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PSEUDO-DIFFERENTIAL OPERATORS AND GEVREY CLASSES

by Louis BOUTET de MONVEL and Paul KRÉE

Singular-integral operators have been studied extensively recently. In this paper we will use the complete symbolic calculus of J. J. Kohn and L. Nirenberg. Our aim is to construct classes of pseudo-differential operators which are continuous on the Gevrey classes and on the associated hyperdistribution spaces. In order to do so, we impose regularity conditions on the symbol of these operators, and " asymptotic " conditions which relate their symbol to their kernels. The results in this paper were announced in [1].

In § 0 we recall briefly the definitions of the Gevrey classes and of the pseudo-differential operators, in order to fix our notations. § 1 describes the symbols and § 2 the operators themselves. It is shown that the composed of two such operators, their adjoints, and their parametrix (when they are elliptic) have the same properties. An easy application to elliptic equations with coefficients in a Gevrey class is made. Our study includes the analytic case, where results are more precise.

0. Preliminaries.

1. In this paper, Ω always denotes an open subset of R^n.

We shall use the notations of L. Hörmander [6] for the classical spaces of differentiable functions and distributions.

We use the integral notation for the duality between functions and distributions : if $T \in \mathcal{D}'(\Omega)$ is a distribution and $\varphi \in C_0^\infty(\Omega)$ a function, the value of T on φ is written

$$\int_\Omega T(x)\,\varphi(x)\,dx \qquad \text{or} \qquad \int_\Omega T\varphi \quad \text{for short.}$$

If f is a distribution with compact support, $\hat{f}$ will denote its Fourier transform :

$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx.$$

(Thus we have $f(x) = (1/2\pi)^n \int e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi$ if $\hat{f} \in L^1(R^n)$.)

If S and T are two distributions, S $*$ T will denote their convolution product (as defined in [6] or [12]).

Finally if P is a continuous linear operator $C_0^\infty(\Omega) \to \mathcal{D}'(\Omega)$, we shall call kernel of P (cf. [13]) the unique distribution $\Omega \times \Omega$ which satisfies

$$\int_\Omega P(\varphi)(x)\, \psi(x)\, dx = \int_{\Omega\times\Omega} T(x, y)\, \psi(x)\, \varphi(y)\, dx\, dy$$

for any test functions $\varphi, \psi \in C_0^\infty(\Omega)$.

2. Gevrey classes.

(For any detail on what follows, the reader may consult [4], [10], and [9], [11] in the analytic case.)

A function $f \in C^\infty(\Omega)$ is said to be of class s if for any compact set $K \subset \Omega$, there exist constants C, A such that for any $x \in K$ and any multi-index α, the following inequality be true :

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha f(x) \right| \leqslant c\, A^{|\alpha|} (\alpha!)^s \tag{0.1}$$

(we have set $(\partial/\partial x)^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$).

The Gevrey class $G^s(\Omega)$ is the set of all functions that are of class s on Ω. It is a vector space ; it is closed under multiplication and derivation ; moreover, $f \circ g$ is of class s if f and g are.

If K is a compact subset of Ω, $G^s(K)$ will denote the space of all functions which are of class s in a neighborhood of K. (It is a quotient space of $G^s(\Omega)$ if $s > 1$, but not if $s = 1$).

If $s > 1$, $G_0^s(\Omega)$ will denote the space of all functions of class s with compact support.

If $s = 1$, $G^s(\Omega) = H(\Omega)$ is well known to be the space of all analytic functions on Ω ; $G^s(K) = H(K)$ is the space of functions which are analytic in a neighborhood of K.

We define a bounded subset of $G^s(\Omega)$ (resp. $G^s(K)$, $G_0^s(\Omega)$) to be a subset B such that all $f \in B$ satisfy uniformly condition (0.1) (i.e. with constants C, A which depend on K but not on f) (resp. if the $f \in B$ are the restrictions to K of the elements of a bounded subset of $G^s(V)$ for a convenient neighborhood V of K - resp. if it is bounded in $G^s(\Omega)$, and every $f \in B$ vanishes outside some fixed compact subset of Ω).

All these spaces are given the strongest locally convex topology for which the subsets previously described are bounded. (It is easily shown that there are no other bounded set for this topology.)

Finally $G_0'^s(\Omega)$ (resp. $G'^s(\Omega)$ (resp. $G'^s(K)$, $G'^s(\Omega)$) will denote the dual space of $G^s(\Omega)$ (resp. $G^s(K)$, $G_0^s(\Omega)$) with the strong topology : its elements are hyper-distributions with compact support (resp. support contained in K, resp. any support).

All these spaces are well known to be nuclear, complete, bornologic and barelled. $G^s(K)$ is a (DF) space, and $G'^s(K)$ is a Frechet space.

Let us recall that even in the analytic case ($s = 1$), the support of an ultra-distribution is well defined : if $T \in G_0'^s(\Omega)$, supp T is the smallest compact subset of Ω such that T can be extended continuously to $G^s(K)$. It is shown that there actually exists one (cf. [9]).

Still in the analytic case, we shall denote by $G'^s(\Omega) = H'(\Omega)$ the space of all « hyperfunctions » on Ω, in the sense of M. Sato (cf. [9], [11]) — one obtains them by « sticking » together real analytic functionals with compact support.

We shall also need spaces of hyper-distributions which are regular in some part of Ω : if K is a compact subset of Ω, we shall denote by $G_0'^s(\Omega) \cap G^s(K)$ the space of all hyper-distributions with compact support which are functions of class s in a neighborhood of K. A subset B of this space is said to be bounded if there exists a neighborhood V of K and a function $\varphi \in C_0^\infty(V)$ equal to 1 in a neighborhood of K such that all $f \in B$ are of class s in V, and their restrictions to V remain in a bounded subset of $G^s(V)$, and if moreover the hyper-distributions

$$(1 - \varphi) f = f - \varphi f, \ f \in B$$

remain in a bounded subset of $G_0'^s(\Omega - K)$ (φf stands for the function equal to $\varphi(x) f(x)$ if $x \in V$ and to 0 if $x \notin V$; $f - \varphi f$ is clearly seen to be vanishing in a neighborhood of K; therefore it belongs to $G_0'^s(\Omega - K)$). The topology of $G_0'^s(\Omega) \cap G^s(K)$ is the strongest locally convex topology for which these sets are bounded. It is easily proved that there are no other bounded sets; and that for this topology, $G_0'^s(\Omega) \cap G^s(K)$ is nuclear, complete, bornologic, and barelled. We emphasize that for $s = 1$, this topology is not the intersection of those of $G^s(K)$ and $G_0'^s(\Omega)$ (for this last topology, our space would not be complete).

Remark. — One could define in the same way a reasonable topology on the space $G'^s(\Omega) \cap G^s(\Omega - K)$ of all hyper-distributions on Ω which are functions of class s on $\Omega - K$. For this topology, it is the strong dual of $G_0'^s(\Omega) \cap G^s(K)$ if we define the duality in the following way: if $f \in G_0'^s(\Omega) \cap G^s(K)$ and $g \in G'^s(\Omega) \cap G^s(\Omega - K)$, we choose $\varphi \in C_0^s(\Omega)$ equal to 1 in a neighborhood of the singular support K′ of f, and vanishing in a neighborhood of K, and we set

$$\int_\Omega fg = \int_{\Omega - K} (f\varphi)\cdot g + \int_{\Omega - K'} f\cdot[g(1-\varphi)].$$

The result does not depend on the choice of φ.

Finally we state without proof.

PROPOSITION 0.1. — *Let* E *denote the space of all functions on* R^n *which are smaller than* $c \exp(-\varepsilon |\xi|^{1/s})$ *for suitable constants* c, ε, *with the evident inductive limit topology. Then the Fourier transform is continuous from* $G_0^s(R^n)$ *to* E *and from* E *to* $G^s(R^n)$.

3. Pseudo-differential operators.

We shall only recall very briefly their definition here. We refer to [7], [8] for the demonstrations.

A pseudo-differential operator $P = P(x, D)$ on Ω is defined by the following formula:

$$P\cdot f(x) = \left(\frac{1}{2\pi}\right)^n \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi)\, d\xi,$$

where $p(x, \xi)$ is a smooth function on $\Omega \times R^n$, and admits when $\xi \to \infty$ an asymptotic expansion in homogeneous functions of ξ :

$$p(x, \xi) \sim \sum_{k=0,1,\ldots} p_k(x, \xi) \tag{0.3}$$

$p_k(x, \xi)$ is a smooth function on $\Omega \times R^n - \{0\}$, homogeneous of degree $r - k$ with respect to ξ (this is a slight restriction to the definition of [7]).

We will call symbol of P the formal series

$$\sigma(P) = \sum p_k(x, \xi). \tag{0.4}$$

The degree of P is the degree r of its principal symbol $p_0(x, \xi)$.

More generally we will call pseudo-differential operator the operator

$$P' = P(x, D) + R$$

if $P(x, D)$ is of the type described above, and R is an operator with a smooth kernel (thus R is continuous $\mathcal{E}'(\Omega) \to C^\infty(\Omega)$).

Following Hörmander, we shall say a pseudo-differential operator P is compactly supported if it is continuous $C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ and can be extended continuously $C^\infty(\Omega) \to C^\infty(\Omega)$ (equivalently if for every compact subset K of Ω, there exists another one K′ such for any $\varphi \in C_0^\infty(\Omega)$, $P \cdot \varphi$ vanishes outside K′ (resp. inside K) if φ vanishes outside K (resp. inside K′)). Such an operator can always be represented by formula (0.2) (although not in a unique way if $\Omega \neq R^n$).

If P and Q are two pseudo-differential operators that can be composed (for instance if P or Q is proper) $P \circ Q$ is a pseudo-differential operator, with symbol

$$\begin{aligned}\sigma(P, Q) &= \sum_\gamma \frac{1}{\gamma!}\left(\frac{\partial}{\partial\xi}\right)^\gamma \sigma(P) \cdot \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\gamma \sigma(Q) \\ &= \sum_{\gamma, k, l} \frac{1}{\gamma!}\left(\frac{\partial}{\partial\xi}\right)^\gamma P_k(x, \xi) \cdot \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\gamma q_l(x, \xi). \end{aligned} \tag{0.5}$$

If P is a pseudo-differential operator, the transposed operator ${}^tP$ (defined by $\int_\Omega {}^tP(\varphi) \cdot \psi = \int_\Omega \varphi \cdot P(\psi)$ for all $\varphi, \psi \in C_0(\Omega)$ is also a pseudo-differential operator, with symbol

$$\sigma({}^{t}\mathrm{P}) = \sum_{\gamma} \frac{(-1)^{\gamma}}{\gamma!} \left(\frac{\partial}{\partial\xi}\right)^{\gamma} \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^{\gamma} \sigma(\mathrm{P})(x, -\xi)$$
$$= \sum_{\gamma, k} \frac{(-1)^{\gamma}}{\gamma!} \left(\frac{\partial}{\partial\xi}\right)^{\gamma} \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^{\gamma} \mathrm{P}_k(x, -\xi). \qquad (0.6)$$

Finally let us recall that an elliptic pseudo-differential operator P (i.e. $p_0(x, \xi) \neq 0$ if $\xi \neq 0$) admits a parametrix — i.e. there exists a pseudo-differential operator E such that $\mathrm{P} \circ \mathrm{E} - 1$ and $\mathrm{E} \circ \mathrm{P} - 1$ have smooth kernels.

4. We end this section by describing some properties of the Fourier transform of some analytic functions — in view of § 1 and § 2. For the sake of brevity, we do not attempt at any general theorem.

For any number ε such that $0 < \varepsilon < 1$, let C_ε design the cone of all $x \in \mathrm{C}^n$ such that $|\mathrm{Im}\, x| < \varepsilon\, |\mathrm{Re}\, x|$; and let U_ε design the set of all matrices $g \in \mathrm{GL}(n, \mathrm{C})$ that can be factored in the following way:

$$g = (l + ih)\cdot g' \qquad (0.7)$$

where $g' \in \mathrm{GL}(n, \mathrm{R})$, $\det g' > 0$ and $h \in gl(n, \mathrm{R})$, $||h|| < \varepsilon$.

U_ε is connected open set in $\mathrm{GL}(n, \mathrm{C})$. It is stable by the symmetry $g \to {}^{t}g^{-1}$ (because ${}^{t}g^{-1} = (l - i\,{}^{t}h)(l + {}^{t}h^2)^{-1}\,{}^{t}g'^{-1}$). Finally $g\cdot x$ describes exactly C_ε when g describes U_ε and x describes $\mathrm{R}^n - \{0\}$.

The following lemma is straightforward:

LEMMA 0.2. — *Let* T *design a hyper-distribution on* R^n *such that* $g \to \mathrm{T}(g\,x)$ *(which is well defined when* $g \in \mathrm{GL}(n, \mathrm{R})$*) can be extended into a holomorphic function on* U_ε *with values in some distribution space. Then the restriction of* T *to* $\mathrm{R}^n - \{0\}$ *can be extended in a holomorphic function on* C_ε.

LEMMA 0.3. — *Let* f *be a bounded holomorphic function on* C_ε. *Then the mapping* $g \to f(g\,x) \in \mathrm{L}^\infty(\mathrm{R}^n)$ *is a bounded holomorphic function on* U_ε, *with values in* $\mathrm{L}^\infty(\mathrm{R}^n)$.

This follows from the Taylor and Cauchy formulas: we have

$$f((l + h)\cdot x) = \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial}{\partial x}\right)^{\alpha} f(x)\,(h\,x)^{\alpha} \qquad (0.8)$$

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} f(x) = \left(\frac{1}{2i\pi}\right)^{n} \iint_{|z_1| = \dots |z_n| = \lambda |x|} \frac{(-1)^{|\alpha|}\, \alpha!\, f(x+z)\, dz_1 \dots dz_n}{z_1^{\alpha_1} \dots z_n^{\alpha_n}} \tag{0.9}$$

(We compute the integral on the polysphere $|z_1| = \dots |z_n| = \lambda\, |x|$ conveniently oriented ; λ is chosen so small that it lies inside of the cone C_ϵ).

The second equality (Cauchy's formula) proves that for large A,

$$\frac{A^{-|\alpha|}}{\alpha\,!} \left(\frac{\partial}{\partial x}\right)^{\alpha} f(x) \cdot |x|^{|\alpha|}$$

is uniformly bounded on R^n (i. e. the bound does not depend on α). Thus the series (0.8) converges in $L^\infty(R^n)$ for small $h \in gl(n, C)$, and the mapping $g \to f(g\,x)$ is holomorphic in a neighborhood of $g = 1$. To prove it also is in a neighborhood of any $g_0 \in C_\epsilon$, we change f to $f(g \circ x)$, and diminish ε conveniently, and apply the first result.

The two following propositions are almost immediate consequences :

PROPOSITION 0.4. — *Design by* $H_{\epsilon,\alpha}$ *the space of all distribution which are homogeneous of degree* α, *and outside the origin can be extended into a holomorphic function on* C_ϵ. *The Fourier transform is continuous from* $H_{\epsilon,\alpha}$ *to* $H_{\epsilon,-n-\alpha}$.

(With Lemma 0.3, it is easy to prove that $g \to f(g\,x)$ is a holomorphic function on U_ϵ with values in a space of temperate distributions if $f \in H_{\epsilon,\alpha}$. Thus the Fourier transform $\widehat{f(g\,x)} = \deg g)^{-1} \hat{f}({}^t g^{-1}\,\xi)$ has the same property; and we have seen that ${}^t g^{-1}$ ranges in the whole U_ϵ when g does the same.)

Finally let $E_{\epsilon,\alpha}$ design the Banach space of all distributions $f \in \mathcal{D}'(R^n)$ which outside of the origin can be extended into a holomorphic function on C_ϵ such that $|f(x)| \leqslant c\,|z|^\alpha$ for suitable c, and near the origin, can be decomposed into

$$f = \sum_{\alpha + k < n} p \cdot f \cdot f_k + f' + f''$$

where f_k is homogeneous, of degree $\alpha + k + 1$, holomorphic on C_ϵ and bounded on the set $x \in C_\epsilon$, $|x| = 1$ (the finite part is defined as in [12]); f' is a sum of derivatives of order $< -n - \alpha$ of the Dirac distribution δ if α is entire ($f' = 0$ if it is not), and f'' is holomorphic in C_ϵ and

$| f'(z) | \leqslant c | z |^{\alpha + k_0}$ for suitable c, where k_0 is the first positive integer such that $\alpha + k_0 > -n$.

PROPOSITION 0.5. — *If $\alpha < -n$, and if $\varepsilon' < \varepsilon$, the Fourier transform is continuous from $E_{\epsilon, \alpha}$ to the space of functions which are holomorphic on C_ϵ, and bounded in a neighborhood of the origin.*

The proof is similar to the preceding one, when one has remarked that if $f \in E_{\epsilon, \alpha}$, then the mapping $g \to f(g\,x)$ is holomorphic in U_ϵ, with values in a space of integrable distributions on R^n.

1. The symbols of class *s*.

As before, Ω designs an open set in R^n. Let $(p) = \sum p_k(x, \xi)$ be a symbol on Ω. Thus $p_k(x, \xi)$ is a smooth function on $\Omega \times (R^n - \{0\})$, homogeneous of degree $r - k$ with respect to ξ.

DEFINITION 1.1. — *The symbol $(p) = \sum p_k(x, \xi)$ is said to be of class $s \geqslant 1$ if for any given compact set $K \subset \Omega$ there exists constants* c, A *such that for any integer k, any multi-indexes α, β, and any $x \in K$, the following inequality holds :*

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta p_k(x, \xi) \right| \leqslant c\, A^{k+|\alpha+\beta|} \, |\xi|^{r-k-|\beta|} (k + |\alpha|)!^s \beta! \tag{1.1}$$

Let us set

$$p^\beta_{k,\alpha} = \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta p_k(x, \xi).$$

Then Definition (1.1) is equivalent to the following : the series

$$N_s((p), T) = \sum_{\alpha, \beta, k} \left(\frac{2\,(2\,n)^{-k}\, k!}{(k + |\alpha|)!^s \, (k + |\beta|)!} \right) |p^\beta_{k,\alpha}| T^{2k+|\alpha+\beta|} \tag{1.2}$$

is a convergent power series, uniformly (with respect to T) when (x, ξ) ranges in a compact subset of $\Omega \times (R^n - \{0\})$. $N_s((p), T)$ will be called the formal norm of the symbol (p). (The monstrous coefficient in (1.2) is justified by Lemma 1.2 below. Of course for the equivalence with Definition 1.1 it could be considerably simplified.)

It follows from Definition 1.1 that $p_k(x, \xi)$ is holomorphic with respect to ξ when $A \mid \mathrm{Im}\,\xi \mid < \mid \mathrm{Re}\,\xi \mid$. In fact Definition 1.1 is equivalent to the following:

DEFINITION 1.1*bis*. — *For any compact set* $K \subset \Omega$ *there exist constants* ε, c, A *such that for any* $x \in K$, p_k *is holomorphic with respect to* ξ *when* $\mid \mathrm{Im}\,\xi \mid < \varepsilon \mid \mathrm{Re}\,\xi \mid$, *and that for any* $x \in K$, $\xi \in C_\epsilon$ (i. e. $\xi \in C^n$ *and* $\mid \mathrm{Im}\,\xi \mid < \varepsilon \mid \mathrm{Re}\,\xi \mid$), *any multi-indexes* α, β *and any integer* k, *the following inequality holds*

$$\left| \left(\frac{\partial}{\partial x} \right)^{\alpha} p_k(x, \xi) \right| \leqslant c\, A^{k+|\alpha|} \mid \xi \mid^{r-k} (k + \mid \alpha \mid)\,!^{s}. \qquad (1.1bis)$$

The fact that (1.1) implies (1.1*bis*) follows from the Taylor formula:

$$\begin{aligned} \mid p_{k,\alpha}(x, \xi + \eta) \mid &= \sum_\beta \frac{1}{\beta\,!} p_{k,\alpha}^{\beta}(x, \xi)\, \eta^\beta \mid \\ &\leqslant c\, A^{k+|\alpha|} \mid \xi \mid^{r-k} (k + \mid \alpha \mid)\,!^{s} \sum_\beta \left(\frac{\beta\,!}{\beta\,!} A^{|\beta|} \mid \xi \mid^{-\beta} \eta^\beta \right) \\ &\leqslant c\, A^{k+|\alpha|} \mid \xi \mid^{r-k} (k + \mid \alpha \mid\,!^{s} \quad (1 - A \mid \eta \mid \mid \xi \mid^{-1})^{-n} \end{aligned}$$

if $A \mid \eta \mid < \mid \xi \mid$.

The implication in the other direction follows from the Cauchy formula, applied as in Lemma 0.3, and we leave the easy proof to the reader.

Recall that if (p) and (q) are two symbols, the composed symbol $(p) \circ (q) = (r)$ is given by formula (0.5):

$$(r) = \sum_\gamma \frac{1}{\gamma\,!} \left(\frac{\partial}{\partial \xi} \right)^{\gamma} (p) \cdot \left(\frac{1}{i} \frac{\partial}{\partial x} \right)^{\gamma} (q).$$

Thus

$$r_m = \sum_{k+l+|\gamma|=m} \frac{1}{\gamma\,!} \left(\frac{\partial}{\partial \xi} \right)^{\gamma} p \cdot \left(\frac{1}{i} \frac{\partial}{\partial x} \right)^{\gamma} q_l$$

and

$$r_{m,\alpha}^{\beta} = \sum \frac{1}{\gamma\,!} \binom{\alpha}{\alpha'} \binom{\beta}{\beta''} p_{l,\alpha'}^{\gamma+\beta'} q_{l,\gamma'+\alpha''}^{\beta''}, \qquad (1.3)$$

where the last sum ranges over all integers and multi-indexes k, l, α', α'', β', β'', γ such that $k + l + |\gamma| = m$, $\alpha' + \alpha'' = \alpha$, $\beta' + \beta'' = \beta$.

$$\binom{\alpha}{\alpha'} \text{ and } \binom{\beta}{\beta''}$$

stand for the " binomial " coefficients

$$\alpha\,!\,/\,\alpha'\,!\,\alpha''\,!\,, \quad \beta\,!\,/\,\beta'\,!\,\beta''\,!\,.$$

LEMMA 1.2. — *The following inequalities are true :*

$$\mathrm{N}_s\,((p) + (q), \mathrm{T}) \ll \mathrm{N}_s\,((p), \mathrm{T}) + \mathrm{N}_s\,((q), \mathrm{T})$$
$$\mathrm{N}_s\,((p) \circ (q), \mathrm{T}) \ll \mathrm{N}_s\,((p), \mathrm{T}) \cdot \mathrm{N}_s\,((q), \mathrm{T})$$

(*the sign* $\ll$ *means that the coefficient of* T^k *in the first member is less than the same in the second member.*)

The first inequality is obvious. To prove the second, let us set $c_{k,\alpha}^{\beta} = 2\,(2\,n)^{-k}\,k\,!\,(k + |\alpha|)\,!^{-s}\,(k + |\beta|)\,!^{-1}$; and let us estimate $|r_{m,\alpha}^{\beta}|$ by the sum of all absolute values in the second member of (1.3). We then get :

$$\mathrm{N}_s\,((r), \mathrm{T}) \ll \sum_{m,\,\alpha,\,\beta} c_{m,\alpha}^{\beta}\,\mathrm{T}^{2m+|\alpha+\beta|}$$
$$\cdot \left\{ \sum_{\substack{k+l+|\gamma|=m \\ \alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} \frac{1}{\gamma\,!} \binom{\alpha}{\alpha'} \binom{\beta}{\beta''} \cdot |p_{k,\alpha'}^{\beta'}|\,|q_{l,\alpha''}^{\beta''}| \right\}$$
$$= \sum_{\substack{k,\,l,\,\alpha' \\ \alpha'',\,\beta'\beta''}} c_{k,\alpha'}^{\beta'}\,|p_{k,\alpha'}^{\beta'}|\,\mathrm{T}^{2k+|\alpha'+\beta'|}\,c_{l,\alpha''}^{\beta''}\,|q_{l,\alpha''}^{\beta''}|\,\mathrm{T}^{2l+|\alpha''+\beta''|}.$$
$$\cdot \left\{ \sum_{\gamma} \frac{1}{\gamma\,!} \binom{\alpha}{\alpha'} \binom{\beta}{\beta''} \frac{c_{m,\alpha}^{\beta}}{c_{k,\alpha'}^{\beta'}\,c_{l,\alpha''}^{\beta''}} \right\} \tag{1.4}$$

In the last sum, we have set $\alpha' + \alpha'' - \gamma$, $\beta = \beta' + \beta'' - \gamma$, $m = k + l + |\gamma|$. After simplification, this is shown to be equal to

$$\sum_{\gamma \leqslant \alpha'',\,\beta'} \frac{1}{2} (2n)^{-|\gamma|} \frac{(k + l + |\gamma|)!}{k!\,l!\,\gamma!} \binom{\alpha}{\alpha'} \binom{\beta}{\beta''} \binom{m + |\alpha|}{k + \alpha'}^{-s} \binom{m + |\beta|}{l + |\beta''|}^{-1}.$$

Using the easy inequalities

$$\binom{\alpha}{\alpha'} \binom{m}{k} \leqslant \binom{m + |\alpha|}{k + |\alpha'|}, \qquad \binom{\beta}{\beta''} \binom{m}{l} \leqslant \binom{m + |\beta|}{l + |\beta''|}$$

and

$$\frac{(k+l+|\gamma|)!}{k!\,l!\,|\gamma|!}\binom{m}{k}^{-s}\binom{m}{l}^{-1} \leqslant \frac{|\gamma|!}{\gamma!}\,\frac{(k+|\gamma|)!\,(l+|\gamma|)!}{|\gamma|!\,(k+l+|\gamma|)!}$$

$$\leqslant \frac{|\gamma|!}{\gamma!} \text{ if } s \geqslant 1$$

we find that the last sum in (1.4) is less than

$$\sum_{\gamma} \frac{1}{2}(2n)^{-|\gamma|}\frac{|\gamma|!}{\gamma!} = \frac{1}{2}\left(1-\frac{1}{2n}-\dots\frac{1}{2n}\right)^{-1} = 1.$$

Therefore it follows from (1.4) that

$$\mathrm{N}_s(r, \mathrm{T}) \ll \sum c^{\beta}_{k,\alpha'}\,|p^{\beta}_{k,\alpha'}|\,\mathrm{T}^{2k+|\alpha'+\beta'|}\,c^{\beta''}_{l,\alpha''}\,|q^{\beta''}_{l,\alpha''}|\,\mathrm{T}^{2l+|\alpha''+\beta''|}$$

which proves Lemma 1.2.

The following propositions follow immediately.

Proposition 1.3. — *Let* (p) *and* (q) *be two symbols of class* (s). *Then the composed symbol* $(r) = (p)\circ(q)$ *is also of class* (s).

Proof. — We have $\mathrm{N}_s((r), \mathrm{T}) \ll \mathrm{N}_s((p), \mathrm{T})\cdot \mathrm{N}_s((q), \mathrm{T})$, so that the formal norm of (r) converges whenever those of (p) and (q) do so.

Proposition 1.4. — *Let* (p) *be an elliptic symbol of class* s (i.e. $p_0(x, \xi) \neq 0$ *when* $\xi \neq 0$). *Then the inverse symbol* (q) *of* (p) *is also of class* s.

Proof. — (q) is the unique symbol such that $(p)\circ(q) = (q)\circ(p) = 1$.

Let (q') design the symbol whose only non vanishing term is the first: $q'_0 = p_0^{-1}$. It is of course of class s.

Define (h) by $(q')\circ(p) = 1 - (h)$. Thus (h) is a symbol of class s, and degree -1.

Now put $(q'') = \sum_{p=0}^{\infty} (h)^p$.

The inverse of (p) is $(q) = (q'')\circ(q')$.

Since (q') is of class s, we only need to prove that (q'') also is. This follows from the inequality

$$N_s((q''), \mathrm{T}) \ll \sum_{0}^{\infty} \mathrm{N}_s((h), \mathrm{T})^p = [1 - \mathrm{N}_s((h), \mathrm{T})]^{-1}.$$

The last series is a convergent one since $N_s((h), T)$ is a convergent series, and has no constant term.

PROPOSITION 1.5. — *Let (p) be a symbol of class s. Then the transposed symbol $({}^t p)$ is also of class s.*

$({}^t p)$ is defined by (0.6):

$$({}^t p) = \sum_\gamma \frac{1}{\gamma!} \left(\frac{\partial}{\partial \xi}\right)^\gamma \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\gamma (p)(x, -\xi).$$

We shall show $N_s(({}^t p), T) \ll 2N_s((p), \sqrt{2}T)$, which will prove the proposition. Effectively we have

$$N_s(({}^t p), T) \ll \sum_{k, \alpha, \beta, \gamma} c_{k+|\gamma|, \alpha}^{\beta} \frac{1}{\gamma!} |p_{k, \alpha+\gamma}^{\beta+\gamma}| T^{2k+2|\gamma|+|\alpha+\beta|}$$
$$= \sum_{k, \alpha, \beta} c_{k, \alpha}^{\beta} |p_{k, \alpha}^{\beta}| T^{2k+|\alpha+\beta|} \left\{ \sum_{\gamma \leqslant \alpha, \beta} \frac{1}{\gamma!} \frac{c_{k+|\gamma|, \alpha-\gamma}^{\beta-\gamma}}{c_{k, \alpha}^{\beta}} \right\}$$

The last sum in this inequality is equal to

$$\sum_{\gamma \leqslant \alpha, \beta} \frac{1}{\gamma!} \frac{2(2n)^{-k-|\gamma|}(k+|\gamma|)!}{(k+|\alpha|)!^s (k+|\beta|)!} \frac{(k+|\alpha|)!^s (k+|\beta|)!}{2(2n)^{-k} k!}$$
$$= \sum_{\gamma \leqslant \alpha, \beta} \frac{(k+|\gamma|)!}{k!\,\gamma!} (2n)^{-|\gamma|} \leqslant \left(1 - \frac{1}{2n} - \dots \frac{1}{2n}\right)^{k+1} = 2^{k+1}.$$

Thus

$$N_s(({}^t p), T) \ll \sum_{k, \alpha, \beta} 2^{k+1} c_{k, \alpha}^{\beta} T^{2k+|\alpha+\beta|} \ll 2N_s((p), \sqrt{2}T).$$

2. The analytic case.

We now make a special study of the analytic case ($s = 1$). Let us first remark that in this case, the proof of the equivalence between Definition 1.1 and 1.1bis, when applied to the x variable, shows that the symbol $(p) = \sum p_k(x, \xi)$ is analytic (of class $s = 1$) if and only if for any given compact set $K \subset \Omega$ there exist constants ε, c, A such that every $p_k(x, \xi)$ is holomorphic in the complex domain $K_\epsilon \times C_\epsilon$ (where K_ϵ stands for the set of all $x \in C^n$ such that $d(x, K) < \varepsilon |\mathrm{Re}\, \xi|$) and further, that the following inequality holds in this domain:

$$|p_k(x, \xi)| \leqslant c A^k |\xi|^{r-k} k! \qquad (1.1)\text{ter}$$

Let now $(p) = \sum p_k(x, \xi)$ be a symbol. The functions $p_k(x, \xi)$ are supposed to be analytic on $\Omega \times (\mathrm{R}^n - \{0\})$. Let $p_k(x, y)$ design the inverse Fourier transform with respect to ξ of the distribution $p \cdot f \cdot p_k(x, \xi)$ (we refer to L. Schwartz [12] for the definition of the finite part).

Proposition 1.5. — *The symbol $(p) = \sum p_k(x, \xi)$ is analytic if and only if for every compact subset $\mathrm{K} \subset \Omega$ there exists a constant $\varepsilon > 0$ such that the series $\sum \hat{p}_k(x, y)$ converges uniformly in the complex domain of all x, y such that $d(x, \mathrm{K}) < \varepsilon$ and $|\mathrm{Im}\, y| < \varepsilon\, |\mathrm{Re}\, y| < \varepsilon^2$.*

Proof. — In regard to (1.1)ter, it is sufficient to prove the equivalence when (p) does not depend on x. Let us first suppose that $(p) = \sum p_k(\xi)$ is an analytic symbol : there exist constants $\varepsilon < 1$, c, A such that every p_k is holomorphic in the complex cone C_ϵ ($|\mathrm{Im}\, \xi| < \varepsilon\, |\mathrm{Re}\, \xi|$), and that in this cone, the following inequality holds :

$$|p_k(\xi)| \leqslant c\, \mathrm{A}^k\, |\xi|^{r-k}\, k! \tag{1.5}$$

It follows from Prop. 0.4 and 0.5 that the distributions $\hat{p}_k(y)$ are holomorphic in the same cone C_ϵ.

Let us now choose an integer m such that $m + r < -n$, and consider the set of all distributions of the form

$$k!\, \mathrm{A}^{-k} \langle x, \xi\rangle^{m+k}\, p \cdot f \cdot p_k(\xi) = k!^{-1}\, \mathrm{A}^{-k}\, p \cdot f \cdot (\langle x, \xi\rangle^{m+k}\, p_k(\xi)),$$
$$x \in \mathrm{C}^n,\ |x| = 1\, ;\quad m + k \geqslant 0.$$

Formula 1.5 clearly shows that this set is bounded in the Banach space $\mathrm{F}_{\epsilon, m+r}$ of Proposition 0.5. It follows from that proposition that the set of Fourier transforms :

$$k!^{-1}\, \mathrm{A}^{-k}\, (x \cdot \mathrm{D})^{m+k}\, \hat{p}_k(y) \qquad (x \in \mathrm{C}^n,\ |x| = 1\, ;\quad m + k \geqslant 0)$$

is a set of functions which are holomorphic and uniformly bounded for $|\mathrm{Im}\, y| < \varepsilon'\, |\mathrm{Re}\, y| < \varepsilon'^2$ (if $\varepsilon' < \varepsilon$). (if $(x_1, \ldots, x_n)$ are the coordinate of x, we set

$$(\mathrm{X} \cdot \mathrm{D})^k = \left(\frac{1}{i} \sum_0^n \mathrm{X}_j \frac{\partial}{\partial y_j}\right)^k .$$

Further, it follows from the definition of finite parts that $p_k(y)$ and its derivatives of order $< -n - r + k$ are continuous and vanish at the origin (even if $p \cdot f \cdot p_k(\xi)$ is not a homogeneous distribution). If we integrate $m + k$ times in the direction of x when $x \in C_{\epsilon'}$, we therefore get the inequality

$$|k!^{-1} A^{-k} p_k(y)| \leqslant c\,(m+k)!^{-1}\,|y|^{m+k} \quad \text{if} \quad |\operatorname{Im} y| < \varepsilon'\,|\operatorname{Re} y| < \varepsilon'^2. \tag{1.6}$$

The convergence of the series $\sum \hat{p}_k(y)$ in a domain such as described in Prop. 1.5 follows immediately (naturally, the first terms, which are eventually not included in inequalities (1.5) and (1.6), have no influence on the result).

To prove the converse implication, we consider the set of all distributions of the form

$$k!^{-1} A^{-k} (X \cdot D)^{m+k} \hat{p}_k(y), \quad k = 0, 1, \dots; \quad x \in C^n, \quad |x| = 1,$$

where m is chosen such that $-n < m + r < 0$. These distributions are homogeneous of degree $-n-m-r$, and it follows from Cauchy's formula (applied as in Lemma 0.3) that for sufficiently large A, their set is bounded in the space $H_{\epsilon,\,-n-m-r}$ of Prop. 0.4 if $\sum \hat{p}_k(y)$ behaves as described in Prop. 1.5.

Prop. 0.4 then proves that the set of all distributions of the form

$$k!^{-1} A^{-k} \langle x, \xi\rangle^{m+k} p_k(\xi), \quad k = 0, 1, \dots; \quad x \in C^n, \quad |x| = 1,$$

is bounded in the space $H_{\epsilon,\,m+r}$, and this of course implies inequality (1.5). This ends the proof of Prop. 1.5.

2. Pseudo-differential operators of class *s*.

The notations are the same as in § 0 and § 1.

DEFINITION 2.1. — *Let* $P = P(x, D)$ *be a pseudo-differential operator on* Ω, *with symbol* $\sigma(P) = \sum p_k(x, \xi)$ *and degree* r. P *will be said to be strictly of class* s *if for any given compact set* $K \subset \Omega$ *there exist constants* c, A *such that for any* $x \in K$, *any multi-indexes* α, β, *and any integer* N, *the following inequality holds :*

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial \xi}\right)^{\beta}\left(p - \sum_{0}^{N-1} p_k(x, \xi)\right)\right| < c\,A^{N+|\alpha+\beta|}\,|\xi|^{r-N-|\beta|}\,(k + |\alpha|)!^{s}\,\beta! \tag{2.1}$$

(*the factor* $|\xi|^{r-N-|\beta|}$ *ought to be replaced by* $(1 + |\xi|^{r-N-|\beta|}$ *without changing anything else when* $r - N - |\beta| > 0$).

DEFINITION 2.1.*bis.* — *A linear continuous operator*

$$P : C_0^\infty(\Omega) \to C^\infty(\Omega)$$

will be called a pseudo-differential operator of class s if its kernel is a function of class s outside of the diagonal of $\Omega \times \Omega$, *and if its restriction to any relatively compact open set* Ω' *in* Ω *can be decomposed into* $P = P' + R$ *where* P' *is strictly of class s and the kernel of* R *is a function of class s on* $\Omega' \times \Omega'$.

(This second definition will be justified by the two propositions below.)

Definition 2.1 implies that the symbol of P is of class s. It also implies that the function $p(x, \xi)$ is holomorphic with respect to ξ in some cone C_ϵ ($|\text{Im}\,\xi| < \varepsilon\,|\text{Re}\,\xi|$), and a formula analogous to (1.1)bis (or (1.1)ter when $s = 1$) could be written out.

We now have

PROPOSITION 2.2. — *Let* P *be a pseudo-differential operator which is strictly of class s* (*Definition* 2.1). *Then the kernel of* P *is a function of class s outside the diagonal of* $\Omega \times \Omega$.

Proof. — We only have to remark that formula (1.1) implies that $x \to p(x, \xi)$ is a function of class s with values in a space of distributions which are holomorphic and bounded by a power of $|\xi|$ in a cone C_ϵ (at least when x remains in a relatively compact subset of Ω). It then follows from Prop. 0.2 to 0.5 that $x \to \hat{p}(x, y)$ is a function of class s with values in a space of distributions which are analytic outside the origin ($\hat{p}(x, y)$ denotes the inverse Fourier transform of $p(x, \xi)$ with respect to ξ). Since the kernel of P is the restriction to $\Omega \times \Omega$ of $\hat{p}(x, x - y)$, this proves Prop. 2.2.

PROPOSITION 2.3. — *Let* P *be a pseudo-differential operator, of class s, and suppose* $\sigma(P) = 0$. *Then the kernel of* P *is a function of class s on* $\Omega \times \Omega$. (*Thus* P *can be extended into a continuous operator* $G_0'^s(\Omega) \to G^s(\Omega)$.)

Proof. — When the symbol of P is zero, inequality (2.1) gives

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} \xi^{\beta} p\right| \leqslant c\, A^{k+|\alpha|}\, |\xi|^{r-k+|\beta|} (k + |\alpha|)!^{s} \quad \text{if} \quad r - k < 0. \tag{2.2}$$

If we take $k = |\beta| + m$ (with $m + r < -n$), then $k = |\beta| + m'$ (with $r + m' \geqslant 0$) and add the two corresponding inequalities, we get (with other constants c', A$'$)

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} \xi^{\beta} p\right| \leqslant c'\, A'^{|\alpha+\beta|} \inf (1, |\xi|^{m+r})\, |\alpha + \beta|!^{s} \tag{2.3}$$

and finally, integrating this last inequality,

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial y}\right)^{\beta} p(z, y)\right| \leqslant c''\, A'^{|\alpha+\beta|}\, |\alpha + \beta|!^{s} \tag{2.4}$$

(where we have set $c'' = c' \cdot \int_{\mathbf{R}^n} \inf (1, |\xi|^{m+r})\, d\xi < \infty$).

This proves the lemma.

Remark 2.4. — The proof of Prop. 2.3 does not depend on the ξ-derivatives of $p(x, \xi)$. It follows that if $\sum p_k(x, \xi)$ is a symbol of class s such that there exists a pseudo-differential operator P$'$, of class s (in the sense of Definition 2.1), with symbol $\sum p_k(x, \xi)$ (this will be shown to be always true in Section 4), if $p(x, \xi)$ is a function on $\Omega \times (\mathbf{R}^n - \{0\})$ such that for any compact set $K \subset \Omega$ there exists constants c, A such that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} \left(p(x, \xi) - \sum_{0}^{N} p_k(x, \xi)\right)\right| \leqslant c\, A^{N+|\alpha|}\, |\xi|^{r-N} (N + |\alpha|)!^{s}\ (^1)$$

then the operator $p(x, D)$ is a pseudo-differential operator of class s, with symbol $\sum p_k(x, \xi)$, in the sense of Definition 2.1bis (because the operator $P' - P(x, D)$ satisfies inequality (2.2), hence also (2.3) and (2.4), which means its kernel is a function of class s).

(1) Here also $|\xi|^{r-N}$ ought to be replaced by $(1 + |\xi|)^{r-N}$ if $r - N > 0$ or if $N = 0$.

2. The analytic case ($s = 1$).

In the analytic case, Proposition 1.5 is completed by the following:

PROPOSITION 2.5. — *Let* $\sum p_k(x, \xi)$ *be an analytic symbol, and* $P = P(x, D)$ *an analytic pseudo-differential operator with symbol* $\sum p_k(x, \xi)$. *Then the kernel of* $P(x, D)$ *differs from the sum of the series* $\sum \hat{p}_k(x, x - y)$ *by a function which is analytic in a neighborhood of the diagonal of* $\Omega \times \Omega$.

Proof. — We can assume that $P = P(x, D)$ is strictly of class s, and further, that $p(x, \xi)$ does not depend on x. We shall then prove Prop. 2.5 by estimating the derivatives of $p(y) - \sum p_k(y)$ (where $\hat{p}(y)$ designs the Fourier transform of $p(\xi)$, and $\hat{p}_k(y)$, that of p. f. $p_k(\xi)$, as in Prop. 1.5): we have

$$\left(\frac{\partial}{\partial y}\right)^{\alpha}(\hat{p}(y) - \sum \hat{p}_k(y)) = \left(\frac{\partial}{\partial y}\right)^{\alpha}\left(\hat{p} - \sum_{0}^{m+|\alpha|-1} \hat{p}_k\right) + \left(\frac{\partial}{\partial y}\right)^{\alpha}\left(\sum_{|\alpha|+m}^{\infty} \hat{p}_k\right),$$

where m is chosen such that $m - n - r > 0$.

The proof of Prop. 1.5 shows that the second sum can be estimated by

$$\left|\sum_{m+|\alpha|}\left(\frac{\partial}{\partial y}\right)^{\alpha} \hat{p}_k(y)\right| \leqslant c\, A^{|\alpha|}\, \alpha! \tag{2.6}$$

for any α, when $|\operatorname{Im} y| < \varepsilon\, |\operatorname{Re} y| < \varepsilon^2$, for sufficiently small ε, and large c, A.

The first term in the second member of (2.5) is the Fourier transform of

$$T_\alpha = \xi^\alpha \left[p(\xi) - \sum_{0}^{m+|\alpha|-1} \text{p.f. } p_k(\xi)\right] = \text{p.f.}\left[\xi^\alpha\left(p - \sum_{0}^{m+|\alpha|-1} p_k\right).\right] \tag{2.7}$$

Now it follows from Definition 1.1 and 2.1 that for suitable ε, c, A, $(c\, A^{|\alpha|}\, \alpha!)^{-1}\, T_\alpha$ remains in a bounded subset of the space $E_{\epsilon, r-m}$ of Prop. 0.5.

Therefore it follows from Prop. 0.5 that for sufficiently small ε and suitable constants c, A, the functions

$$\left(\frac{\partial}{\partial y}\right)^{\alpha}\left(\hat{p}(y) - \sum_{0}^{m+|\alpha|-1} \hat{p}_k(y)\right)$$

are holomorphic in the set $|\operatorname{Im} y| < \varepsilon \, |\operatorname{Re} y| < \varepsilon^2$, and we have in this set

$$\left| \left(\frac{\partial}{\partial y}\right)^{\alpha} \left(\hat{p}(y) - \sum_{0}^{m+|\alpha|-1} \hat{p}_k(y)\right) \right| \leqslant c \, \mathrm{A}^{|\alpha|} \, \alpha!$$

We now only have to add up inequalities (2.6) and (2.8) to finish the proof.

3. Continuity properties.

In this section we show that our pseudo-differential operators are continuous on some of the hyper-distribution spaces introduced in § 0, No. 2.

THEOREM 2.6. — *Let* P *be a pseudo-differential operator of class s on* Ω. *Then* P *has a unique continuous extension :*

$$\mathrm{G}_0'^{s}(\Omega) \cap \mathrm{G}^{s}(\mathrm{U}) \to \mathrm{G}'^{s}(\Omega) \cap \mathrm{G}^{s}(\mathrm{U})$$

for any open subset $\mathrm{U} \subset \Omega$ *such that* $\Omega - \mathrm{U}$ *be compact.*
(The hyper-distribution spaces involved are defined in § 0, No. 2). In other words, if T is a hyper-distribution with compact support, $\mathrm{P}(x, \mathrm{D}) \cdot \mathrm{T}$ is well defined ; and it is a function of class s in any open set where T is.

We give the proof in two steps : first suppose $\mathrm{P} = \mathrm{P}(x, \mathrm{D})$ is a convolution operator (i.e. $p(x, \xi)$ does not depend on x). We then have $\mathrm{P}(x, \mathrm{D}) \cdot \mathrm{T} = \mathrm{E} * \mathrm{T}$, where E is a function of class s outside of the origin. Theorem 2.6 then follows from

LEMMA 2.7. — *Let* S, T *be two hyper-distributions of class s, one of which has a compact support. We then have*

$$\text{sing supp } (\mathrm{S} * \mathrm{T}) \subset \text{sing supp } (\mathrm{S}) + \text{sing supp } (\mathrm{T}).$$

(Here sing supp T designates the smallest closed set outside of which T is a function of class s. If A and B are two sets in R^n, A + B designates the set of all $x + y$, $x \in \mathrm{A}$, $y \in \mathrm{B}$. In our case, A + B is closed since A and B are, and one of them is compact.)

Proof. — We first remark that

$$\text{sing supp } (\mathrm{S} * \mathrm{T}) \subset \text{sing supp } (\mathrm{S}) + \text{supp } \mathrm{T}.$$

This follows from the equality $S * T(x) = \int S(x - t)\, T(t)\, dt$ if x lies outside of sing supp S + supp T, and from the fact that if K is a compact set and has void intersection with supp sing (S) + supp T, then $t \to S(x - t)/K$ is a function of class s with values in $G^s(K)$ in a neighborhood of sing supp S.

In the general case, let x_0 lie outside sing supp (S) + sing supp (T). We can choose two neighborhoods K, K′ of sing supp (S) and sing supp (T) which are unions of cubes with small side length such that $x_0 \notin K + K'$. We then write $S = S_1 + S_2$, $T = T_1 + T_2$, where S_1 (resp. T_1) is the function equal to zero in K (resp. K′) and to $S(x)$ (resp. $T(x)$) outside K (resp. K′). It follows from the remark above that $S_1 * T_2$, $S_2 * T_1$ and $S_2 * T_2$ are of class s in a neighborhood of x_0. Thus it is sufficient to prove that $S_1 * T_1(x) = \int_{\substack{x-t \in K \\ t \in K'}} S(x - t)\, T(t)\, dt$ also is. When K, K′ have this particular shape, this is quite elementary to prove, and we leave the proof to the reader [1]. (When $s > 1$, it is easier to prove Lemma 2.7 by means of a partition of unity of class s. This of course cannot be done when $s = 1$.)

The following corollary is an immediate consequence of Lemma 2.7 (also cf. [13], Lectures No. 4, 5, 6).

COROLLARY 2.8. — *Let* T_i *be a bounded set of hyperdistributions in* $G'^s(R^n) \cap G^s(R^n - \{0\})$ (cf. § 0, *No.* 2). *Then the convolutions* $u \to T_i * u$ *form an equi-bounded and therefore equi-continuous set of linear mappings* $G_0'^s(R^n) \cap G^s(U) \to G'^s(R^n) \cap G^s(U)$ *for any open set* U *such that* $R^n - U$ *is compact.*

We now prove Theorem 2.6 in the general case. Since an operator R whose kernel is a function of class s is continuous from $G_0'^s(\Omega)$ to $G^s(\Omega)$, we will assume that $P = P(x, D)$ is strictly of class s (Definition 2.1). Let us now denote by E_α the inductive limit when $\varepsilon \to 0$ of the Banach spaces $E_{\epsilon,\alpha}$ of Proposition 0.5, and by F_α its isomorphic image by the Fourier transform : it is a complete bornological space of type (DF) (cf. (5)). It follows from inequality (2.1) that $x \to \hat{p}(x, y)$ is a function of class s in Ω, with values in F_r. It also follows from Prop. 0.5 that F_α is a topological subspace of $G'^s(R^n) \cap G^s(R^n - \{0\})$. Theorem 2.6 then follows from the three remarks below :

1. The bilinear mapping $B(\varphi, T) = P$, where the operator P is defined by $P(u) = \varphi \cdot (T * u)$, is continuous from $G^s(\Omega) F_\alpha$ to

$$\mathcal{L}[G_0'^s(\Omega) \cap G^s(U),\ G'^s(\Omega) \cap G^s(U)]$$

(on this last space we put the topology of uniform convergence on bounded sets of $G_0'^s(\Omega) \cap G^s(U)$). In order to prove this, it is sufficient to prove the same when one replaces Ω by any compact set $K \subset \Omega$, containing $\Omega - U$, and $G^s(\Omega)$ (resp. $G_0'^s(\Omega) \cap G^s(U)$, $G'^s(\Omega) \cap G^s(U)$) by $G^s(K)$ (resp. $G^s(K) \cap G^s(U)$, resp. the space of hyperdistributions which are defined in some neighborhood of K and of class s in U). This last assertion follows from the fact that B is clearly bounded on those spaces (Cor. 2.8), and $G^s(K)$ and F_α are both bornological, of type (DF) (cf. [5]).

2. Since $G_0'^s(\Omega) \cap G^s(U)$ is bornological, and $G'^s(\Omega) \cap G^s(U)$ complete, $\mathcal{L}[G_0'^s(\Omega) \cap G^s(U), G'^s(\Omega) \cap G^s(U)]$ is complete, and B has a continuous extension to the Grothendieck product $G^s(\Omega) \hat{\otimes} F_\alpha$.

3. By Grothendieck's theorem, this last space is identified to the space of functions of class s in Ω, with values in F_α. And through this identification, the operator corresponding to the function $\hat{p}(x, y)$ is the operator with kernel $\hat{p}(x, x - y)$ which is precisely $P(x, D)$. This completes the proof.

4. An existence theorem.

THEOREM 2.9. — *Let $(p) = \sum p_k(x, \xi)$ be any symbol of class $s \geqslant 1$ on Ω. Then there exists a pseudo-differential operator of class s on Ω with symbol (p).*

We shall first prove the theorem when (p) satisfies inequality (1.1) uniformly. It is clear that we can suppose that the degree of (p) is zero — which we will do. The problem is to construct a function $p(x, \xi)$ satisfying (2.1) uniformly on Ω. We first reduce this problem to a problem concerning functions of one complex variable t:

Let us designate by $B_{\varepsilon, A, s}$ (resp. $B^+_{\varepsilon, A, s}$) the space of all smooth functions f on the closed complex cone $|\operatorname{Im} t| \leqslant \varepsilon |\operatorname{Re} t|$ (resp. and $t \geqslant 0$), which are holomorphic in the interior, and satisfy

$$\left| \left(\frac{d}{dt}\right)^k f \right| \leqslant c\, A^k\, k!^s \quad \text{for suitable } c. \tag{2.9}$$

In order that the function $p(x, \xi)$ satisfy inequality (2.1) it is sufficient that the function $(x, \xi) \to f(t) = p(x, t^{-1}\xi)$ be partially analytic in ξ and of class s in x on the manifold $\Omega \times \sum$ (where $\sum$ designs the unit cotangent sphere $|\xi| = 1$), with values in $B_{\epsilon, A, s+1}$, and satisfy

$$\left(\frac{d}{dt}\right)^k f(0) = k!\, p_k(x, \xi). \tag{2.10}$$

(We leave the easy proof of the reader.)

Let us now denote by $\hat{B}_{\epsilon, A, s}$ the space of all functions which are holomorphic in the cone C_ϵ : $|\operatorname{Im} t| < \varepsilon\, |\operatorname{Re} t|$ and satisfy the inequality

$$|f(t)| \leqslant c \exp - (2A\, |t|^{1/s}) : \tag{2.11}$$

Lemma 0.3, and the end of No. 2 in § 0 imply that the Fourier transform is continuous from $\hat{B}_{\epsilon, A, s}$ to $B_{\epsilon', A', s'}$ for sufficiently large A' and small ε'. And of course if $f \in \hat{B}_{\epsilon, A, s}$ we have

$$\left(\frac{d}{dt}\right)^p \hat{f}(0) = i^p \int_{-\infty}^{+\infty} t^p f(t)\, dt.$$

Finally let us designate by $S_{A, s}$ the space of all sequences

$$s = (s_k)_{k=0, 1, \ldots}$$

such that

$$||s||^2 = \sum |A^{-k} k!^{-s} s_k|^2 = \sum \left|\frac{s_k}{\lambda_k}\right|^2 < \infty. \tag{2.12}$$

Definition 1.1 implies that $(x, \xi) \to (p_k(x, \xi) \cdot k!)_{k=0, 1, \ldots}$ is partially of class s in x and analytic in ξ on $\Omega \times \sum$, with values in the Hilbert space $S_{A, s+1}$, for sufficiently large A.

Theorem 2.9 will now readily follow from.

LEMMA 2.10. — *For any* $s \geqslant 2$, *there exists a continuous linear mapping* $U : S_{A, s} \to \hat{B}_{\epsilon, A', s}$ *(for sufficiently small* ε *and* A'*) such that if* $f(t) = U(s)$ *we have*

$$\int_{-\infty}^{+\infty} t^k f(t)\, dt = s_k. \tag{2.13}$$

(The lemma is also true, in fact, for any $s > 1$. We only give a proof when $s \geqslant 2$).

The following proof only adds minor details to the article of L. Carleson [3], whose notations we adopt. We do not repeat here the proofs that can be found in [3] :

Let us designate by $P_m(t) = \sum_0^m \alpha_{m,p} t_p$ the sequence of real polynomials which is uniquely determined (up to the factor ± 1) by the fact that P_m is real, its degree is m and the relations

$$\int P_m(t) P_n(t) \exp(-2 \mid t \mid^{1/s}) dt = \begin{cases} 0 \text{ if } m \neq n \\ 1 \text{ if } m = n. \end{cases} \tag{2.14}$$

We try to construct a solution $f = U(s)$ by the formula

$$f = \exp(2 - 2 A \mid t \mid^{1/s}) \left(\sum_0^\infty b_m P_m \right) \tag{2.15}$$

Formulae (2.13) and (2.14) then imply

$$b_m = \int f(t) p_m(t) dt = \sum_{p=0}^{u} \alpha_{m,p} s_p . \tag{2.16}$$

Now it follows from L. Carleson's article that the coefficients $\alpha_{m,p}$ satisfy the following estimates : let $w(z)$ be the unique harmonic function defined for $\operatorname{Im} z \geqslant 0$, infinitely small compared to z when $z \to \infty$, with boundary value $c \mid z \mid^{1/s}$ for real z :

$$w(z) = \operatorname{Re}\left[c e^{\frac{i\pi}{2s}} \exp\left(\frac{1}{s} \log z \right) \right]. \tag{2.17}$$

Put

$$\mu(r) = \sup_{|z|=r} w(z) = c \left(\cos \frac{\pi}{2s} \right)^{-1} r^{1/s} = \alpha r^{1/s}. \tag{2.18}$$

(Here we have $\alpha = c(\cos \pi/2s)^{-1} < 2c$ since $s \geqslant 2$.)

Finally we define

$$M_p = \sup_{r>0} r^{p+1/2} e^{-\mu(r)} = \left[\frac{s(p + 1/2)}{\alpha e} \right]^{s(p+1/2)} . \tag{2.19}$$

Then we have for suitable constant k

$$\sum_{m=0}^{\infty} \mid \alpha_{m,p} \mid^2 \leqslant k^2 M_p^{-2}. \tag{2.20}$$

Let us now choose c so small that $\sum_{,} \lambda_p/M_p |^2 = k'^2 < \infty$. We then have

$$\sum | b_m |^2 \leqslant \sum_{m=0}^{\infty} \left(\sum_{p=0}^{m} \left| \frac{s_p}{\lambda_p} \right|^2 \right) \left(\sum_{p=0}^{m} | \lambda_p \alpha_{m,p} |^2 \right)$$

$$\leqslant || s ||^2 \left[\sum_{\infty}^{0} \lambda_p^2 \cdot \left(\sum_{m=0}^{\infty} | \alpha_{m,p} |^2 \right) \right] \leqslant k^2 || s ||^2 \left(\sum_{0}^{\infty} \left| \frac{\lambda_p}{M_p} \right|^2 \right)$$

$$= (k, k')^2 || s ||^2. \tag{2.21}$$

Further it follows from (2.16), (2.17) and (2.21) that

$$\varphi (z) = \sum b_m P_m (z) = \sum a_p z^p$$

is an entire analytic function.

In fact we have

$$| a_p | = \left| \sum_m b_m \alpha_{m,p} \right| \leqslant k^2 k' || s || M_p \quad ,$$

and

$$| \varphi (z) | \leqslant k^2 k' || s || \sum \frac{|z|^p}{M_p} \leqslant k'' || s || \left| \frac{1}{M_0} + \sqrt{| z |} \sum_{\infty}^{0} \frac{z^{p+1/2}}{M_p} \cdot \frac{M_p}{M_{p+1}} \right| ,$$

it follows from (2.19) that we have $z^{p+1/2} / M_p \leqslant e^{\mu(|z|)}$ and

$$\sum M_p/M_{p+1} < \infty$$

so that finally we get

$$| \sum b_m P_m (z) | \leqslant k (1 + \sqrt{| z |} e^{\alpha |z|^{1/s}}) \tag{2.22}$$

Since we have $\alpha < 2c$, $f = \exp(-2c \, | t |^{1/s}) (\sum b_m P_m (t))$ lies in the space $\hat{B}_{\epsilon, A'}$ if ε and A' are small enough. Moreover it is quite evident that the operator U thus defined is continuous from $S_{A,s}$ to $\hat{B}_{\epsilon, A', s}$. This ends the proof of Lemma 2.10.

Since we are not able to vary the constant A in Lemma 2.10 (from one point x, ξ to another), Lemma 2.10 only permits us to construct a solution (a pseudo-differential operator of class s with symbol (p)) locally. If $s > 1$, it is easy to achieve the construction globally by means of partitions of unity. If $s = 1$, Lemma 2.10 and Prop. 1.5 and 2.5 prove that the operator which is defined in small open subsets of Ω by the distribution kernel $k' (x, y) = \sum \hat{p}_k (x, x-y)$ is an analytic pseudo-differential operator in these small open sets. Next it is a well known property of real analytic functions that there exists a function $k''(x, y)$ defined and analytic in $\Omega \times \Omega - \Delta$ (where Δ designates the diagonal) such that $k''(x, y) - k'(x, y)$ is analytic in a neighborhood of Δ. It is then clear that the operator with kernel

$$k(x, y) = \begin{cases} k''(x, y) \text{ outside of } \Delta \\ k'(x, y) + (k''(x, y) - k'(x, y)) \text{ near } \Delta \end{cases}$$

is an analytic pseudo-differential operator with symbol $\sum p_k(x, \xi)$, according to Definition 2.1*bis*.

5. In this section, we examine the composed of two pseudo-differential operators of class s.

PROPOSITION 2.11. — *Let* P *and* Q *be two pseudo-differential operators of class s, one of which is compactly supported* (cf. § 0, *No.* 3). *Then* $\mathrm{P} \circ \mathrm{Q} = \mathrm{R}$ *is also a pseudo-differential operator of class s.*

We give separate proofs in the analytic and non-quasianalytic cases. In the first case, the hypothesis of Prop. 2.11 implies that either P or Q is a differential operator. The result is evident when P is (by Leibniz' formula), or when Q is a differential operator with constant coefficients. The only case that remains to be examined is therefore when Q is the multiplication by an analytic function $q(x)$. By Prop. 1.5 and 2.5, the kernel of P∘Q then differs from $(\sum \hat{p}_k(x, x\text{-}y)) \cdot q(y)$ by a function which is analytic in a neighborhood of the diagonal (here we use the notations of Prop. 1.5 and 2.5). Next we remark that for small $x - y$ we have:

$$\left(\sum p_k(x, x-y)\right) \cdot q(y) = \sum_{k, \alpha} \frac{1}{\alpha!} q^{(\alpha)}(x)\, \hat{p}_k(x, x-y) \cdot (y-x)^\alpha.$$

This last series is exactly the series which corresponds to the symbol of $\mathrm{P} \circ \mathrm{Q} = \mathrm{R}$ (i.e. $\sum \hat{r}_k(x, x\text{-}y)$), and Prop. 1.3 and show that it is convergent in a neighborhood of the diagonal if the degree of P is not a integer.

If the degree of P is integer, $\hat{p}_k(x, x\text{-}y)(y\text{-}x)^\alpha$ differs from the corresponding term in the series that corresponds to the symbol of P ∘ Q by

$$\left(\frac{1}{i}\right)^\alpha \bar{\mathcal{F}} \left[\left(\frac{\partial}{\partial \xi}\right) \text{p. f. } p_k(x, \xi) - \text{p. f. } \left(\frac{\partial}{\partial \xi}\right)^\alpha p_k(x, \xi)\right] = p_{k, \alpha}(x, x\text{-}y).$$

Here $\bar{\mathfrak{F}}$ designates the inverse Fourier transform. It is easy to prove that for fixed x, $p_{k,\alpha}(x, x\text{-}y)$ is a polynomial with respect to $(x\text{-}y)$, which is homogeneous of degree $|\alpha| - r - n + k$, where r is the degree of P (so that the degree of p_k is $r - k$). Further, the coefficients of this polynomial only depend on the spherical moments of $p_k(x, \xi)$. An exact computation shows that for suitable constants c, A, they are bounded by

$$\frac{C}{k!} A^{k+|\alpha|} \int_{|\xi|=1} |p_k(x, \xi)| \, d\sigma(\xi)$$

(where $d\sigma$ is the superficial measure of the sphere $|\xi| = 1$).

It follows that if p and q are analytic, the series

$$\sum_{k,\alpha} \frac{(-1)^\alpha}{\alpha!} p(x, x\text{-}y)(x\text{-}y)^\alpha q^{(\alpha)}(x)$$

is convergent for small complex $(x\text{-}y)$, so that it sum is analytic in a neighborhood of the diagonal $x = y$. Prop. 2.5 then shows that $P \circ Q$ is an analytic pseudodifferential operator.

So that Prop. 2.11 is proved in this case.

In the other case ($s > 1$) we will suppose that both $P = p(x, D)$ and $Q = q(x, D)$ are strictly of class s (Definition 2.1); we will also suppose that $p(x, \xi)$ and $q(x, \xi)$ vanish for large s (if they do not, we can replace P and Q by $P' \circ \varphi P$, $Q' = \varphi Q$ where $\varphi \in G_0^s(\Omega)$ and $\varphi = 1$ in an open bounded subset U of Ω : the restrictions of P' (resp. Q', $P' \circ Q'$) and P (resp. Q, $P \circ Q$) to $G_0^s(U)$ differ by an operator whose kernel is a function of class s). We shall show that $R = P \circ Q$ satisfies the inequality (2.1 *bis*) of Remark 2.4, wherefore it is a pseudo-differential operator of class s. To do so we first remark that we have $R = r(x, D)$ where

$$r(x, \xi) = (2\pi)^{-n} \int e^{ix.\xi} p(x, \xi + \eta) \tilde{q}(\eta, \xi) \, d\eta, \qquad (2.23)$$

where $\tilde{q}$ designates the Fourier transform of q with respect to x. We now wish to estimate the x-derivatives of

$$r_N(x, \xi) = r(x, \xi) - \sum_{|\gamma|<N} \frac{1}{\gamma!} \left(\frac{\partial}{\partial \xi}\right)^\gamma p(x, \xi) \left(\frac{1}{i}\frac{d}{dx}\right)^\gamma q(x, \xi). \qquad (2.24)$$

We have

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} r_{\mathrm{N}}(x, \xi) = \sum_{\alpha'+\alpha''=\alpha} \binom{\alpha}{\alpha'} \int e^{ix.\eta}\, \eta^{\alpha'} \left(\frac{\partial}{\partial x}\right)^{\alpha''} \cdot \left[p(x, \xi+\eta) - \sum_{|\gamma|<\mathrm{N}} \left(\frac{\partial}{\partial \xi}\right)^{\gamma} p(x, \xi) \frac{\eta^{\gamma}}{\gamma\,!} \right] \hat{q}(\eta, \xi)\, d\eta. \tag{2.25}$$

It follows from (2.1) that the Taylor expansion of $p(x, \xi + \eta)$ is convergent if $|\eta| < \varepsilon |\xi|$ for small ε. On account of this, we separate the integral (2.24) into

$$\int_{|\eta|<\epsilon|\xi|} + \int_{|\eta|>\epsilon|\xi|} .$$

In the following estimates, c, A designate constants which do not depend on P, Q, x, ξ, but which can vary from one line to the next. We designate by r_1, r_2 the degree of P and Q.

It immediately follows from (2.1) that we have for suitable c, A

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} p(x, \xi+\eta)\right| \leqslant c\,\mathrm{A}^{|\alpha|}\, |\alpha|\,!^{s} (1+|\xi|)^{r_1} (1+|\eta|)^{r_2} \tag{2.26}$$

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} \frac{1}{\gamma\,!} \left(\frac{\partial}{\partial \xi}\right)^{\gamma} p(x, \xi)\right| \leqslant c\,\mathrm{A}^{|\alpha+\gamma|}\, |\alpha|\,!^{s}\, |\xi|^{r_1-|\gamma|} \tag{2.26 bis}$$

and if $|\eta| < \varepsilon |\xi|$ (when ε is small enough).

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} \left[p(x, \xi+\eta) - \sum_{|\gamma|<\mathrm{N}} \frac{\eta^{\gamma}}{\gamma\,!} \left(\frac{\partial}{\partial \xi}\right)^{\gamma} p(x, \xi) \right]\right| \leqslant c\,\mathrm{A}^{\mathrm{N}+|\alpha|}\, |\xi|^{r_1-\mathrm{N}}\, |\eta|^{\mathrm{N}}.$$

Further we have $|(\partial/\partial x)^{\alpha} q(x, \xi)| \leqslant c\,\mathrm{A}^{|\alpha|}\, |\alpha|\,!^{s} (1+|\xi|)^{r_2}$, and since $q(x, \xi)$ vanishes for large x, we get

$$|\eta^{\alpha}\, \tilde{q}(\eta, \xi)| \leqslant \mathrm{c}\,\mathrm{A}^{|\alpha|}\, |\alpha|\,!^{s} (1+|\xi|)^{r_2}. \tag{2.28}$$

If we multiply (2.28) by (2.26) or (2.27) and add the result for convenient values of α in (2.28), we then get

$$\left|\eta^{\alpha'} \left[\left(\frac{\partial}{\partial x}\right)^{\alpha''} p(x, \xi+\eta)\right] \tilde{q}(\eta, \xi)\right| \leqslant c\,\mathrm{A}^{\mathrm{N}+|\alpha'+\alpha''|} (1+|\xi|)^{r_1+r_2} (\mathrm{N} + |\alpha'+\alpha''|)\,!^{s}\, |\eta|^{-\mathrm{N}-n} \tag{2.29}$$

$$\left|\frac{1}{\gamma!}\,\eta^{\alpha'+\gamma}\left[\left(\frac{\partial}{\partial x}\right)^{\alpha''}\left(\frac{\partial}{\partial \xi}\right)^{\gamma} p(x,\xi)\right]\tilde{q}(\eta,\xi)\right|$$
$$\leqslant c\,\mathrm{A}^{\mathrm{N}+|\alpha'+\alpha''|}\,|\xi|^{r_1+r_2-|\gamma|}\,(\mathrm{N}+|\alpha'+\alpha''|)!^s\,|\eta|^{-\mathrm{N}-n-|\gamma|} \tag{2.29 bis}$$

and if $|\eta|<\varepsilon|\xi|$

$$\left|\eta^{\alpha'}\left(\frac{\partial}{\partial x}\right)^{\alpha''}\left[p(x,\xi+\eta)-\sum_{|\gamma|<\mathrm{N}}\frac{\eta^{\gamma}}{\gamma!}\left(\frac{\partial}{\partial \xi}\right)p(x,\xi)\right]\tilde{q}(\eta,\xi)\right|$$
$$\leqslant c\,\mathrm{A}^{\mathrm{N}+|\alpha'+\alpha''|}\,|\xi|^{r_1+r_2-\mathrm{N}}\,(\mathrm{N}+|\alpha'+\alpha''|)!^s\,(1+|\eta|)^{-n-1}. \tag{2.30}$$

We now introduce inequality (2.30) (resp. (2.29) and (2.29 *bis*) in the first (resp. second) integral into which we have broken up (2.25), and add up. Since there are less than 2^{N} terms in the sum (2), and also we have the estimate

$$\int_{|\eta|>\epsilon|\xi|}|\eta|^{-p-n}\,d\eta\leqslant c\,|\xi|^{-p}\quad\text{if}\quad p\geqslant 1,$$

where the constant c does not depend on p, we finally get

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} r_{\mathrm{N}}(x,\xi)\right|\leqslant c\,\mathrm{A}^{\mathrm{N}+|\alpha|}\,|\xi|^{r_1+r_2-\mathrm{N}}\,(\mathrm{N}+|\alpha|)!^s. \tag{2.31}$$

Estimate (2.1 *bis*) for R then follows immediately from estimate (2.1) for P and Q. This ends the proof.

In a similar way (and we do not repeat the proof, which is completely analogous to the preceding one) we have.

PROPOSITION 2.12. — *Let* P *be a pseudo-differential operator of class* $s\geqslant 1$. *Then the adjoint* P* *of* P (*which is defined by the equality*

$$\int_{\Omega}\varphi\cdot\mathrm{P}^*(\psi)=\int_{\Omega}\mathrm{P}(\varphi)\cdot\psi$$

for all $\varphi,\psi\in C_0^{\infty}(\Omega)$) *is also a pseudo-differential operator of class s.*

Finally we state.

(2) And the binominal coefficients $\binom{\alpha}{\alpha'}$ which figure in it are less than an exponential of $(|\alpha|+\mathrm{N})$.

PROPOSITION 2.13. — *Let* P *be an elliptic pseudo-differential operator of class* s. *There exists a pseudo-differential operator of class* s, E, *such that* $E \circ P - 1$ *and* $P \circ E - 1$ *have for kernel a function of class* s.

Proof. — It follows from Prop. 1.4 and Theorem 2.9 that there exists a pseudo-differential operator of class s, E, with symbol inverse to that of P. If P is proper (cf. § 0) (this means that P is an ordinary differential operator in the analytic case), $E \circ P$ and $P \circ E$ are well defined. And it follows from Prop. 2.11 that $E \circ P - 1$ and $P \circ E - 1$ are pseudo-differential operators of class s with symbol zero. Prop. 2.13 then follows from Prop. 2.3.

COROLLARY 2.14. — *Let* P *be an elliptic differential operator with coefficients in* $G^s(\Omega)$, *and* $T \in G'^s(\Omega)$ *any hyperdistribution. Then* T *is a function of class* s *in any open set where* P (T) *is.*

This follows immediately from the existence of a parametrix E of class s : $E(P(T)) - T$ is a function of class s, and Theorem 2.6 shows that E (P (T)) is one in any open set where P (T) is.

It seems that this corollary has only been proved for differential operators with constant coefficients (cf. [2]). In the usual proofs of the regularity of the solutions of an elliptic equation, it is usually assumed that T is (locally) a finite sum of derivatives of continuous functions.

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