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ON THE LIFTING PROPERTY (IV). DISINTEGRATION OF MEASURES [1].

by A. IONESCU TULCEA and C. IONESCU TULCEA.

CONTENTS

In Section 1 we introduce in particular the notion of strong lifting. In Section 2 (Theorem 1) we consider a lifting T and we give several conditions (concerning T) equivalent with the assertion that T is a strong lifting. In Section 3 (Theorem 2) we show in particular that the existence of a strong linear lifting implies the existence of a strong lifting. The results of Section 4 are auxiliary; they are used in Section 5.

The principal results of this paper are contained in Section 5. These results are Theorems 3 and 4 on the disintegration of measures. The notion of strong lifting is essentially used in the formulation and proof of these theorems. We also introduce and use here the notion of « appropriate family of measures » which replaces Bourbaki's notion of « adequate family ».

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In Section 6 (Theorem 5) we prove, in a special setting, the existence of a strong lifting satisfying a supplementary condition. The Appendix contains a result concerning the integration of appropriate families.

1. Notations and terminology.

Let Z be a locally compact space and $\mu \neq 0$ a positive Radon measure on Z. Let $M_R^\infty(Z, \mu)$ be the Banach algebra of all bounded real-valued μ-measurable functions defined on Z, endowed with the norm $f \to \|f\|_\infty = \sup_{z \in Z}|f(z)|$. Let $C_R^\infty(Z)$ be the subalgebra of $M_R^\infty(Z, \mu)$ consisting of all bounded continuous functions on Z and $\mathcal{K}(Z)$ the subalgebra of all $f \in C_R^\infty(Z)$ having compact support. For two functions f and g defined on Z we shall write $f \equiv g$ whenever f and g coincide locally almost everywhere.

Let now $T: f \to T_f$ be a mapping of $M_R^\infty(Z, \mu)$ into $M_R^\infty(Z, \mu)$. Properties of T such as those listed below will be considered in what follows:

(I) $\quad T_f \equiv f;$

(II) $\quad f \equiv g$ implies $T_f = T_g;$

(III) $\quad T_1 = 1;$

(IV) $\quad f \geqslant 0$ implies $T_f \geqslant 0;$

(V) $\quad T_{\alpha f + \beta g} = \alpha T_f + \beta T_g;$

(VI) $\quad T_{fg} = T_f T_g;$

(VII) $\quad T_f = f$ if $f \in C_R^\infty(Z).$

A mapping $T: f \to T_f$ of $M_R^\infty(Z, \mu)$ into $M_R^\infty(Z, \mu)$ satisfying (I)-(V) will be called a *linear lifting* of $M_R^\infty(Z, \mu)$; if the condition (VI) is also verified the mapping will be called a *lifting* [2] of $M_R^\infty(Z, \mu)$. A *strong linear lifting* [*strong lifting*] of $M_R^\infty(Z, \mu)$ is a linear lifting [lifting] which verifies also (VII).

Denote by $\mathcal{T}(\mu)$ the tribe (= σ-algebra) of all μ-measurable sets. We shall write $A \equiv B$ whenever $(A - B) \cup (B - A)$ is locally μ-negligible; it is obvious that $A \equiv B$ if and only if $\varphi_A \equiv \varphi_B$(for $X \subset Z$, φ_X denotes the characteristic function of X).

[2] Condition (III) above can be replaced by: (III*) $T_1 = 1$ or $T_1 = 0$, if we want to avoid the restriction $\mu \neq 0$. In this paper, however, we always assume $\mu \neq 0$, whenever we consider a lifting of $M_R^\infty(Z, \mu)$ or $M_R(Z, \mu)$.

Let now $T : f \to T_f$ be a lifting of $M_R^\infty(Z, \mu)$ and let ρ_T be the mapping of $\mathcal{T}(\mu)$ into $\mathcal{T}(\mu)$ defined by the equations $T_{\varphi_A} = \varphi_{\rho_T(A)}$ for $A \in \mathcal{T}(\mu)$. It is easy to verify that ρ_T has the following properties :

(I') $\qquad \rho_T(A) \equiv A;$

(II') $\qquad A \equiv B \quad \text{implies} \quad \rho_T(A) = \rho_T(B);$

(III') $\qquad \rho_T(\emptyset) = \emptyset, \qquad \rho_T(Z) = Z;$

(IV') $\qquad \rho_T(A \cap B) = \rho_T(A) \cap \rho_T(B);$

(V') $\qquad \rho_T(A \cup B) = \rho_T(A) \cup \rho_T(B).$

Conversely if ρ is a mapping of $\mathcal{T}(\mu)$ into $\mathcal{T}(\mu)$ having the properties (I')-(V') then there is a lifting T of $M_R^\infty(Z, \mu)$ such that $\rho = \rho_T$ (this is proved in [5], Proposition 2, when μ has finite total mass; the same proof is valid in the general case). A mapping ρ of $\mathcal{T}(\mu)$ into $\mathcal{T}(\mu)$ having the properties (I')-(V') is called a *lifting* of $\mathcal{T}(\mu)$.

Let ρ be a lifting of $\mathcal{T}(\mu)$; denote by τ the set of all parts $\rho(A)$-N where $A \in \mathcal{T}(\mu)$ and N is locally μ-negligible. Then τ is a topology on Z (this result is essentially due to J. Oxtoby and was given by him in a lecture at Yale in the fall of 1960).

We shall denote below by $f \to N_\infty(f)$ the essential supremum semi-norm on $M_R^\infty(Z, \mu)$. Let us remark that if $T : f \to T_f$ is a linear lifting of $M_R^\infty(Z, \mu)$, then $\|T_f\|_\infty = N_\infty(f)$ for each $f \in M_R^\infty(Z, \mu)$.

It is known that there is always a lifting of $M_R^\infty(Z, \mu)$ (see [5] and [7]).

2. Properties characterizing a strong lifting.

Let T be a lifting of $M_R^\infty(Z, \mu)$. For each $z \in Z$ we shall denote by $\mathcal{V}_T(z)$ the set of all parts $\rho_T(V)$ where V belongs both to $\mathcal{V}(z)$ and to $\mathcal{T}(\mu)$. We shall prove now the following :

THEOREM 1. — *Let* $T : f \to T_f$ *be a lifting of* $M_R^\infty(Z, \mu)$. *Then the following assertions are equivalent*: 1.1) T *is a strong lifting*; 1.2) *There is* $V \subset M_R^\infty(Z, \mu)$ *dense in* $\mathcal{K}(Z)$ *for the topology induced by the norm* $f \to \|f\|_\infty$ *such that* $T_f = f$ *for all* $f \in V$; 1.3) $\rho_T(U) \supset U$ *for all* $U \subset Z$ *open*; 1.4) $\rho_T(F) \subset F$ *for all* $F \subset Z$ *closed*; 1.5) τ*(= the topology corresponding to the lifting* ρ_T*)*

is stronger than the topology of Z; 1.6) *For each* $z \in Z$, $\mathfrak{V}_T(z)$ *is a fundamental system of* z.

It is obvious that 1.1) implies 1.2). Suppose now that 1.2) is true. Then we deduce that $T_f = f$ for all $f \in \mathcal{K}(Z)$. Let now $U \subset Z$ open and let $\mathcal{F}_U$ be the set of all $f \in \mathcal{K}(Z)$ such that $0 \leqslant f \leqslant \varphi_U$; we have $\varphi_U = \sup \mathcal{F}_U$. But $T_{\varphi_U} \geqslant T_f = f$ for each $f \in \mathcal{F}_U$, whence $T_{\varphi_U} \geqslant \sup \mathcal{F}_U = \varphi_U$. Therefore $\rho_T(U) \supset U$ and hence 1.2) implies 1.3). Suppose now that 1.3) is true. We shall show first that $T_g \geqslant g$ for each $g \in C_R^\infty(Z)$, $g \geqslant 0$. In fact, let $\mathcal{U}_g$ be the set of all functions $\alpha\varphi_U \leqslant g$, with $\alpha \geqslant 0$ and $U \subset Z$ open. It is easy to see that $\sup \mathcal{U}_g = g$. As in the proof of the implication 1.2) $\Rightarrow$ 1.3) we deduce that $T_g \geqslant g$. Let now $f \in C_R^\infty(Z)$ and let λ be a constant such that $\lambda + f \geqslant 0$, $\lambda - f \geqslant 0$. Since $T_\lambda = \lambda$ (by (III) and (V)) we deduce $\lambda + T_f = T_{\lambda+f} \geqslant \lambda + f$ and $\lambda - T_f = T_{\lambda-f} \geqslant \lambda - f$; hence $T_f = f$ and thus 1.1), 1.2) and 1.3) are equivalent.

By duality we deduce that 1.3) and 1.4) are equivalent.

Suppose now that 1.3) is true. Then $U = \rho_T(U) - N$ if $N = \rho_T(U) - U$; since N is locally μ-negligible, $U \in \tau$. Therefore 1.3) implies 1.5). Conversely, if 1.5) is true then for each open set U there is $A_U \in \mathcal{C}(\mu)$ and a locally μ-negligible set N_U such that $U = \rho_T(A_U) - N_U$. We deduce that $\rho_T(U) = \rho_T(A_U)$ and hence $U \subset \rho_T(U)$. Thus 1.3) and 1.5) are equivalent.

Suppose again that 1.3) ($\equiv$ 1.4) is true. Let $z \in Z$, $V \in \mathfrak{V}(z)$ and let $W \in \mathfrak{V}(z)$ such that $W = \overline{W} \subset V$. Then

$$\mathring{W} \subset \rho_T(\mathring{W}) \subset \rho_T(W) \subset W \subset V;$$

hence 1.3) implies 1.6). Conversely, suppose 1.6) true and let $U \subset Z$ open. For each $z \in U$ let $U_z \in \mathfrak{V}_T(z)$ such that $U_z \subset U$. Then $U = \bigcup_{z \in U} U_z$, whence $\rho_T(U) \supset \bigcup_{z \in U} \rho_T(U_z) = \bigcup_{z \in U} U_z = U$. Therefore 1.3 and 1.6) are equivalent.

Hence Theorem 1 is completely proved.

Remarks. — 1) Let $\mathcal{B}$ be a basis for the topology of Z. As above it can be shown that T is a strong lifting if and only if $\rho_T(U) \supset U$ for every $U \in \mathcal{B}$. 2) One can also show that T is a strong lifting if and only if for every $A \in \mathcal{C}(\mu)$ such that $A = \rho_T(A)$, every $z \in A$ and $V \in \mathfrak{V}(z)$ we have $\mu^*(A \cap V) \neq 0$.

3. On the existence of a strong lifting.

We shall use below the following result:

PROPOSITION 1. — *Let* $T: f \to T_f$ *be a linear lifting of* $M_R^\infty(Z, \mu)$. *For each* $A \in \mathcal{T}(\mu)$ *define* $\theta'(A) = \{z | T_{\varphi_A}(z) = 1\}$ *and* $\theta''(A) = \{z | T_{\varphi_A}(z) \neq 0\}$. *Let* $\mathfrak{D}$ *be the set of all linear liftings* S *of* $M_R^\infty(Z, \mu)$ *such that*

$$\varphi_{\theta'(A)} \leqslant S_{\varphi_A} \leqslant \varphi_{\theta''(A)}, \qquad A \in \mathcal{T}(\mu).$$

Then $\mathfrak{D}$ *contains a lifting of* $M_R^\infty(Z, \mu)$.

This result was proved in [5] (see the proof of Proposition 4) in the case when μ has finite total mass; the general case can be proved exactly in the same way.

Let $M_R(Z, \mu)$ be the algebra of all locally bounded (= bounded on every compact $K \subset Z$) real-valued μ-measurable functions defined on Z. Denote with $C_R(Z)$ the subalgebra of real-valued continuous functions on Z. A mapping $T: f \to T_f$ of $M_R(Z, \mu)$ into $M_R(Z, \mu)$ which verifies properties (I)-(VI) will be called a *lifting* of $M_R(Z, \mu)$. A *strong lifting* of $M_R(Z, \mu)$ is a lifting $T: f \to T_f$ such that $T_f = f$ for all $f \in C_R(Z)$.

THEOREM 2. — *The following assertions are equivalent:* 2.1) *There is a strong linear lifting of* $M_R^\infty(Z, \mu)$; 2.2) *There is a strong lifting of* $M_R^\infty(Z, \mu)$; 2.3) *There is a strong lifting* of $M_R(Z, \mu)$.

Assume 2.1) true. Let $T: f \to T_f$ be a strong linear lifting of $M_R^\infty(Z, \mu)$ and define $\theta'(A)$, $\theta''(A)$ (for each $A \in \mathcal{T}(\mu)$) and $\mathfrak{D}$ as in Proposition 1. Let S be a lifting of $M_R^\infty(Z, \mu)$ belonging to $\mathfrak{D}$. Then

$$\varphi_{\theta'(A)} \leqslant S_{\varphi_A} \leqslant \varphi_{\theta''(A)}$$

for all $A \in \mathcal{T}(\mu)$. Let now U be an open set. Then $\varphi_U \leqslant T_{\varphi_U}$ [3] and thus $\varphi_U \leqslant \varphi_{\theta'(U)}$. We deduce $\varphi_U \leqslant S_{\varphi_U}$ whence $\rho_S(U) \supset U$. By 1.3) of Theorem 1, S is a strong lifting. Hence 2.1) implies 2.2). Since 2.2) obviously implies 2.1) it follows that 2.1) and 2.2) are equivalent.

[3] One can show as in Theorem 1 that a linear lifting $T: f \to T_f$ of $M_R^\infty(Z, \mu)$ is strong if and only if $T_{\varphi_U} \geqslant \varphi_U$ for every open set $U \subset Z$.

Suppose now that 2.2) is true and let $T : f \to T_f$ be a strong lifting of $M_R^\infty(Z, \mu)$. Let $f \in M_R(Z, \mu)$ and let $t \in Z$. Let $V \in \mathfrak{V}_T(t)$ be relatively compact. We shall define (remark that $\varphi_V f \in M_R^\infty(Z, \mu)$)

$$T'_f(t) = T_{\varphi_V f}(t).$$

If $U \in \mathfrak{V}_T(t)$ is relatively compact, then $T_{\varphi_U f}(t) = T_{\varphi_V f}(t)$. In fact let $W = U \cap V \in \mathfrak{V}_T(t)$; then

$$T_{\varphi_U f}(t) = T_{\varphi_W}(t) T_{\varphi_U f}(t) = T_{\varphi_W \varphi_U f}(t) \\ = T_{\varphi_W \varphi_V f}(t) = T_{\varphi_W}(t) T_{\varphi_V f}(t) = T_{\varphi_V f}(t).$$

We deduce that $T'_f(t)$ is well defined, for $f \in M_R(Z, \mu)$ and $t \in Z$. Let us remark that T'_f belongs to $M_R(Z, \mu)$ for each $f \in M_R(Z, \mu)$. In fact, let $K \subset Z$ be a compact set and let $U \supset K$ be open and relatively compact. Since T is a strong lifting $\overline{U} \supset \rho_T(U) \supset U$; hence $\rho_T(U)$ is relatively compact. For $t \in K$ we have $T'_f(t) = T_{\varphi_{\rho_T(U)} f}(t)$; this shows that $\varphi_K T'_f$ belongs to $M_R^\infty(Z, \mu)$. Since K is arbitrary we deduce that $T'_f \in M_R(Z, \mu)$. Thus T' is a mapping of $M_R(Z, \mu)$ into $M_R(Z, \mu)$. It is easy to see that this mapping is a strong lifting of $M_R(Z, \mu)$. To prove (I) for instance we may reason as follows: Let

$$f \in M_R(Z, \mu),$$

and let U be open and relatively compact. Then for

$$t \in U, \ T'_f(t) = T_{\varphi_{\rho_T(U)} f}(t);$$

whence $\varphi_U T'_f = \varphi_U T_{\varphi_{\rho_T(U)} f} \equiv \varphi_U \varphi_{\rho_T(U)} f = \varphi_U f$. Since U was arbitrary we deduce $T'_f \equiv f$. The proof of the properties (II)-(VI) is left to the reader. Below we shall show only that $T'_f = f$ for each $f \in C_R(Z)$. In fact let $f \in C_R(Z)$ and $t \in Z$. Let $V \in \mathfrak{V}_T(t)$ be relatively compact, and $g \in \mathcal{K}(Z)$ such that $g(z) = 1$ for $z \in V$. Then

$$T'_f(t) = T_{\varphi_V f}(t) = T_{\varphi_V g f}(t) = T_{\varphi_V}(t) T_{g f}(t) = (gf)(t) = f(t).$$

Since $f \in C_R(Z)$ and $t \in Z$ were arbitrary, T' is a strong lifting. Therefore 2.2) implies 2.3) .Conversely if $T : f \to T_f$ is a strong lifting of $M_R(Z, \mu)$, T maps $M_R^\infty(Z, \mu)$ into $M_R^\infty(Z, \mu)$ (since T is positive and $T_1 = 1$) and hence the restriction of T to $M_R^\infty(Z, \mu)$ is a strong lifting of $M_R^\infty(Z, \mu)$. Thus 2.2) and 2.3) are equivalent. Hence the theorem is completely proved.

Remark. — We proved above that if T is a strong lifting of $M_R^\infty(Z, \mu)$ then there is a strong lifting $T' : f \to T'_f$ of $M_R(Z, \mu)$ such that the restriction of T' to $M_R^\infty(Z, \mu)$ coincides with T.

4. Remarks on measure-valued mappings.

For an arbitrary locally compact space E and a positive Radon measure β on E we shall denote by $\mathcal{C}(E, \beta)$ the set of all locally countable families $\mathcal{K} = (K_j)_{j \in J}$ of disjoint compact parts of E such that the complement of $\bigcup_{j \in J} K_j$ is locally β-negligible. We shall use below the following properties which are easy to prove:

a) If $(K'_j)_{j \in J'} \in \mathcal{C}(E, \beta)$ and $(K''_j)_{j \in J''} \in \mathcal{C}(E, \beta)$ then

$$(K'_{j'} \cap K''_{j''})_{(j', j'') \in J' \times J''} \in \mathcal{C}(E, \beta);$$

b) Suppose $\beta \neq 0$ and let T be a lifting of $M_R^\infty(E, \beta)$. Then given $(K_j)_{j \in J} \in \mathcal{C}(E, \beta)$, there is $(L_i)_{i \in I} \in \mathcal{C}(E, \beta)$ such that every L_i is contained in some $\rho_T(K_j)$.

Let now Z and B be two locally compact spaces. For each mapping $\lambda : b \to \lambda_b$ of B into $M_+(Z)$ [4] and for each $g \in \mathcal{K}(Z)$ we denote by $\langle g, \lambda \rangle$ the mapping $b \to \langle g, \lambda_b \rangle$ of B into R.

Let now α be a positive Radon measure on B. Denote by $\mathcal{F}(B, M_+(Z), \alpha)$ the cone of all mappings $\lambda : b \to \lambda_b$ of B into $M_+(Z)$ having the following property:

c) There is $\mathcal{K} = (K_j)_{j \in J} \in \mathcal{C}(B, \alpha)$ such that

$$\varphi_{K_j} \langle g, \lambda \rangle \in M_R^\infty(B, \alpha)$$

for every $g \in \mathcal{K}(Z)$ and $j \in J$.

From *c*) it follows that $\langle g, \lambda \rangle$ is α-measurable if $\lambda \in \mathcal{F}(B, M_+(Z), \alpha)$ and $g \in \mathcal{K}(Z)$.

Suppose now $\alpha \neq 0$ and let $T : f \to T_f$ be a lifting of $M_R^\infty(B, \alpha)$. To shorten the notation we shall sometimes write $\rho_T(f)$ instead of T_f for $f \in M_R^\infty(B, \alpha)$. If $\lambda \in \mathcal{F}(B, M_+(Z), \alpha)$ we shall write

$$\rho_T[\lambda] = \lambda$$

[4] M(Z) is the vector space of all Radon measures on Z and $M_+(Z)$ the cone of all positive Radon measures on Z.

whenever there is $(K_j)_{j\in J} \in \mathcal{C}(B, \alpha)$ such that

$$\rho_T(\varphi_{K_j}\langle g, \lambda \rangle) = \varphi_{\rho_T(K_j)}\langle g, \lambda \rangle$$

for all $g \in \mathcal{K}(Z)$ and $j \in J$ (since the lifting T is defined on $M_R^\infty(B, \alpha)$ this presupposes implicitely that

$$\varphi_{K_j}\langle g, \lambda \rangle \in M_R^\infty(B, \alpha)$$

for all $g \in \mathcal{K}(Z)$ and $j \in J$).

For $\lambda' \in \mathcal{F}(B, M_+(Z), \alpha)$ and $\lambda'' \in \mathcal{F}(B, M_+(Z), \alpha)$ we shall write $\lambda' \equiv \lambda''$ whenever $\langle g, \lambda' \rangle \equiv \langle g, \lambda'' \rangle$ for every $g \in \mathcal{K}(Z)$; in this way we define an equivalence relation in $\mathcal{F}(B, M_+(Z), \alpha)$.

Proposition 2. — 2.1) *For every* $\lambda \in \mathcal{F}(B, M_+(Z), \alpha)$ *there is* $\lambda' \in \mathcal{F}(B, M_+(Z), \alpha)$ *such that* $\rho_T[\lambda'] = \lambda'$ *and* $\lambda' \equiv \lambda$; 2.2) *If* $\lambda' \in \mathcal{F}(B, M_+(Z), \alpha)$, $\lambda'' \in \mathcal{F}(B, M_+(Z), \alpha)$ *and* $\rho_T[\lambda'] = \lambda'$, $\rho_T[\lambda''] = \lambda''$, $\lambda' \equiv \lambda''$ *then* λ'_b *coincides with* λ''_b *locally almost everywhere for* α; 2.3) *If* $\lambda \in \mathcal{F}(B, M_+(Z), \alpha)$ *and* $\rho_T[\lambda] = \lambda$ *then* $b \to \lambda^*_b(1)$ *is* α*-measurable.*

Let us prove 2.1). Let $(K_j)_{j\in J} \in \mathcal{C}(B, \alpha)$ such that

$$\varphi_{K_j}\langle g, \lambda \rangle \in M_R^\infty(B, \alpha)$$

for all $g \in \mathcal{K}(Z)$ and $j \in J$. For each $g \in \mathcal{K}(Z)$ define

$$\lambda'_b(g) = \begin{cases} \rho_T(\varphi_{K_j}\langle g, \lambda \rangle)(b) & \text{if } b \in \rho_T(K_j) \\ 0 & \text{if } b \notin \bigcup_{j\in J} \rho_T(K_j). \end{cases}$$

Since $\rho_T(K_{j'}) \cap \rho_T(K_{j''}) = \emptyset$ if $j' \neq j''$ it follows that $\lambda'_b(g)$ is well defined for $g \in \mathcal{K}(Z)$. The properties of the lifting T imply that λ'_b is a positive linear mapping of $\mathcal{K}(Z)$ into R. Therefore λ'_b is a positive Radon measure on Z; whence $\lambda' : b \to \lambda'_b$ is a mapping of B into $M_+(Z)$.

We shall show now that $\lambda' \in \mathcal{F}(B, M_+(Z), \alpha)$, $\lambda' \equiv \lambda$ and $\rho_T[\lambda'] = \lambda'$.

Let $(L_i)_{i\in I} \in \mathcal{C}(B, \alpha)$ such that each L_i is contained in some $\rho_T(K_j)$. Let $i \in I$ and $j \in J$ such that $L_i \subset \rho_T(K_j)$; then

$$\varphi_{L_i}\langle g, \lambda' \rangle = \varphi_{L_i}\rho_T(\varphi_{K_j}\langle g, \lambda \rangle) \in M_R^\infty(B, \alpha).$$

Since $g \in \mathcal{K}(Z)$ and $i \in I$ were arbitrary we deduce

$$\lambda' \in \mathcal{F}(B, M_+(Z), \alpha).$$

Remark now that for $i \in \mathrm{I}$ and $j \in \mathrm{J}$ as above, we have

$$\varphi_{\mathrm{L}_i}\langle g, \lambda' \rangle = \varphi_{\mathrm{L}_i}\rho_{\mathrm{T}}(\varphi_{\mathrm{K}_j}\langle g, \lambda \rangle) \equiv \varphi_{\mathrm{L}_i}\varphi_{\mathrm{K}_j}\langle g, \lambda \rangle \equiv \varphi_{\mathrm{L}_i}\langle g, \lambda \rangle.$$

Since $i \in \mathrm{I}$ was arbitrary, $\langle g, \lambda' \rangle \equiv \langle g, \lambda \rangle$; since g was also arbitrary it follows that $\lambda \equiv \lambda'$. Let again $i \in \mathrm{I}$ ans $j \in \mathrm{J}$ as above. Then

$$\rho_{\mathrm{T}}(\varphi_{\mathrm{L}_i}\langle g, \lambda' \rangle) = \rho_{\mathrm{T}}(\varphi_{\mathrm{L}_i}\varphi_{\mathrm{K}_j}\langle g, \lambda' \rangle) = \rho_{\mathrm{T}}(\varphi_{\mathrm{L}_i}\varphi_{\mathrm{K}_j}\langle g, \lambda \rangle)$$
$$= \varphi_{\rho_{\mathrm{T}}(\mathrm{L}_i)}\rho_{\mathrm{T}}(\varphi_{\mathrm{K}_j}\langle g, \lambda \rangle).$$

But since $\rho_{\mathrm{T}}(\mathrm{L}_i) \subset \rho_{\mathrm{T}}(\mathrm{K}_j)$, we deduce

$$\rho_{\mathrm{T}}(\varphi_{\mathrm{K}_j}\langle g, \lambda \rangle)(b) = \langle g, \lambda'_b \rangle$$

for $\mathrm{b} \in \rho_{\mathrm{T}}(\mathrm{L}_i)$; whence

$$\varphi_{\rho_{\mathrm{T}}(\mathrm{L}_i)}\rho_{\mathrm{T}}(\varphi_{\mathrm{K}_j}\langle g, \lambda \rangle) = \varphi_{\rho_{\mathrm{T}}(\mathrm{L}_i)}\langle g, \lambda' \rangle.$$

We deduce

$$\rho_{\mathrm{T}}(\varphi_{\mathrm{L}_i}\langle g, \lambda' \rangle) = \varphi_{\rho_{\mathrm{T}}(\mathrm{L}_i)}\langle g, \lambda' \rangle$$

and hence ($g \in \mathcal{K}(\mathrm{Z})$ and $i \in \mathrm{I}$ being arbitrary) $\lambda' = \rho_{\mathrm{T}}[\lambda']$. Thus 2.1) is proved.

Let us prove now 2.2). Let $(\mathrm{K}'_j)_{j \in \mathrm{J}'} \in \mathcal{C}(\mathrm{B}, \alpha)$ and $(\mathrm{K}''_j)_{j \in \mathrm{J}''} \in \mathcal{C}(\mathrm{B}, \alpha)$ such that

$$\rho_{\mathrm{T}}(\varphi_{\mathrm{K}'_{j'}}\langle g, \lambda' \rangle) = \varphi_{\rho_{\mathrm{T}}(\mathrm{K}'_{j'})}\langle g, \lambda' \rangle$$

and

$$\rho_{\mathrm{T}}(\varphi_{\mathrm{K}''_{j''}}\langle g, \lambda'' \rangle) = \varphi_{\rho_{\mathrm{T}}(\mathrm{K}''_{j''})}\langle g, \lambda'' \rangle$$

for all $g \in \mathcal{K}(\mathrm{Z})$ and $j' \in \mathrm{J}'$, $j'' \in \mathrm{J}''$. Let now $(j', j'') \in \mathrm{J}' \times \mathrm{J}''$. We deduce that for every $g \in \mathcal{K}(\mathrm{Z})$

$$\varphi_{\rho_{\mathrm{T}}(\mathrm{K}'_{j'} \cap \mathrm{K}''_{j''})}\langle g, \lambda' \rangle = \varphi_{\rho_{\mathrm{T}}(\mathrm{K}'_{j'} \cap \mathrm{K}''_{j''})}\langle g, \lambda'' \rangle.$$

Thus $\lambda'_b = \lambda''_b$ for every $b \in \rho_{\mathrm{T}}(\mathrm{K}'_{j'} \cap \mathrm{K}''_{j''})$ and hence $\lambda'_b = \lambda''_b$ for almost every $b \in \mathrm{K}'_{j'} \cap \mathrm{K}''_{j''}$. Since

$$(\mathrm{K}'_{j'} \cap \mathrm{K}''_{j''})_{(j', j'') \in \mathrm{J}' \times \mathrm{J}''} \in \mathcal{C}(\mathrm{B}, \alpha),$$

2.2) is proved.

It remains to prove 2.3). Let $(\mathrm{K}_j)_{j \in \mathrm{J}} \in \mathcal{C}(\mathrm{B}, \alpha)$ such that

$$\varphi_{\rho_{\mathrm{T}}(\mathrm{K}_j)}\langle g, \lambda \rangle = \rho_{\mathrm{T}}(\varphi_{\mathrm{K}_j}\langle g, \lambda \rangle)$$

for all $g \in \mathcal{K}(\mathrm{Z})$ and $j \in \mathrm{J}$. Let $(\mathrm{L}_i)_{i \in \mathrm{I}} \in \mathcal{C}(\mathrm{B}, \alpha)$ such that each L_i is contained in some $\rho_{\mathrm{T}}(\mathrm{K}_j)$. Let $i \in \mathrm{I}$ and $j \in \mathrm{J}$ such that $\mathrm{L}_i \subset \rho_{\mathrm{T}}(\mathrm{K}_j)$. Then for $b \in \mathrm{L}_i$ we have

$$\lambda^*_b(1) = \sup_{0 \leqslant f \leqslant 1,\, f \in \mathcal{K}(\mathrm{Z})} \langle f, \lambda_b \rangle = \sup_{0 \leqslant f \leqslant 1,\, f \in \mathcal{K}(\mathrm{Z})} \rho_{\mathrm{T}}(\varphi_{\mathrm{K}_j}\langle f, \lambda \rangle)(b).$$

But $\sup_{0 \leqslant f \leqslant 1, f \in \mathcal{K}(Z)} \rho_T(\varphi_{K_j}\langle f, \lambda \rangle)$ is α-measurable (see A.II. of the Appendix). Hence the restriction of $b \to \lambda_b^*(1)$ to L_i is α-measurable and thus 2.3) is proved.

Remark. — Proposition 2 above is, in a certain sense, similar to Proposition 1 in [6].

Denote by $\mathcal{F}^\infty(B, M_+(Z), \alpha)$ the cone of all mappings $\lambda : b \to \lambda_b$ of B into the cone of bounded positive Radon measures on Z such that:

d) $\langle g, \lambda \rangle$ is α-measurable for each $g \in \mathcal{K}(Z)$;

e) The mapping $b \to \|\lambda_b\|$ is bounded.

Obviously $\mathcal{F}^\infty(B, M_+(Z), \alpha) \subset \mathcal{F}(B, M_+(Z), \alpha)$.

Suppose now that $\alpha \neq 0$ and let T be a lifting of $M_R^\infty(B, \alpha)$. If $\lambda \in \mathcal{F}^\infty(B, M_+(Z), \alpha)$ we shall write

$$\rho_T(\lambda) = \lambda$$

whenever

$$\rho_T(\langle g, \lambda \rangle) = \langle g, \lambda \rangle$$

for all $g \in \mathcal{K}(Z)$. It is clear that, if $\lambda \in \mathcal{F}^\infty(B, M_+(Z), \alpha)$ and $\rho_T(\lambda) = \lambda$ then $\rho_T[\lambda] = \lambda$.

PROPOSITION 3. — 3.1) *For every* $\lambda \in \mathcal{F}^\infty(B, M_+(Z), \alpha)$ *there is* $\lambda' \in \mathcal{F}^\infty(B, M_+(Z), \alpha)$ *such that* $\rho_T(\lambda') = \lambda'$ *and* $\lambda' \equiv \lambda$; 3.2) *If* $\lambda \in \mathcal{F}^\infty(B, M_+(Z), \alpha)$ *and* $\rho_T(\lambda) = \lambda$ *then* $b \to \|\lambda_b\|$ *is* α*-measurable.*

The first assertion is a particular case of (4.1) of Proposition 4 in [6]. It can also be proved directly as 2.1) of Proposition 2 above. The second assertion is a consequence of 2.3) of Proposition 2.

PROPOSITION 4. — *Let* $\lambda \in \mathcal{F}^\infty(B, M_+(Z), \alpha)$ *such that* $\rho_T(\lambda) = \lambda$. *Let* $K \subset Z$ *be compact and let* $\lambda_K : b \to \lambda_{K,b}$ *be the element of* $\mathcal{F}^\infty(B, M_+(K), \alpha)$ *defined* by $\lambda_{K,b} = (\lambda_b)_K$ *for each* $b \in B$. *Then if* $\lambda'_K : b \to \lambda'_{K,b}$ *belongs to* $\mathcal{F}^\infty(B, M_+(K), \alpha)$ *and has the properties:*

4.1) $$\rho_T(\lambda'_K) = \lambda'_K;$$

4.2) $$\lambda'_K \equiv \lambda_K;$$

we deduce that $\lambda'_{K,b} = \lambda_{K,b}$ *locally almost everywhere for* α.

For $g \in \mathcal{K}_+(K)$ and $b \in B$

$$\langle g, \lambda_{K,b} \rangle = \langle \varphi_K h, \lambda_b \rangle$$

if $h \in \mathcal{K}_+(Z)$ is such that $h|K = g$; it follows that

$$\langle \varphi_K h, \lambda_b \rangle = \inf_{f \in \mathcal{K}(Z),\, f \geqslant \varphi_K} \langle fh, \lambda_b \rangle;$$

since $\rho_T(\langle fh, \lambda \rangle) = \langle fh, \lambda \rangle$ for each $f \in \mathcal{K}(Z)$ we deduce (see A.II of the Appendix) that $\langle g, \lambda_K \rangle$ is α-measurable; hence $\langle g, \lambda_K \rangle$ is α-measurable for each $g \in \mathcal{K}(K)$. For each $b \in B$, $||\lambda_{K,b}|| \leqslant ||\lambda_b||$; hence $b \to ||\lambda_{K,b}||$ is bounded. It follows that $\lambda_K \in \mathcal{F}^\infty(B, M_+(K), \alpha)$.

Let now $g \in \mathcal{K}_+(K)$. Then

$$\langle g, \lambda'_K \rangle = \rho_T(\langle g, \lambda'_K \rangle) = \rho_T(\langle g, \lambda_K \rangle).$$

But for each $b \in B$

$$\langle g, \lambda_{K,b} \rangle = \langle \varphi_K h, \lambda_b \rangle = \inf_{f \in \mathcal{K}(Z),\, f \geqslant \varphi_K} \langle fh, \lambda_b \rangle$$

if $h \in \mathcal{K}_+(Z)$ is such that $h|K = g$; then

$$\rho_T(\langle g, \lambda_K \rangle) \leqslant \inf_{f \in \mathcal{K}(Z),\, f \geqslant \varphi_K} \rho_T(\langle fh, \lambda \rangle) = \inf_{f \in \mathcal{K}(Z),\, f \geqslant \varphi_K} \langle fh, \lambda \rangle = \langle g, \lambda_K \rangle.$$

We obtain $\lambda'_{K,b} \leqslant \lambda_{K,b}$ for each $b \in B$. Since $\langle 1, \lambda'_K \rangle \equiv \langle 1, \lambda_K \rangle$, we deduce that λ'_K and λ_K coincide locally almost everywhere for α.

5. On the disintegration of measures.

Let Z and B be two locally compact spaces and $\alpha \neq 0$ a positive Radon measure on B. Let T be a lifting of $M_R^\infty(B, \alpha)$.

We shall say that a mapping $\lambda : b \to \lambda_b$ of B into $M_+(Z)$ is *appropriate with respect to* (α, T) if:

A_1) $\lambda \in \mathcal{F}(B, M_+(Z), \alpha)$ and $\rho_T[\lambda] = \lambda$;

A_2) $\langle g, \lambda \rangle$ is essentially α-integrable for each $g \in \mathcal{K}(Z)$ (that is λ is scalarly essentially α-integrable when $M(Z)$ is endowed with the topology $\sigma(M(Z), \mathcal{K}(Z))$).

If the mapping $\lambda : b \to \lambda_b$ of B into $M_+(Z)$ is appropriate with respect to (α, T), then $\int_B \lambda_b d\alpha(b)$ is a positive Radon measure on Z.

Before proceeding further we shall prove the:

Proposition 5. — *Suppose that* $\alpha \neq 0$ *and that there is a strong lifting* T *of* $M_R^\infty(B, \alpha)$. *Let* $\lambda : b \to \lambda_b$ *be an* α*-adequate mapping* ([1], *chap.* V, p. 18) *of* B *into* $M_+(Z)$. *Then* λ *is appropriate with respect to* (α, T).

Since λ is α-adequate, the mapping $b \to \lambda_b$ of B into M(Z) (endowed with the topology $\sigma(M(Z), \mathcal{K}(Z))$ is α-measurable. There is then $(K_j)_{j \in J} \in \mathcal{C}(B, \alpha)$ such that $\langle g, \lambda \rangle | K_j$ is continuous, for each $g \in \mathcal{K}(Z)$. This shows in particular that $\lambda \in \mathcal{F}(B, M_+(Z), \alpha)$. Let now $g \in \mathcal{K}(Z)$, $j \in J$ and let $h \in C_R^\infty(B)$ such that

$$h|K_j = \langle g, \lambda \rangle | K_j.$$

Then

$$\rho_T(\varphi_{K_j} \langle g, \lambda \rangle) = \rho_T(\varphi_{K_j} h) = \varphi_{\rho_T(K_j)} h = \varphi_{\rho_T(K_j)} \langle g, \lambda \rangle$$

since $\rho_T(K_j) \subset K_j$ (see 1.4) of Theorem 1). Thus $\rho_T[\lambda] = \lambda$ and hence λ satisfies A_1). Since A_2) is verified by any α-adequate mapping, we deduce that λ is appropriate with respect to (α, T).

We shall prove below three propositions which prepare the ground for Theorem 3. To avoid repetition we shall introduce now several notations (and assumptions) which will be used throughout the Propositions 6, 7, 8.

Let:

(1) Z and B be two locally compact spaces;

(2) μ a positive Radon measure on Z;

(3) p a μ-proper mapping of Z into B;

(4) $\alpha \neq 0$ a positive Radon measure on B such that $\nu = p(\mu)$ is absolutely continuous with respect to α(that is $\nu = \psi . \alpha$ for some locally α-integrable function ψ);

(5) T a lifting of $M_R^\infty(B, \alpha)$;

(6) $\lambda : b \to \lambda_b$ a mapping of B into $M_+(Z)$ appropriate with respect to (α, T) such that

$$\int_Z g(f \circ p)\, d\mu = \int_B f \langle g, \lambda \rangle\, d\alpha$$

for every $g \in \mathcal{K}(Z)$, $f \in \mathcal{K}(B)$.

Remarks. — 1) We have

$$(6') \qquad \mu = \int_B \lambda_b \, d\alpha(b).$$

In fact, let $g \in \mathcal{K}_+(Z)$ and let $\mathcal{F}_1$ be the set of all $f \in \mathcal{K}(B)$ satisfying $0 \leqslant f \leqslant 1$. Then

$$\begin{aligned}\int_B \langle g, \lambda \rangle \, d\alpha &= \sup_{f \in \mathcal{F}_1} \int_B f \langle g, \lambda \rangle \, d\alpha \\ &= \sup_{f \in \mathcal{F}_1} \int_Z g(f \circ p) \, d\mu = \sup_{f \in \mathcal{F}_1} \int_B f \, dp(g \cdot \mu) = \int_Z g \, d\mu;\end{aligned}$$

thus (6′) is proved.

2) For $f : B \rightarrow R$, α-measurable, bounded and with compact support, $g : Z \rightarrow R$, μ-measurable, bounded and with compact support we have

$$(6'') \qquad \int_Z g(f \circ p) \, d\mu = \int_B f(b) \, d\alpha(b) \int_Z g \, d\lambda_b.$$

Formula (6″) follows easily by approximating f and g with functions from $\mathcal{K}(B)$ and $\mathcal{K}(Z)$ and using the Corollary of the Appendix (this formula remains obviously valid under more general conditions, but the above form is sufficient for our purposes).

PROPOSITION 6. — *The measures* λ_b *are bounded and* $\|\lambda_b\| = \psi(b)$, *locally almost everywhere for* α.

Let $(K_j)_{j \in J} \in \mathcal{C}(B, \alpha)$ such that

$$\rho_T(\varphi_{K_j} \langle g, \lambda \rangle) = \varphi_{\rho_T(K_j)} \langle g, \lambda \rangle$$

for every $g \in \mathcal{K}(Z)$ and $j \in J$. For each $H \in \mathcal{F}(J)$ (= the set of all finite parts of J) denote

$$K_H = \bigcup_{j \in H} K_j;$$

then for $g \in \mathcal{K}(Z)$ and $H \in \mathcal{F}(J)$ we have

$$\rho_T(\varphi_{K_H} \langle g, \lambda \rangle) = \varphi_{\rho_T(K_H)} \langle g, \lambda \rangle.$$

Let now $f \in \mathcal{K}_+(B)$ and let $\mathcal{F}_1$ be the set of all $g \in \mathcal{K}(Z)$ satsifying $0 \leqslant g \leqslant 1$. Then $(\varphi_{\rho_T(K_H)} \langle g, \lambda \rangle)_{g \in \mathcal{F}_1, \; H \in \mathcal{F}(J)}$ is a directed family of functions belonging to $M_R^\infty(B, \alpha)$. By A.II.

of the Appendix we deduce (use also Remark 2) above):

$$\begin{aligned}\int_B f(b)\lambda_b^*(1)\,d\alpha(b) &= \sup_{g\in\mathcal{F}_1,\ H\in\mathcal{F}(J)} \int_B f\varphi_{\rho_T(K_H)}\langle g,\lambda\rangle\,d\alpha \\ &= \sup_{g\in\mathcal{F}_1,\ H\in\mathcal{F}(J)} \int_Z g(f\varphi_{K_H})\circ p\,d\mu = \sup_{H\in\mathcal{F}(J)} \int_Z (f\varphi_{K_H})\circ p\,d\mu \\ &= \sup_{H\in\mathcal{F}(J)} \int_B f\varphi_{K_H}\,dp(\mu) = \int_B f\,dp(\mu) = \int_B \psi\, f\,d\alpha.\end{aligned}$$

Since $f\in\mathcal{K}_+(B)$ was arbitrary, we deduce $\|\lambda_b\| = \psi(b)$ locally almost everywhere for α.

PROPOSITION 7. — *Suppose that*: 7.1) T *is a strong lifting of* $M_R^\infty(B,\alpha)$; 7.2) Z *is compact*; 7.3) p *is continuous*; 7.4)

$$\lambda\in\mathcal{F}^\infty(B, M_+(Z),\alpha)$$

and $\rho_T(\lambda)=\lambda$. *Then* Supp $\lambda_b \subset \overset{-1}{p}(\{b\})$ *for every* $b\in B$.

Let $b_0\in B$; then

$$\bigcap_{V=\mathring{V},\ V\in\mathcal{V}(b_0)} \overset{-1}{p}(V) = \overset{-1}{p}\Big(\bigcap_{V=\mathring{V},\ V\in\mathcal{V}(b_0)} V\Big) = \overset{-1}{p}(\{b_0\}).$$

Hence given a neighborhood W of $\overset{-1}{p}(\{b_0\})$ there is $V=\mathring{V}$, $V\in\mathcal{V}(b_0)$ such that $\overset{-1}{p}(V)\subset W$.

Let $g\in\mathcal{K}(Z)$ such that Supp $g\cap \overset{-1}{p}(\{b_0\})=\emptyset$. Then by the above remark there is $V=\mathring{V}$, $V\in\mathcal{V}(b_0)$ such that

$$\text{Supp } g\cap \overset{-1}{p}(V)=\emptyset.$$

Let $f\in\mathcal{K}(B)$ such that Supp $f\subset V$. Then $f(p(z))=0$ for $z\notin \overset{-1}{p}(V)$, whence

$$\int_B f\varphi_V\langle g,\lambda\rangle\,d\alpha = \int_Z g(f\circ p)\,d\mu = 0.$$

Since $f\in\mathcal{K}(B)$ is arbitrary, except for the condition Supp $f\subset V$, we deduce $\varphi_V\langle g,\lambda\rangle\equiv 0$. Then

$$\varphi_{\rho_T(V)}\langle g,\lambda\rangle = \rho_T(\varphi_V\langle g,\lambda\rangle) = 0.$$

But $\rho_T(V)\supset V\ni b_0$ (see 1.3) of Theorem 1). Therefore $\langle g,\lambda_{b_0}\rangle = 0$. Since $g\in\mathcal{K}(Z)$ was arbitrary, except for the condition Supp $g\cap \overset{-1}{p}(\{b_0\})=\emptyset$, the proposition is proved.

PROPOSITION 8. — *Suppose that* T *is a strong lifting of* $M_R^\infty(B, \alpha)$. *Then* λ_b *is concentrated on* $\overset{-1}{p}(\{b\})$ *locally almost everywhere for* α.

We shall divide the proof into four parts:

(I) Let $(K_j)_{j\in J} \in \mathcal{C}(B, \alpha)$ such that

$$1)\qquad \rho_T(\varphi_{K_j}\langle g, \lambda\rangle) = \varphi_{\rho_T(K_j)}\langle g, \lambda\rangle$$

for each $g \in \mathcal{K}(Z)$ and $j \in J$. We may suppose that $\psi|K_j$ is bounded for every $j \in J$ and that $\|\lambda_b\| = \psi(b)$ for every $b \in \bigcup_{j\in J} K_j$. For each $j \in J$ and $b \in B$ define

$$\lambda_b^j = \varphi_{\rho_T(K_j)}(b)\lambda_b.$$

Since the lifting T is strong, $\rho_T(K_j) \subset K_j$ (see 1.4) of Theorem 1); we deduce that $b \to \|\lambda_b^j\|$ is a bounded mapping.

To prove the proposition it is enough to show that, for each $j \in J$, λ_b^j is concentrated on $\overset{-1}{p}(\{b\})$ locally almost everywhere for α. In fact suppose that this is true; for each $j \in J$, let H_j be the set of all $b \in B$ such that λ_b^j is not concentrated on $\overset{-1}{p}(\{b\})$. Since $\lambda_b^j = 0$ for $b \notin \rho_T(K_j)$, it follows that $H_j \subset \rho_T(K_j) \subset K_j$. Therefore $H_\infty = \bigcup_{j\in J} H_j$ is locally α-negligible. If $b \in \bigcup_{j\in J} \rho_T(K_j)$ and $b \notin H_\infty$, then $\lambda_b = \lambda_b^j$ is concentrated on $\overset{-1}{p}(\{b\})$.

(II) Let now $j \in J$ be fixed. By the definition of $\lambda^j : b \to \lambda_b^j$ and formula 1) we have

$$\rho_T(\varphi_{K_j}\langle g, \lambda\rangle) = \langle g, \lambda^j\rangle$$

for every $g \in \mathcal{K}(Z)$. Since (as we remarked in (I)) $b \to \|\lambda_b^j\|$ is bounded, these formulas show that λ^j belongs to $\mathcal{F}^\infty(B, M_+(Z), \alpha)$, $\rho_T(\lambda^j) = \lambda^j$ and that $\langle g, \lambda^j\rangle$ is essentially α-integrable for every $g \in \mathcal{K}(Z)$. Thus λ^j is appropriate with respect to (α, T).

By (6″) of Remark 2) above, we have

$$2)\qquad \int_Z g(f\varphi_{K_j})\circ p\, d\mu = \int_B f\varphi_{\rho_T(K_j)}\langle g, \lambda\rangle\, d\alpha$$

for $g \in \mathcal{K}(Z)$ and $f \in \mathcal{K}(B)$. Denote by μ_j the measure $(\varphi_{K_j}\circ p).\mu$; remark that μ_j is bounded and $\mu_j \leqslant \mu$. With this notation we may rewrite formula 2) as follows

$$\int_Z g(f \circ p)\, d\mu_j = \int_B f \langle g, \lambda^j \rangle \, d\alpha$$

for $g \in \mathcal{K}(Z)$ and $f \in \mathcal{K}(B)$.

(III) Since μ_j is bounded and $\mu_j \leqslant \mu$ there is a countable family $(L_n)_{n\in N} \in \mathcal{C}(Z, \mu_j)$ such that $p_n = p|L_n$ is continuous for each $n \in N$. For every $n \in N$, let $\lambda^j_{L_n} : b \to \lambda^j_{L_n, b}$ be the element of $\mathcal{F}^\infty(B, M_+(L_n), \alpha)$ (see Proposition 4) defined by

$$\lambda^j_{L_n, b} = (\lambda^j_b)_{L_n},$$

for $b \in B$. By Propositions 3 and 4 there is $\gamma_n : b \to \gamma_{n,b}$ belonging to $\mathcal{F}^\infty(B, M_+(L_n), \alpha)$ such that $\rho_T(\gamma_n) = \gamma_n$ and $\lambda^j_{L_n, b} = \gamma_{n,b}$ except for b in a locally α-negligible set D_n. We have (note that $Z, B, \mu_j, p, \alpha, T, \lambda^j$, satisfy obviously (1)-(6) and use correspondingly the formula (6″) of Remark 2) above):

$$\int_{L_n} g(f \circ p_n)\, d(\mu_j)_{L_n} = \int_Z^* g'(f \circ p)\, d\mu_j$$
$$= \int_B f(b)\, d\alpha(b) \int_Z g'\, d\lambda^j_b = \int_B f \langle g, \gamma_n \rangle \, d\alpha$$

for $g \in \mathcal{K}(L_n)$, $f \in \mathcal{K}(B)$; here g' is defined by $g'(z) = g(z)$ for $z \in L_n$ and $g'(z) = 0$ for $z \notin L_n$. By Proposition 7 (note that $L_n, B, (\mu_j)_{L_n}, p_n, \alpha, T, \gamma_n$ satisfy again (1)-(6) and the hypotheses of Proposition 7)

$$\operatorname{Supp} \gamma_{n,b} \subset \overset{-1}{p_n}(\{b\}), \qquad b \in B.$$

But then $\operatorname{Supp} \lambda^j_{L_n, b} \subset \overset{-1}{p_n}(\{b\})$ if $b \notin D_n$. If $u_n : L_n \to Z$ is the canonical injection we have $u_n(\lambda^j_{L_n, b}) = \varphi_{L_n} . \lambda^j_b$ for every $b \in B$ and hence (see [1], chap. V, p. 75)

$$\operatorname{Supp} (\varphi_{L_n} . \lambda^j_b) \subset \overset{-1}{p_n}(\{b\}) \subset \overset{-1}{p}(\{b\}) \qquad \text{if} \qquad b \notin D_n.$$

(IV) The family $(\varphi_{L_n} . \lambda^j_b)_{n\in N}$ is obviously summable for every $b \in B$ and

$$\lambda^j_b = \sum_{n\in N} \varphi_{L_n} . \lambda^j_b$$

locally almost everywhere for α (we use the Corollary of the Appendix to deduce that the complement of $\bigcup_{n\in N} L_n$ is λ^j_b-negligible, locally almost everywhere for α). We conclude (see [1], chap. V, p. 27) that λ_b^f is concentrated on $\overset{-1}{p}(\{b\})$ locally almost everywhere for α.

THEOREM 3. — *Let* Z *and* B *be two locally compact spaces, μ a positive Radon measure on* Z, *p a μ-proper mapping of* Z *into* B *and $\nu = p(\mu)$. Let now α be a positive Radon measure on* B *such that $\nu = \psi . \alpha$ for some locally α-integrable function ψ. Suppose $\alpha \neq 0$ and let* $T : f \to T_f$ *be a lifting of* $M_R^\infty(B, \alpha)$. *Then:*

3. 1) *There is a mapping $\lambda : b \to \lambda_b$ of* B *into* $M_+(Z)$, *appropriate with respect to* (α, T), *such that:*

(*i*) $\|\lambda_b\| = \psi(b)$ *locally almost everywhere for α;*

(*ii*) $\int_Z g(f \circ p)\, d\mu = \int_B f \langle g, \lambda \rangle\, d\alpha$ *for every* $f \in \mathcal{K}(B)$ *and* $g \in \mathcal{K}(Z)$.

3.2) *Moreover, if* T *is a strong lifting then λ_b is concentrated on $\overset{-1}{p}(\{b\})$ locally almost everywhere for α.*

3.3) *Let $\lambda' : b \to \lambda'_b$ and $\lambda'' : b \to \lambda''_b$ be two mappings of* B *into* $M_+(Z)$, *appropriate with respect to* (α, T) (5) *and such that:*

(*j*) *λ'_b and λ''_b are concentrated on $\overset{-1}{p}(\{b\})$ locally almost everywhere for α;*

(*jj*)
$$\mu = \int_B \lambda'_b\, d\alpha(b) = \int_B \lambda''_b\, d\alpha(b).$$

Then $\lambda'_b = \lambda''_b$ locally almost everywhere for α.

Existence of λ. — Consider the mapping $f \to (f \circ p) . \mu$ of $\mathcal{K}(B)$ into the Banach space $M^1(Z)$ of bounded Radon measures on Z endowed with the usual norm; $M^1(Z)$ is the dual of the Banach space of all continuous real-valued functions on Z, vanishing at infinity. We have

$$\|(f \circ p) . \mu\| = \int_Z |f \circ p|\, d\mu = \int_B |f|\, d\nu.$$

We deduce that $f \to (f \circ p) . \mu$ is a *positive* continuous linear mapping of $\mathcal{K}(B) \subset \overset{1}{\mathcal{L}}_R(B, \nu)$ into $M^1(Z)$. By the Dunford-Pettis theorem, which is valid without any countability

(5) Here the lifting T is not necessarily supposed to be strong.

hypotheses (see [5] and [6]), there is $\delta : b \to \delta_b$ belonging to $\mathcal{F}^\infty(\mathrm{B}, \mathrm{M}_+(\mathrm{Z}), \nu)$ verifying

$$\int_{\mathrm{Z}} g(f \circ p)\, d\mu = \int_{\mathrm{B}} f \langle g, \delta \rangle\, d\nu$$

for all $g \in \mathcal{K}(\mathrm{Z})$ and $f \in \mathcal{K}(\mathrm{B})$. Henceforth

$$\int_{\mathrm{Z}} g(f \circ p)\, d\mu = \int_{\mathrm{B}} f \langle g, \delta \rangle\, \psi\, d\alpha$$

for all $g \in \mathcal{K}(\mathrm{Z})$ and $f \in \mathcal{K}(\mathrm{B})$. Define $\gamma_b = \psi(b)\delta_b$ for $b \in \mathrm{B}$; it is clear that $\gamma : b \to \gamma_b$ belongs to $\mathcal{F}(\mathrm{B}, \mathrm{M}_+(\mathrm{Z}), \alpha)$. Let $\lambda : b \to \lambda_b$ in $\mathcal{F}(\mathrm{B}, \mathrm{M}_+(\mathrm{Z}), \alpha)$ be such that $\gamma \equiv \lambda$ and $\rho_{\mathrm{T}}[\lambda] = \lambda$ (by 2.1) of Proposition 2 such a λ exists). Obviously

$$\int_{\mathrm{Z}} g(f \circ p)\, d\mu = \int_{\mathrm{B}} f \langle g, \lambda \rangle\, d\alpha \tag{3}$$

for $g \in \mathcal{K}(\mathrm{Z})$ and $f \in \mathcal{K}(\mathrm{B})$. By an argument similar to that used in the proof of Remark 1) (preceding Proposition 6) we obtain, for $g \in \mathcal{K}_+(\mathrm{Z})$

$$\int_{\mathrm{B}}^{*} \langle g, \lambda \rangle\, d\alpha = \int_{\mathrm{Z}} g\, d\mu;$$

hence $\langle g, \lambda \rangle$ is essentially α-integrable for $g \in \mathcal{K}_+(\mathrm{Z})$ and therefore for each $g \in \mathcal{K}(\mathrm{Z})$. Thus λ is appropriate with respect to (α, T) and satisfies (*ii*) (see formulas 3) above). By Proposition 6, λ verifies also (*i*). Hence 3.1) is proved. The assertion 3.2) is a consequence of 3.1) and Proposition 8.

Uniqueness of λ. — Let $\mathrm{A} \subset \mathrm{B}$ be the set of all $b \in \mathrm{B}$ such that λ'_b is not concentrated on $\overset{-1}{p}(\{b\})$; then A is locally α-negligible. Let $f \in \mathcal{K}(\mathrm{B})$ and $g \in \mathcal{K}(\mathrm{Z})$. There is then a set $\mathrm{A}_{f,g} \subset \mathrm{B}$ locally α-negligible such that $g(f \circ p)$ is λ'_b-integrable if $b \notin \mathrm{A}_{f,g}$ (use the Theorem in the Appendix and remark that $g(f \circ p)$ is μ-integrable since it has compact support). Then, if $b \notin \mathrm{A} \cup \mathrm{A}_{f,g}$,

$$\begin{aligned}\int_{\mathrm{Z}} g(f \circ p)\, d\lambda'_b &= \int_{\mathrm{Z}} \varphi_{\overset{-1}{p}(\{b\})}(z) g(z) f(p(z))\, d\lambda'_b(z) \\ &= f(b) \int_{\mathrm{Z}} \varphi_{\overset{-1}{p}(\{b\})}(z) g(z)\, d\lambda'_b(z) = f(b) \langle g, \lambda'_b \rangle.\end{aligned}$$

Therefore, given $f \in \mathcal{K}(\mathrm{B})$ and $g \in \mathcal{K}(\mathrm{Z})$, we have (we use again the Theorem in the Appendix)

$$\int_{\mathrm{B}} f \langle g, \lambda' \rangle\, d\alpha = \int_{\mathrm{B}} d\alpha(b) \int_{\mathrm{Z}} g(f \circ p)\, d\lambda'_b = \int_{\mathrm{Z}} g(f \circ p)\, d\mu.$$

In the same way we prove that

$$\int_B f \langle g, \lambda'' \rangle \, d\alpha = \int_Z g(f \circ p) \, d\mu$$

for $f \in \mathcal{K}(B)$ and $g \in \mathcal{K}(Z)$.

From the above formulas we deduce that $\lambda' \equiv \lambda''$ and hence, by 2.2) of Proposition 2, that $\lambda'_b = \lambda''_b$ locally almost everywhere for α. Hence 3.3) is proved.

This completes the proof of Theorem 3.

Remarks. — 1) Theorem 3 above generalizes Theorem 1 in [1], chap. VI, p. 58-63 (see also [2]). In fact (with the notations of the latter theorem), if $\mathrm{Supp}\, \nu = B$ then (B, ν) has the « strong lifting property » (see Section 6) and hence Theorem 1 in [1], chap. VI, p. 58-63 follows directly from Theorem 3. The case $\mathrm{Supp}\, \nu \neq B$ can be reduced to the previous one. 2) Theorem 4 (6) in [2], p. 40-41 (see also [4]) and the results in [3], section 6, are also particular cases of Theorem 3. 3) Theorem 2 in [1], chap VI, p. 64-65, for instance, can be generalized using the strong lifting. For certain other methods and results concerning the disintegration of measures see also [8].

The next result is in a certain sense converse (7) to Theorem 3 :

THEOREM 4. — *Let* B *be a locally compact space,* $\alpha \neq 0$ *a positive Radon measure on* B *with* $\mathrm{Supp}\, \alpha = B$ *and* $T : f \to T_f$ *a lifting of* $\mathcal{M}^\infty_R(B, \alpha)$. *Then the assertions* 4.1) *and* 4.2) *below are equivalent:*

4.1) *There is a locally* α*-negligible set* $B_T \subset B$ *such that* $T_f(b) = f(b)$ *for each* $f \in C^\infty_R(B)$ *and* $b \notin B_T$;

4.2) *For every locally compact space* Z, *positive Radon measure* μ *on* Z *and* μ*-proper mapping* p *of* Z *into* B *such that* $\nu = p(\mu)$ *is absolutely continuous with respect to* α, *there is a mapping* $\lambda : b \to \lambda_b$ *belonging to* $\mathcal{F}(B, M_+(Z), \alpha)$, *appropriate with respect to* (α, T), *and having the properties:*

(h) $\mu = \int_B \lambda_b \, d\alpha(b)$;

(hh) λ_b *is concentrated on* $\overset{-1}{p}(\{b\})$ *locally almost everywhere for* α.

(6) This is in fact a consequence of Proposition 7 in this paper.

(7) Concerning the implication 4. 2) $\Longrightarrow$ 4. 1) see the remark in (X_3) of the Example at the end of Section 6.

The proof will be divided into three parts:

(I) Suppose that the lifting T satisfies 4.1). For each $b \notin B_T$ let χ_b be a character of $M_R^\infty(B, \alpha)$ such that $\chi_b(f) = f(b)$ for $f \in C_R^\infty(B)$ and $\chi_b(f) = \chi_b(g)$ if $f \in M_R^\infty(B, \alpha)$, $g \in M_R^\infty(B, \alpha)$ and $f \equiv g$. For $f \in M_R^\infty(B, \alpha)$ define $T'_f(b) = T_f(b)$ if $b \notin B_T$ and $T'_f(b) = \chi_b(f)$ if $b \in B_T$. It is easily seen that $T' : f \to T'_f$ is a strong lifting of $M_R^\infty(B, \alpha)$ and it is obvious that for each $f \in M_R^\infty(B, \alpha)$ we have $T'_f(b) = T_f(b)$ if $b \notin B_T$.

(II) Suppose 4.1) valid and let T′ be the lifting constructed in (I). By Theorem 3 there is a mapping $\lambda' : b \to \lambda'_b$ belonging to $\mathcal{F}(B, M_+(Z), \alpha)$, appropriate with respect to (α, T') and such that (h) and (hh) of 4.2) are verified. Let now $\lambda \in \mathcal{F}(B, M_+(Z), \alpha)$ such that $\lambda' \equiv \lambda$ and $\rho_T[\lambda] = \lambda$. Then λ is appropriate with respect to (α, T) and obviously satisfies (h). It remains to show that λ satisfies also (hh). For this it will be enough to show that $\lambda'_b = \lambda_b$ locally almost everywhere for α. Let

$$(K_j)_{j \in J} \in \mathcal{C}(B, \alpha)$$

such that

$$\rho_{T'}(\varphi_{K_j} \langle g, \lambda' \rangle) = \varphi_{\rho_{T'}(K_j)} \langle g, \lambda' \rangle$$

and

$$\rho_T(\varphi_{K_j} \langle g, \lambda \rangle) = \varphi_{\rho_T(K_j)} \langle g, \lambda \rangle$$

for $g \in \mathcal{K}(Z)$ and $j \in J$. But

$$\rho_{T'}(K_j) - B_T = \rho_T(K_j) - B_T (= C_j)$$

and

$$\rho_{T'}(\varphi_{K_j} \langle g, \lambda' \rangle)(b) = \rho_T(\varphi_{K_j} \langle g, \lambda \rangle)(b)$$

if $b \notin B_T$. Hence for $b \in C_j$ we have $\langle g, \lambda'_b \rangle = \langle g, \lambda_b \rangle$; whence, since g was arbitrary, $\lambda'_b = \lambda_b$. Since $j \in J$ was also arbitrary, ir follows that $\lambda'_b = \lambda_b$ locally almost everywhere for α.

(III) Suppose now 4.2) valid. Let $Z = B$, $p =$ the identity mapping and $\mu = \alpha$; then $\nu = p(\mu) = \alpha$. By 4.2) there is a mapping $\lambda : b \to \lambda_b$ belonging to $\mathcal{F}(B, M_+(Z), \alpha)$, appropriate with respect to (α, T), such that (h) and (hh) of 4.2) are verified. From (hh) of 4.2) it follows that Supp $\lambda_b \subset \{b\}$ locally almost everywhere for α and by Proposition 6, $\|\lambda\|_b = 1$ locally almost everywhere for α. Hence there is $A \subset B$ locally α-negligible such that $\lambda_b = \varepsilon_b$ for $b \notin A$. We deduce that for every $g \in \mathcal{K}(B)$, $\langle g, \lambda_b \rangle = g(b)$ if $b \notin A$.

Let now $(K_j)_{j \in J} \in \mathcal{C}(B, \alpha)$ such that

$$\rho_T(\varphi_{K_j} \langle g, \lambda \rangle) = \varphi_{\rho_T(K_j)} \langle g, \lambda \rangle$$

for every $g \in \mathcal{K}(B)$ and $j \in J$. We shall show that for every $g \in \mathcal{K}(B)$, $T_g(b) = g(b)$ if $b \in \bigcup_{j \in J} \rho_T(K_j) - A$. In fact, let $b \in \rho_T(K_j) - A$ for some $j \in J$. Then for each $g \in \mathcal{K}(B)$ we have

$$\begin{aligned}\rho_T(g)(b) = \rho_T(g)(b)\varphi_{\rho_T(K_j)}(b) = \rho_T(\varphi_{K_j}g)(b) = \rho_T(\varphi_{K_j} \langle g, \lambda \rangle)(b) \\ = \varphi_{\rho_T(K_j)}(b) \langle g, \lambda_b \rangle = \langle g, \lambda_b \rangle = g(b)\end{aligned}$$

and the assertion is proved. Denote the complement of $\bigcup_{j \in J} \rho_T(K_j) - A$ with B_T; then B_T is locally α-negligible and for every $g \in \mathcal{K}(B)$ we have

$$T_g(b) = g(b) \qquad \text{if} \qquad b \notin B_T.$$

Let now $g \in C_R^\infty(B)$, $g \geqslant 0$ and let $\mathcal{F}_g$ be the set of all $f \in \mathcal{K}(B)$ satisfying $0 \leqslant f \leqslant g$. Then, if $b \notin B_T$, we have

$$T_g(b) \geqslant T_f(b) = f(b)$$

for every $f \in \mathcal{F}_g$, whence $T_g(b) \geqslant g(b)$ if $b \notin B_T$. Let now $h \in C_R^\infty(B)$ and choose a constant c such that $c + h \geqslant 0$ and $c - h \geqslant 0$; we deduce $T_h(b) \geqslant h(b)$ and $-T_h(b) \geqslant -h(b)$ if $b \notin B_T$, whence $T_h(b) = h(b)$ for $b \notin B_T$. Thus 4.1) is proved.

This completes the proof of Theorem 4.

6. Various examples and remarks.

Let Z be a locally compact space and μ a positive Radon measure on Z. To simplify some of the following statements we shall say that the couple (Z, μ) has the *strong lifting property* whenever there is a strong lifting of $M_R^\infty(Z, \mu)$.

We shall state here several results without proof; some of them are quite easy to prove. In the statements below Z is a locally compact space and $\mu \neq 0$ a positive Radon measure on Z with Supp $\mu = Z$.

A) The couple (Z, μ) has the strong lifting property in each of the following cases: *i*) Z is metrizable; *ii*) (Z, μ) is hyperstonean (= Z is stonean and every rare set is locally μ-negligible); *iii*) μ is atomic.

B) If (Z, μ) has the strong lifting property and $K \subset Z$, $K \neq \emptyset$, is a compact such that $\operatorname{Supp} \mu_K = K$, then (K, μ_K) has the strong lifting property.

C) If $(K_j)_{j \in J} \in \mathcal{C}(Z, \mu)$ is such that (K_j, μ_{K_j}) has the strong lifting property for each $j \in J$, then (Z, μ) has the strong lifting property.

D) Let Z_1, Z_2 be two locally compact spaces and $\mu_1 \neq 0$, $\mu_2 \neq 0$ two positive Radon measures on Z_1, Z_2, respectively. Suppose that (Z_1, μ_1) has the strong lifting property, Z_2 is metrizable and $\operatorname{Supp} \mu_2 = Z_2$. Then $(Z_1 \times Z_2, \mu_1 \otimes \mu_2)$ has the strong lifting property.

E) Let $(Z_j)_{j \in J}$ be a family of metrizable compact spaces and for each $j \in J$ let μ_j be a positive Radon measure on Z_j with $\mu_j(Z_j) = 1$ and $\operatorname{Supp} \mu_j = Z_j$; Let $Z_\infty = \prod_{j \in J} Z_j$ and $\mu_\infty = \bigotimes_{j \in J} \mu_j$. Then (Z_∞, μ_∞) has the strong lifting property [8].

Let R be the real line and β the Lebesgue measure on R. Denote by $C_R^\infty(R, +)$ the algebra of all bounded real-valued functions defined on R, *continuous on the right* and by $C_R^\infty(R, -)$ the algebra of all bounded real-valued functions defined on R, *continuous on the left.* With this notation we may state and prove the following:

THEOREM 5. — *There is a lifting* $T : f \to T_f$ of $M_R^\infty(R, \beta)$ *such that* $T_f = f$ *for every* $f \in C_R^\infty(R, +)$ [*for every* $f \in C_R^\infty(R, -)$].

We shall consider only the case of $C_R^\infty(R, +)$; the case of $C_R^\infty(R, -)$ can be treated similarly.

For each $x \in R$ and $n \in N^* = \{1,2, \ldots\}$ let

$$I_n(x) = [x, x + 1/n].$$

For $f \in M_R^\infty(R, \beta)$ and $n \in N^*$ define the function f_n on R by the equations

$$f_n(x) = n \int_{I_n(x)} f(t)\, d\beta(t), \qquad x \in R;$$

(8) This result is essentially contained in [7].

it is obvious that $f_n \in M_R^\infty(R, \beta)$ (in fact $f_n \in C_R^\infty(R)$) and that

$$\sup_{(n,\, x) \in N^* \times R} |f_n(x)| \leqslant N_\infty(f).$$

Let $\mathfrak{U}$ be an ultrafilter on N^* finer than the Fréchet filter on N^*. Then for each $x \in R$, $n \to f_n(x)$ has a limit $f_\infty(x)$ with respect to $\mathfrak{U}$. Since (see for instance [9], p. 132) $(f_n(x))_{n \in N^*}$ converges almost everywhere to $f_\infty(x)$ and since f_∞ is bounded on R it follows that $f_\infty \in M_R^\infty(R, \beta)$. Define $S_f = f_\infty$ for $f \in M_R^\infty(R, \beta)$. Then $S : f \to S_f$ is a linear lifting of $M_R^\infty(R, \beta)$ and $S_f = f$ if $f \in C_R^\infty(R, +)$. Let $\mathfrak{D}$ be the set of all linear liftings V of $M_R^\infty(R, \beta)$ such that

$$\varphi_{\theta'(A)} \leqslant V_{\varphi_A} \leqslant \varphi_{\theta''(A)}, \qquad \text{for} \qquad A \in \mathcal{C}(\beta);$$

here $\theta'(A) = \{x | S_{\varphi_A}(x) = 1\}$ and $\theta''(A) = \{x | S_{\varphi_A}(x) \neq 0\}$. Then (see Proposition 1) there is a lifting $T : f \to T_f$ belonging to the set $\mathfrak{D}$. It is obvious that $\varphi_{\theta'(A)} = \varphi_{\theta''(A)}$ whenever S_{φ_A} is a characteristic function; hence $T_{\varphi_A} = S_{\varphi_A}$ whenever S_{φ_A} is a characteristic function. We deduce that $T_{\varphi_A} = S_{\varphi_A} = \varphi_A$ if $A = [a, b)$ where $a \in R$, $b \in R$, $a < b$ (since in this case $\varphi_A \in C_R^\infty(R, +)$).

Let us now prove that $T_g \geqslant g$ for each $g \in C_R^\infty(R, +)$, $g \geqslant 0$. In fact let $\mathfrak{V}_g$ be the set of all functions $\alpha\varphi_{[a,\, b)} \leqslant g$ with $\alpha \geqslant 0$, $a \in R$, $b \in R$, $a < b$. It is easy to see that $\sup \mathfrak{V}_g = g$. But $T_g \geqslant T_f = f$ for each $f \in \mathfrak{V}_g$; whence $T_g \geqslant \sup \mathfrak{V}_g = g$. Let now $f \in C_R^\infty(R, +)$ and let c be a constant such that $c + f \geqslant 0$, $c - f \geqslant 0$. Since $T_c = c$ we deduce $c + T_f = T_{c+f} \geqslant c + f$ and $c - T_f = T_{c-f} \geqslant c - f$; hence $T_f = f$ and the theorem is proved.

Remarks. — 1) The use of the ultrafilter $\mathfrak{U}$ was suggested by [3]. 2) Theorem 5 can be stated and proved in more general forms; however we shall not consider here such generalizations. 3) From Theorem 5 we deduce that if

$$\mathcal{F} \subset C_R^\infty(R, +)[\mathcal{F} \subset C_R^\infty(R, -)]$$

is a directed set of positive functions, then $f_\infty = \sup \mathcal{F}$ is β-measurable and, for every positive Radon measure ν on R absolutely continuous with respect to β, we have

$$\sup_{f \in \mathcal{F}} \int_R^* f\, d\nu = \int_R^* f_\infty\, d\nu.$$

An example. — For each $x \in \mathrm{R}$ and $y > 0$ let $\mathrm{U}_y(x)$ be the union of the intervals $[x, x+y)$ and $(-x-y, -x)$. Then there is a topology Σ on R such that $\{\mathrm{U}_y(x) | y > 0\}$ is a fundamental system of x, for each $x \in \mathrm{R}$. Let Z be the interval $[-1, 1)$ endowed with the toplogy induced by Σ. Then Z is non-metrizable, compact and $f : \mathrm{Z} \to \mathrm{R}$ belongs to $\mathrm{C_R(Z)}$ if and only if $f(z) = f(z+) = f((-z)-)$ for each $z \in \mathrm{Z}$. We shall denote below by B and H the intervals $[0, 1)$ and $[-1, 1)$ respectively, when endowed with the usual topological structures. Let β be the Lebesgue measure on R and α the restriction of β to B. Let μ be the Radon measure on Z defined by the equations

$$\mu(f) = \int_{\mathrm{B}} (f|\mathrm{B})\, d\alpha, \qquad f \in \mathrm{C_R(Z)}.$$

For a set $\mathrm{A} \subset \mathrm{Z}$ the equations $\beta^*(\mathrm{A}) = 0$ and $\mu^*(\mathrm{A}) = 0$ are equivalent (for the above assertions see N. Bourbaki, *Topologie*, chap. IX, p. 49 (1958) and [1], chap. VI, p. 99). It follows that if $f : \mathrm{Z} \to \mathrm{R}$ is μ-measurable then f is β_{H}-measurable; in particular $\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu) \subset \mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{H}, \beta_{\mathrm{H}})$. Also if T is a lifting of $\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{H}, \beta_{\mathrm{H}})$ then $\mathrm{T}|\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu)$ is a lifting of $\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu)$.

(X_1) *The couple* (Z, μ) *has the strong lifting property.*

In fact let $\mathrm{U} : f \to \mathrm{U}_f$ be a lifting of $\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{R}, \beta)$ such that $\mathrm{U}_f = f$ if $f \in \mathrm{C}_{\mathrm{R}}^{\infty}(\mathrm{R}, +)$ (see Theorem 5). Let $\mathrm{U}' : f \to \mathrm{U}'_f$ be the lifting of $\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{H}, \beta_{\mathrm{H}})$ defined by the equations $\mathrm{U}'_f(z) = \mathrm{U}_{f'}(z)$ if $f \in \mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{H}, \beta_{\mathrm{H}})$ and $z \in \mathrm{H}$; here (and further below) $f'(z) = f(z)$ if $z \in \mathrm{H}$ and $f'(z) = 0$ if $z \notin \mathrm{H}$. Since every function in $\mathrm{C_R(Z)}$ is in particular continuous on the right, it follows that

$$\mathrm{T} = \mathrm{U}'|\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu)$$

is a strong lifting of $\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu)$.

(X_2) *There is a lifting* $\mathrm{S} : f \to \mathrm{S}_f$ *of* $\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu)$ *such that*

$$(*) \qquad \bigcup_{f \in \mathrm{C_R(Z)}} \{z | \mathrm{S}_f(z) \neq f(z)\} = \mathrm{Z}.$$

Let $\mathrm{Y} : f \to \mathrm{Y}_f$ be a lifting of $\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{R}, \beta)$ such that $\mathrm{Y}_f = f$ if $f \in \mathrm{C}_{\mathrm{R}}^{\infty}(\mathrm{R}, -)$. Let χ be a character of $\mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu)$ such that $\chi(f) = f(1/2)$ if $f \in \mathrm{C_R(Z)}$ and $\chi(f) = \chi(g)$ if $f \in \mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu)$, $g \in \mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu)$ and $f \equiv g$. For $f \in \mathrm{M}_{\mathrm{R}}^{\infty}(\mathrm{Z}, \mu)$ define

$$\mathrm{S}_f(z) = \begin{cases} \mathrm{Y}_{f'}(z) & \text{if} \quad z \neq 0 \quad \text{and} \quad z \in \rho_{\mathrm{Y}}(\mathrm{Z}) \cap \mathrm{Z} = (-1, 1) \\ \chi(f) & \text{if} \quad z = 0 \quad \text{or} \quad z = -1. \end{cases}$$

Then $S : f \to S_f$ is a lifting of $M_R^\infty(Z, \mu)$. If $f = \varphi_{[-a, a)}$, $0 < a < 1$, we have $f \in C_R(Z)$ and $S_f(z) \neq f(z)$ for $z = -a$ or $z = a$. We deduce (use also the definition of χ) the relation $(*)$.

This shows in particular that *a lifting of* $M_R^\infty(Z, \mu)$ *can not necessarily be modified on a set of measure zero so as to become a strong lifting.*

(X_3) Let $\lambda : z \to \lambda_z$ be the mapping of Z into $M_+(Z)$ defined by the equations (S is the lifting defined in (X_2)) :

$$\langle f, \lambda_z \rangle = S_f(z), \qquad f \in C_R(Z), \qquad z \in Z.$$

We have $\operatorname{Supp} \lambda_z \cap \{z\} = \emptyset$ *for each* $z \in Z$, *although* λ *is appropriate with respect to* (μ, S). Moreover Z, Z, μ, p = the identity mapping, μ, S, λ, satisfy (1)-(6) formulated at the beginning of Section 5.

The main interest of (X_3) is that a situation is exhibited here, where for every b, $\operatorname{Supp} \lambda_b$ and $\overset{-1}{p}(\{b\})$ are disjoint. That such a situation can occur was first remarked to us by Professor Gustave Choquet. It was this remark which suggested to us some of the considerations in this Example (including (X_3)) as well as the formulation of a « converse » to Theorem 3 (namely the implication 4.2) $\Longrightarrow$ 4.1) of Theorem 4).

(X_4) Let Z, B, μ, α be as in the introduction of this Example. Let p be a mapping of Z into B such that $p(z) = |z|$ for $z \in (-1, 1)$. Let $T : f \to T_f$ be a lifting of $M_R^\infty(B, \alpha)$ such that $T_f = f$ if $f \in M_R^\infty(B, \alpha)$ is continuous on the right. Finally let $\lambda : b \to \lambda_b$ be the mapping of B into $M_+(Z)$ defined by $\lambda_b = \varepsilon_b$ for $b \in B$. *Then* λ *is appropriate with respect to* (α, T) *but* λ *is not* α*-adequate (even though* T *is a strong lifting).* Moreover Z, B, μ, p, α, T, λ satisfy (1)-(6).

It is known that for every couple (Z, μ) with $\mu \neq 0$, there is a lifting of $M_R^\infty(Z, \mu)$. *Problem:* Decide whether or not every couple (Z, μ) with $\mu \neq 0$ and $\operatorname{Supp} \mu = Z$ has the strong lifting property.

Appendix.

A.I. — Let Z be a locally compact space and let M(Z) be endowed with the topology $\sigma(M(Z), \mathcal{K}(Z))$. Let B be another locally compact space and let α be a positive Radon measure on B. If $\lambda : b \to \lambda_b$ is a *scalarly essentially α-integrable mapping* ([1]), chap. VI, p. 8) of B into $M_+(Z)$ then

$$\mu = \int_B \lambda_b \, d\alpha(b)$$

is a positive Radon measure on Z. Let us recall that λ is scalarly essentially α-integrable if $\langle g, \lambda \rangle$ essentially α-integrable for every $g \in \mathcal{K}(Z)$; μ is defined by the equations

$$\langle g, \mu \rangle = \int_B \langle g, \lambda_b \rangle \, d\alpha(b), \qquad g \in \mathcal{K}(Z).$$

Let us now formulate the following condition:

(C) If $f : Z \to R_+$ is lower semi-continuous then

$$\int_Z^* f(z) \, d\mu(z) = \overline{\int_B^*} d\alpha(b) \int_Z^* f(z) \, d\lambda_b(z).$$

Recall that if λ is α-adequate (see [1], chap. V, p. 18-19) the condition (C) is verified. We shall prove now the following:

Proposition. — *Assume* $\alpha \neq 0$ *and let* $T : f \to T_f$ *be a lifting of* $M_R^\infty(B, \alpha)$. *Suppose that* $\lambda : b \to \lambda_b$ *is appropriate with respect to* (α, T). *Then the condition* (C) *is verified.*

Let $(K_j)_{j \in J} \in \mathcal{C}(B, \alpha)$ such that

$$\rho_T(\varphi_{K_j} \langle g, \lambda \rangle) = \varphi_{\rho_T(K_j)} \langle g, \lambda \rangle$$

for all $g \in \mathcal{K}(Z)$ and $j \in J$. For each $H \in \mathcal{F}(J)$ (= the set of all finite parts of J) denote

$$K_H = \bigcup_{j \in H} K_j;$$

then for $g \in \mathcal{K}(Z)$ and $H \in \mathcal{F}(J)$ we have

$$\rho_T(\varphi_{K_H} \langle g, \lambda \rangle) = \varphi_{\rho_T(K_H)} \langle g, \lambda \rangle.$$

Let $f : Z \to R_+$ be lower semi-continuous and let $\mathcal{F}_f$ be the set of all $g \in \mathcal{K}(Z)$ satisfying $0 \leqslant g \leqslant f$. Then

$$\sup_{g \in \mathcal{F}_f, \, H \in \mathcal{F}(J)} \rho_T(\varphi_{K_H} \langle g, \lambda \rangle)(b) = \int_Z^* f(z) \, d\lambda_b(z)$$

for each $b \in \bigcup_{j \in J} \rho_T(K_j)$. It follows (see A.II. below)

$$\overline{\int_B^*} d\alpha(b) \int_Z^* f(z)\, d\lambda_b(z) = \sup_{g \in \mathcal{F}_f,\, H \in \mathcal{F}(J)} \int_B \varphi_{\rho_T(K_H)} \langle g, \lambda \rangle\, d\alpha$$
$$= \sup_{g \in \mathcal{F}_f} \left(\sup_{H \in \mathcal{F}(J)} \int_B \varphi_{\rho_T(K_H)} \langle g, \lambda \rangle\, d\alpha \right) = \sup_{g \in \mathcal{F}_f} \int_B \langle g, \lambda \rangle\, d\alpha$$
$$= \sup_{g \in \mathcal{F}_f} \int_Z g\, d\mu = \int_Z^* f\, d\mu.$$

Therefore condition (C) is satisfied.

Exactly as in [1], chap. V, pp. 21-23, one can prove the following:

Theorem. — *Suppose that* $\lambda : b \to \lambda_b$ *satisfies condition* (C). *Let* f *be a real-valued* μ*-integrable function on* Z. *Then there is a locally* α*-negligible set* $H \subset B$ *such that* f *is* λ_b*-integrable for* $b \notin H$, $b \to \int_Z f\, d\lambda_b$ *is essentially* α*-integrable and*

$$\int_Z f\, d\mu = \int_B d\alpha(b) \int_Z f\, d\lambda_b.$$

From the above theorem we deduce immediately the:

Corollary. — *Let* $A \subset Z$ *be a* μ*-negligible set. Then there is a locally* α*-negligible set* $H \subset B$ *such that* A *is* λ_b*-negligible for each* $b \notin H$.

A.II. — Let B be a locally compact space, $\alpha \neq 0$ a positive Radon measure on B and T a lifting of $M_R^\infty(B, \alpha)$. The following result was used several times in the text:

(P) *Let* $\mathcal{F}$ *be a directed set of functions belonging to* $M_R^\infty(B, \alpha)$, *such that* $f \geqslant 0$ *and* $\rho_T(f) = f$ *for each* $f \in \mathcal{F}$. *Let* f_∞ *be defined by* $f_\infty(b) = \sup_{f \in \mathcal{F}} f(b)$, *for each* $b \in B$. *Then* f_∞ *is* α*-measurable and for every positive Radon measure* ν *on* B *absolutely continuous with respect to* α *we have*

$$\sup_{f \in \mathcal{F}} \overline{\int_B^*} f\, d\nu = \overline{\int_B^*} f_\infty\, d\nu.$$

This is (essentially) the Theorem (P) of the Appendix in [6].

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