly $\underline{m}E \geqslant \underline{m} \bigcup_1^n E_r = m \bigcup_1^n E_r = \sum_1^n mE_r$ except on a negligible set. So $\underline{m}E \geqslant mE$ except on a negligible set. Thus $mE = \overline{m}E = \underline{m}E$ except on a negligible set and so $E \in \mathfrak{R}$. So $\mathfrak{R}$ is the σ-field of Baire sets of Z.

It follows that m is a regular Baire measure.

Corollary P. — *When* V *is weakly* σ*-distributive then* V *has the measure extension property.*

By Theorem N the Baire measure m is regular and hence near regular. So, by Corollary F of Theorem E, V has the measure extension property.

Theorem Q. — *When* V *has the measure extension property each* V*-valued Baire measure* m *on a totally disconnected. compact Hausdorff space is near regular.*

Let T be a compact Hausdorff totally disconnected space and let m be a V-valued Baire measure on T. Suppose that m is not near regular. Then there exists a Baire set $E \subset T$ such that $mE \neq 0$ and $mF = 0$ for each closed Baire set $F \subset E$. Since $mE \neq 0$ the set E is not empty and, since T is a closed set, E is a proper subset of T. Let $X = T - E$.

Let $\mathcal{B}$ be the field of clopen subsets of T and $\mathcal{B}^\infty$ the σ-field generated by $\mathcal{B}$, that is, the Baire sets of T. Let $\mathcal{F} = \{B \cap X : B \in \mathcal{B}\}$ so that $\mathcal{F}$ is a field of subsets of X. When O_1, O_2 are clopen sets such that $O_1 \cap X = O_2 \cap X$ then $(O_1 - O_2) \cup (O_2 - O_1)$ is a closed Baire set contained in E and so $m(O_1 - O_2) + m(O_2 - O_1) = 0$. Thus $l : \mathcal{F} \to V$ can be properly defined by $l(O \cap X) = mO$ for each clopen set $O \subset T$. Then l is positive and finitely additive on $\mathcal{F}$.

When $\{S_n\}$ $(n = 1, 2, \ldots)$ is a monotone decreasing sequence in $\mathcal{F}$ with $\bigcap_1^\infty S_n = \emptyset$ then there exists a monotone decreasing sequence of clopen sets $\{K_n\}$ $(n = 1, 2, \ldots)$ with $K_n \cap X = S_n$ for each n. So $\bigcap_1^\infty K_n$ is a closed subset of E and thus

$$0 = m\bigcap_{n=1}^\infty K_n = \bigwedge_{n=1}^\infty mK_n = \bigwedge_{n=1}^\infty lS_n.$$

So l is a V-valued measure on $(X, \mathcal{F})$.

Since V has the measure extension property there exists an extension l^* of l to $\mathcal{F}^\infty$. Consider the family of sets

$$\mathcal{S} = \{B \in \mathcal{B}^\infty : B \cap X \in \mathcal{F}^\infty \quad \text{and} \quad mB = l^*B \cap X\}.$$

Since l^* is an extension of l it follows that $\mathcal{B} \subset \mathcal{S}$. Let $\{B_n\}$ $(n = 1, 2, \ldots)$ be an increasing sequence on $\mathcal{S}$ then

$$m\bigcup_1^\infty B_n = \bigvee_1^\infty mB_n = \bigvee_1^\infty l^*B_n \cap X = l^*\left(\bigcup_1^\infty B_n\right) \cap X$$

so that $\bigcup_1^\infty B_n \in \mathcal{S}$. Similarly the intersection of a monotone decreasing sequence of elements of $\mathcal{S}$ is itself an element of $\mathcal{S}$. It follows from Theorem 21.6 [4] or Theorem B § 6 Chapter 1 [3] that $\mathcal{S}$ contains $\mathcal{B}^\infty$.

But

$$0 = l^*\emptyset = l^*E \cap X = mE \neq 0.$$

This contradiction establishes the theorem.

Corollary R. — *Each V-valued Baire measure on a compact Hausdorff totally disconnected space is near regular if, and only if,* V *has the measure extension property.*

This follows from Corollary F to Theorem E and Theorem Q.

When C(S) is a σ-Stone algebra the idempotents of C(S) (the characteristic functions of the clopen subsets of S) form a Boolean σ-algebra. It follows from a theorem proved independently by Loomis and Sikorski, see § 29 [12], that there exists a Boolean σ-homomorphism k of the Baire sets of S onto the idempotents in C(S) whose kernel is the σ-ideal of meagre Baire sets. We call k the Loomis-Sikorski Baire measure on S.

Theorem S. — *Let* C(S) *be a σ-Stone algebra such that the Loomis-Sikorski Baire measure k is near regular. Then* C(S) *is weakly σ-distributive.*

Let N be the union of a countable family of closed nowhere dense Baire subsets of S so that N is a σ-meagre Baire set. Let K be a clopen subset of $\overline{\mathrm{N}}$. Then K — N is a Baire set. When O is any open subset of K — N then $\mathrm{O} \cap \mathrm{N} = \varnothing$ and so $\mathrm{O} \cap \overline{\mathrm{N}} = \varnothing$. Thus $\mathrm{O} \cap \mathrm{K} = \varnothing$ and hence $\mathrm{O} = \varnothing$. So each closed subset of K — N has empty interior and so is nowhere dense. Thus, by hypothesis, $k(\mathrm{K} - \mathrm{N}) = 0$ so that K — N is meagre.

But $\mathrm{K} \subset (\mathrm{K} - \mathrm{N}) \cup \mathrm{N}$ and so K is a meagre open set. By the Baire category theorem for compact spaces $\mathrm{K} = \varnothing$. Since S is totally disconnected each open subset of $\overline{\mathrm{N}}$ is the union of clopen sets. Thus $\overline{\mathrm{N}}$ has empty interior, that is, N is nowhere dense. So each σ-meagre subset of S is nowhere dense. Then it follows from Lemma L that C(S) is weakly σ-distributive.

We now prove the main theorem which characterises the measure extension property.

Theorem T. — *Let* V *be a boundedly σ-complete vector lattice then the following are equivalent*:

(i) V *has the measure extension property.*

(ii) *Each* V*-valued Baire measure on a totally disconnected compact space is near regular.*

(iii) *Each* V*-valued Baire measure on a compact Hausdorff space is regular.*

(iv) V *is weakly σ-distributive.*

By Corollary R to Theorem Q, (i) and (ii) are equivalent.

Theorem S shows that (ii) implies (iv). By Theorem N, (iv) implies (iii) and, trivially, (iii) implies (ii).

COROLLARY U. — *Let $\mathcal{S}$ be a Boolean σ-algebra. If there exists a separating family of σ-finite σ-additive real measures on $\mathcal{S}$ then $\mathcal{S}$ is weakly σ-distributive.*

Let S be the Boolean space of $\mathcal{S}$ and use the Loomis-Sikorski σ-homomorphism of the Baire sets of S onto $\mathcal{S}$ to transfer the measures on $\mathcal{S}$ to Baire measures on S. Then C(S) has a separating family of σ-normal functionals and so, by Corollary H of Theorem E, C(S) has the measure extension property. Theorem T implies that C(S) and hence $\mathcal{S}$ is weakly σ-distributive.

COROLLARY V. — (Horn and Tarski [5]) *Let $\mathcal{S}$ be a Boolean σ-algebra. If there exists a strictly positive σ-finite real measure on $\mathcal{S}$ then $\mathcal{S}$ is weakly σ-distributive.*

In § 5 of McShane [7] *normal* partially ordered sets are defined.

COROLLARY W. — *When V is a normal vector lattice then V is weakly σ-distributive and each V-valued Baire measure on a compact Hausdorff space is regular.*

From Chapter 5 of McShane [7] V has the measure extension property.

Corollaries U, V and the first part of Corollary W can be obtained from very general results of Matthes [8] but it seems woth noticing that they follow so easily from Theorem T.

It might appear that the results in Chapter V of McShane [7] imply Theorem 4.1 of [15]. This is not so because McShane needs the hypothesis, Postulate 22.1 [σ], that V is normal which, by Corollary W, implies that V is weakly σ-distributive. On page 74 we give a simple example of a complete vector lattice, satisfying the countable chain condition, which does not have the measure extension property and so is not weakly σ-distributive and hence is certainly not normal.

If V does not have the measure extension property it is still possible for some V-valued measure on $(X, \mathcal{F})$ to have an extension to a σ-measure on $(X, \mathcal{F}^{\infty})$. When this extension does exist it is given by Theorem E. Several attractive conjectures are demolished by the example on page 74 for in that example X is a countable set and also V is boundedly complete and satisfies the countable chain condition.

The σ-extension problem for Boolean algebras seems to have first been considered by Sikorski [13]. A Boolean σ-algebra $\mathcal{G}$ has the *weak σ-extension property*, see § 34 Sikorski [12], if, when $\mathcal{B}$ is any Boolean σ-algebra, $\mathcal{B}_0$ is a subalgebra of $\mathcal{B}$ which σ-generates $\mathcal{B}$, and $h_0 : \mathcal{B}_0 \rightarrow \mathcal{G}$ is a Boolean homomorphism such that $h_0\left(\bigwedge_1^\infty A_n\right) = 0$ whenever $\{A_n\}$ $(n = 1, 2, \ldots)$ is a monotone decreasing sequence in $\mathcal{B}_0$ with null intersection, then there exists a σ-homomorphism $h : \mathcal{B} \rightarrow \mathcal{G}$ which is an extension of h_0. A theorem of Matthes, Theorem 4 § 34 [12], shows that when $\mathcal{G}$ is weakly σ-distributive then it has the weak σ-extension property.

Sikorski § 34 [12] poses the problem of whether there exist Boolean algebras with the weak σ-extension property which are not weakly σ-distributive. As a consequence of the results of this paper it can be shown that no such Boolean algebras exist. The analagous problem for the weak $\aleph$-extension property where $\aleph > \aleph_0$ will be dealt with in a later work.

THEOREM X. — *Each Boolean σ-algebra with the weak σ-extension property is weakly σ-distributive.*

When $\mathcal{G}$ is a Boolean σ-algebra with the weak σ-extension property then it follows from the argument of Theorem Q that the Loomis-Sikorski σ-homomorphism of the Baire sets of the Boolean space of $\mathcal{G}$ onto $\mathcal{G}$ is near regular. It then follows from Theorem $\mathcal{G}$ that $\mathcal{G}$ is weakly σ-distributive.

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