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HYPOELLIPTIC DIFFERENTIAL OPERATORS [1]

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1. Introduction.

A differential operator $P(x, D)$ with coefficients in C^∞ is called *hypoelliptic* if the equation

$$(1.1) \qquad P(x, D)u = f$$

only has solutions $u \in C^\infty$ when $f \in C^\infty$. (For the notations see section 2.) When the coefficients are constant, a complete algebraic characterization of hypoelliptic operators was given in [1]. For variable coefficients a sufficient condition for hypoellipticity has been given by several authors (see [2], [3], [4], [6]), namely that the operators with constant coefficients $P(x, D)$ obtained by giving x fixed values shall be hypoelliptic and equally strong in the sense defined in [1]. In fact, the latter condition enables one to carry over most results known in the case of constant coefficients at least locally by means of a perturbation argument (see [4]). A weaker sufficient condition has also been given by Trèves [6], but it is extremely implicit and difficult to verify for a given operator. His proofs depend on the construction of a parametrix for the adjoint operator by the method of successive approximations, in an abstract and very intricate form. We shall here use the same idea but in a technically different and really straight-forward

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way. This will yield sufficient conditions for hypoellipticity which are weaker than those of [2], [3], [4] and are satisfied by the only other example of a hypoelliptic operator given in [6].

2. The plan for constructing a parametrix.

In this section we shall only give a formal outline of the construction of a parametrix which will be carried out in section 4. Accordingly, we shall postpone discussing the convergence of the integrals occurring here.

We first introduce some notations. Differential operators will be written in the form

$$P(x, D) = \Sigma a_\alpha(x) D^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$, called a multi-index, is a sequence of integers between 1 and the dimension n of the space, and

$$D^\alpha = (-i\partial/\partial x_{\alpha_1}) \ldots (-i\partial/\partial x_{\alpha_k}).$$

The empty multi-index will be denoted by 0; we set $D^0 = 1$. The length k of the multi-index is denoted by $|\alpha|$. If

$$\xi = (\xi_1, \ldots, \xi_n)$$

is a real vector, we write

$$P(x, \xi) = \Sigma a_\alpha(x) \xi_\alpha$$

where $\xi_\alpha = \xi_{\alpha_1} \ldots \xi_{\alpha_k}$. The derivatives of $P(x, \xi)$ with respect to ξ are sometimes denoted by $P^{(\alpha)}(x, \xi)$;

$$P^{(\alpha)}(x, \xi) = \partial^{|\alpha|} P(x, \xi)/\partial\xi_{\alpha_1} \ldots \partial\xi_{\alpha_k}.$$

Derivatives with respect to ξ or x will be denoted by D_ξ^α or D_x^α. Finally, we shall use the notation

$$\tilde{P}(x, \xi) = (\Sigma|P^{(\alpha)}(x, \xi)|^2)^{\frac{1}{2}}.$$

A fundamental solution $E(x, y)$ of a differential operator $P(x, D)$ is a kernel (in fact a distribution) such that

$$\varphi(x) = P(x, D) \int E(x, y)\varphi(y)\, dy \tag{2.1}$$

if φ has compact support. In the case where the coefficients are constant, it is convenient to construct a fundamental solution by means of the Fourier transformation, that is, to set

$$\int \mathrm{E}(x, y)\varphi(y)\, dy = (2\pi)^{-n}\int e^{i\langle x, \xi\rangle}\mathrm{P}(\xi)^{-1}\hat{\varphi}(\xi)\, d\xi,$$

where

$$\hat{\varphi}(\xi) = \int e^{-i\langle x, \xi\rangle}\varphi(x)\, dx \tag{2. 2}$$

is the Fourier transform of φ. We shall imitate this in the case of variable coefficients, thus try to find a kernel K such that

$$\varphi(x) = \mathrm{P}(x, \mathrm{D})(2\pi)^{-n}\int e^{i\langle x, \xi\rangle}\mathrm{K}(x, \xi)\hat{\varphi}(\xi)\, d\xi \tag{2. 3}$$

when φ has compact support. Operating under the integral sign we find that this is equivalent to

$$\varphi(x) = (2\pi)^{-n}\int e^{i\langle x, \xi\rangle}\, \mathrm{P}(x, \mathrm{D}_x + \xi)\, \mathrm{K}(x, \xi)\, \hat{\varphi}(\xi)\, d\xi. \tag{2. 4}$$

Thus we have to find a kernel K such that

$$\mathrm{P}(x, \mathrm{D}_x + \xi)\, \mathrm{K}(x, \xi) = 1,$$

or, which is equivalent in view of Taylor's formula,

$$\mathrm{P}(x, \xi)\, \mathrm{K}(x, \xi) + \sum_{\alpha\neq 0} \mathrm{P}^{(\alpha)}(x, \xi)\, \mathrm{D}_x^\alpha \mathrm{K}(x, \xi)/|\alpha|\,! = 1. \tag{2. 5}$$

The equation (2. 5) can be solved approximately in the following way. First we neglect the sum since it would have been absent if the coefficients were constant. Thus we define a kernel K_0 by the equation

$$\mathrm{P}(x, \xi)\, \mathrm{K}_0(x, \xi) = 1. \tag{2. 6}$$

To compensate the error committed in solving (2. 5) in this way we then define successively kernels K_j by means of the recursion formula

$$\mathrm{P}(x, \xi)\, \mathrm{K}_{j+1}(x, \xi) + \sum_{\alpha\neq 0} \mathrm{P}^{(\alpha)}(x, \xi)\, \mathrm{D}_x^\alpha \mathrm{K}_j(x, \xi)/|\alpha|\,! = 0, \quad j \geqq 0. \tag{2. 7}$$

Adding the equations (2. 7) and (2. 6) we obtain

$$\mathrm{P}(x, \mathrm{D}_x + \xi)\, (\mathrm{K}_0 + \cdots + \mathrm{K}_j) + \mathrm{P}(x, \xi)\, \mathrm{K}_{j+1} = 1. \tag{2. 8}$$

Hence, formally, we obtain instead of (2. 3)

(2. 9)

$$\varphi(x) = P(x, D)(2\pi)^{-n}\int e^{i\langle x,\xi\rangle}(K_0(x, \xi) + \cdots + K_j(x, \xi))\hat{\varphi}(\xi)\,d\xi$$
$$+ (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}\,P(x, \xi)\,K_{j+1}(x, \xi)\,\hat{\varphi}(\xi)\,d\xi.$$

If the polynomial $P(x, \xi)$ has real zeros, the kernels K_j become singular. However, if the zeros are all contained in a fixed compact set, as will be the case here, it is easy to avoid the singularities in the following way. Choose a function $\psi_0 \in C_0^\infty$ which is equal to 1 in a neighborhood of the zeros of $P(x, \xi)$ (as a function of ξ). Set $\psi_1 = 1 - \psi_0$, so that $\psi_1 = 1$ outside a compact set. Since

$$1 = \psi_1 + \psi_0 = P(x, D_x + \xi)\,(K_0 + \cdots + K_j)\,\psi_1$$
$$+ P(x, \xi)\,K_{j+1}\psi_1 + \psi_0,$$

we may then replace (2. 9) by

(2. 10)

$$\varphi(x) = P(x, D)(2\pi)^{-n}\int e^{i\langle x,\xi\rangle}(K_0(x, \xi) + \cdots + K_j(x, \xi))\psi_1(\xi)\hat{\varphi}(\xi)\,d\xi$$
$$+ (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}(P(x, \xi)\,K_{j+1}(x, \xi)\,\psi_1(\xi) + \psi_0(\xi))\,\hat{\varphi}(\xi)\,d\xi.$$

In the next section we shall introduce certain conditions which ensure that the kernel $K_{j+1}(x, \xi)$ decreases very rapidly as $\xi \to \infty$ when j is large. The last term in (2. 10) will then be an integral operator with a very smooth kernel when expressed in terms of φ instead of $\hat{\varphi}$. The first term on the right hand side of (2. 10), on the other hand, will be easy to study with the same methods that are used in the case of constant coefficients.

3. The condition HE.

The recursion formula (2. 7) indicates that to be sure that K_{j+1} decreases faster than K_j at infinity, we need to know that differentiation of $P(x, \xi)$ with respect to ξ will decrease the growth at infinity more than the corresponding differentiation with respect to x will increase the growth. This leads to the condition posed in the following definition.

Definition. — *The operator* $P(x, D)$ *will be said to satisfy the condition* HE *in* Ω *if* $P(x, D)$ *is not identically* 0 *in any component of* Ω *and*

a) *The coefficients are in* $C^\infty(\Omega)$;

b) *There are functions* $M_j(x, \xi)$ *defined in* $\Omega \times R^n$ *such that for all* α *and* β

$$(3.1) \quad |D_\xi^\alpha D_x^\beta P(x, \xi)| \leq C_{\beta,x}(1+|\xi|)^{-d|\alpha|} M^{\beta-\alpha}(x, \xi)\tilde{P}(x, \xi), \ \xi \in R^n,$$
$$(3.2) \quad 1 \leq M_j(x, \xi) \leq C_x(1 + |\xi|)^{1-d}.$$

Here d *is a positive constant,* $C_{\beta, x}$ *and* C_x *are bounded when* x *varies in compact subsets of* Ω, *and we have used the notation*

$$M^{\beta-\alpha} = M_{\beta_1} M_{\beta_2} \ldots M_{\alpha_1}^{-1} M_{\alpha_2}^{-1} \ldots$$

Since this condition, although very convenient in the proofs, may seem involved and difficult to check, we also give a simpler but more restrictive condition.

THEOREM 3.1. — *Let the coefficients of* $P(x, D)$ *be in* $C^\infty(\Omega)$ *and set*

$$M(\xi) = \sup_{x, y \in \Omega} \tilde{P}(y, \xi)/\tilde{P}(x, \xi).$$

Assume that the coefficients only depend on $x_1, \ldots, x_k$ *and set* $|\alpha|_k =$ *the number of indices* $\leq k$ *in* α. *If there are positive constants* C *and* d *such that*

$$(3.3) \quad |P^{(\alpha)}(y, \xi)|/\tilde{P}(y, \xi) \leq C(1 + |\xi|)^{-d|\alpha|} M(\xi)^{-|\alpha|_k}, \quad (y, \xi) \in \Omega \times R^n,$$
$$(3.4) \quad M(\xi) \leq C(1 + |\xi|)^{1-d},$$

it then follows that $P(x, D)$ *satisfies the condition* HE.

PROOF. — Set $M_j(\xi) = M(\xi)$ when $j \leq k$ and $M_j(\xi) = 1$ when $j > k$. If $\beta = 0$, the inequality (3.1) then follows from (3.3), and (3.1) is trivial if $|\beta|_k = 0$ but $|\beta| \neq 0$, the left hand side being 0. Hence we may assume that $|\beta|_k \neq 0$.

It is clear that (3.4) implies that the degree of $P(x, \xi)$ in ξ is bounded when $x \in \Omega$. Hence there exists a maximal set of points x_j such that the polynomials $P_j(\xi) = P(x_j, \xi)$ are linearly independent. We can thus write

$$P(x, \xi) = \Sigma c_j(x) \ P_j(\xi)$$

with uniquely determined coefficients $c_j \in C^\infty(\Omega)$. From the formula

$$D_x^\beta P^{(\alpha)}(x, \xi) = \Sigma D_x^\beta c_j(x) P_j^{(\alpha)}(\xi),$$

we obtain by using (3. 3) and the definition of $M(\xi)$, with y replaced by x_j, that

$$|D_x^\beta P^{(\alpha)}(x, \xi)| \leqq C_{\beta, x}(1 + |\xi|)^{-d|\alpha|} M(\xi)^{1-|\alpha|_k} \tilde{P}(x, \xi).$$

When $|\beta|_k \neq 0$, this implies (3. 1). The proof is complete.

Remark. — The sufficient condition for hypoellipticity given in [2], [3] and [4] is that M shall be bounded and (3. 3) valid with the factor $M^{-|\alpha|_k}$ omitted. Hence Theorem 5. 1 below will contain the results of those papers. In section 6 we shall also give some other examples, in particular one studied in [6].

Before proceeding we shall write (3. 1) in a more useful form. First note that taking $\beta = 0$ in (3. 1), squaring and adding over all $\alpha \neq 0$, we obtain, since $M_j(x, \xi) \geqq 1$,

$$\tilde{P}(x, \xi)^2 \leqq |P(x, \xi)|^2 + C_x \tilde{P}(x, \xi)^2 (1 + |\xi|)^{-2d}.$$

Here C_x is bounded on compact subsets of Ω. (From now on this will be the case whenever we indicate that a constant depends on x. The same notation will be used for different constants.) Hence there is a constant A_x such that

$$\tilde{P}(x, \xi) \leqq 2|P(x, \xi)|, \quad |\xi| \geqq A_x, \tag{3. 5}$$

so we may replace (3. 1) by

(3. 1)′

$$|D_\xi^\alpha D_x^\beta P(x, \xi)| \leqq C_{\beta, x}(1 + |\xi|)^{-d|\alpha|} M^{\beta-\alpha}(x, \xi) |P(x, \xi)|, \quad |\xi| \geqq A_x.$$

We note that if follows from (3. 1)′ that $P(x, \xi) \neq 0$ when $|\xi| \geqq A_x$. For if $P(x_0, \xi_0) = 0$ and $|\xi_0| \geqq A_{x_0}$, it follows from (3. 1)′ that $D_\xi^\alpha P(x_0, \xi_0) = 0$ for all α, hence $P(x_0, \xi) = 0$ for all ξ. Since (3. 1)′ with $\alpha = 0$ and $|\beta| = 1$ shows that

$$|\mathrm{grad}_x P(x, \xi)| \leqq C_{x, \xi} |P(x, \xi)| \quad \text{when} \quad |\xi| \geqq A_x,$$

it would follow that $P(x, \xi) = 0$ for all ξ and all x in the same component of Ω as x_0, which contradicts the definition. We can thus define the kernels $K_j(x, \xi)$ by means of (2. 6) and (2. 7) when $|\xi| \geqq A_x$.

Theorem 3. 2. — *If* $P(x, D)$ *satsifies the condition* HE *and the kernels* K_j *are defined by* (2. 6) *and* (2. 7), *we have the estimates*

(3. 6)

$$|D_\xi^\alpha D_x^\beta K_j(x, \xi)| \leqq C_{\alpha, \beta, x, j}(1 + |\xi|)^{-d(|\alpha|+j)} M^{\beta-\alpha}(x, \xi)/|P(x, \xi)|, \ |\xi| \geqq A_x.$$

REMARK. — Note that for $j = 0$ this differs from (3. 1)′ only in the fact that $P(x, \xi)$ has been replaced by $1/P(x, \xi) = K_0(x, \xi)$. It is thus clear that (3. 6) for $j = 0$ implies (3. 1)′ so that (3. 6) is in fact equivalent to the assumptions.

PROOF OF THEOREM 3. 2. — When $j = |\alpha| = |\beta| = 0$, the estimate (3. 6) is trivial. We shall prove it in general by induction. With the convention $K_{-1} = 0$, it follows from (2. 6) and (2. 7) by application of the differential operator $D_\xi^\alpha D_x^\beta$ that

$$(3.7)\quad \begin{aligned} P(x,\xi)D_\xi^\alpha D_x^\beta K_j(x,\xi) = &-\Sigma'(D_\xi^{\alpha'}D_x^{\beta'}P(x,\xi))(D_\xi^{\alpha''}D_x^{\beta''}K_j(x,\xi)) \\ &-\Sigma \sum_{|\gamma|\neq 0} i^{|\gamma|}(D_\xi^{\alpha'+\gamma}D_x^{\beta'}P(x,\xi))(D_\xi^{\alpha''}D_x^{\beta''+\gamma}K_{j-1}(x,\xi))/|\gamma|\,! \end{aligned}$$

if $|\alpha| + |\beta| + j \neq 0$. The sums are extended over all α', α'', β', β'' with $\alpha' + \alpha'' = \alpha$ and $\beta' + \beta'' = \beta$, except in the sum denoted by Σ', where the term $\alpha' = \beta' = 0$ shall be omitted.

Assume that (3. 6) is already proved when j is replaced by a smaller number or the multi-indices α, β are replaced by multi-indices of smaller total length. In view of (3. 1)′ we can then estimate the right hand side of (3. 7) by a constant times

$$(1+|\xi|)^{-d(|\alpha|+j)}M^{\beta-\alpha}(x,\xi) + \sum_{|\gamma|\neq 0}(1+|\xi|)^{-d(|\alpha|+|\gamma|+j-1)}M^{\beta-\alpha}(x,\xi).$$

Since $|\gamma| - 1 \geqq 0$ in the last sum, this proves (3. 6).

Two corollaries of Theorem 3. 2 will be useful in the next section.

COROLLARY 3. 1. — *If* $P(x, D)$ *satisfies the condition* HE, *we have*

$$(3.8)\quad |D_x^\beta(P(x,\xi)K_j(x,\xi))| \leqq C_{\beta,x,j}(1+|\xi|)^{|\beta|-dj},\ |\xi| \geqq A_x.$$

PROOF. — The inequality follows immediately from Leibniz′ formula, (3. 1)′ and (3. 6) if we estimate $M^\beta(x, \xi)$ by $C_x(1+|\xi|)^{|\beta|}$, which is possible in view of (3. 2).

COROLLARY 3. 2. — *If* $P(x, D)$ *satisfies the condition* HE *and we denote by* m_x *the order of* $P(x, D)$ *we have*

$$(3.9)\quad |D_\xi^\alpha D_x^\beta K_j(x,\xi)| \leqq C_{\beta,\alpha,x,j}(1+|\xi|)^{m_x+|\beta|-d(j+|\alpha|+|\beta|)},\qquad |\xi| \geqq A_x.$$

PROOF. — In (3. 6) we estimate $M^{\beta-\alpha}$ by $(C_x(1+|\xi|)^{1-d})^{|\beta|}$, which is possible in view of (3. 2). It then only remains to prove that

$$(3.\ 10)\qquad 1/|P(x, \xi)| \leqq C_x(1+|\xi|)^{m_x},\ |\xi| \geqq A_x.$$

By forming the Taylor expansion of $P(x, \xi)$ at ξ we find that

$$\tilde{P}(x,\ 0) \leqq C_{m_x}(1+|\xi|)^{m_x}\tilde{P}(x,\ \xi),$$

hence

$$(3.\ 11)\qquad 1/\tilde{P}(x, \xi) \leqq C_{m_x}(1+|\xi|)^{m_x}/\tilde{P}(x, 0).$$

As proved after (3. 1)′ the polynomial $P(x, \xi)$ is not identically 0 for any x, hence $\tilde{P}(x,\ 0)$ is continuous and $\neq 0$ everywhere. The inequality (3. 10) is therefore a consequence of (3. 11) and (3. 5).

For reference in section 5 we end this section by proving

THEOREM 3. 3. — *If* $P(x,\ D)$ *satisfies the condition* HE, *it follows that the adjoint operator* $P^*(x,\ D)$ *also satisfies the condition* HE.

PROOF. — First recall that the adjoint is defined by the identity

$$\int (P(x,\ D)u)\varphi\, dx = \int uP^*(x,\ D)\varphi\, dx$$

when u and φ are in $C_0^\infty(\Omega)$. This means that if we write

$$P(x,\ D) = \Sigma a_j(x)P_j(D)$$

where $P_j(D)$ are differential operators with constant coefficient (for example the operators D^α indexed as a sequence), then

$$P^*(x,\ D)\varphi = \Sigma P_j(-D)(a_j\varphi) = \Sigma((-D_x)^\alpha a_j)(P_j^{(\alpha)}(-D)\varphi)/|\alpha|!$$

Thus

$$(3.\ 12)\qquad P^*(x,\ -\xi) = \Sigma(-1)^{|\alpha|}D_x^\alpha P^{(\alpha)}(x,\ \xi)/|\alpha|!$$

which gives in view of (3. 1)′

$$(3.\ 13)\qquad |P^*(x, -\xi) - P(x, \xi)| \leqq C_x(1+|\xi|)^{-d}|P(x, \xi)|,\ |\xi| \geqq A_x.$$

Hence we can find B_x so that

$$(3.\ 14)\qquad |P(x,\ \xi)| \leqq 2|P^*(x,\ -\xi)| \quad \text{when} \quad |\xi| \geqq B_x.$$

From (3. 12) and (3. 1)′ we now immediately obtain using (3. 14) that

$$(3.\ 15)\quad |D_\xi^\alpha D_x^\beta P^*(x, -\xi)| \leq C_{\beta,x}(1+|\xi|)^{-d|\alpha|}M^{\beta-\alpha}(x, \xi)|P^*(x, -\xi)|, \quad |\xi| \geq B_x,$$

which proves the theorem.

4. Regularity properties of the parametrix.

We assume once for all in this section that $P(x, D)$ satisfies the condition HE. By Ω' we denote a relatively compact open subset of Ω and by A' an upper bound for A_x when $x \in \Omega'$. We shall study the integral operators in (2. 10) for $x \in \Omega'$, taking $\psi_0(\xi) = 1$ when $|\xi| \leq A'$. Note that if $x \in \Omega'$ it follows from (3. 9) that all integrals in (2. 10) converge, hence that (2. 10) is valid.

THEOREM 4. 1. — *The integral*

(4. 1)

$$F_j(x, y) = (2\pi)^{-n}\int e^{i\langle x-y, \xi\rangle}(\psi_1(\xi)P(x, \xi)K_{j+1}(x, \xi) + \psi_0(\xi))\, d\xi, \quad x \in \Omega',$$

converges absolutely and F_j *is in* $C^k(\Omega' \times R^n)$ *if* $d(j+1) > (n+k)$. *If* $\varphi \in C_0^\infty(R^n)$, *we then have*

$$(4.\ 2)\quad \int F_j(x, y)\varphi(y)\, dy = (2\pi)^{-n}\int e^{i\langle x, \xi\rangle}(P(x, \xi)K_{j+1}(x, \xi)\psi_1(\xi) + \psi_0(\xi))\hat{\varphi}(\xi)\, d\xi.$$

PROOF. — From Corollary 3. 1 it follows that when $|\beta| \leq k$ and $x \in \Omega'$, $|\xi| \geq A'$,

$$(4.\ 3)\quad (1+|\xi|)^{k-|\beta|}|D_x^\beta(P(x, \xi)K_{j+1}(x, \xi))| \leq C_{k,j}(1+|\xi|)^{k-d(j+1)}.$$

Since the exponent $k - d(j+1)$ is $< -n$ by assumption, the inequality (4. 3) shows that the integral (4. 1) and the integrals obtained by at most k differentiations under the integral sign are absolutely and uniformly convergent. This proves the theorem since $\int f\, \hat{g}\, dx = \int \hat{f}\, g\, d\xi$ for arbitrary integrable functions f and g.

In general, the other terms in the right hand side of (2. 10) cannot be written as integral operators on φ with functions as kernels. However, we can introduce distribution kernels in the following way. If F is in $C_0^\infty(\Omega' \times R^n)$, we set

$$\hat{F}(x, \xi) = \int e^{-i\langle y, \xi\rangle} F(x, y)\, dy$$

and write

$$(4.\ 4) \quad E_j(F) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} K_j(x, \xi)\psi_1(\xi)\hat{F}(x, \xi)\, d\xi\, dx.$$

Since K_j is bounded by a power of $|\xi|$ at infinity (Corollary 3. 2), it is clear that this does define a distribution in $\Omega' \times R^n$.

Theorem 4. 2. — *The distribution E_j in $\Omega' \times R^n$ defined by (4. 4) is an infinitely differentiable function outside the diagonal. If $\varphi \in C_0^\infty(R^n)$ and x is in Ω' but not in the support of φ, we have*

$$(4.\ 5) \quad \int E_j(x, y)\varphi(y)\, dy = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} K_j(x, \xi)\psi_1(\xi)\hat{\varphi}(\xi)\, d\xi.$$

Proof. — To prove that E_j is smooth outside the diagonal we shall study the product $(x - y)_\alpha E_j$. Since

$$e^{i\langle x, \xi\rangle} \int e^{-i\langle y, \xi\rangle}(x - y)_\alpha \ F(x, y)\ dy = D_\xi^\alpha(e^{i\langle x, \xi\rangle}\hat{F}(x, \xi)),$$

we have

$$((x - y)_\alpha E_j)(F) = E_j((x - y)_\alpha F(x, y)) =$$
$$= (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} \hat{F}(x, \xi)(- D_\xi)^\alpha (K_j(x, \xi)\psi_1(\xi))\, d\xi\, dx.$$

If we choose α soi large that

$$d(j + |\alpha|) > n + m, \ m = \sup_{x \in \Omega'} m_x,$$

it follows from (3. 9) that $(- D_\xi)^\alpha\ (K_j(x, \xi)\psi_1(\xi))$ is integrable, uniformly in x. Hence we obtain

$$((x - y)_\alpha E_j)(F) =$$
$$\iint F(x, y)\, dx\, dy\, (2\pi)^{-n} \int e^{i\langle x-y, \xi\rangle}(- D_\xi)^\alpha (K_j(x, \xi)\, \psi_1(\xi))\, d\xi.$$

This means that $(x - y)_\alpha E_j$ is equal to a continuous function

$$(4.\ 6) \quad (x - y)_\alpha E_j = (2\pi)^{-n} \int e^{i\langle x-y, \xi\rangle}(- D_\xi)^\alpha (K_j(x, \xi)\, \psi_1(\xi))\, d\xi.$$

If the inequality

$$(4.\ 7) \qquad d(j + |\alpha|) > n + m + |\beta| + |\gamma|$$

is valid, we still obtain a uniformly convergent integral if we apply the differential operator $D_x^\beta D_y^\gamma$ to the integral (4. 6), for in view of (3. 9) the integrand obtained after differentiating will have a bound of the form

$$C(1 + |\xi|)^{m+|\beta|+|\gamma|-d(j+|\alpha|)}$$

which is integrable in view of (4. 7). Hence E_j is infinitely differentiable outside the diagonal.

To prove the last statement we observe that if $g \in C_0^\infty(\Omega')$ has a support disjoint from that of φ, the definition (4. 4) of E_j applied to $F(x, y) = g(x)\varphi(y)$ shows that

$$\iint E_j(x, y)\, g(x)\, \varphi(y)\, dx\, dy$$
$$= (2\pi)^{-n} \int g(x)\, dx \int e^{i\langle x, \xi\rangle} K_j(x, \xi)\, \psi_1(\xi)\, \hat{\varphi}(\xi)\, d\xi$$

since $g(x)\varphi(y)$ vanishes in a neighborhood of the diagonal. This implies (4. 5). The proof is complete.

If $g \in C_0^\infty(\Omega')$ and $\varphi \in C_0^\infty(R^n)$ we shall, following Schwartz [5], denote the function $(x, y) \to g(x)\varphi(y)$ defined in $\Omega' \times R^n$ by $g \times \varphi$. We also recall that a distribution E in $\Omega' \times R^n$ is called *regular* in x if for every *fixed* $\varphi \in C_0^\infty(R^n)$ there is a function $E\varphi \in C^\infty(\Omega')$ such that

$$(4.\ 8) \qquad E(g \times \varphi) = \int g E\varphi\, dx, \quad g \in C_0^\infty(\Omega)';$$

similarly E is called regular in y if to every *fixed* $g \in C_0^\infty(\Omega')$ there is a function $E^*g \in C^\infty(R^n)$ such that

$$(4.\ 9) \qquad E(g \times \varphi) = \int (E^*g)\, \varphi\, dy, \quad \varphi \in C_0^\infty(R^n).$$

(Obviously $E\varphi$ and E^*g are uniquely determined by these identities.)

THEOREM 4. 3. — *The distribution* E_j *in* $\Omega' \times R^n$ *defined by* (4. 4) *is regular both in x and in y.*

PROOF. — To prove the regularity in x, which we do not strictly need in the next section, we only have to note that the function

$$(4.\ 10) \quad (E_j\varphi)(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} K_j(x, \xi)\psi_1(\xi)\hat{\varphi}(\xi)\, d\xi$$

satisfies (4. 8) and is in $C^\infty(\Omega')$ if $\varphi \in C_0^\infty(R^n)$ In fact, since $\hat{\varphi}$

tends to 0 at infinity faster than $(1+|\xi|)^{-N}$ for every N, it follows from (3. 9) that the integral (4. 10) is absolutely and uniformly convergent and remains so after any number of differentiations with respect to x.

The regularity with respect to y is less trivial. Writing

$$G(\xi) = (2\pi)^{-n}\int e^{i\langle x, \xi\rangle} K_j(x, \xi)\psi_1(\xi) g(x)\, dx, \tag{4. 11}$$

we have

$$E_j(g \times \varphi) = \int G(\xi)\hat{\varphi}(\xi)\, d\xi = \int \hat{G}(x)\varphi(x)\, dx$$

provided that G is integrable. We shall prove this and, moreover, that $G(\xi)(1+|\xi|)^N$ is bounded for every N. This will show that $\hat{G} \in C^\infty(R^n)$ and since $E_j^* g = \hat{G}$, the proof of the theorem will then be complete.

To estimate G we multiply (4. 11) by $|\xi|^{2k}$ and integrate by parts. If we denote the Laplace operator by Δ, this gives

$$|\xi|^{2k} G(\xi) = (2\pi)^{-n}(-1)^k \int e^{i\langle x, \xi\rangle} \Delta_x^k (K_j(x, \xi) g(x))\psi_1(\xi)\, dx.$$

Now g is in $C_0^\infty(\Omega')$, so we obtain by using the estimate (3. 9) that

$$|\xi|^{2k}|G(\xi)| \leq C(1+|\xi|)^{m+2k(1-d)-dj}.$$

Since $G = 0$ in a neighborhood of 0, this gives with another constant

$$|G(\xi)| \leq C(1+|\xi|)^{m-d(j+2k)}.$$

This completes the proof of the theorem, for k is an arbitrary positive integer.

Remark. — The proof of this theorem is essentially the same as that of proposition 1. 18 in [6].

5. Hypoellipticity of operators satisfying the condition HE.

It is very well known how regularity theorems can be proved when one has a parametrix with the properties obtained in the preceding paragraph. However, we shall supply the proof here for the convenience of the reader. (See also Schwartz [5].)

THEOREM 5. 1. — *Every differential operator satisfying the condition* HE *in* Ω *is hypoelliptic in* Ω.

PROOF. — Denote the adjoint operator by $P(x, D)$. According to Theorem 3. 3 the operator $P(x, D)$ also satisfies the condition HE. We have to prove that if $u \in \mathcal{D}'(\Omega)$ and $P^*(x, D)u = f$, then $u \in C^\infty(\omega)$ if ω is an open subset of Ω such that $f \in C^\infty(\omega)$. There is no restriction in assuming that ω is relatively compact, and multipliyng u by a function in $C_0^\infty(\Omega)$ which equals 1 in ω we may then reduce ourselves to the case where $u \in \mathcal{E}'(\Omega)$. We choose a relatively compact subdomain Ω' of Ω such that the support of u is contained in Ω'. Replacing ω by a smaller domain we may finally write $f = g + h$ where $g \in C_0^\infty(\Omega')$ and h vanishes in ω.

Summing up, we have to prove that if $u \in \mathcal{E}'(\Omega')$,

$$P^*(x, D)u = g + h,$$

where $g \in C_0^\infty(\Omega')$ and h vanishes in ω, then $u \in C^m(\omega)$ for an arbitrary integer m.

With $\varphi \in C_0^\infty(\omega)$ we now apply (2. 10) in combination with (4. 2) and (4. 10). This gives

$$\varphi(x) = P(x, D) \sum_0^j (E_k\varphi)(x) + \int F_j(x, y)\, \varphi(y)\, dy$$

Let μ be the order of the distribution u and choose j so large that $F_j \in C^{\mu+m}(\Omega' \times R^n)$. Since u has compact support we then obtain

$$(5.\ 1) \quad u(\varphi) = (P^*u)\Big(\sum_0^j E_k\varphi\Big) + \int (u(F_j(\cdot, y))\varphi(y)\, dy.$$

Here we have used the properties of the direct product of distributions (Schwartz [5]); the notation $u(F_j(\cdot, y))$ means that the distribution u operates on the variable indicated by a dot. Since F_j is in $C^{m+\mu}$, this a function in $C^m(\omega)$. The other terms in (5. 1) we rewrite in the following way

$$\begin{aligned}(P^*u)(E_k\varphi) = (g + h)(E_k\varphi) &= E_k(g \times \varphi) + h(E_k\varphi)\\ &= \int (E_k^*g)\varphi\, dy + \int h(E_k(\cdot, y))\varphi(y)\, dy.\end{aligned}$$

The last computation follows again from the fact that

$$(E_k\varphi)(x) = \int E_k(x, y)\varphi(y)\, dy, \quad x \notin \omega,$$

in view of Theorem 4. 2, by using the properties of the direct product and the fact that the support of h belongs to $C\omega$. Hence u is in ω equal to the function

$$y \to \sum_0^j (E^*g)(y) + \sum_0^j h(E_k(.,\, y)) + u(F_j(.,\, y)).$$

Since all terms except the last are in $C^\infty(\omega)$ and the last is in $C^m(\omega)$, this completes the proof.

6. Examples.

We first give an example which proves that an operator which is elliptic except at a point where the principal part degenerates may still be hypoelliptic if the principal part vanishes so rapidly that the strength of the operator does not change too fast.

Example 1. — *The operator corresponding to*

$$P(x,\, \xi) = 1 + |x|^{2\nu}|\xi|^{2\mu} \tag{6. 1}$$

satisfies the condition HE, *hence is hypoelliptic, if* $\nu > \mu$.

Proof. — We shall choose for $j = 1, \ldots, n$

$$M_j(x,\, \xi) = M(x,\, \xi) = ((1 + \rho^{2\mu})/(1 + r^{2\nu}\rho^{2\mu}))^{1/2\nu}$$

where $\rho = |\xi|$ and $r = |x|$. We may of course assume that $r < 1$; the inequality (3. 2) is then fulfilled since $\nu > \mu$ and

$$1 \leq M(x,\, \xi) \leq (1 + \rho^{2\mu})^{1/2\nu}.$$

For $0 \leq j \leq 2\nu$ and $0 \leq k \leq 2\mu$ we shall prove that

$$r^{2\nu-j}\rho^{2\mu-k}M^{k-j}(1 + \rho)^{dk}/(1 + r^{2\nu}\rho^{2\mu}) \tag{6. 2}$$

is bounded if $0 < d \leq 1 - \mu/\nu$. That (6. 2) is bounded for $0 < \rho < 1$ is trivial. If the number c defined by

$$\mu c = 2\mu - k + (k - j)\mu/\nu + dk$$

is negative, the boundedness of (6. 2) is trivial also for $\rho > 1$ since the total order in ρ of the factors in (6. 2) which do not contain r is μc. On the other hand, if $c \geq 0$ we can estimate (6. 2) for $\rho > 1$ by a constant times

$$r^{2\nu-j-\nu c}(r^\nu\rho^\mu)^c/(1 + r^{2\nu}\rho^{2\mu})^{1+(k-j)/2\nu}.$$

Since $2\nu - j - \nu c = k\nu(1 - d - \mu/\nu)/\mu \geqq 0$ and

$$2 + (k - j)/\nu - c = k(1 - d)/\mu \geqslant 0,$$

this proves the boundedness.

We shall also give two examples of operators in two variables, related to the example of Trèves [6].

EXAMPLE 2. — *The operator corresponding to*

$$P(x, \xi) = \xi_1^{2m} + \xi_2^{2n} + ic(x)\xi_1^a\xi_2^b + 1 \tag{6.3}$$

satisfies the condition HE, *hence is hypoelliptic, if* $c \in C^\infty$ *is real valued and*

$$(a - 1)/2m + b/2n < 1, \quad a/2m + (b - 1)/2n < 1. \tag{6.4}$$

Note that (6. 4) means that the first order derivatives of the polynomial $P_2(\xi) = \xi_1^a\xi_2^b$ shall be strictly weaker than $P_1(\xi) = \xi_1^{2m} + \xi_2^{2n} + 1$. The operator is of constant strength if, and for general c only if, $a/2m + b/2n \leqq 1$. The example then gives nothing new.

The example studied by Trèves [6] is of the form (6. 3) with $m = 1$, $n = 2$, $a = 1$, $b = 3$ and c depending only on x_1. The inequalities (6. 4) are not satisfied in this example, so we give another containing the example of [6] where we use in an essential way that c only depends on x_1.

EXAMPLE 3. — *The operator given by* (6. 3) *satisfies* HE, *hence is hypoelliptic, if* $c \in C^\infty$ *is a real valued function of* x_1 *only, and*

$$(a - 1)/2m + b/2n < 1, \quad 0 < a < 2m. \tag{6.5}$$

We leave it to the reader to verify that the hypotheses of Theorem 3. 1 are fulfilled in examples 2 and 3.

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