ANNALES DE L'INSTITUT FOURIER

RALPH E. EDWARDS

On functions whose translates are independent

Annales de l'institut Fourier, tome 3 (1951), p. 31-72

<http://www.numdam.org/item?id=AIF_1951__3__31_0>

ON FUNCTIONS WHOSE TRANSLATES ARE INDEPENDENT

by **R. E. EDWARDS** (London).

1. — Introduction and generalities.

It is the object of this paper to study some special cases of an apparently new problem concerning the translates of functions on a group. In order to ennunciate the problem in its general form and to indicate its origin, some definitions will be made at once. Let G be a group, assumed topological, abelian, and locally compact: of these conditions the first is essential for our problem to have meaning, and the second and third appear to be temporarily inevitable in so far as a fairly complete and detailed theory of harmonic analysis seems to be an almost indispensable tool. G will be written additively, and its elements denoted by x, y, ... Let $\mathcal{E}$ be a translation-invariant, topological vector space of functions $f = f(x)$, $g = g(x)$, ... defined on G. If $f \in \mathcal{E}$, and if A is a subset of G, let $\mathfrak{J}(f, \mathrm{A}) = \mathfrak{J}(f, \mathrm{A}, \mathcal{E})$ denote the closed vector subspace of $\mathcal{E}$ generated by the translates $f_a = f_a(x) = f(x + a)$ of f when a ranges over A; for brevity we shall write $\mathfrak{J}(f) = \mathfrak{J}(f, \mathcal{E})$ in place of $\mathfrak{J}(f, \mathrm{G}, \mathcal{E})$.

For many special choices of G and of $\mathcal{E}$, the problem of determining the extent of $\mathfrak{J}(f)$ (in particular, the problem of determining when $\mathfrak{J}(f) = \mathcal{E}$, the problem of the fundamentality or totality of translates) has been discussed in considerable detail. The following cases are well known: for $\mathcal{E} = \mathrm{L}^1\mathrm{G}$ we have the theorem of Wiener-Godement (1), (2), [1]; for $\mathcal{E} = \mathrm{L}^2\mathrm{G}$ the discussion given by Wiener (1) for the group R of reals is easily extended; for $\mathcal{E} = \mathrm{L}^1\mathrm{G} \cap \mathrm{L}^p\mathrm{G}$ we have the results of Segal (3) and Pollard (4); for $\mathcal{E} = \mathrm{L}^\infty\mathrm{G}$ we have the theorem of Beurling-Godement (2), (5).

[1] Numbers in brackets refer to the list of references at the end of the paper.

Interest has also been shown in the case $\mathcal{E} = CG$, the space of all continuous functions on G, the topology being that of convergence uniform on every compact subset of G : little difficulty attaches to this case when G is compact, but problems of great interest arise when G is not compact. Thus, when $G = R$, we have the work of L. Schwartz (6).

Whilst in many of these cases the interest has centred on a study of $\mathfrak{J}(f)$, thus using translates of f corresponding to all the group elements, little has been written about the study of $\mathfrak{J}(f, A)$ for comparatively sparse sets A. Fragmentary results are to be found in Edwards (7), (8), (9), (10) and various other results are included in an unpublished thesis of the present author. These results are all concerned with showing that, for special functions f, $\mathfrak{J}(f, A) = \mathfrak{J}(f)$ for sparse sets A, *i. e.* that some translates of f are « approximately linearly dependent » on certain others. And this brings us to the problem to be discussed in this paper, namely, the discussion of those functions for which no such approximate linear dependence is possible.

To the best of my knowledge, the only existing results of this nature occur in Edwards (8), (10) : in (8) the question concerns functions of a complex variable, but in (10) we have a veritable special case of the problem for functions on groups (namely the case $\mathcal{E} = CG$ and $G = T = R/2\pi$). This special case exhibits the general difficulties : the problem is a rather delicate constructional one involving the relationship between harmonic analysis on the one hand and quasi-analytic classes of functions on the other. The relevance of this latter notion is the main difficulty in the path of discussing the problem for a general group, and the case of a general non-discrete group remains almost entirely untouched.

In this paper we confine our main attention to the case in which G is either a discrete abelian group, or T, or R, or again finite direct products of these latter groups, whilst the choice of $\mathcal{E}$ is usually L^2G. This latter choice is technically the simplest in view of the symmetrical theory of harmonic analysis available. However, as will be indicated, the methods employed in this case yield non-trivial results for certain other choices of $\mathcal{E}$.

Notation. — The symbols G and $\mathcal{E}$ will have the meaning already explained. In addition to this, $\widehat{G}$ will denote the dual group of bounded, continuous characters of G ; elements of $\widehat{G}$ will be denoted by χ, ..., the value at the point $x \varepsilon G$ of the character χ being written $\chi(x)$.

We denote by $\mathcal{E}'$ the topological dual of $\mathcal{E}$, the symbols f', g', ... being used to denote elements of $\mathcal{E}'$; the bilinear functional expressing the duality between $\mathcal{E}$ and $\mathcal{E}'$ is written $< f, f' >$.

The notation in connection with Fourier transforms will be explained a little later.

Summary. — Section 2 is devoted to the reduction of the problem to a form suitable for analysis on the basis of the Hahn-Banach theorem, and to a few general remarks. This leads to a complete solution of the problem for $\mathcal{E} = L^2G$, G discrete, given in Section 3. The solution for the case $G = N$, the discrete additive group of integers, may be interpreted in terms of translational bases in the space of Paley-Wiener functions on the real axis.

Section 4 is concerned mainly with the case $G = T$ and $\mathcal{E} = L^2G$, the results obtained being supplementary to those given in Edwards (10). Section 5 is devoted largely to the analogous problem for $G = R$. In either case there is no difficulty in deducing non-trivial results for the groups T^m and R^m. At the end of Section 4 we derive criteria for a periodic distribution to have its translates independent; the analogous problem for distributions on R or R^m is discussed briefly at the end of Section 5.

Section 6 contains some remarks on the case $G = R$ or R^m and $\mathcal{E} = CG$. For functions in CR^m of slow growth [2], Schwartz's theory of generalised Fourier transforms proves to be useful, but it is to be hoped that the restriction to such functions may be ultimately removed. For this reason, we have confined ourselves to broad indications and to some examples. The independence of translates of distributions of slow growth on R^m is also discussed briefly at the end of this section.

The method employed throughout is much the same whatever the space $\mathcal{E}$ in question, the essential step being the construction, or proof of the existence of, continuous functions on the group G which (*i*) are supported by small neighbourhoods of zero, and (*ii*) have Fourier transforms which are as small as possible at infinity on G. As is indicated in Section 7, this problem is more or less closely connected with that of deciding the regularity of a suitably chosen

[2] The phrase « of slow growth » is used throughout as an equivalent of Schwartz's « à croissance lente » or « tempérée » ; see L. Schwartz (11).

normed ring. Unfortunately, this approach has a strictly limited practical value outside the case in which G is compact owing to the difficulty experienced in reducing the known sufficient conditions of regularity to a sufficiently simple form.

Finally, in Section 8 we mention some extensions of the problem to non-abelian groups and interpret the problem as one concerning the continuous functions of positive type on G, or, equivalently, the bounded, positive Radon measures on G when G is abelian. Some other extensions are also mentioned.

My debt to Mr J. Deny will be obvious at many points: I wish to express here my sincere gratitude to him for his many helpful suggestions. The problem dealt with in this paper is a natural consequence of work undertaken in connection with a thesis approved for the degree of Ph. D. in the University of London. This thesis was written under the direction of Professor J. L. B. Cooper; to him and to Dr. F. Smithies I wish to offer my thanks for their help and encouragement during the early stages of my work on this problem. I am also particularly grateful to Professor G. W. Mackey in connection with parts of the substance of Section 8. Finally, my thanks are due to Dr. P. Vermes for drawing my attention to the recent paper (4) of Pollard.

2. — Reduction of the general problem.

The notion of independence of translates is specified by the following

Definition 1. — *If* G *and* $\mathcal{E}$ *are as in Section* 1, *an* $f \in \mathcal{E}$ *has its translates independent if, whenever* A *as in a closed subset of* G *and* a *is a point of* G *not in* A, *we have* f_a *non*-$\in$ $\mathfrak{I}(f, A, \mathcal{E})$.

The notion is thus dependent on the topology on $\mathcal{E}$, but this will be taken for granted once the space $\mathcal{E}$ has been fixed in any given instance. Further, although we speak of the elements of $\mathcal{E}$ as functions, these elements may strictly speaking be equivalence classes of functions (as will be the case if, for example, $\mathcal{E}$ is one of the Lebesgue spaces built over G) : this licence will be taken without further comment. When we come to speak of the independence of translates of entities other functions, we shall not explicitly reformulate Definition 1 since the necessary amendments are quite trivial.

The following assumptions are valid in all the cases we shall consider and bring with them certain simplifications :

(I) *The space* $\mathcal{E}$ *is locally convex.*

(II) *For fixed* $a \in G$, *the mapping* $f \rightarrow f_a$ *is continuous from* $\mathcal{E}$ *into* $\mathcal{E}$.

These hypotheses allow one to frame the definition of independence translates in a form more suitable for analysis, namely :

THEOREM 1. — *For* $f \in \mathcal{E}$ *to have its translates independent the following condition is necessary and sufficient*[3]. *Denoting by* $\mathcal{E}'(f)$ *the vector space of functions on* G *having the form* $\varphi(x) = \langle f_x, f' \rangle$ *when* f' *ranges over* $\mathcal{E}'$, *to every member* U *of a basis of neighbourhoods of zero in* G *shall correspond a fonction* $\varphi \in \mathcal{E}'(f)$, *depending on* U, *with the properties*

$$(2.1) \qquad \varphi(0) \neq 0, \qquad \varphi(x) = 0 \qquad \text{for } x \text{ non-}\in U.$$

Proof. — It is a consequence of (II) that the translates of f are independent if and only if, whenever A is a closed subset of G not containing zero, f is not in $\mathfrak{J}(f, A, \mathcal{E})$. Hypothesis (I) ensures that the Hahn-Banach theorem is valid for $\mathcal{E}$. By virtue of this theorem, the assertion that f non-$\in \mathfrak{J}(f, A, \mathcal{E})$ is equivalent to the existence of $f' \in \mathcal{E}'$ such that $\langle f, f' \rangle \neq 0$ and $\langle f_x, f' \rangle = 0$ for $x \in A$. Whence the theorem.

It is also true that in all cases considered here the following assumption is satisfied :

(III) *For a fixed* $\chi \in \widehat{G}$, *the mapping* $f(x) \rightarrow f(x) . \chi(x)$ *leaves* $\mathcal{E}$ *invariant and is continuous from* $\mathcal{E}$ *into* $\mathcal{E}$.

Granted this, we shall have.

THEOREM 2. — *For a fixed* $\chi \in \widehat{G}$ *and* $f \varepsilon \mathcal{E}$, *the functions* f *and* $f . \chi$ *together have their translates indépendent or not.*

The result is useful in certain cases since it shows that we can assume at will that the Fourier transform of f is non-zero at any particular one point of $\widehat{G}$.

Theorem 1 also makes apparent the relevance of relations between harmonic analysis and quasi-analyticity. In many cases, the functions of $\mathcal{E}'(f)$ take the form of a convolution $f * f'$ of f with f', f' being a function, a measure, or a Schwartzian distribution. And, assuming that the Fourier transforms of f and f' exist as functions, the transform of $f * f'$ will be small at infinity to much the same

(3) The sufficiency of the condition is clearly independent of hypothesis (1).

degree as is the transform of f. But a function cannot have a Fourier transform which is « very small » at infinity without being a member of a quasi-analytic class : in particular, such a function cannot vanish outside small neighbourhoods of zero without vanishing identically. One is thus led to expect that a restriction on the smallness at infinity of the Fourier transform of f will be a necessary condition for f to have its translates independent : this point is illustrated by the results of Edwards (10) and by most of the results to follow.

Again, in many cases the equation

$$f * f' = \varphi$$

is equivalent to

$$\mathrm{F} . \mathrm{F}' = \Phi \qquad \left(\text{p. p. on } \widehat{\mathrm{G}}\right),$$

capital letters denoting passage to the Fourier transform. It therefore appears that conditions of a local character restricting the number or density of zeros of the Fourier transform F of f will also be necessary for the translates of f to be independent. These conditions appear to be much more difficult to make precise than those involving the behaviour of F at infinity. Nearly all the results to follow are concerned with establishing the sufficiency of certain sets of conditions and it would be of great interest to develop some necessary conditions.

Some general negative results can be formulated at least for all spaces $\mathcal{E}$ built over the groups R^m or T^m. Unlike the positive results we are able to prove, these are quite naturally formulated in terms of the function itself rather than its Fourier transform. For simplicity, let us consider the case of a space $\mathcal{E}$ built over the group R. We may define a function $f \in \mathcal{E}$ to be weakly differentiable in $\mathcal{E}$ if there is a function $g \in \mathcal{E}$ such that

$$\lim_{a \to 0} (f_a - f)/a = g$$

weakly in $\mathcal{E}$; g will then be called the weak derivative of f in $\mathcal{E}$ and will be denoted by Df. The successive weak derivatives Df, $\mathrm{D}^2 f$, ... may then be defined inductively. When all these weak derivatives exist, every function φ in $\mathcal{E}'(f)$ has derivative of all orders in the usual sense. If, in addition, we impose restrictions on the rate of growth of the nth derivative $\varphi^{(n)}(x)$, we shall be able to affirm that

$\varphi(x)$, if zero on any non-void open set, is identically zero. We shall have in fact for every $n = 0, 1, 2, \ldots,$

$$\varphi^{(n)}(x) = \langle (D^n f)_x, f' \rangle = \langle D^n f_x, f' \rangle.$$

Accordingly, if we suppose that the semi-norms $p_i (i \in I)$ define the topology on $\mathcal{E}$, and if we write for any integer $n \geqslant 0$, any $i \in I$, and any neighbourhood V of zero in R,

$$M(n; i, V) = \sup_{x \in V} p_i(D^n f_x),$$

then we can impose such restrictions on the rate of increase of $M(n; i, V)$ with respect to n as will ensure that the restrictions to V of the functions in $\mathcal{E}'(f)$ form a quasi-analytic class. For example, write

$$T(r; i, V) = \sup_{n \geqslant 1} r^n / M(n; i, V)$$

for $r > 0$. Then we may assert that: *if, for some* V *and each i the integral*

$$\int_1^\infty \log T(r; i, V) dr/r^2$$

is divergent, then the translates of f are not independent.

In fact, if U is a neighbourhood of zero contained in V and such that V-$\bar{U}$ is not void, and if $\varphi \in \mathcal{E}'(f)$ is supported by U, then all derivatives of φ vanish at all points of V-$\bar{U}$; consequently, by well known results on quasi-analytic classes, φ vanishes identically on V and hence, in particular, at the origin. This suffices to show that the translates of f are not independent.

Naturally, there are close connections between conditions of this type and the behaviour at infinity of the Fourier transforms of functions belonging to $\mathcal{E}'(f)$. For, if φ belongs to $\mathcal{E}'(f)$ and has a compact support $\subset V$, we shall have a system of inequalities of the form

$$\int |\varphi^{(n)}(x)| dx \leqslant M(n; i, V)$$

holding for a suitable fixed i depending on φ and for all n. If then $\Phi(\chi)$ is the Fourier transform of $\varphi(x)$:

$$\Phi(\chi) = \int \varphi(x) e^{-2\pi i \chi x} dx,$$

integration by parts n times gives

$$|(2\pi\chi)^n \cdot \Phi(\chi)| \leqslant M(n; i, V).$$

Choosing n suitably, we deduce that

$$|\Phi(\chi)| \leqslant 1/T(2\pi|\chi|;\ i,\ V),$$

which is generally enough to show that $\varphi(x)$, known to vanish outside a neighbourhood of zero, is identically zero. For example, if $M(n;\ i,\ V) = n!\,.\,Q^n$ for some Q independent of n, then we can conclude in this manner that $\Phi(\chi)$ is of order at most $\exp(-|\chi|/Q)$ for large $|\chi|$, hence that $\varphi(x)$ is regular-analytic in a strip containing the real axis.

We adopt an approach in which the Fourier transform plays the basic role for the obvious reason that the operation of point-wise multiplication is easier to handle than that of convolution and so leads most easily to the positive results which constitute our main aim in this paper.

It is convenient at this stage to explain the notation concerning Fourier transforms. If f is a function or a measure (or a distribution in the case of special groups) on G, we shall denote by $\mathcal{F}(f)$ the Fourier transform of f; the sense in which this transform is to be taken will usually be obvious from the context. When no confusion can arise, the transform of f will be denoted by F, and likewise for other functions, measures or distributions.

3. — The case $\mathcal{E} = L^2G$, G discrete.

The simplicity of this case is due solely to the fact that a basis of neighbourhoods of zero in G is comprised of the single set $\{0\}$. Accordingly Theorem 1 tells us that $f \in L^2G$ will have its translates independent if and only if there is $f' \in L^2G$ such that

$$\varphi = f * f'$$

has the properties

$$\varphi(0) = 1 \qquad \text{and} \qquad \varphi(x) = 0 \qquad \text{if} \qquad x \neq 0.$$

This is equivalent to

$$F\,.\,F' = 1 \quad p.p. \ \text{ on } \widehat{G},$$

where $F = \mathcal{F}(f)$ and $F' = \mathcal{F}(f')$, and this is soluble for $F' \in L^2G$ if and only if $1/F \in L^2\widehat{G}$. Thus we may state

THEOREM 3. — *If G is discrete, a necessary and sufficient condition* *for* $f \in L^2G$ *to have its translates independent in this space is that* $1/F \in L^2\widehat{G}$ $\big(F = \mathcal{F}(f)\big)$.

The hypothesis of Theorem 3 implies that F vanishes on a null set at most in $\widehat{G}$ and hence, as is easily seen to follow from the Hahn-Banach theorem, that the translates of f are fundamental in L^2G : these translates therefore form a sort of basis for L^2G. We may legitimately digress here to the extent of a brief discussion of this last fact.

Suppose that $f \in L^2G$ satisfies the hypothesis of Theorem 3. The translates of f being fundamental in L^2G, one may hope to expand any given $h \in L^2G$ in the form

$$(3.\ 1) \qquad h \sim \sum_a \lambda_a f_a,$$

the numbers λ_a depending upon h. Since the translates of f are independent, we may choose $\psi \in L^2G$ such that

$$(3.\ 2) \qquad \int f_a(x)\overline{\psi(x)}\,dx = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise}; \end{cases}$$

by Parseval's formula, this is equivalent to

$$\int \chi(a)F(\chi)\overline{\psi(\chi)}\,d\chi = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise}, \end{cases}$$

where $F = \mathcal{F}(f)$ and $\Psi = \mathcal{F}(\psi)$. Thus (3. 2) is equivalent to

$$(3.\ 3) \qquad F(\chi)\overline{\Psi(\chi)} = 1 \ \text{ p. p. on } \widehat{G}.$$

Now we know that, given $\varepsilon > 0$, there is a finite subset S of G and numbers λ_a $(a \in S)$ such that

$$\|h - \sum_{a \in S} \lambda_a f_a\|_{L^2G} \leqslant \varepsilon.$$

From this relation follows

$$\left|\int h(x)\overline{\psi(x+a)}\,dx - \sum_{b \in S} \lambda_b \int f_b(x)\overline{\psi(x+a)}\,dx\right| \leqslant \varepsilon \,.\, \|\psi\|_{L^2G},$$

that is by, (3. 1),

$$\left|\lambda_a - \int h(x)\overline{\psi_a(x)}\,dx\right| \leqslant \varepsilon \,.\, \|\psi\|_{L^2G} (a \in S).$$

It is therefore natural to study the formal development (3. 1) with the following choice of the constants:

$$\lambda_a = \int h(x)\overline{\psi_a(x)}\,dx, \tag{3. 4}$$

or equivalently, by Parseval's formula and (3. 3),

$$\lambda_a = \int H(\chi)\overline{\chi(a)}\,d\chi/F(\chi). \tag{3. 4'}$$

where $H = \mathcal{F}(h)$. This shows that λ_a is, as a function of a, the inverse Fourier transform of the function $H/F \in L^1\widehat{G}$, hence that $\lambda_a \to 0$ as $a \to \infty$ on G, and so that $\lambda_a = 0$ except for a countable set $\{a_n : n = 1, 2, \ldots\}$ of values of a. The right member of (3. 1) is thus extended over this countable set only.

In general, the question of the convergence of the development (3. 1) is rather delicate, but it is easy to see that convergence will take place in the L^2-sense under suitable restrictions on h. For example, if

$$\sum_a |\lambda_a| < +\infty \tag{3. 5}$$

(the sum extending in reality only over the aforesaid countable set of values of a), then the right member of (3. 1) is absolutely convergent in norm and so defines an element h^* of L^2G. Further, for every $a \in G$, (3. 2) yields

$$\begin{aligned}\int h^*(x)\overline{\psi_a(x)}\,dx &= \sum_n \lambda_{a_n}\int f_{a_n}(x)\overline{\psi_a(x)}\,dx \\ &= \sum_n \lambda_{a_n}\int f(y+a_n-a)\overline{\psi(y)}\,dy \quad (y = x+a) \\ &= \begin{cases}\lambda_{a_n} & \text{if } a = a_n \text{ for some } n, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus

$$\int h^*(x)\overline{\psi_a(x)}\,dx = \int h(x)\overline{\psi_a(x)}\,dx$$

for all $a \in G$, that is

$$\int H^*(\chi)\overline{\chi(a)}\,d\chi/F(\chi) = \int H(\chi)\overline{\chi(a)}\,d\chi/F(\chi)$$

for all $a \in G$, and so, since $1/F \neq 0$ p. p. on $\widehat{G}$, $H^* = H$ p. p. on $\widehat{G}$ and $h^* = h$ everywhere on G.

Condition (3. 5) will be satisfied whenever H/F is equal p. p. on $\widehat{G}$ to a linear combination of summable and continuous functions of positive type. Of course, for special groups, the validity of (3. 5) can be ensured in others ways. For example, if $G = N$, the discrete additive group of integers, then $\widehat{G} = T$, and (3. 5) will hold at least whenever $H(\chi)/F(\chi)$ is equal p. p. to a function having two continuous derivatives. A special example arises when $f(x) = \exp(-a|x|)(a > 0)$, in which case

$$F(\chi) = (1 - r^2)/(1 - 2r\cos\chi + r^2)\left(r = \exp(-a)\right),$$
$$\lambda_a = (1/2\pi)\int_{-\pi}^{\pi} H(\chi)e^{-ia\chi}(1 - 2r\cos\chi + r^2)dx/(1 - r^2),$$
$$H(\chi) = \sum_x h(x)e^{-ix\chi}(x = 0, \pm 1, \pm 2, \ldots);$$

in this case, (3. 5) will be satisfied provided

$$\sum_x x^2|h(x)| < +\infty.$$

In case $G = N$, the discrete additive group of integers, Theorem 3 may be restated in an equivalent form concerning th class of Paley-Wiener functions on the real axis. Recall that a function $p(x)(x$ real) is a Paley-Wiener function if it is continuous and of summable square over the real axis, and if the Fourier transform $P(\chi)$ of $p(x)$ is zero p. p. outside $(-\pi, \pi)$. It is well known that the class (PW) of such functions forms *a* Hilbert space with the scalar product

$$(p, q) = \int_{-\pi}^{\pi} P(\chi)\overline{Q(\chi)}d\chi \qquad \left(P = \mathcal{F}(p),\ Q = \mathcal{F}(q)\right).$$

It is also true that there is an isomorphism between (PW) and the usual Hilbert sequence space $l^2 = L^2N$ in which a function $p(x) \in (PW)$ is mapped onto the sequence whose nth term is $p(n)(n = 0, \pm 1, \pm 2, \ldots)$. The inverse mapping determines $p(x)$ as the mean-square limit of the series

$$\sum_{-\infty}^{\infty} p(n) . \sin\pi(x - n)/\pi(x - n).$$

Theorem 3 proves then to be equivalent to

THEOREM 3'. — *For a function* $p(x) \in (PW)$ *to have its integral translates independent in* (PW), *it is necessary and sufficient that* $1/P(\chi)$ *be of summable square over* $(-\pi, \pi)$. *The Hahn-Banach theorem shows that for these integral translates to be fundamental in* (PW), *it is necessary and sufficient that* $P(\chi)$ *be non-zero p. p. on* $(-\pi, \pi)$.

Returning to the case of the general discrete group G, it is not difficult to obtain sufficient conditions for the independence of translates for other choices of $\mathcal{E}$, though it is very much more difficult to decide the necessity of these same conditions. For example, if

$\mathcal{E} = L^1G$, $\mathcal{E}'(f)$ comprises all convolutions $f * f'$ with $f' \in L^\infty G$. It may be shown that a sufficient condition for $f \in L^1G$ to have its translates independent in this space is that $F \neq 0$ everywhere on $\widehat{G}$. This follows from an extension of a result due to Wiener (1) concerning absolutely convergent Fourier series. The extension is well known as an application of the methods of Gelfand (12), (13), and reads : If G is discrete, if $f \in L^1G$, and if $F = \mathcal{F}(f)$ is $\neq 0$ everywhere on $\widehat{G}$, then there is $f' \in L^1G$ (a fortiori in $L^\infty G$, G being discrete) such that $F . F' = 1$ everywhere on $\widehat{G}$.

4. — The case $\mathcal{E} = L^2G$, $G = T$.

This section may be taken in conjunction with, and is supplementary to, the relevant portions of Edwards (10). There are several respects in which the presentation here differs from that in (10), but in only one instance is the divergence of any great importance. Our choice of $\mathcal{E}$ is different : here we take $\mathcal{E} = L^2T$, whereas in (10) the choice is $\mathcal{E} = CT$. Besides this, the results borrowed from the theory of quasi-analytic classes are manipulated a little differently. However, the important addition is the correction of an oversight on the part of the present writer which led to a neglect in (10) of any discussion of the influence on the independence of translates of vanishing Fourier coefficients. This oversight is rectified here.

The elements of T being real numbers modulo 2π, $\widehat{T}$ may be identified with the discrete additive group N of integers. Accordingly, χ will in this section denote an integer. The character functions are

$$\chi(x) = e^{i\chi x},$$

x denoting a real number modulo 2π. The Fourier transform of a function $f(x)$ on T is now

$$F(\chi) = (1/2\pi) \int_{-\pi}^{\pi} f(x) e^{-i\chi x} dx.$$

In what follows, h will denote a positive real number, the emphasis being laid on its possible smallness.

The main theorem here is.

THEOREM 4. — *If* $f \in L^2T$, *the following conditions are sufficient for the translates of* f *to be independent in* L^2T. *There exists a func-*

tion $p(t)(t$ real), even in t, such that $t.\ p'(t)\uparrow +\infty$ with t, and such that

$$(4.\ 1) \qquad \int_1^\infty p(t)dt/t^2 < +\infty,$$

and a sequence $\{\chi_k\}(k = 0, 1, 2, \ldots)$ of integers satisfying

$$(4.\ 2) \qquad \chi_0 = 0, \quad \chi_k > 0 \quad \text{if} \quad k > 0, \quad \sum_k 1/\chi_k < +\infty$$

for which

$$(4.\ 3) \qquad \sum_{\chi \neq \pm \chi_k} |\exp(-p(\chi))/F(\chi)|^2 < +\infty.$$

Proof. — According to Theorem 2, we may assume that $F(0) \neq 0$. According to the results of Mandelbrojt (14), paragraph 59, p. 134, for any small $h > 0$ we can construct an infinitely derivable periodic function $\alpha(x)$, zero outside $(-h, h)$ modulo 2π, and having Fourier coefficients $A(\chi)$ satisfying

$$A(0) \neq 0, \qquad A(\chi) = 0 \quad \text{for} \quad \chi = \pm\chi_k (k = 1, 2, \ldots).$$

Again, by the arguments of pp. 78-83 of this same reference, we can construct a function $\beta(x)$, infinitely derivable, periodic, non-negative, such that $\beta(x) = 0$ outside $(-h, h)$ modulo 2π, and having Fourier coefficients $B(\chi)$ satisfying

$$B(0) \neq 0, \qquad |B(\chi)| < \exp(-p(\chi)).$$

Now consider the function $\varphi = \alpha * \beta$, the convolution being taken over a period (that is, in the sense of the group T); this has Fourier coefficients

$$\Phi(\chi) = A(\chi) . B(\chi),$$

which vanish for $\chi = \pm\chi_k (k = 1, 2, \ldots)$ and which satisfy

$$|\Phi(\chi)| < \text{const. } \exp(-p(\chi)),$$

the constant depending on h. Further, $\varphi(x) \not\equiv 0$ since $\Phi(0) \neq 0$. On the other hand, $\varphi(x) = 0$ outside $(-2h, 2h)$ modulo 2π. Hence, by translating φ by an amount $2h$ at most, we can arrange that $\varphi(0) \neq 0$ and $\varphi(x) = 0$ outside $(-4h, 4h)$ modulo 2π.

It results that f will have its translates independent provided we can solve the equation

$$f * f' = \varphi$$

for $f' \in L^2T$. This is equivalent to

$$F . F' = \Phi$$

with $F = \mathcal{F}(f)$ and $F' = \mathcal{F}(f')$, and we require that $F' \in L^2N$. However, since $\Phi(\chi) = 0$ for $\chi = \pm \chi_k (k = 1, 2, \ldots)$ and since $F(0) \neq 0$, this entails simply that (4. 3) be valid. This completes the proof.

Remarks. — It follows from Theorem 4 that the independance of translates of f in L^2T is compatible with $F(\chi)$ being at infinity as small as, say,

$$\exp(-|\chi|/\log^{1+\varepsilon}|\chi|)$$

for any fixed $\varepsilon > 0$; in fact, we could here replace $\log^{1+\varepsilon}|\chi|$ by the usual succession of refinements :

$$\log|\chi| \cdot \log\log^{1+\varepsilon}|\chi|,$$

et cetera. On the other hand, it is trivial that if $F(\chi)$ is at infinity as small $\exp(-a|\chi|)$ for any $a > 0$, then the translates of f are not independent, since in this case every function $f * f'$ would be regular-analytic in a strip containing the real axis. Indeed, the conditions of Theorem 4 cease to imply the independence of the translates of f as soon as the integral in (4. 1) is allowed to diverge : see Mandelbrojt (14), pp. 78-83.

Again, the conditions cease to be sufficient as soon as the series in (4. 2) is allowed to diverge, since the conditions would then be satisfied by functions f having period π, say, which are obviously inadmissible. However, this constitutes a very crude infringement of (4. 2) and it would be interesting to study more refined counter-examples.

I do not know the form of the analogue of Theorem 4 for a general compact group. As regards the behaviour at infinity of F, the methods outlined in Section 7 may prove to be useful. But, as regards the local behaviour of F, I have been able to prove only that, under one subsidiary condition on G, the behaviour of F on any given *finite* subset of G has no influeuce on the independence of translates of f.

Naturally, there is no difficulty in passing from T to T^m : one has only to utilise the functions on T^m of the form

$$\varphi_m(x) = \varphi_m(x_1, \ldots, x_m) = \varphi(x_1) \ldots \varphi(x_m)$$

having as Fourier transforms the functions

$$\Phi_m(\chi) = \Phi_m(\chi_1, \ldots, \chi_m) = \Phi(\chi_1) \ldots \Phi(\chi_m).$$

We shall conclude this section by considering briefly the problem of independence of translates of distributions on T in the sense of

Schwartz (11), Tome II, Chapitre VII. Notations introduced here without explanation will have the same significance as in (11). Since, by the theorem of Ascoli, the bounded sets in $(\mathcal{D})_T$ are relatively compact, a theorem of Mackey-Arens tells us that the space $(\mathcal{D})_T$ is reflexive. Consequently, the weak and strong closures of vector subspaces of $(\mathcal{D}')_T$ coincide; in particular, the problem of the independence of translates of a distribution $s \in (\mathcal{D}')_T$ is the same for the weak as for the strong topology.

If $s \in (\mathcal{D}')_T$ has the Fourier coefficients

$$S(\chi) = s(e^{-i\chi x}),$$

then $S(\chi)$ is of polynomial order at infinity, and the series

$$\sum_{\chi} S(\chi) e^{i\chi x}$$

of functions converges strongly to s itself. Conversely, given any sequence $\{a_\chi\}$ $(\chi = 0, \pm 1, \pm 2, \ldots)$ of polynomial order at infinity, the series

$$\sum_{\chi} a_\chi e^{i\chi x}$$

of functions converges strongly to *a* distribution $s \in (\mathcal{D}')_T$ for which $S(\chi) = a_\chi$ for every χ.

If $\psi \in (\mathcal{D})_T$ has the Fourier coefficients $\Psi(\chi)$, the effect of applying to ψ the distribution s is

$$s(\psi) = \sum_{\chi} S(\chi)\Psi(-\chi),$$

the series being automatically absolutely convergent. In particular, defining the translates s_a of s by

$$s_a(\psi) = s(\psi_{-a}) \qquad \left(\psi \in (\mathcal{D})_T\right),$$

we shall have

$$s_a(\psi) = \sum_{\chi} S(\chi)\Psi(-\chi)e^{i\chi a}.$$

Now, according to the proof of Theorem 4, we can find a continuous periodic function $\varphi(x)$ such that $\varphi(0) \neq 0$ and $\varphi(x) = 0$ outside any preassigned neighbourhood of zero in T, and such that the Fourier coefficients $\Phi(\chi)$ of $\varphi(x)$ satisfy the conditions

$$|\Phi(\chi)| < \exp\left(-p(\chi)\right), \qquad \Phi(\chi) = 0 \text{ for } \chi = \pm \chi_k.$$

We can therefore arrange that $\varphi(x) = s_x(\psi)$ for a suitable $\psi \in (\mathcal{D})_T$ provided we can arrange that

$$S(\chi)\Psi(-\chi) = \Phi(\chi)$$

for all χ. But this merely entails that, neglecting the values $\chi = \pm \chi_k$, $\Phi(\chi)/S(\chi)$ must be smaller at infinity than any power of $1/|\chi|$: this we may express briefly by saying that $\Phi(\chi)/S(\chi)$ must be « of rapid decrease » (« à décroissance rapide » in Schwartz's terminology) as $\chi \neq \pm \chi_k$ tends to infinity. Thus we may state

Theorem 5. — *Let $s \in (\mathcal{D}')_T$ have the Fourier coefficients $S(\chi)$. A sufficient condition for the translates of s to be independent in $(\mathcal{D}')_T$ is that for some function $p(t)$ and some sequence $\{\chi_k\}$ satisfying the conditions of Theorem 4,*

$$\exp\left(-p(\chi)\right)/S(\chi)$$

shall be of rapid decrease as $\chi \neq \pm \chi_k$ tends to infinity.

5. — The case $\mathcal{E} = L^2G$, $G = R$.

The results obtained in this section bear close formal resemblance to those of Section 4 in so far as is concerned the behaviour at infinity of the Fourier transform in relation to the independence of translates. However, there are important differences when one considers the influence of the local behaviour of the Fourier transform.

Throughout this section, all integrals are taken over the entire real axis unless the contrary is explicitly indicated; h denotes a positive number, emphasis again being laid on its possible smallness. Fourier transforms of functions concerned are to be taken in an appropriate classical sense which will be obvious from the membership of the functions to L^1R or L^2R.

Our starting point is the Theorem XII of Paley-Wiener (15), which we restate in the following form convenient for our present purpose.

Theorem A. — *Let $p(t)$ (t real) be real and measurable and satisfy*

(5. 1)
$$\int \exp\left(-p(t)\right) dt < +\infty, \qquad \int \exp\left(-2p(t)\right) dt < +\infty.$$

A necessary and sufficient condition for there to exist a function $\alpha(x) \in L^2R$, *zero on a neignbourhood of* $+\infty$, *and having a Fourier transform*

$$A(\chi) = \int \alpha(x)\, e^{-2\pi i \chi x}\, dx$$

satisfying

$$|A(\chi)| = \exp(-p(\chi)), \tag{5. 2}$$

is that

$$\int |p(t)|\, dt/(1+t^2) < +\infty. \tag{5. 3}$$

Observe that the relation (5. 2) is invariant with respect to translations of α, so that the neighbourhood of $+\infty$ on which α vanishes may be varied at will. A direct corollary of this theorem which has immediate application is

Corollary ([4]). — *Let* (5. 1) *and* (5. 3) *be true, and let*

$$q(t) = \int \exp\{-p(t+u) - p(u)\}\, du.$$

Let $h > 0$ *be arbitrary. Then there is a function* $\varphi(x)$, *continuous, zero for* $|x| \geqslant h$, *satisfying* $\varphi(0) \neq 0$, *and having a Fourier transform* $\Phi(\chi)$ *satisfying*

$$|\Phi(\chi)| \leqslant q(\chi). \tag{5. 4}$$

Proof. — Let $p_1(t)$ be any function satisfying (5. 1) and (5. 3). By Theorem A, we can find a function $\alpha_1(x) \in L^2R$ such that $\alpha_1(x) = 0$ for $x > 1$ and $|A_1(\chi)| = \exp(-p_1(\chi))$, where $A_1 = \mathcal{F}(\alpha_1)$. This α_1 is certainly not equivalent to zero, and so we can choose $a > 1$ so large that α_1 is not equivalent to zero on $(-a, a)$. Put

$$\beta_1(x) = \alpha_1(2ax/h);$$

this vanishes for $x > \frac{1}{2}h$ and is not equivalent to zero on $\left(-\frac{1}{2}h, \frac{1}{2}h\right)$. Further, its transform $B_1 = \mathcal{F}(\beta_1)$ is

$$B_1(\chi) = (h/2a)A_1(h\chi/2a),$$

([4]) Cf. Paley-Wiener (15), pp. 24-25.

so that

$$|B_1(\chi)| = (h/2a).\exp\{-p_1(h\chi/2a)\} = C.\exp\{-p_1(C\chi)\},$$

where $C = h/2a$. Next put $\beta_2(x) = \beta_1(-x)$: this vanishes for $x < -\frac{1}{2}h$, is not equivalent to zero on $\left(-\frac{1}{2}h, \frac{1}{2}h\right)$, and has a Fourier transform $B_2(\chi) = \mathcal{F}(\beta_2) = B_1(-\chi)$.

Consider for fixed b the function

$$\gamma(x) = \beta_1(x)\beta_2(x-b).$$

Since the transform of $\beta_2(x-b)$ is $\exp(-2\pi i b\chi)B_2(\chi)$, that of $\gamma(x)$ is

$$\Gamma(\chi) = \int B_1(\chi - \xi)\, B_2(\xi)e^{-2\pi i b\xi}d\xi.$$

Now this cannot be identically zero in ξ for every b: for otherwise we should have $B_1(\chi - \xi)\, B_2(\xi)$ equivalent to zero in ξ for each χ, and in particular (taking $\chi = 0$), $B_1(-\xi)$ equivalent to zero, which would imply that $\beta_1(x)$ is equivalent to zero. Moreover a simple argument shows that any value of b making $\Gamma(\chi)$ not equivalent to zero must be in the interval $(-h, h)$. For example, since $\beta_1(x) = 0$ if $x > \frac{1}{2}h$ and $\beta_2(x-b) = \beta_1(b-x) = 0$ if $x < b - \frac{1}{2}h$, if b were greater than h, $\beta_2(x-b)$ would be zero for $x < \frac{1}{2}h$, in which case $\gamma(x)$ would be zero save perhaps for $x = \frac{1}{2}h$, hence would be equivalent to zero, contrary to the assumption that $\Gamma(\chi)$ is not equivalent to zero. A similar argument shows that b must be greater than $-h$.

Now we have

$$|\Gamma(\chi)| \leqslant \int |B_1(\chi - \xi)|.|B_2(\xi)|\,d\xi = \int |B_1(\chi + \eta)|.|B_1(\eta)|\,d\eta$$
$$= C^2 \int \exp\{-p_1[C(\chi + \eta)]\}.\exp\{-p_1(C\eta)\}\,d\eta;$$

the last member here is $q(\chi)$ provided we put

$$p(t) = p_1(Ct) - \log C,$$

which satisfies (5. 1) and (5. 3) at the same time as $p_1(t)$.

In view of (5. 1) it follows in particular that $\Gamma(\chi)$ is summable, so that the inversion formula yields

$$\gamma(x) = \int \Gamma(\chi) e^{2\pi i x\chi} d\chi \equiv \varphi_0(x),$$

say, holding $p.p.$ in x and certainly at all points of continuity of $\gamma(x)$. It therefore holds if $x < b - \frac{1}{2}h$ or if $x > \frac{1}{2}h$. Thus $\varphi_0(x)$ is not identically zero and vanishes for $x < b - \frac{1}{2}h$ and for $x > \frac{1}{2}h$. Choose x_0 so that $b - \frac{1}{2}h < x_0 < \frac{1}{2}h$ and $\varphi_0(x_0) \neq 0$. Then the function

$$\varphi(x) = \varphi_0(x + x_0)$$

is continuous, is non-zero for $x = 0$, is zero if $x + x_0 > \frac{1}{2}h$, a fortiori if $x > h$, or if $x + x_0 < b - \frac{1}{2}h$, a fortiori if $x < -2h$ (recall that b lies in the interval $(-h, h)$). Further, the transform of φ is equal in modulus to that of φ_0, and so is equal in modulus to $|\Gamma|$. This completes the proof.

The corollary just proved implies non trivial sufficient conditions for a function $f \in L^2R$ to have its translates independent. Indeed, for this to be the case, it is clearly enough that the equation $f * f' = \varphi$ be soluble for $f' \in L^2R$, φ denoting the function constructed in the above proof. Since φ is continuous and summable, this equation is equivalent to

$$F . F' = \Phi \text{ p. p.}$$

were $F = \mathcal{F}(f)$, $F' = \mathcal{F}(f')$ and $\Phi = \mathcal{F}(\varphi)$. This is soluble for $F' \in L^2R$ provided $\Phi/F \in L^2R$ and so certainly provided $q/F \in L^2R$. Thus we have.

THEOREM 6. — *For $f \in L^2R$ to have its translates independent in this space, it is sufficient that, for some $p(t)$ satisfying (5. 1) and (5. 3), we have $q(\chi)/F(\chi) \in L^2R$, were $F = \mathcal{F}(f)$ and where $q(t)$ is defined in terms of $p(t)$ as in the Corollary.*

It is readily verified that the condition of Theorem 6 is compatible with $F(\chi)$ being at infinity as small as, say, $\exp(-|\chi|^{1-\varepsilon})$ for any fixed $\varepsilon > 0$. As in section 4, the translates of f fail to be independent if $F(\chi) = o \exp(-a|\chi|)$ for any $a > 0$; indeed, more than this is true, as can be seen by using well known results on quasi-analytic classes.

Observe however that there is an important difference between the cases $G = T$ and $G = R$. In the former case a function with independent translates can have a Fourier transform vanishing on a set of infinite Haar measure in the dual group; but in the case $G = R$, the transform can vanish on a set of zero measure at most. For if f and f' are in L^1R, and if φ is continuous and has a compact support, then from $f * f' = \varphi$ follows $F . F' = \Phi$ p. p. Hence, if F vanishes on a set of positive measure, the same is true of Φ ; and, Φ being an entire function, this would imply that Φ, an hence φ also, is identically null. This observation has a general significance : if φ, continuous and having a compact support, has the transform Φ, then Φ is in some sense analytic and is determined throughout any connected set by its values on a relatively sparce subset thereof. When G is compact and $\widehat{G}$ discrete, this has no importance since the only connected sets in $\widehat{G}$ are single points. But when $\widehat{G}$ is connected, one can expect results approximating those for the simple case $G = R$.

Another point of interest is that the above arguments yield non-trivial results for the space $\mathcal{E} = L^1R$. In this case, for $f \in L^2R$ to have its translates independent in this space, it is enough that we can solve $f * f' = \varphi$ for $f' \in L^\infty R$.

However, if $q/F \in L^1R \cap L^2R$, the same is true of Φ/F, and we can then define

$$f'(x) = \int \{\Phi(\chi)/F(\chi)\} e^{2\pi i x \chi} d\chi :$$

this f' will be bounded and in L^2R, hence in $L^\infty R$. So we have

THEOREM 7. — *For $f \in L^1R$ to have its translates independent in this space, it is sufficient that, for some $p(t)$ satsifying* (5. 1) *and* (5. 3), *we have $q(\chi)/F(\chi) \in L^1R \cap L^2R$, where $F = \mathcal{F}(f)$ and where $q(t)$ is defined in terms of $p(t)$ as in the Corollary.*

The remainder of this section is devoted to some remarks on the independence of translates of a distribution on R^m. This question is considerably more complicated than the problem for periodic distributions discussed at the end of Section 4, and we shall confine ourselves to a brief survey of the problem. To begin with, it seems likely that distinctly stronger results can be obtained for the distributions of slow growth than for distributions in general, and that therefore the problem may well be treated in two distinct parts. Using the notations of Schwartz (11), Tome II, Chapitre VII, the strong topology of $(\mathcal{S}')$ is finer than that induced on it by the strong topology of $(\mathcal{D}')$. Hence, a priori, the notion of independence of translates of a distribution s of slow growth will depend on whether we regard s as a member of $(\mathcal{S}')$ or of $(\mathcal{D}')$. The independence of translates of s in $(\mathcal{D}')$ implies their independence in $(\mathcal{S}')$.

On the other hand, since both spaces $(\mathcal{S})$ and $(\mathcal{D})$ are reflexive (the bounded sets being relatively compact), the weak and strong independence of translates in $(\mathcal{S}')$ are equivalent, and likewise in $(\mathcal{D}')$.

As M. Deny has pointed out to me, a sufficient condition for a distribution s to have its translates independent in $(\mathcal{D}')$ is that it be inversible with a distribution having a compact support: by this it is meant that there exists a distribution t with a compact support such that

$$s * t = \delta,$$

δ being the Dirac measure (mass $+1$ at the origin). Indeed, it is enough that a distribution t with a compact support and a distribution $u \neq 0$ with a point support at the origin exist such that

$$s * t = u. \tag{5. 5}$$

For, if this is the case, whenever $\alpha \in (\mathcal{D})$,

$$\theta = t * \alpha, \qquad \varphi = u * \alpha$$

are in $(\mathcal{D})$, and (5, 5) yields

$$s * \theta = \varphi; \tag{5. 6}$$

that we have here a case of associativity and commutativity of the convolution follows from p. 14, Tome II, of Schwartz (11). Now if α is supported by a neighbourhood of zero which becomes smaller and smaller, the same is true of φ; also

$$\varphi(0) = u(\check{\alpha}) \qquad \big(\check{\alpha}(x) = \alpha(-x)\big),$$

and this can be made non-zero for suitably chosen $\alpha \in (\mathcal{D})$ supported by arbitrarily small neighbourhoods of zero (since u, supported at the origin, is not identically zero). This is plainly enough to show than the translates of s are independent in $(\mathcal{D}')$.

If $s \in (\mathcal{S}')$, (5. 5) is equivalent to $S. T = U$, where $S = \mathcal{F}(s)$, $T = \mathcal{D}(t)$, $U = \mathcal{F}(u)$. Taking, for example, $u = \delta$, this last equation will be soluble for T the transform of a distribution t with a compact support provided S is a function $S(\chi)$ and $1/S(\chi)$ is entire of exponential type and of polynomial order at infinity on the real axis. These conditions are therefore certainly sufficient for a distribution $s \in (\mathcal{S}')$ to have its translates independent, but they almost certainly not necessary. In fact, by utilising functions φ, continuous and supported by arbitrarily small neighbourhoods of zero (constructed, say, as in the Corollary to Theorem A), it is enough for independence in $(\mathcal{S}')$ that we can solve $s * \psi = \varphi$ for $\psi \in (\mathcal{S})$. And this will be possible at least whenever $S = \mathcal{F}(s)$ is a function $S(\chi)$ such that $1/S(\chi)$ is infinitely derivable and $O\{\exp(|\chi|^{1-\varepsilon})\}$ as $|\chi| \to \infty$ on the real axis for some $\varepsilon > 0$.

6. — The case $\mathcal{E} = CG$, $G = R^m$.

Recall that CG is the vector space of all continuous functions on G, the topology being that of convergence uniform on every compact set in G. As is well known, the dual space is isomorphic as a vector space with the set $\mathcal{M}$ of all Radon measures on G having compact supports, and we can arrange that the duality is expressed by the bilinear functional

$$< f,\ \mu > = \int f(-x)\, d\mu(x) = \int \check{f} d\mu,$$

f being in CG and μ a measure with a compact support. In this case we shall have therefore

$$(6.1) \qquad < f_x,\ \mu > = \int f(x-y)\, d\mu(y) = f * \mu(x).$$

The criteria I am able to give as sufficient to ensure the independence of translates in CR^m are somewhat crude and, although they are useful for fabricating certain examples which we shall discuss shortly, their theoretical interest is strictly limited. For this reason, I have refrained from elevating these criteria to the status of theorems. Instead, they will be included in a list of remarks to follow. A detailed and systematic study of the independence of translates in CR^m promises to be both long and difficult and will not be undertaken here.

Any $f \in CR^m$ can be regarded as a distribution $\in (\mathcal{D}')$ and arguments similar to those at the end of Section 5 lead to the following criterion :

1° *A sufficient condition for $f \in CR^m$ to have its translates independent in this space is that to every given neighbourhood V of zero in R^m shall* *correspond a distribution t with a compact support and a distribution $u \neq 0$ supported by* V *such that*

$$f * t = u.$$

If $f \in \mathrm{CR}^m$ is of slow growth, that is if

$$\int |f(x)|\,dx/(1+|x|)^k < +\infty,$$

$$x = (x_1, \ldots, x_m) \in \mathrm{R}^m, \quad |x| = (x_1^2 + \cdots + x_m^2)^{\frac{1}{2}}, \quad dx = dx_1 \ldots dx_m,$$

for some k depending on f, we can from 1°) deduce sufficient conditions involving the Fourier transform $\mathrm{F} = \mathcal{F}(f)$. It is a corollary of the generalised form of the Paley-Wiener theorem (due to L. Schwartz (11), Tome II, pp. 127-129) that, for a distribution q to be the Fourier transform of a distribution with a compact support, it is necessary and sufficient that it be a function $q(\chi)$, entire of exponential type, and of polynomial order at infinity on the « real axis » (that is, for real values of the coordinates of $\chi \in \mathrm{R}^m$). So if φ is a continuous function with a compact support, we can solve $f * t = \varphi$ for t a distribution with a compact support, provided that F is a function, that $1/\mathrm{F}$ is entire of exponential type, and provided $\Phi/\mathrm{F}\big(\Phi = \mathcal{F}(\varphi)\big)$ is of polynomial order at infinity on the « real axis ». For, if this is so, $\mathrm{T} = \Phi/\mathrm{F}$, which is then a function, entire of exponential type and of polynomial order at infinity on the « real axis », is the transform of some distribution t with a compact support; and from $\mathrm{T}.\,\mathrm{F} = \Phi$ follows $t * f = \varphi$. If we take φ to be of the type constructed in the Corollary to Theorem A, we are led to.

2° *For $f \in \mathrm{CR}^m$ of slow growth to have its translates independent, it is enough that $\mathrm{F} = \mathcal{F}(f)$ be a function such that $1/\mathrm{F}$ is entire of exponential type and $\mathrm{O}\,\{\exp(|\chi|^{1-\varepsilon})\}$ for some $\varepsilon > 0$ as $\chi \to \infty$ on the « real axis ».*

Now we shall illustrate the use of these criteria in connection with a few examples.

To begin with, it is easy to see that no monomial

$$f(x) = x_1^{n_1} \ldots x_m^{n_m}$$

has its translates independent in CR^m : on the contrary, the translates of such a function are very strongly coherent since, if U is any non-void open set, then $f \in \mathfrak{I}(f, \mathrm{U})$. For, if $a \in \mathrm{U}$, we can take derivatives (which are limits of linear combinations of translates) and thus discover that

$$(x_1+a_1)^{j_1}\dots(x_m+a_m)^{j_m}\in \mathfrak{J}(f,\ \mathrm{U}),$$

where $a=(a_1,\ \dots,\ a_m)$ and the j_i are integers satisfying $0\leqslant j_i\leqslant n_i$ for $1\leqslant i\leqslant m$. Then the binomial theroem shows that
$f(x)=x_1^{n_1}\dots x_m^{n_m}=\{(x_1+a_1)-a_1\}^{n_1}\dots\{(x_m+a_m)-a_m\}^{n_m}\in\mathfrak{J}(f,\ \mathrm{U})$.

This situation can also be predicted by use of the Fourier transform. For the transform of $x_1^{n_1}\dots x_m^{n_m}$ is

$$(-1/2\pi i)^{n_1+\dots+n_m}\cdot\frac{\partial^{n_1+\dots+n_m}\delta}{\partial\chi_1^{n_1}\dots\partial\chi_m^{n_m}}.$$

Thus if μ is a measure $\in\mathcal{M}$, and if φ is a function with a compact support, and if

$$\mu * x_1^{n_1}\dots x_m^{n_m}=\varphi,$$

then, taking transforms, we shall have

$$\mathrm{M}(\chi)\cdot(-1/2\pi i)^{n_1+\dots+n_m}\frac{\partial^{n_1+\dots+n_m}\delta}{\partial\chi_1^{n_1}\dots\partial\chi_m^{n_m}}=\Phi(\chi),$$

where $\mathrm{M}=\mathcal{F}(\mu)$ and $\Phi=\mathcal{F}(\varphi)$. Since the left member here is a distribution supported at the origin, equality can hold only if both sides are zero distributions, in which case $\varphi=0$. The same argument yields the more general proposition :

3° *For* $f\in\mathrm{CR}^m$ *of slow growth to have its translates independent, it is necessary that* $\mathrm{F}=\mathcal{F}(f)$ *have as support the entire space* R^m (which implies that $\mathfrak{J}(f)=\mathrm{CR}^m$).

A difference, at first sight a little surprising, appears in the case $m=1$ and $f(x)=|x|^n$ where $n>0$ is and *odd* integer. Here we can show that f is inversible as a distribution (that is, satisfies the hypothesis of 1°) with $u=\delta$), so that the translates of f are independent. For $f(x)$ has derivatives of all orders in the usual sense for $x\neq 0$, and $f^{(n+1)}(x)=0$ for $x\neq 0$. For $0\leqslant p\leqslant n$, $f^{(p)}(x)$ has a jump J_p at $x=0$ given by $\mathrm{J}_p=0$ if $0\leqslant p\leqslant n-1$, and $\mathrm{J}_n=2\,.\,n!$. The formula (11,28) of Schwartz (11), Tome I, gives (with $p=n+1$) :

$$\delta^{(n+1)}*|x|^n=2\,.\,n\,!\,.\,\delta,$$

which proves our assertion.

There are numerous other functions in CR which are inversible in the sense just explained, all of which therefore have their translates independent. M. Deny communicated to me the example $f(x) = \exp(a|x|)$ $(a \neq 0)$, which is inversible with the distribution $t = (1/2a)(\delta'' - a^2\delta)$. When $a < 0$, this function is of slow growth and has as Fourier transform the function

$$F(\chi) = -2a/(a^2 + 4\pi^2\chi^2),$$

so that f satisfies the criterion 2°). More generally, one can start from a function $g(x)$, continous for $x \geqslant 0$, having derivatives of all orders in the usual sense for $x > 0$, and satisfying there a linear differential equation with constant coefficients of the form

$$\sum_r c_r(d^2/dx^2)^r g(x) = 0;$$

one then considers the function $f(x) = g(|x|) \in \mathrm{CR}$. Thus, starting from $g(x) = \mathrm{A}e^{ax} + \mathrm{B}e^{bx}$, it could be shown that if $a \neq 0$, $b \neq 0$, and if A and B are suitably related, then

$$\{\delta^{\mathrm{IV}} - (a^2 + b^2)\,\delta'' - a^2b^2\delta\} * f$$

is a non-vanishing linear combination of δ, δ', δ''' and δ^{IV}, so that 1°) is satisfied and f has its translates independent.

We shall now give a proof of the following fact :

4° *Consider the continuous function on* R^m *defined by*

$$\mathrm{K}_{a,p}(x) = \mathcal{F}[\mathrm{C}_{a,p}/(a^2 + 4\pi^2|\chi|^2)^p]$$

where $a \neq 0$ *and the integer* p *is* $> \frac{1}{2}m$, *the constant* $\mathrm{C}_{a,p}$ *being chosen so that* $\mathrm{K}_{a,p}(0) = 1$. *Then, if* $\mathrm{P}(x) = \mathrm{P}(x_1, \ldots, x_m) \not\equiv 0$ *is a polynomial,* $f(x) = \mathrm{K}_{a,p}(x)\,.\,\mathrm{P}(x)$ *has its translates independent in* CR^m. *Further,* $\mathrm{K}_{a,p}(x) \to 1$ *as* $a \to 0$, *and this in the sense of* CR^m, *so that the functions in* CR^m *having independent translates are everywhere dense.*

I had originally constructed a proof of this for the case $m = 1$ only, using the function $\mathrm{K}_{a,1}(x) = \exp\,(a\,|x|)$, but M. Deny afterwards communicated to me an extension and simplification of the arguments which cover also the case $m > 1$. He has kindly granted me permission to reproduce his proof here.

Let n be the degree of P, and denote by Δ the Laplace distribution on R^m :

$$\Delta = \frac{\partial^2\delta}{\partial x_1^2} + \cdots + \frac{\partial^2\delta}{\partial x_m^2}.$$

Consider the distribution

$$u = (\Delta - a^2\delta)_*^{(p+n)*}(K_{a,p}p):$$

I assert that u is a distribution $\neq 0$ having a point support at the origin. Once this proved, 1°) serves to show that $f = K_{a,p}P$ has its translates independent. Now, if $q_1 + \cdots + q_m \leqslant n$,

$$\mathcal{F}[(\Delta - a^2\delta)^{*(p+n)*}(x_1^{q_1}\ldots x_m^{q_m}K_{a,p})]$$

$$= \text{const}.(a^2 + 4\pi^2|\chi|^2)^{p+n}.\frac{\partial^{q_1+\cdots+q_m}}{\partial\chi_1^{q_1}\ldots\partial\chi_m^{q_m}}\{(a^2+4\pi^2|\chi|^2)^{-p}\},$$

which is easily seen to be a polynomial. By addition the same is therefore true of $\mathcal{F}(u)$, so that u must have a point support at the origin. Also, $u \neq 0$, since, otherwise, we should, have on taking transforms,

$$(a^2 + 4\pi^2|\chi|^2)^{p+n}.F = 0 \qquad (F = \mathcal{F}(K_{a,p}P)),$$

hence $F = 0$ and so $K_{a,p}P = 0$, which is not the case.

Next, since $p > \frac{1}{2}m$,

$$\int d\chi/(a^2 + 4\pi^2|\chi|^2)^p \to +\infty$$

as $a \to 0$ (by the theorem on the term-wise integration of monotone sequences of functions). Hence $C_{a,p} \to 0$ as $a \to 0$. Further,

$$K_{a,p}(x) - 1 = K_{a,p}(x) - K_{a,p}(0)$$
$$= C_{a,p}\int(e^{-2\pi i\chi x} - 1)d\chi/(a^2 + 4\pi^2|\chi|^2)^p.$$

Given any $\eta > 0$ and any compact set C, we can choose a neighbourhood V of zero such that

$$|e^{-2\pi i\chi x} - 1| \leqslant \eta$$

whenever $x \in C$ and $\chi \in V$. Thus, if $\chi \in V$,

$$\sup_{\chi\in C}|K_{a,p}(x) - 1| \leqslant \eta C_{a,p}\int_V d\chi/(a^2 + 4\pi^2|\chi|^2)^p$$
$$+ C_{a,p}\int_{R^m - V} 2d\chi/(a^2 + 4\pi^2|\chi|^2)_p \leqslant \eta + 2C_{a,p}\int_{R^m - V} d\chi/(4\pi^2|\chi|^2)^p.$$

Keeping η, and hence V also, fixed and letting $a \to 0$, we conclude that

$$\limsup_{a\to 0}\left\{\sup_{\chi\in C}|K_{a,p}(x) - 1|\right\} \leqslant \eta.$$

Since η is arbitrary, this completes the proof.

5° *If* $f \in CR^m$ *has independent translates, if* $q \neq 0$ *is a distribution with a point support at the origin, and if* $g = f * q \in CR^m$, *then* g *has its translates independent in this space.*

To see this, we know that for every neihgourhood V of zero in R^m we can choose $\mu \in \mathcal{M}$ such that

$$\varphi = f * \mu$$

satisfies $\varphi(0) \neq 0$ and $\varphi = 0$ outside V. Then

$$g * \mu = q * \varphi = \psi,$$

say. Here ψ is supported by V. And $\psi \not\equiv 0$ since otherwise a Fourier transformation would yield

$$Q \cdot \Phi \equiv 0 \qquad \big(Q = \mathcal{F}(q), \quad \Phi = \mathcal{F}(\varphi)\big).$$

But this is impossible since both Q and Φ are entire functions, neither identically zero, so that each can vanish on a non-dense set at most. Consequently we can choose $a \in V$ such that the translate $\theta = \psi_a$ satisfies $\theta(0) \neq 0$ and $\theta = 0$ outside $W = V \oplus V$. Since we have then $\theta = g * \nu$, where ν is a translate of μ, and since W is arbitrarily small with V, this shows that the translates of g are independent in CR^m.

It follows in particular that no function which is a solution of any linear partial differential equation with constant coefficients and having an analytic second member can have its translates independent in CR^m.

Finally, let us interpret for the space $\mathcal{E} = CR$ the remarks in Section 2. To begin with, it is easy to see that the weak differentiability of $f \in CR$ is equivalent in this case to the existence of a continuous derivative in the ordinary (point-wise) sense. So we are to consider those functions $f \in CR$ which have derivatives of all orders in the ordinary sense. The numbers $M(n; i, V)$ defined on p. 11 may here be replaced by the numbers

$$M(n; K) = \sup_{x \in K} |f^{(n)}(x)|,$$

K denoting a general compact subset of R and $f^{(n)}(x)$ being the ordinary n th derivative of f: this number corresponds to the semi-norm

$$p_H(f) = \sup_{x \in H} |f(x)| \ (H \subset R, \text{ compact})$$

and the symmetric and compact neighoourhood V of zero in R whenever $H \oplus V = K$. Accordingly we introduce the numbers

$$T(r;\ K) = \sup_{n \geqslant 1} r^n / M(n;\ K),$$

and we may then assert

6° *If* $f \in CR$ *has derivatives of all orders in the usual sense, and if for every compact* $K \subset R$ *the integral*

$$\int_1^\infty \log T(r;\ K)\, dr/r^2 = +\infty,$$

then the translates of f *are not independent in* CR.

In particular, if f is analytic on the real axis its translates are not independent.

It is in fact the case that the hypotheses of 6° are strong enough to imply much more than that the translates of f are not independent since they ensure that $\mathcal{E}'(f)$ is quasi-analytic over every compact interval of the real axis. A consequence of this is that, if f satisfies the hypotheses of 6°, then $f \in \mathfrak{I}(f, S)$ whenever the set S having a finite limiting point: for if $\mu \in \mathfrak{M}$ is orthogonal to $\mathfrak{I}(f, S)$ the function $\varphi = f * \mu$ is zero at all points of S. If s is a limiting point of S, it results from successive application of Rolle's theorem that $\varphi^{(n)}(s) = 0$ for $n = 0, 1, 2, \ldots$; whence, since the class $\mathcal{E}'(f)$ is quasi-analytic over every compact interval, φ is identically zero throughout any such interval containing S and hence everywhere. This shows in particular that μ is orthogonal to f, whence our assertion. That all this is a consequence of hypotheses designed merely to ensure the non-independence of translates appears to be due in the last analysis to the fact that convergence in the space $\mathcal{E}$ concerned (namely CR) is at least as strong as convergence uniform on every compact set. It seems highly likely that a different situation would arise if we took on the vector space CR a less fine topology (that defined by the simple convergence, for example).

7. Connections with the theory of normed rings.

It is possible in theory to use some results concerning the regularity of normed rings given in Silov (19) in such a manner as to derive sufficient conditions for a function $f \in L^2G$ to have its translates independent. In principle the method is applicable to an arbitrary group G, but the calculations appear to be much more complicated when G is non-compact, and we shall illustrate the method in the compact case. Such results would apply in particular to the groups T or T^m considered in Section 4, but the proofs given there are more direct and the results more manageable and deeper in so far as some account is taken of vanishing Fourier coefficients. What is more, results of the type of Theorem B to follow concerning the regularity of normed rings presuppose those results on quasi-analytic classes which are used directly for the study of the problem of independence of translates for the spaces L^2T and L^2R.

The notion of regularity of a normed ring is defined in the following manner. Let $\mathfrak{R}$ be a normed ring in the sense of Gelfand (that is, a commutative Banach algebra with unit); denote by M a typical maximal ideal in $\mathfrak{R}$, and by (φ, M) the image of the element φ of $\mathfrak{R}$ under the canonical homomorphism of $\mathfrak{R}$ onto the complex field defined by M; finally, suppose the set $\mathfrak{M}$ of all maximal ideals in $\mathfrak{R}$ to be topologised after the manner of Gelfand. This last means that we consider on $\mathfrak{M}$ the « strong topology », characterised as the least fine for which all the functions (φ, M) of M are continuous, or, again, as the topology induced on $\mathfrak{M}$ as a subset of the dual space of the Banach space $\mathfrak{R}$ by the weak topology on the latter. $\mathfrak{R}$ is then said to be regular if, given $M_0 \in \mathfrak{M}$ and any neighbourhood $\mathfrak{V}$ of M_0, there exists $\varphi \in \mathfrak{R}$ such that

$$(\varphi, M_0) \neq 0, \qquad (\varphi, M) = 0 \text{ if } M \text{ non} - \in \mathfrak{V}.$$

In other words, $\mathfrak{R}$ is regular if and only if the strong topology on $\mathfrak{M}$ coincides with the « weak topology » of Wallman-Stone.

Silov (16) gives sufficient conditions for a normed ring having real generators to be regular, an element φ of $\mathfrak{R}$ being termed real if (φ, M) is real for every M. In view of the fact that reference (16) is very difficult to obtain, and for the sake of completeness, I include here a proof of a result similar to that of Silov, whose article has been available to me in the form of an abstract only.

Let us term « admissible » any function $N(t)$ (t real), non-negative and measurable, having the property that there exists for every $\delta > 0$ a function $h(t)$, summable over the real axis, and such that

(7.1) $$\int N(t)|h(t)|\,dt < +\infty;$$

(7.2) $\int h(t)e^{-2\pi i\chi t}\,dt$ satisfies $H(0) = 1$ and $H(\chi) = 0$ for $|\chi| \geq \delta$.

The emphasis here rests on the possibility of choosing $N(t)$ very large at infinity. The Corollary to Theorem A shows that we may take, for example, $N(t) = \exp(-|t|^a)$ for any $a < 1$. Since (7. 1) is invariant when we replace $h(t)$ by $\exp(2\pi i\lambda t).\ h(t)$ (λ real), if $N(t)$ is admissible and if $\delta > 0$ and any real λ are given, we can find a summable function $h(t)$ satisfying (7. 1) and such $H(\lambda) = 1$ and $H(\chi) = 0$ if $|\chi - \lambda| \geqslant \delta$.

Returning to the normed ring $\mathfrak{R}$, suppose that $\mathfrak{R}$ has a system $\theta_k (k \in K)$ of real generators; let M_0 be a maximal ideal in $\mathfrak{R}$, and let $\mathfrak{V}$ be any given neighbourhood of M_0. Since the θ_k generate $\mathfrak{R}$, we may assume that $\mathfrak{V}$ has the form

$$\mathfrak{V} = \{M : |(\theta_k, M) - (\theta_k, M_0)| < \delta \text{ for } k \in J\},$$

J being a *finite* subset of K. We take as an hypothesis that for each K the function

$$N_k(t) = ||\exp(-2\pi i t\theta_k)||$$

is admissible, $\exp(-2\pi i t\theta_k)$ being defined as an element of $\mathfrak{R}$ by the usual power series. Since θ_k is real, we shall have

$$N((t) \geqslant \sup_{M \in \mathfrak{M}} |\exp\{-2\pi i t.(\theta_k, M)\}| = 1.$$

Granted this, we choose for each $k \in J$ a function $h_k(t)$, summable and such that

$$\int N_k(t)|h_k(t)|\,dt < +\infty,$$

and having a transform $H_k(\chi)$ such that

$$H_k\{(\theta_k, M_0)\} = 1, \qquad H_k(\chi) = 0 \text{ if } |\chi - (\theta_k, M_0)| \geqslant \delta.$$

Consider then the element of $\mathfrak{R}$ defined by

$$\varphi = \prod_{k \in J} \varphi_k,$$

where

$$\varphi_k = \int h_k(t) e^{-2\pi i t.\,\theta_k}\,\mathrm{dt},$$

the integral on the right being taken in the sense of Bochner (an integral of a vector-valued function). For any M, one has

$$(\theta_k, M) = \int h_k(t) e^{-2\pi i t\,(\theta_k, M)}\,dt = H_k\{(\theta_k, M)\},$$

and so

$$(\varphi, M) = \prod_{k \in J} H_k\{(\theta_k, M)\}.$$

By choice of the functions $h_k(t)$ we shall therefore have

$$(\varphi, M_0) = \prod_{k \in J} H_k\{(\theta_k, M_0)\} = \prod_{k \in J} 1 = 1;$$

on the other hand, if M non-$\in \mathfrak{V}$, we have

$$|(\theta_k, M) - (\theta_k, M_0)| \geqslant \delta$$

for at least one $k \in J$; for any such k, $H_k\{(\theta_k, M)\} = 0$, and so $(\varphi, M) = 0$ whenever M non-$\in \mathfrak{V}$. This shows that $\mathfrak{R}$ is regular, and we have thereby proved

THEOREM B. — *Let $\mathfrak{R}$ be a normed ring having the $\theta_k (k \in K)$ as real generators. A sufficient condition for $\mathfrak{R}$ to be regular is that for each $k \in K$ the function*

(7. 3) $$N_k(t) = \|\exp(-2\pi i t \theta_k)\|$$

be admissible.

We shall now see how this result can be applied to the problem in hand. Take a compact group G and consider the problem of independence of translates in L^2G. As will be amply clear from the foregoing work, the crux of the problem is to construct continuous functions φ on G, supported by arbitrarily small pre-assigned neighbourhoods of zero in G, and having Fourier transforms as small as possible at infinity on $\widehat{G}$. This, as we intend to show, is equivalent to deciding the regularity of a class of suitably chosen normed rings offunctions on G.

Take on G a real-valued function $Q(\chi)$ having the properties

(7. 4)
$$Q(\chi + \chi') \leqslant Q(\chi) + Q(\chi'), \qquad Q(0) = 0, \qquad \liminf_{\chi \to \infty} Q(\chi) > 0.$$

Denote by $\mathfrak{R} = \mathfrak{R}_Q(G)$ the class of all continuous functions $\varphi(x)$ on G having a Fourier transform $\Phi(\chi)$ satisfying

(7. 5) $$\|\varphi\|_Q \equiv \int \exp\big(Q(\chi)\big) \cdot |\Phi(\chi)| \, d\chi < +\infty.$$

Thanks to the conditions (7. 4), $\mathfrak{R}$ proves to be a normed ring under pointwise multiplication. Besides this, it is easy to see that the maximal ideals in $\mathfrak{R}$ are in a bi-unique correspondence, $x \longleftrightarrow M_x$, with the points x of G, M_x being defined by the relations $(\varphi, M_x) = \varphi(x)$ for all φ in $\mathfrak{R}$. This correspondence is bi-unique since $\mathfrak{R}$ always contains the continuous characters of G and these separate the points of G. If we therefore identify (as sets) G and $\mathfrak{M}$, the Gelfand topology on G is none other than the natural topology on G : this follows at once from the fact that, whatever the normed ring $\mathcal{R}$, the Gelfand topology on $\mathfrak{M}$ is uniquely characterised as being the compact Hausdorff topology on $\mathfrak{M}$ for which all the functions (φ, M) of M are continuous.

As a set of real generators in $\mathcal{R} = \mathcal{R}_Q(G)$ it is natural to take the functions

$$\theta_\xi(x) = \frac{1}{2}\xi(x) + \frac{1}{2}\overline{\xi(x)},$$
$$\theta'_\xi(x) = \xi(x)/2i - \overline{\xi(x)}/2i,$$

ξ ranging over $\widehat{G}$. Define therefore

$$(7.6) \quad N_{Q,\xi}(t) = \|\exp(-2\pi i t\theta_\xi)\|_Q, \qquad N'_{Q,\xi}(t) = \|\exp(-2\pi i t\theta'_\xi)\|_Q$$

for real t. According to Theorem B, the conditions

(7. 7) For each $\xi \in \widehat{G}$, $N_{Q,\xi}(t)$ and $N'_{Q,\xi}(t)$ are admissible ensure that $\mathcal{R}$ is regular.

Consequently, assuming (7. 7) to be true, given any neighbourhood U of zero in G, there is φ in $\mathcal{R}$ such that $\varphi(0) \neq 0$ and $\varphi(x) = 0$ for x non $- \in U$. And then, if $f \in L^2G$ is given, we can solve the equation $f * f' = \varphi$ for $f' \in L^2G$ provided $\Phi/F \in L^2\widehat{G}(F = \mathcal{F}(f), \Phi = \mathcal{F}(\varphi))$, and so surely if

$$\exp(-Q(\chi))/F(\chi) \in L^2\widehat{G}.$$

We have therefore established the result

Theorem 8. — *Let* G *be compact. Let* $Q(\chi)$, $N_{Q,\xi}(t)$ *and* $N'_{Q,\xi}(t)$ *be defined as above in* (7. 6), *and let* (7. 7) *be true. Then for* $f \in L^2G$ *to have its translates independent in this space, it is enough that*

$$\int |\exp(-Q(\chi))/F(\chi)|^2 d\chi < +\infty.$$

As has been remarked before, the weakness of Theorem 8 lies in the fact that the sufficient condition prescribed therein is not compatible with the existence of any zeros of the Fourier transform $F = \mathcal{F}(f)$, whereas we know already that for $G = T^m$, the non-vanishing of F is not necessary for the translates of f to be independent. It would appear that the above method cannot be adapted in such a manner as to permit zeros of F since $Q(\chi)$, if it is zero anywhere, is identically zero by virtue of (7. 4) (which condition is necessary to ensure that the class $\mathcal{R}_Q(G)$ shall be a ring).

8. — Various extensions.

Since the several extensions to be mentioned are somewhat disconnected, we shall list them under separate headings.

A. — Non-linear spans of translates. — In a brief note which is to appear in the Journal of the London Mathematical Society, I have discussed the independence problem in a modified form. Given a space $\mathcal{E}$, a function $f \in \mathcal{E}$, and a subset $A \subset G$, let us denote by $\mathcal{M}(f, A) = \mathcal{M}(f, A, \mathcal{E})$ the closure in $\mathcal{E}$ of the set of all finite sums

$$(8.\ 1) \qquad \sum_m \rho_m f_{a_m}$$

where

$$(8.\ 2) \qquad a_m \in A$$

and

$$(8.\ 3) \qquad \sum_m |\rho_m| \leqslant 1.$$

A sum of the form (8. 1) with the a_m and ρ_m subject to (8. 2) and (8. 3) respectively will be termed a « mean » of the translates of f corresponding to the set A. I have termed the translates of f « mean-independent » if, whenever A is a closed subset of G with O non-ε A, f non-$\in$ $\mathcal{M}(f, A)$. In the note already mentioned I have dealt with the problem of mean-independence of translates for the space $\mathcal{E} = L^2G$ (G locally compact and abelian) : the solution for this space is complete. The only functions $f \in L^2G$ whose translates are *not* mean-independent are those whose failure can be predicted a priori, namely those which are periodic (having for period a non-trivial, closed subgroup of G). Further, if $\widehat{G}$ is connected, the only function of this space whose translates fail to be mean-independent is the null function.

The problem of determining the extent of $\mathcal{M}(f) = \mathcal{M}(f, G)$ also presents some interest. A related problem concerns the study of the convex span $\mathcal{K}(f)$ of translates of f [defined to be the closure in $\mathcal{E}$ of the set of all finite sums (8. 1) with the a_m arbitary in G and (8. 3) replaced by

$$(8.\ 3') \qquad \rho_m \geqslant 0, \qquad \sum_m \rho_m = 1].$$

Both of these problems appear to be new : in fact, I do not know of any literature dealing with non-linear spans of translates whatever. The solution of certain such problems is at hand. For a number of interesting spaces $\mathcal{E}$ (including L^1G), making the restriction to real functions in the case of $\mathcal{K}(f)$, it turns out that the membership of a function h to $\mathcal{K}(f)$ or to $\mathcal{M}(f)$ depends solely on the behaviour of the quotient H/F $\big(H = \mathcal{F}(h),\ F = \mathcal{F}(f)\big)$: in the case of $\mathcal{K}(f)$, for example, for $h \in \mathcal{K}(f)$, it is necessary and sufficient that H/F be *a* normalised (continuous) function of positive type on $\widehat{G}$. These results are not surprising when one recalls the crucial role played by the quotient H/F in the study of linear spans of translates.

B. — **The L^2G problem for non-abelian groups.** — I owe much of the substance to be given under this heading to Professor G. W. Mackey, who has very kindly granted me permission to record his suggestions here.

The problem of independence of translates in L^2G over an abelian group has a natural extension to non-abelian groups in which L^2G is replaced by the Hilbert space $\mathcal{H}$ underlying some continuous unitary representation $(\mathcal{H},\ U_x)$ of G [the notation is that of Godement (17)]. The problem here is to determine those vectors $X \in \mathcal{H}$ with the property that $U_{x_0}X$ is not in the closed vector subspace of $\mathcal{H}$ generated by the U_xX with $x \in A$ (A a closed subset of G) unless $x_0 \in A$. In particular, if we restrict ourselves to those X for which the $U_xX (x \in G)$ generate $\mathcal{H}$ [so that the system $(\mathcal{H},\ U_x,\ X)$ is a simple unitary representation of G : vide Godement (17), p. 16], since any such representation is uniquely determined by its « caracteristic function »

$$\varphi(x) = (X,\ U_xX),$$

we may view the problem as one concerning the functions of positive type on G. Namely : *for what (continuous) functions* φ *of positive type on* G *is it true that from*

a) $x_1, \ldots, x_m \in A$, *with* A *closed and fixed*,

b) x_0 *fixed and non-* $\in A$,

c) $\alpha_0 = -1,\ \alpha, \ldots, \alpha_m$ *arbitrary complex numbers*,

follows

$$(8\cdot 4) \qquad \inf \sum_{i,\,j=0}^{m} \alpha_i \bar{\alpha}_j \varphi(x_i x_j^{-1}) > 0\,?$$

It should however be observed that the assumption that the U_xX

generate $\mathcal{H}$ is somewhat irrelevant a priori (and even eventually in some cases): in terms of the original problem, it is tantamount to assuming that the functions f considered have their translates fundamental and this, as we have seen in Section 4, is not necessary for the translates of f to be independent.

When G is abelian, there is a bi-unique correspondence between the continuous functions φ of positive type on G and the bounded, positive Radon measures μ on $\widehat{G}$: to each φ corresponds a unique μ such that

$$\varphi(x) = \int \chi(x) d\mu(\chi).$$

In this case, (8. 4) becomes

$$(8.\ 5) \qquad \inf \int \left| \chi(x_0) - \sum_{i=1}^{m} \alpha_i \cdot \chi(x_i) \right|^2 d\mu(\chi) > 0.$$

Thus the problem can also stated thus: *For each bounded positive measure* μ *on* $\widehat{G}$, *form the Hilbert space* $L^2(\widehat{G}, \mu)$; *for which* μ *is it true that a), b) and c) imply* (8. 5)? This last inequality states that the function $\chi(x_0)$ of χ is not a member of the closed vector subspace of $L^2(G, \mu)$ generated by the functions $\chi(x)$ of χ with $x \in A$. The problem is thus displayed as one concerning approximation by trigonometric polynomials on $\widehat{G}$. In the original problem, this formulation arises more directly from the Parseval formula and the measures μ considered are all absolutely continuous with respect to the Haar measure on $\widehat{G}$: in fact, $d\mu(\chi) = |F(\chi)|^2 d\chi$ where $F(\chi) \in L^2\widehat{G}$ is $\mathcal{F}(f)$. Within these limits, I had conceived this formulation of the problem prior to Professor Mackey's communication.

If we denote by $I(\mu) = I(\mu, V)$ the infinum of

$$\int | 1 - \Sigma\alpha_i \cdot \chi(x_i)|^2 d\mu(\chi)$$

for all finite sums $\Sigma\alpha_i \cdot \chi(x_i)$ with the x_i non-$\in$ V, our problem is to determine those positive μ of finite total mass for which $I(\mu, V) > 0$ for all V. It is almost obvious that a necessary condition is that μ shall be not too small at infinity on $\widehat{G}$. For example, if $G = R$, and if

$$\int \exp(a|\chi|) d\mu(\chi) < +\infty$$

for some $a > 0$, then $I(\mu, V) = 0$ for *every* $V (\neq G)$: this is easily demonstrated by using the Hahn-Banach theorem to show that in this case we can choose the constants α_n and the points x_n non-$\in$ V such that

$$\sup_{\chi} \exp(-a|\chi|) . |1 - \Sigma\alpha_n \exp(2\pi i x_n \chi)|$$

is arbitrarily small. Stronger results of the same type follow from the theory of qasi-analytic classes.

It is interesting to observe that if μ and ν are two positive measures on G of finite total mass, and if $\lambda = \mu * \nu$, then for every V we have $I(\lambda, V) \geqslant \int d\mu(\chi) . I(\nu, V)$. For

$$\begin{aligned} &\int |1 - \sum \alpha_n . \chi(x_n)|^2 \, d\lambda(\chi) \\ &\quad = \int d\mu(\xi) \Big\{ \int |1 - \sum \alpha_n . \xi(x_n) . \eta(x_n)|^2 \, d\nu(\eta) \Big\} \\ &\quad \geqslant \int I(\nu, V) \, d\mu(\xi). \end{aligned}$$

In particular therefore, $\mu * \nu$ has the desired property whenever either μ or ν has the property and the other is not the zero measure. In other words, if φ and ψ are two continuous functions of positive type on G, *a*), *b*) and *c*) imply (8. 4) for the product function $\varphi . \psi$ whenever this implication is valid for one of φ or ψ and the other function is not identically zero.

A study of the variation of $I(\mu)$ as a function of μ would be of great interest. Although it is obvious that $I(\mu_n) \to I(\mu)$ whenever $\mu_n \to \mu$ weakly in the dual of the dual of the space of bounded continuous functions on $\widehat{G}$, it is important to know whether this assertion can be sharpened since it is on the basis of such results that one might hope to study the adherence under limiting processes of the property of a function on G having its translates independent.

Finally, I am indebted to Professor Mackey for the following suggested extension in the case of discrete groups. This extension consists of the passage from the case in which G is discrete and abelian to the case in which it is discrete and non-abelian. As we have seen, the problem of the independence of translates in L^2G with G abelian is entirely equivalent to a certain problem of trigonometric approximation in the Hilbert space $L^2(\widehat{G}, \mu)$, μ being a bounded, positive measure on $\widehat{G}$ which is absolutely continuous with respect to Haar measure. When G is non-abelian and discrete (so that $\widehat{G}$ is compact), a natural extension of this problem arises on replacing the characters of $\widehat{G}$ by the minimal, translation-invariant, finite dimensional vector subspaces $\mathcal{J}$ of $L^2\widehat{G}$; see Mackey (18). Consequently, the problem about compact abelian groups which is equivalent to our original problem for the space L^2G, G discrete and abelian, may be formulated thus : *Given a compact group K, for which bounded, absolutely continuous, positive measures* μ *on* K *is it true that the closed vector subspace of* $L^2(K, \mu)$ *generated by a set* $\mathfrak{S}$ *of* $\mathfrak{J}$*'s contains no* $\mathfrak{J}$*'s except those in* $\mathfrak{S}$? In discussing this, it would be natural to define a priori a notion of closure for the sets $\mathfrak{S}$ (corresponding to that following from the topology of G in the abelian case) and then to discuss when the closure of $\mathfrak{S}$ in this sense is enough to ensure the above property.

C. — **The degree of dependence of translates.** — If a function has translates which fail to be independent, it may yet be of interest to ascertain some measure of the extent of this failure. In other words, it may be of interest to measure the degree of dependence of the translates of the function in question. How this is to be done is fairly obvious for the metrisable groups. For any number $r > 0$, let S_r be the exterior of the sphere centre O and radius r; one might then define the degree D_f of dependence of the translates of f as the infinum of numbers $r > 0$ such that f non-$\in \mathfrak{J}(f, S_r)$. Naturally, there is the difficulty that a number of uniformly equivalent metrics may be available but, whichever one is used, the number D_f will retain its essential property of measuring the relative degree of dependence of the translates of f.

For simplicity, let us confine our attention to the group R and the space L^2R. It is easy to give examples of functions f for which D_f is 0, or positive and finite, or $+\infty$: the first case corresponds to that in which the translates of f are independent, and the last case to that in which $f \in \mathfrak{J}(f, S_r)$ for every $r > 0$. Thus, examples of functions f with $D_f = 0$ are discussed in Section 5 ; examples of functions f with $D_f = +\infty$ are easy to find. In fact, $D_f = +\infty$ whenever the Fourier transform $F = \mathcal{F}(f)$ is exponentially small at infinity. Finally, as a simple example of the intermediate case, consider the function

$$f(x) = \begin{cases} 1 \text{ for } |x| \leqslant L, \\ 0 \text{ elsewhere,} \end{cases}$$

L being any positive number : it is not difficult to show that $D_f = L$. For, to begin with, $f \in \mathfrak{J}(f, S_L)$ and so $D_f \geqslant L$. To see this, suppose that $\psi \in L^2R$ is orthogonal to $f(x - a)$ whenever $|a| \geqslant L$, so that

$$\int f(x - a)\, \psi(x)\, dx = 0 \qquad \text{if} \qquad |a| \geqslant L.$$

This means that

$$\int_{a-L}^{a+L} \psi(x)\,dx = 0 \qquad \text{if} \qquad |a| \geqslant L.$$

Putting

$$\theta(x) = \int_0^{\chi} \psi(t)\,dt,$$

we have

$$\theta(a+L) - \theta(a-L) = 0 \qquad \text{if} \qquad a \geqslant L \qquad \text{or if} \qquad a \leqslant -L.$$

Taking first the case $a \geqslant L$, we see that $\theta(x+2L) = \theta(x)$ if $x \geqslant 0$ and so, differentiating with respect to x, $\psi(x+2L) = \psi(x)$ $p.p.$ for $x \geqslant 0$. Since $\psi \in L^2$, this implies $\psi(x) = 0$ $p.p.$ for $x \geqslant 0$. Likewise, considering $a \leqslant -L$, we see that $\psi(x) = 0$ $p.p.$ for $x \leqslant 0$. So $\psi(x) = 0$ $p.p.$ so that ψ is orthogonal to f: this shows that $f \in \mathfrak{I}(f, S_L)$, as asserted. On the other hand, if $L' > L$, f non $- \in \mathfrak{I}(f, S_{L'})$, whence $D_f \leqslant L$. To see this, note that the transform of f is

$$F(\chi) = \sin 2\pi L\chi/\pi\chi\,;$$

the function

$$\Phi(\chi) = \sin 2\pi L\chi/\pi\chi \,.\, \sin 2\pi(L' - L)\chi/\pi\chi$$

is entire of exponential type $2\pi L'$ and belongs to L^2 over the real axis. By the Paley-Wiener theorem, $\Phi = \mathcal{F}(\varphi)$ where $\varphi \in L^2$ and is zero for $|\chi| \geqslant L'$; further, it is easy to see that φ is equivalent to a continuous function non-vanishing at the origin. Finally, $F' = \Phi/F \in L^2$ and so $F' = \mathcal{F}(f')$ with $f' \in L^2$ satisfying $f * f' = \varphi$ $p.p.$ Replacing φ by the continuous function to which it is equivalent, we see that $f * f'(x) = 0$ for $|x| \geqslant L'$ and $f * f'(0) \neq 0$: this proves our assertion.

The argument just completed may be generalised so as to yield a method of evaluating D_f which is based upon the behaviour of the Fourier transform F of f. We consider the class E_f of all functions $\Phi = \Phi(\chi)$ which satisfy the conditions

(a) $\Phi(\chi)$ is entire of exponential type and of class L^2 over the real axis;

(b) $$\Phi/F \in L^2R\,;$$

by (a) and (b), $\Phi = F.(\Phi/F) \in L^1R$, and we impose finally the condition

(c) $$\int \Phi(\chi)d\chi \neq 0.$$

Then we shall have

$$D_f = (1/2\pi).\inf(\text{type of } \Phi)$$

with Φ ranging over E_f. Of course, E_f may be void in wich case the usual convention yields $D_f = +\infty$. This evaluation of the number D_f is a consequence of the Paley-Wiener theorem.

It is clear that similar definitions and results could be formulated for the groups R^m and T^m. For these groups, the function-space under consideration could also be varied within limits. For more general groups, it seems unlikely that any reasonable *numerical* measure of the degree of dependence of translates can be formulated, though this may be done if the group satisfies the first countability axiom by using an « écart » defining the uniform structure of the group. At this level of generality there is at present no means of effecting such a numerical estimate in terms of the Fourier transform parallel to that based on the Paley-Wiener theorem and indicated above.

D. — **Relations with mean-periodicity.** — Among the examples we have treated there appear instances of some relationship between mean-periodicity in the sense of Schwartz (6) and the independence of translates, this relationship varying greatly from one case to the next. Thus, for the space L^2T we have seen that a function may be mean-periodic and simultaneously have its translates independent. On the other hand, for the space L^2R, we have seen that these two properties cannot be possessed simultaneously by any function. In this respect, the spaces L^2T^m and CT^m behave like L^2T, whilst L^2R^m and CR^m behave like L^2R. For example, in CR^m, a function cannot be mean-periodic and simultaneously have its translates independent: indeed, if $f \in CR^m$ is mean-periodic, there is a measure μ on R^m having a compact support such that

$$(8.\ 6) \qquad \mu \neq 0, \qquad f * \mu = 0;$$

consequently, if ν is a measure with a compact support and φ a continuous function with a compact support, and if

$$(8.\ 7) \qquad f * \nu = \varphi,$$

from (8. 6) and (8. 7) would follow $\mu * \varphi = 0$ and hence, taking transforms, $\Phi . M = 0$. Since $\mu \neq 0$, and since M is an entire function, M can vanish on a non-dense set at most, and so Φ, which is also entire, must be identically zero.

Both the grouping of the various cases and the arguments employed show fairly clearly that it is the group G itself (rather than the particular function-space over G which is under consideration) which determines the form of the relationship between mean-periodicity and independence of translates : in fact, it is the dual group $\widehat{G}$ which has direct influence. I think it would be of great interest to be able to make predictions on the basis of the structure of G. To do this would involve, I think, developing some notion of quasi-analyticity of functions on G and relating this notion to harmonic analysis, which programme would be desirable even for the study of independence of translates on its own account.

E. — **The class of functions** $\mathcal{E}'(f)$. — Theorem 1 tells us that if the translates of f are independent, then the functions of the class $\mathcal{E}'(f)$ separate the points of G. By analogy with the theorem of Weierstrass-Stone, this suggests the study of connections between the independence of translates of f on the one hand and density theorems concerning the class $\mathcal{E}'(f)$ on the other. Of course, the theorem of Weierstrass-Stone is not directly applicable to $\mathcal{E}'(f)$ since this class is not an algebra under pointwise multiplication of its members. Moreover, Theorem 4 shows that, even when $\mathcal{E} = L^2G$ and G is compact, a function f may have its translates independent without the class $\mathcal{E}'(f)$ being dense in the sense of compact convergence amongst all continuous functions on G. The converse assertion is also certainly false.

Nevertheless, if $\mathcal{E} = L^2G$, we can link the following propositions in certain cases :

(*a*) $F = \mathcal{F}(f)$ is p. p. non-zero on $\widehat{G}$.

(*b*) $\mathcal{E}'(f)$ is uniformly dense amongst all continuous functions which tend to zero at infinity on G.

(*c*) If μ is a bounded measure such that $\int_G f_a d\mu(a) = 0$, then $\mu = 0$.

(*d*) The translates of f are independent in L^2G.

In (*c*), $\int_G f_a d\mu(a)$ denotes the integral of the bounded, continuous vector-valued function $a \rightarrow f_a$ from G into L^2G ; this integration process commutes with all continuous, linear operations from L^2G into itself so that, in particular, the Fourier transform of the element of L^2G represented by this integral is none other than $F(\chi) . M(-\chi)$ where $F = \mathcal{F}(f)$ and $M = \mathcal{F}(\mu)$.

It is almost immediate that (*b*) and (*c*) are equivalent. In fact, by the last remark we have for any $f' \in \mathcal{E}'$

$$< \int_G f_a \, d\mu(a), f' > = \int_G \varphi(a) \, d\mu(a)$$

where $\varphi(a) = < f_a, f' >$ belongs to $\mathcal{E}'(f)$. Also, every member of $\mathcal{E}'(f)$ belongs to the Banach space C_0G formed of all continuous functions on G which tend to zero at infinity. Finally, the general continuous linear functional on C_0G has the form

$$\psi \rightarrow \int_G \psi(x) \, d\nu(x) \qquad (\text{all } \psi \in C_0G),$$

ν being a certain bounded Radon measure on G fixed by the functional in question. Piecing these facts together, and using the Hahn-Banach theorem, yields the postulated equivalence.

The implication $(a) \Rightarrow (c)$ follows at once from the closing remark of the last paragraph but one.

The relationship between (*a*) and (*d*) is more complicated and depends to begin with on G. For example, $(d) \Rightarrow (a)$ if G is discrete (Theorem 3) or if $G = R$ (remarks following Theorem 6), but the implication is not valid if $G = T$ (Theorem 4).

We know that $(a) \Rightarrow (b)$; as regards the converse, it is easy to show at any rate that if $F = 0$ on a non-void open set (modulo null sets), then (*c*), and hence also (*b*), is false. To see this, observe first of all that for any fixed character χ_0 the classes $\mathcal{E}'(f)$ and $\mathcal{E}'(\chi_0 . f)$ are together dense in C_0G or not. Hence, if F vanishes on a non-void open set, we may assume (by replacing f by a function of the form $\chi_0 . f$ if necessary) that $F = 0$ on a symetric neighbourhood $\widehat{U}$ of zero in $\widehat{G}$. We can then take a compact and symmetric neighbourhood $\widehat{V}$ of zero such that $\widehat{V} \oplus \widehat{V} \subset \widehat{U}$, let $R(\chi)$ be the characteristic function of $\widehat{V}$, and consider the function $M(\chi) = R * \tilde{R}(\chi)$. $M(\chi)$ is continuous, $\not\equiv 0$, of positive type, and vanishes outside $\widehat{U}$. Consequently, $M(\chi)$ is the Fourier transform a bounded, positive measure μ on G, and from $F(\chi) . M(-\chi) = 0$ p.p. follows

$$\int_G f_a d\mu(a) = 0.$$

Since $\mu \neq 0$, this shows that (*c*) is false.

LITERATURE CITED

[1] N. Wiener, The Fourier integral and certain of its applications, Cambridge, 1933.
[2] R. Godement, Théorèmes tauberiens et théorie spectrale, *Annales École Normale Supérieure*, 64 (1947).
[3] I. E. Segal, The group algebra of a locally compact group, *Trans. Amer. Math. Soc.*, vol. 61 (1947).
[4] H. Pollard, The closure of translations in L^p, *Proc. Amer. Math. Soc.*, vol. (1951).
[5] A. Beurling, Un théorème sur les fonctions bornées et uniformément continues sur l'axe réel, *Acta Mathematica,* vol 77 (1945).
[6] L. Schwartz, Théorie générale des fonctions moyenne-périodiques, *Annals of Maths.*, vol. 48 (1947).
[7] R. E. Edwards, A property of the class of functions regular in the unit circle and a theorem on translations, *Journal London Math. Soc.* vol. 25 (1950).
[8] R. E. Edwards, The translations of a function holomorphic in a half-plane and related problems in the real domain, *Proc. London Math. Soc.*, 3 (1) 1951.
[9] R. E. Edwards, The translations and affine transforms of special functions, to appear in the *Journal London Math. Soc.*
[10] R. E. Edwards, Derivative and translational bases for periodic functions, to appear in the *Proc. Amer. Math. Soc.*
[11] L. Schwartz, *Théorie des distributions,* Tomes I and II, Paris, 1950 and 1951.
[12] I. Gelfand, Normierte Ringe, *Recueil Math.* 9 (51), 1941.
[13] I. Gelfand, Über absolut konvergente trigonometrische Reihen und Integrale, *Ibidem.*
[14] S. Mandelbrojt, *Séries de Fourier et classes quasi-analytiques de fonctions,* Borel Collection, Paris, 1935.
[15] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain, Amer. Math. Soc. Coll. Pubs.* vol. XIX, 1934.
[16] G. Silov, On regular normed rings, *Travaux de l'Inst. Math. Stekloff,* 1947.
[17] R. Godement, Les fonctions de type positif et la théorie des groupes, *Trans. Amer. Math. Soc.* vol. 63 (1), 1948.
[18] G. W. Mackey, Functions on locally compact groups, *Bull. Amer. Math. Soc.* 56 (5), 1950.

Birbeck College University of London.

(Parvenu aux Annales le 23 octobre 1951).