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THEOREMS OF KREIN-MILMAN TYPE FOR CERTAIN CONVEX SETS OF FUNCTIONS AND OPERATORS

by Robert R. PHELPS

Let X be a compact Hausdorff space and E a real [or complex] locally convex Hausdorff vector space. Denote by C(X, E) the real [or complex] linear space of all continuous functions from X to E, provided with the topology of uniform convergence. (Thus, a typical neighborhood of 0 has the form

$$\{f \in C(X, E) : \sup_X p(f(x)) \leqslant 1\}$$

where p is a continuous seminorm on E.)

For any subset A of E we let

$$C(X, A) = \{f \in C(X, E) : f(X) \subset A\}.$$

It follows that if B is a bounded closed convex subset of E, then C(X, B) is a bounded closed convex subset of C(X, E). We denote by exts the set of extreme points of a given convex set S; it is readily verified that

$$C(X, \text{ext } B) \subset \text{ext } C(X, B).$$

[It is known [2, p. 755] that this inclusion can be proper, even for four dimensional E. There are also examples where ext C(X, B) is empty for every X; for instance, if $E = c_0$ in the norm topology and B is its unit ball.] The main purpose of this note is to exhibit conditions under which the set C(X, B) will be the closed convex hull $\overline{\text{co}}$ C(X, ext B) of this subset of extreme points.

[Note that, even for one dimensional B, the set C(X, B) need not be compact, so the Krein-Milman theorem does not apply.] Our main result was proved in two special cases in [6] (Theorems 2.1 and 4.1), where applications were made to various convex sets of bounded (or of compact) linear operators from a Banach space into C(X). The more general result of the present note may be applied to analogous sets of *weakly* compact operators. We give one such application, as well as two results which were overlooked in [6].

As in [6], the problem is handled in two steps. First, we consider a condition (D) (below) on a pair of spaces (X, A), with X compact Hausdorff and A bounded in E, which implies that

$$C(X, \overline{\text{co}}\ A) = \overline{\text{co}}\ C(X, A).$$

[This formulation was first considered by G. Seever [7].] We then apply this to bounded closed convex subsets B of E such that (with A = ext B), the pair (X, A) satisfies condition (D) and $B = \overline{\text{co}}\ A$.

Definition. — *A pair of Hausdorff spaces* (X, A) *is said to satisfy condition* (D) *if the following holds for each* $n > 0$:

Given nonempty open sets $U_1, U_2, \ldots, U_n$ *in* A *and pairwise disjoint nonempty compact sets* $K_1, K_2, \ldots, K_n$ *in* X, *there exists* $f \in C(X, A)$ *such that* $f(K_i) \subset U_i$, $i = 1, 2, \ldots, n$.

Condition (D) is a sort of density property for the subspace C(X, A) in the space A^X of *all* functions from X to A. Indeed, condition (D) implies that C(X, A) is dense in the pointwise topology on the space A^X, while density of C(X, A) in the compact-open topology implies condition (D). As noted in [6], if X is a totally disconnected compact space, then (X, A) satisfies condition (D) for any A. On the other hand, if A is arcwise connected (or even only « almost arcwise connected » [6]), then (X, A) satisfies (D) for any compact X.

Theorem 1. — *Let* E *be a real or complex locally convex Hausdorff vector space,* X *a compact Hausdorff space and* A *a bounded subset of* E. *If* (X, A) *satisfies condition* (D), *then*

$$C(X, \overline{\text{co}}\ A) = \overline{\text{co}}\ C(X, A).$$

The proof of the theorem depends on the following two technical lemmas.

LEMMA 1. — *Suppose that* L *is a continuous linear functional on* $C(X, E)$. *Then there exists a continuous seminorm* p *on* E *and a regular Borel positive measure* μ *on* X *such that* $\mu(X) \leqslant 1$ *and*

$$|L(f)| \leqslant \int_X p(f(x))\, d\mu(x) \quad (\textit{for each } f \in C(X, E)).$$

LEMMA 2. — *Suppose that* X *and* A *are as described in the statement of the theorem. Given a continuous seminorm* p *on* E, $\varepsilon > 0$, *a regular Borel probability measure* μ *on* X *and* $f \in C(X, \overline{\text{co}}\ A)$, *there exist* $g \in \overline{\text{co}}\ C(X, A)$ *and a compact subset* $K \subset X$ *such that*

$$p(g(x) - f(x)) < \varepsilon \quad \textit{for} \quad x \in K \quad \textit{and} \quad \mu(X \setminus K) < \varepsilon.$$

Assuming that these lemmas have been proved, the theorem follows readily. Indeed, since $C(X, A) \subset C(X, \overline{\text{co}}\ A)$ and since the latter is closed and convex, we have

$$\overline{\text{co}}\ C(X, A) \subset C(X, \overline{\text{co}}\ A).$$

To show equality, it suffices to show that for each $\varepsilon > 0$, each $L \in C(X, E)^*$ and each $f \in C(X, \overline{\text{co}}\ A)$, there exists $g \in \overline{\text{co}}\ C(X, A)$ with

$$\text{Re}\ L(g) > \text{Re}\ L(f) - \varepsilon.$$

Choose p and μ according to Lemma 1, and let $M = \sup\{p(a) : a \in A\}$. Choose $K \subset X$ and $g \in \overline{\text{co}}\ C(X, A)$ according to Lemma 2, with ε replaced by $\varepsilon/2(M + 1)$. It follows that

$$\text{Re}\ L(f) - \text{Re}\ L(g) \leqslant |L(f - g)| \leqslant \int_X p(f(x) - g(x))\, d\mu(x).$$

The integral on the right is the sum of the integral over K and the integral over $X \setminus K$. From Lemma 2, the first summand is at most $\varepsilon/2(M + 1)$, while the second is at most $M\varepsilon/2(M + 1)$, hence the total is at most ε.

We now turn to the proof of Lemma 1. Since L is continuous on $C(X, E)$ it is bounded in absolute value by 1 on a neighborhood of the form

$$\{f \in C(X, E) : p(f(x)) \leqslant 1, x \in X\},$$

where p is a continuous seminorm on E. Thus,

$$(*) \qquad |L(f)| \leqslant \sup\{p(f(x)) : x \in X\}, \quad f \in C(X, E).$$

Let N denote the closed subspace $p^{-1}(0)$ and consider the space E/N, normed by the quotient norm $\|\cdots\|$ defined by p. Let φ denote the quotient map from E into $F = E/N$; the composition $f \to \varphi \circ f$ defines a linear mapping of C(X, E) into C(X, F) which satisfies

$$\|\varphi(f(x))\| = p(f(x))$$

for all $f \in C(X, E)$, $x \in X$. The space C(X, F) has the norm

$$\|g\| = \sup\{\|g(x)\| : x \in X\}.$$

It follows from $(*)$ that the formula $J(\varphi \circ f) = L(f)$ defines a continuous linear functional J of norm at most 1 on the subspace $\varphi \circ C(X, E)$ of C(X, F), and we can extend J to a functional of norm at most 1 on all of C(X, F). At this point we could apply known results, which represent $C(X, F)^*$ in terms of dominated vector valued measures [4, p. 387], but we prefer to use the following direct (and simple) proof which was kindly furnished us by Dr. Erik Thomas. Let us define, for $h \in C(X)$, $h \geqslant 0$,

$$(**) \quad \mu(h) = \sup\{|J(g)| : g \in C(X, F), \; \|g(x)\| \leqslant h(x) \text{ for } x \text{ in } X\}$$

It is straightforward to verify that $\mu(h) < \infty$, that $\mu(\lambda h) = \lambda\mu(h)$ for $\lambda > 0$, and that $\mu(h_1 + h_2) \geqslant \mu(h_1) + \mu(h_2)$ if $h_1, h_2 \geqslant 0$ are in C(X). The reverse inequality follows easily once we have the following fact: If $h = h_1 + h_2 (h_i \geqslant 0)$ and $\|g(x)\| \leqslant h(x)$ for all x in X, then there exists g_1, g_2 in C(X, F) such that $g = g_1 + g_2$ and $\|g_i(x)\| \leqslant h_i(x)$, $i = 1, 2$ and $x \in X$. Indeed, let $V = \{x \in X : \|g(x)\| > 0\}$ and for x in V let

$$\alpha_1(x) = \min(1, h_1(x)/\|g(x)\|), \qquad \alpha_2(x) = 1 - \alpha_1(x).$$

If we define $g_i(x) = \alpha_i(x)g(x)$ for $x \in V$, $= 0$ for $x \in X \setminus V$, then $g = g_1 + g_2$, $\|g_i(x)\| \leqslant h_i(x)$ and $\|g_i(x)\| \leqslant \|g(x)\|$ ($x \in X$, $i = 1, 2$). (The last inequality shows that each g_i is continuous.) Thus, μ is additive, non negative and positive homogeneous on the positive cone in $C(X)$, hence can be considered as an integral with respect to a finite positive regular Borel measure, say μ, on X. Furthermore, from $(**)$ it is obvious that $|J(g)| \leqslant \int \|g(x)\| \, d\mu(x)$ for all $g \in C(X, F)$ and that $\mu(1) = \|J\| \leqslant 1$. Finally, for $f \in C(X, E)$ we have $|L(f)| = |J(\varphi \circ f)| \leqslant \int \|\varphi(f(x))\| \, d\mu(x) = \int p(f(x)) \, d\mu(x)$, which completes the proof of Lemma 1.

We next give the proof of Lemma 2. For $x \in X$, let

$$V_x = \{y \in X : p(f(x) - f(y)) < \varepsilon/3\};$$

this is an open neighborhood of x, and we can choose $x_1, \ldots, x_n$ such that the collection $\{V_{x_1}, \ldots, V_{x_n}\}$ covers X, and such that no proper subcollection covers X. An easy induction argument, using the regularity of μ, shows that we can find another cover $\{V_1, \ldots, V_n\}$ of open sets V_i such that $V_i \subset V_{x_i}$ and such that $\mu(D) < \varepsilon$, where $D = \cup \{V_i \cap V_j : i, j = 1, 2, \ldots, n; i \neq j\}$. Let

$$K_i = V_i \setminus \cup \{V_j : j \neq i\} = X \setminus \cup \{V_j : j \neq i\}, i = 1, 2, \ldots, n.$$

Then each K_i is compact, nonempty and $K_i \cap K_j$ is empty if $i \neq j$. Furthermore, if $K = \cup K_i$, then K is compact and $X \setminus K \subset D$, hence $\mu(X \setminus K) < \varepsilon$. Now, for each $i = 1, 2, \ldots, n$ we have $f(x_i) \in \overline{\text{co}}\, A$, hence we can find $u_i \in \text{co}\, A$, with $p[u_i - f(x_i)] < \varepsilon/3$, of the following form:

$$u_i = \sum_{k=1}^{m_i} \lambda_{ik} a_{ik}, \quad \{a_{ik}\}_{k=1}^{m_i} \subset A, \quad \lambda_{ik} > 0, \quad \sum_{i=1}^{m_i} \lambda_{ik} = 1$$

where each λ_{ik} is a rational number, $k = 1, 2, \ldots, m_i$. We can assume that the numbers λ_{ik} have a common denominator $Q > 0$, so by allowing repetitions of the points a_{ik} and by relabelling, we have

$$u_i = Q^{-1} \sum_{k=1}^{Q} b_{ik}, \{b_{ik}\}_{k=1}^{Q} \subset A, i = 1, 2, \ldots, n.$$

By property (D), for each $k = 1, 2, \ldots, Q$, we can choose

$g_k \in C(X, A)$ such that

$$g_k(K_i) \subset \{v \in E : p(v - b_{ik}) < \varepsilon/3\}.$$

Let $g = Q^{-1} \sum_{k=1}^{Q} g_k$, so that $g \in \text{co}\, C(X, A)$.

Suppose that $x \in K$; then $x \in K_i$ for some i and

$$p[g(x) - u_i] = p[Q^{-1}\Sigma g_k(x) - Q^{-1}\Sigma b_{ik}] \leqslant Q^{-1}\Sigma p[g_k(x) - b_{ik}] < \varepsilon/3.$$

Since $K_i \subset V_i \subset V_{x_i}$, we have $p[f(x) - f(x_i)] < \varepsilon/3$. Thus,

$$p[g(x) - f(x)] \leqslant p[g(x) - u_i] + p[u_i - f(x_i)] + p[f(x_i) - f(x)] < \varepsilon,$$

which completes Lemma 2.

Corollary 1. — *Suppose that* B *is a bounded closed convex subset of the locally convex space* E, *and that* X *is a compact Hausdorff space. Let* $A \subset \text{ext}\, B$. *If* $B = \overline{\text{co}}\, A$ *and if* (X, A) *satisfies condition* (D), *then*

$$\overline{\text{co}}\, C(X, A) = C(X, B);$$

in particular, the latter set is the closed convex hull of its extreme points.

The hypothesis in Corollary 1 that $B = \overline{\text{co}}\, A$ is obviously a necessary one for the conclusion; indeed, since

$$C(X, A) \subset C(X, \overline{\text{co}}\, A)$$

and since the latter is closed and convex, it contains $\overline{\text{co}}\, C(X, A)$. Thus, if $C(X, B) = \overline{\text{co}}\, C(X, A)$, then $C(X, B) \subset C(X, \overline{\text{co}}\, A)$, whence $B = \overline{\text{co}}\, A$.

In general, condition (D) is not a necessary one for the validity of the equality $C(X, \overline{\text{co}}\, A) = \overline{\text{co}}\, C(X, A)$. Consider, for instance, $X = [0, 1]$, $E = C$ (complex plane) and

$$A = \{z : |z| < 1/4\} \cup \{z : 3/4 < |z| < 1\}.$$

Then $\overline{\text{co}}\, A = \{z : |z| \leqslant 1\}$ is compact, and the above equality holds, but it is easily seen that C(X, A) is not even pointwise dense in A^X. If, however, A is the set of extreme points of $\overline{\text{co}}$ A — this is the situation we are mainly interested in — then there is a partial converse to Theorem 1.

THEOREM 2. — *If* B *is a compact convex subset of the locally convex space* E *and* $A = \text{ext } B$ (*so* $B = \overline{\text{co}}\, A$), *then the equality*

$$\overline{\text{co}}\, C(X, A) = C(X, \overline{\text{co}}\, A)$$

implies that $C(X, A)$ *is pointwise dense in* A^X.

We omit the proof, since it, closely parallels that of Theorem 3.1 of [6], in which E is a dual Banach space (in the weak* topology) and B is its unit ball. The same argument works in the general case, using the fact that each extreme point of B has a neighborhood base in B consisting of « slices » [3, p. 108].

We now consider some applications of the foregoing results to spaces of linear operators. Suppose that M is a real (resp. complex) Banach space and let C(X) denote the real (resp. complex) continuous functions on the compact Hausdorff space X. The space $\mathcal{L}(M, C(X))$ (or simply $\mathcal{L}$) of all bounded linear operators from M into C(X) is linearly isomorphic to the space C(X, E), where $E = M^*_{w^*}$ is the space M^* in its weak* topology [5, p. 490]. The correspondence between an operator T in $\mathcal{L}$ and a function f in C(X, E) is defined by

$$(Tm)(x) = \langle m, f(x)\rangle, \qquad (x \in X,\ m \in M).$$

Moreover, $\|T\| = \sup\{f(x)\| : x \in X\} = \|f\|$.

Thus, the unit ball $\mathcal{U}$ of $\mathcal{L}$ may be identified with the subset $C(X, U^*)$ of C(X, E), where U^* is the unit ball of M^*. This correspondence was used in [6] to obtain various corollaries to Theorem 1, which was proved there for this particular choice of E. Similarly, the subspace

$$\mathcal{L}_c = \mathcal{L}_c(M,\ C(X))$$

of all compact operators in $\mathcal{L}$ can be identified with the subspace $C(X, M^*_n)$, of C(X, E), where M^*_n is M^* in its norm topology [5], and Theorem 1 was also proved in [6] for this case. It is readily verified that the uniform topology on C(X, E) carries over (under the correspondence indicated above) to the strong operator topology on $\mathcal{L}$, and that in $C(X, M^*_n)$ the uniform topology is the norm topology (norm defined as above) and this identifies on $\mathcal{L}_c$ with the norm (or « uniform operator ») topology. The fact that Theorem 1 was proved for arbitrary E allows us to consider the case where $E = M^*_w$, the space M^* in its weak (i.e. $\sigma(M^*, M^{**})$) topology. Under the above correspondence, $C(X, M^*_w)$ is exactly the space $\mathcal{L}_{wc} = \mathcal{L}_{wc}(M, C(X))$ of all weakly compact operators from M into C(X). The topology induced on $\mathcal{L}_{wc}$ by the uniform topology on $C(X, M^*_w)$ is not one of the usual « operator » topologies, but is easily seen to be between the strong operator and norm topologies on $\mathcal{L}_{wc}$.

We will denote by $\mathcal{U}$, $\mathcal{U}_c$ and $\mathcal{U}_{wc}$ the unit ball of $\mathcal{L}$, $\mathcal{L}_c$ and $\mathcal{L}_{wc}$ respectively. These are, of course, the same as the sets $C(X, U^*)$, $C(X, U^*_n)$ and $C(X, U^*_w)$. An operator which corresponds to an element f of one of these sets such that $f(X) \subset \text{ext } U^*$ is called a *nice* (resp. nice compact, nice weakly compact) operator. They are of course, extreme points of the sets $\mathcal{U}$, $\mathcal{U}_c$ and $\mathcal{U}_{wc}$ respectively.

The next result is almost a direct application of Corollary 1 to the ball of weakly compact operators. The main point is to account for the difference between the two topologies involved.

PROPOSITION 1. — *Let* M *and* C(X) *be as above, and let* U^* *be the unit ball of* M^*. *Suppose that there is a subset* $A \subset \text{ext } U^*$ *such that:*

(i) *The pair* (X, A_w) *satisfies condition* (D).

(ii) U^* *is the norm closed convex hull of* A.

Then the unit ball $\mathcal{U}_{wc}$ *of* $\mathcal{L}_{wc}$ *is the strong operator closed convex hull of the nice weakly compact operators.*

Proof. — Hypotheses (*i*) and (ii) allow us to apply Corollary 1 to obtain the equality $C(X, U^*_w) = \overline{\text{co}}\, C(X, A_w)$, where the closure is in the uniform topology of $C(X, M^*_w)$. Since $C(X, M^*_w) \subset C(X, M^*_{w^*})$, the uniform topology on the latter space induces a topology on $C(X, M^*_w)$ which is weaker than the original; we will call it the « strong » topology since it corresponds exactly to the strong operator topology on $\mathcal{L}_{wc}$. Thus, we want to show that $C(X, U^*_w)$ is the strong closed convex hull of $C(X, A_w)$, since the latter is clearly a subset of the nice weakly compact operators. But it is easily verified that (since U^* is weak* closed) $C(X, U^*_w)$ is strongly closed in $C(X, M^*_w)$, hence contain the strong closure of co $C(X, A_w)$, which in turn contains $\overline{\text{co}}\, C(X, A_w) = C(X, U^*_w)$.

The fact that in hypothesis (ii) above we used the norm closure instead of the weak closure (which Corollary 1 would have allowed) is no loss in generality, of course, since the set involved is convex.

Recall that a real or complex Banach space M is said to be *smooth* if for each point $x \in S(M) = \{x \in M : \|x\| = 1\}$ there exists a unique functional f_x in the unit sphere $S(M^*)$ of M^* such that $\operatorname{Re} f_x(x) = 1$. This is equivalent to Gateaux differentiability of the norm (at each nonzero point), and the functional f_x is the Gateaux differential of the norm at x.

THEOREM 3. — *Let* M *be a real or complex Banach space and* X *a compact Hausdorff space. In the real case, we assume that* $\dim M > 1$.

(*a*) *If* M *is smooth, then* $\mathfrak{U}$ *is the strong operator closed convex hull of the nice operators.*

(*b*) *If the norm in* M *is Fréchet differentiable at each nonzero point, then* $\mathfrak{U}_{wc}$ [resp. $\mathfrak{U}_c$] *is the strong operator* [resp. norm] *closed convex hull of its nice operators.*

Proof. — (*a*) It is well known (and easily proved) that if M is smooth, then the map $x \to f_x$ defined above is continuous from S(M) in its norm topology into $S(M^*)$ in its weak* topology. It is readily verified that U^* is the weak* closed convex hull of the image A of S(M) under this map, and that $A \subset \operatorname{ext} U^*$. [In fact, A is known [1] to be norm dense in $S(M^*)$.] Since S(M) is arcwise connected (in the real case this assertion obviously requires that $\dim M > 1$), the set A is arcwise connected in the weak* topology. Thus, (X, A) satisfies condition (D) so Corollary 1 yields the desired conclusion.

(*b*) The Féchet differentiability of the norm in M implies that the derivative map $x \to f_x$ defined above is continuous from the norm topology on S(M) into the norm topology on $S(M^*)$. With the same notation as in (*a*), the set A is norm arcwise connected and norm dense in $S(M^*)$, hence U^* is the norm closed convex hull of A and Proposition 1 [resp. Corollary 1] applies.

In the case when $M = C(X)$ for some compact Hausdorff space X, it is possible to obtain necessary and sufficient conditions on X and Y that $\mathcal{U}_{wc} \subset \mathcal{L}_{wc}(C(X), C(Y))$ be the strong operator closed convex hull of the nice weakly compact operators. These conditions are the same as those in Theorem 4.6 of [6], and the methods for obtaining them are essentially the same. (We don't know, in this case, whether every extreme element of $\mathcal{U}_{wc}$ is a nice operator.) Similar results hold in the real case for the set of positive normalized weakly compact operators.

The following problem arises in the context of Corollary 1: Suppose that $C(X, B) = \overline{\text{co}}$ ext $C(X, B)$. Must ext B be nonempty?

[*Note added in proof:* J. Lindenstrauss (private communication) has answered this question in the negative by showing that there exists a normed linear space E, a nonempty convex closed and bounded subset $B \subset E$ and a nonempty compact Hausdorff space X such that ext B is empty, but $C(X, B) = \overline{\text{co}}$ ext $C(X, B)$.]

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