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LOCAL AND GLOBAL DYNAMICS IN A NEOCLASSICAL GROWTH MODEL WITH NONCONCAVE PRODUCTION FUNCTION AND NONCONSTANT POPULATION GROWTH RATE*

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Abstract. In this paper we analyze the dynamics shown by the neoclassical one-sector growth model with differential savings as in Bohm and Kaas [*J. Econom. Dynam. Control*, 24 (2000), pp. 965–980] while assuming a sigmoidal production function as in [V. Capasso, R. Engbers, and D. La Torre, *Nonlinear Anal.*, 11 (2010), pp. 3858–3876] and the labor force dynamics described by the Beverton–Holt equation (see [R. J. H. Beverton and S. J. Holt, *Fishery Invest.*, 19 (1957), pp. 1–533]). We prove that complex features are exhibited, related both to the structure of the coexisting attractors (which can be periodic or chaotic) and to their basins (which can be simple or nonconnected). In particular we show that complexity emerges if the elasticity of substitution between production factors is low enough and shareholders save more than workers, confirming the results obtained with concave production functions. Anyway, in contrast to previous studies, the use of the S-shaped production function implies the existence of a poverty trap: by performing a global analysis we study the properties of the regions generating trajectories converging to it.

Key words. economic growth, nonconcave production function, nonconstant population growth rate, global dynamics, fluctuations and chaos

AMS subject classifications. 37C, 37G, 37N40

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1. Introduction.

Dynamic economic growth models have often considered the standard one-sector neoclassical model by Ramsey (1928) or the Solow–Swan model (see Solow (1956) and Swan (1956)). Both of these dynamic models show that the system monotonically converges to the steady state (i.e., the capital per capita equilibrium), so neither cycles nor complex dynamics can be observed (see also Dechert (1984)). However, while Ramsey's assumption on savings behavior corresponds to maximization of the infinite discounted sum of utility of a representative consumer who lives infinitely, in the Solow–Swan model constant average propensity to save is assumed.

In order to investigate the possibility of complex dynamics to be exhibited in optimal growth models, many authors (i.e., Kaldor (1956, 1957), Pasinetti (1962), and Samuelson and Modigliani (1966)) have studied the question of whether the different saving propensities of two groups (labor and capital) might influence the final dynamics of the system. The question of differential savings between groups of agents was originally posed within the Harrod–Domar model of fixed portion (Harrod (1939)). Stiglitz (1969) took Solow's model to another level by analyzing how different savers' wealth and income evolve. In his model each agent follows his or her private decision rule and the economy approaches a balanced growth solution. Obviously, different

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but constant saving propensities make the aggregate saving propensity nonconstant and dependent on income distribution so that multiple and unstable equilibria can occur. However, qualitative dynamics are still simple.

Bohm and Kaas (2000) investigated the discrete time neoclassical growth model with constant but different saving propensities between capital and labor (as proposed by Kaldor (1956)) using a generic production function satisfying the *weak Inada conditions*. The authors showed that instability and topological chaos can be generated in this kind of model.

Differently from their assumption, several papers consider other production functions. Brianzoni, Mammana, and Michetti (2007, 2008, 2009) investigated the neoclassical growth model in discrete time with differential savings and endogenous labor force growth rate while assuming constant elasticity of substitution (CES) production function. The authors proved that multiple equilibria are likely to emerge and that complex dynamics can be exhibited if the elasticity of substitution between production factors is sufficiently low. In fact, the elasticity of substitution between production factors plays a crucial role in the theory of economic growth since it represents one of the determinants of the economic growth level. Later, Tramontana, Gardini, and Agliari (2011) introduced the Leontief technology, representing the limit case as the elasticity of substitution tends to zero; they obtained a discontinuous one-dimensional piecewise linear model able to exhibit cycles of any period. As a further step in this field, Brianzoni, Mammana, and Michetti (2012b) first introduced the variable elasticity of substitution (VES) production function in the form given by Revankar (1971), while Cheban, Mammana, and Michetti (2013) extended the model to consider the case of nonconstant population growth rate. The authors proved that the model can exhibit unbounded endogenous growth (differently from CES) and that the production function elasticity of substitution is responsible for the creation and propagation of complicated dynamics, as in models with explicitly dynamic optimizing behavior by the private agents. Furthermore, Capasso, Engbers, and La Torre (2010) focused on a parametric class of nonconcave production functions which can be considered as an extension of the standard Cobb–Douglas production function; the authors study the Solow growth model in continuous time and show the existence of rich dynamics by mainly using numerical techniques. More recently, Brianzoni, Mammana, and Michetti (2012a) considered the nonconcave production function, as first formulated by Clark (1971), proving that, similarly to what happens with the CES and VES production functions, if shareholders save more than workers and the elasticity of substitution between production factors is low, then the model can exhibit complexity. Following this contribution, in the present paper we consider a sigmoidal production function in a discrete time setup.

Consider now that in Bohm and Kaas (2000) the labor force is assumed to grow at a constant rate $n \geq 0$. This last hypothesis is usually assumed in standard economic growth theory. However, one implication of a constant population growth rate is that population grows exponentially, which is clearly unrealistic. In fact a logical consideration is that there is a limit, called the carrying capacity of the environment. As described by Maynard Smith (1974), a more realistic economic growth model has the following properties: when population is small in proportion to environmental carrying capacity, then it grows at a positive constant rate; when population is larger in proportion to environmental carrying capacity, the resources become relatively more scarce, and as a result this must affect the population growth rate negatively. Since the logistic map in continuous time satisfies both properties, many authors consider it more realistic to describe the population growth using the logistic growth function rather than the exponential growth function. The first to propose modeling population growth by logistic equation is Schtickzelle and Verhulst (1981). Other authors made the following choices: Accinelli and Brida (2005) analyze the neoclassical Solow model with growth of population described by a generalized logistic equation (Richard's law), and Faria (2004) studies the Ramsey model with logistic growth. Since we consider the discrete time Solow model, we assume that population growth rate is described by the Beverton–Holt equation, since the Beverton–Holt model in discrete time is equivalent to the logistic model.

Summarizing, in the present paper we study the discrete time one-sector Solow–Swan growth model with differential savings as in Bohm and Kaas (2000), while assuming that the technology is described by a nonconcave production function in the form proposed by Capasso, Engbers, and La Torre (2010) and the population growth dynamics is formalized by the Beverton–Holt equation. Our main goal is to describe the qualitative and quantitative long run dynamics of the growth model to show that complex features can be observed and to compare the results obtained with those reached while considering the CES or the VES technology and the constant population growth rate.

On the basis of our assumptions, the resulting model is a two-dimensional autonomous dynamic system; we prove that multiple equilibria emerge, and we provide conditions on the parameters for their local stability. Furthermore, we show that our model can exhibit complexity related both to the structure of the attractors of the system (passing from locally stable fixed points to bounded fluctuations, or even to chaotic patterns), to the coexistence of attractors giving rise to multistability phenomena, and, finally, to the structure of the basins of attraction (from a simply connected to a nonconnected one).

The role of the production function elasticity of substitution has been related to the creation and propagation of complicated dynamics. In fact, similarly to what happens with the CES and VES production functions, if shareholders save more than workers and the elasticity of substitution between production factors is low, then fluctuations may arise. These results are important in the economic growth theory since they confirm the central role of the production function elasticity of substitution as in models with explicitly dynamic optimizing behavior by private agents (see Becker (2006) for a survey about these models). Moreover, differently from previous studies, the use of the S-shaped production function implies the existence of a poverty trap eliminating any possibility of economic growth. In the economics literature this fact is interesting since one should expect that there is a critical level of physical capital having the property that if the initial value of physical capital is lesser than such a level, then the dynamic of physical capital will descend to the zero level, thus eliminating any possibility of economic growth.

The paper is organized as follows. In section 2 we introduce the model. In section 3 we perform the dynamic analysis. Section 4 concludes the paper.

2. The economy. Let us consider a standard neoclassical one-sector growth model (see Kaldor (1956, 1957) and Pasinetti (1962)) where, as in Bohm and Kaas (2000), the two types of agents, workers and shareholders, have different but constant saving rates. The one-dimensional map describing the evolution of the capital per capita k_t is given by

$$k_{t+1} = \frac{1}{1+n}[(1-\delta)k_t + s_w(F(k_t) - k_tF'(k_t)) + s_rk_tF'(k_t)], \tag{1}$$

where $\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are the constant saving rates for workers and shareholders, respectively, while n is the constant population growth rate. Function $y = F(k)$ is the production function in intensive form.

The economic growth models are used to consider the hypothesis of a production function satisfying the following standard economic properties: $F(k) > 0$, $F'(k) > 0$, and $F''(k) < 0\ \forall k > 0$; observe that such properties hold for the Cobb–Douglas, CES, and VES production functions. In addition, both the VES and the Cobb–Douglas production functions verify one of the *Inada conditions*, that is, $\lim_{k\to 0} F'(k) = +\infty$.

According to the last condition, an economy with no physical capital can gain infinitely high returns by investing only a small amount of money. This obviously cannot be realistic since before getting returns it is necessary to create prerequisites by investing a certain amount of money. After establishing a basic structure for production, one might still get only small returns until reaching a threshold where returns increase greatly to the point where the law of diminishing returns takes effect. In the literature this fact is known as a *poverty trap*. In other words, one should expect that there is a critical level of physical capital (i.e., $\bar{k} > 0$) having the property that if the initial value of physical capital is less than such a level, then the dynamic of physical capital will descend to the zero level, thus eliminating any possibility of economic growth. Following this argument, concavity assumptions provide a good approximation of a high level of economic development, but they are not always applicable to less-developed countries. Thus it makes sense to assume that only an amount of money larger than some threshold will lead to returns.

Following Capasso, Engbers, and La Torre 2010, we consider a sigmoidal production function (that is, it shows an S-shaped behavior) given by

$$F(k) = \frac{\alpha k^p}{1+\beta k^p}, \tag{2}$$

where $\alpha > 0$, $\beta > 0$, and $p \geq 2$. Observe that

$$F'(k) = \frac{\alpha p k^{p-1}}{(1+\beta k^p)^2} \quad \text{and} \quad F''(k) = \frac{(p\alpha k^{p-2})[p(1-\beta k^p) - (1+\beta k^p)]}{(1+\beta k^p)^3}. \tag{3}$$

Hence function (2) is positive $\forall k > 0$, strictly increasing, and it is a convex-concave production function. In fact, $F'(k) > 0\ \forall k > 0$, while a $\bar{k} > 0$ exists such that $F''(k) >$ (resp., $<$) 0 if $0 < k < \bar{k}$ (resp., $k > \bar{k}$), with $\bar{k} = \left(\frac{p-1}{\beta(p+1)}\right)^{\frac{1}{p}}$ the inflection point of F. Furthermore, the production function (2) does not satisfy one of the Inada conditions since $\lim_{k\to 0} F'(k) = 0$.

The elasticity of substitution between production factors of function (2) is mathematically defined as

$$\sigma(k) = -\frac{F'(k)\cdot(F(k) - k\cdot F'(k))}{k\cdot F(k)\cdot F''(k)}; \tag{4}$$

see Sato and Hoffman (1968). Hence it depends on the level of the capital per capita k, as it is given by

$$\sigma(k) = 1 + \frac{\beta p k^p}{p(1-\beta k^p) - (1+\beta k^p)}, \tag{5}$$

so that also the sigmoidal production function belongs to the class of VES production functions. Observe the role played by the constant p: if p is great enough, then $\sigma(k)$ decreases with respect to p.

Concerning the second ingredient of the study herewith proposed, we consider the labor force growth rate as not being constant and described by a model for density-dependent population growth formalized by the Beverton–Holt equation (see Beverton and Holt (1957)),

$$n_{t+1} = \frac{rh}{h + (r-1)n_t} n_t, \tag{6}$$

where $h > 0$ is the carrying capacity (for example, resource availability) and $r > 1$ is the inherent growth rate (this rate being determined by life cycle and demographic properties such as birth rates, etc.). Equation (6) is a continuous function and is the equivalent in discrete time of the continuous time logistic model, which is frequently used as an application in population dynamics, as previously underlined. Such a dynamic map was extensively studied in Cushing and Henson (2001, 2002).

The final two-dimensional dynamic system $T : R^2_+ \to R^2_+$ describing the capital per capita (k) and the population growth rate (n) evolution is then given by

$$T := \begin{cases} n' = f(n) = \frac{rh}{h+(r-1)n} n, \\ \\ k' = g(n,k) = \frac{1}{1+n}\left[(1-\delta)k + \frac{\alpha k^p}{1+\beta k^p}\left(s_w + p\frac{s_r - s_w}{1+\beta k^p}\right)\right]. \end{cases} \tag{7}$$

We also assume $s_r > s_w$, i.e., $\Delta s = s_r - s_w > 0$, that is, shareholders save more than workers, for function g not being negative.

System T is a discrete time dynamical system described by the iteration of a triangular map of the plane, with g and f continuous and smooth functions for all $k \geq 0$ and $n \geq 0$.

3. Local and global dynamics. The equilibrium points of map T are all the solutions of the algebraic system $T(n,k) = (n,k)$, where T is given by (7). The first equation says that the fixed points belong to the lines $n = 0$ and $n = h$. From the second equation we have that the corresponding k-values are the fixed points of the one-dimensional maps $g_0(k) := g(0,k)$ and $g_h(k) := g(h,k)$. About the number of steady states of such one-dimensional maps, we consider the one-dimensional map $g_n(k) := g(n,k)$ for any given n constant value, and we recall the following result proved in Brianzoni, Mammana, and Michetti (2012a), which applies generically to systems of the same form of T (see, for instance, Mammana and Michetti (2004)).

PROPOSITION 1. *Let*

$$G(k) := \frac{k^{p-1}}{1+\beta k^p}\left(s_w + p\frac{s_r - s_w}{1+\beta k^p}\right), \quad k > 0. \tag{8}$$

Then a $\tilde{k} > 0$ does exist such that

(i) *if $\frac{n+\delta}{\alpha} > G(\tilde{k})$, $g_n(k)$ has a unique fixed point given by $k = 0$;*

(ii) *if $\frac{n+\delta}{\alpha} = G(\tilde{k})$, $g_n(k)$ has two fixed points given by $k = 0$ and $k = k^* > 0$;*

(iii) *if $\frac{n+\delta}{\alpha} < G(\tilde{k})$, $g_n(k)$ has three fixed points given by $k = 0$, $k = k_1$, and $k = k_2$, $0 < k_1 < k_2$.*

According to the previous proposition it follows that map $g_n(k)$ always admits the equilibrium $k = 0$; moreover, up to two additional (positive) fixed points can exist according to the parameter values, and hence multiple equilibria are exhibited. Since

$$\tilde{k} = \left(\frac{\bar{M}}{\beta}\right)^{\frac{1}{p}}, \tag{9}$$

where $\bar{M} = \frac{-b-\sqrt{b^2-4ac}}{2a} > 0$, with $a = -s_w < 0$, $b = s_w(p-2) - \Delta s(p^2+p)$ and $c = s_w(p-1) + \Delta s(p^2-p) > 0$, then $G(\tilde{k})$ does not depend on parameters n, δ, and α.

The above-mentioned arguments prove the following proposition, which states the number of fixed points of the two-dimensional system T.

Proposition 2. *Let $G(k)$ be given by (8) and $\tilde{k} > 0$ be given by (9).*

(i) *If $G(\tilde{k}) < \frac{\delta}{\alpha}$, then T admits two fixed points $E_{00} = (0,0)$ and $E_{h0} = (h,0)$;*

(ii) *if $\frac{\delta}{\alpha} < G(\tilde{k}) < \frac{h+\delta}{\alpha}$, then T admits four fixed points $E_{00} = (0,0)$, $E_{0k_A} = (0,k_A)$, $E_{0k_B} = (0,k_B)$, and $E_{h0} = (h,0)$, $0 < k_A < k_B$;*

(iii) *if $G(\tilde{k}) > \frac{h+\delta}{\alpha}$, then T admits six fixed points $E_{00} = (0,0)$, $E_{0k_A} = (0,k_A)$, $E_{0k_B} = (0,k_B)$ and $E_{h0} = (h,0)$, $E_{hk_1} = (h,k_1)$, $E_{hk_2} = (h,k_2)$, $0 < k_A < k_1 < k_2 < k_B$.*

At $G(\tilde{k}) = \frac{\delta}{\alpha}$ and $G(\tilde{k}) = \frac{h+\delta}{\alpha}$ a tangent bifurcation occurs.

System T is such that $T(0,k) = (0,k')$ and $T(h,k) = (h,k')$ $\forall k \geq 0$; furthermore $T(n,0) = (n',0)$ $\forall n \geq 0$. These properties prove the following proposition (hence, from a mathematical point of view, the following result applies to systems with the same properties).

Proposition 3. *The following sets are invariant for system $T : R_+^2 \to R_+^2$: $N_0 = \{(n,k) \in R_+^2 : n = 0\}$, $N_h = \{(n,k) \in R_+^2 : n = h\}$, $K_0 = \{(n,k) \in R_+^2 : k = 0\}$.*

In order to study the local stability of the fixed points owned by T, we consider the Jacobian matrix (denoting the matrix of first partial derivatives), given by

$$DT(n,k) = \begin{pmatrix} \frac{\partial f}{\partial n}(n,k) & 0 \\ \frac{\partial g}{\partial n}(n,k) & \frac{\partial g}{\partial k}(n,k) \end{pmatrix}. \tag{10}$$

Let $(n^\star, k^\star)$ be a fixed point of T; then the eigenvalues of $DT(n^\star, k^\star)$ are real and given by

$$\lambda_1(n^\star, k^\star) = \frac{\partial f}{\partial n}(n^\star, k^\star) = \frac{hh^2}{(h+(r-1)h)^2},$$

$$\lambda_2(n^\star, k^\star) = \frac{\partial g}{\partial k}(n^\star, k^\star) = \frac{1}{1+n}\left[1 - \delta + \alpha(G(k) + KG'(k))\right],$$

respectively.

Notice that if $n^\star = h$, then $|\lambda_1| < 1$, while if $n^\star = 0$, then $\lambda_1 > 1$. This fact implies that for initial conditions (n_0, k_0) with $n_0 > 0$ the asymptotic dynamics of the two-dimensional system T can be investigated along the invariant line $n = h$. More precisely, once the invariant sets defined in Proposition 3 are considered, it is straightforward to observe that N_0 is a repellor while N_h is an attracting set. With regard to the set K_0, the following proposition holds.

Proposition 4. *Set K_0 attracts all trajectories starting from initial conditions (n_0, k_0) having k_0 sufficiently small. If $G(\tilde{k}) < \frac{\delta}{\alpha}$, then set K_0 attracts all trajectories.*

Proof. The second eigenvalue of T restricted to set K_0 is $\lambda_2(n,0) = g_n(0)$. First notice that function $g_n(k)$ may be written in terms of function $G(k)$ as

$$g_n(k) = \frac{1}{1+n}[(1-\delta)k + \alpha k G(k)].$$

Therefore $g_n(k) = \frac{1}{1+n}[1-\delta+\alpha(G(k)+kG'(k))]$. Since $\lim_{k\to 0^+} G(k) = 0$ and $\lim_{k\to 0^+} kG'(k) = 0$, then $g_n(0) = \frac{1-\delta}{1+n} \in (0,1)\ \forall n \geq 0$. Hence set K_0 is locally attracting. If $G(\tilde{k}) < \frac{\delta}{\alpha}$, T has two fixed points both located on K_0, and hence set K_0 is globally attracting. □

According to Proposition 4, trajectories starting close to the invariant set K_0 are mapped into K_0 after a finite number of iterations; such a result holds for all initial conditions in R^2_+ if T admits only two fixed points (both located on K_0). This evidence is due to the fact that, as in Capasso, Engbers, and La Torre (2010), the use of the S-shaped production function implies the existence of a poverty trap. Recall that in the models previously proposed in which the production function is concave, set K_0 is always a locally unstable set, hence the economy will converge in the long run to positive growth rates (eventually with periodic or even aperiodic dynamic features). Differently, in this new setup, set K_0 is locally stable; hence economies starting from a sufficiently low level of capital per capita may be captured by the poverty trap. The restriction of system T to the set K_0 generates trajectories which converge monotonically to $n^\star = h$ for every initial condition $n_0 > 0$.

Observe also that since $G(\tilde{k})$ does not depend on α and that $\alpha > 0$, we can conclude that $\exists\bar{\alpha}$ such that K_0 attracts all trajectories $\forall\alpha \in (0,\bar{\alpha})$ (this means that the production function upper bound is small enough). In this case the poverty trap cannot be avoided and the system will converge to a zero growth rate.

In order to draw a conclusion about the local stability of fixed points of T, recall the following result about the local stability of the fixed points of the one-dimensional map $g_n(k)$ proved in Brianzoni, Mammana, and Michetti (2012a).

PROPOSITION 5. *The equilibrium $k = 0$ is locally stable for $g_n(k)$; if $g_n(k)$ admits three fixed points, $0 < k_1 < k_2$, then k_1 is locally unstable while $g_n'(k_2) < 1$.*

Taking into account the dynamics of the Beverton–Holt equation (see Cushing and Henson (2001, 2002)) and Proposition 5, it follows that the equilibrium E_{00} is a saddle point, while E_{h0} is a stable node; similarly E_{0k_A} is an unstable node, while E_{hk_1} is a saddle point.

Obviously the fixed point E_{0k_B} can be a saddle point or an unstable node, while E_{hk_2} can be a node or a saddle point depending on the eigenvalues $\lambda_2(E_{0k_B}) = {g'}_0(k_B)$ and $\lambda_2(E_{hk_2}) = {g'}_h(k_2)$ that can be discussed while considering the one-dimensional map $g_n(k)$. Following Brianzoni, Mammana, and Michetti (2012a), $g_n(k)$ can be strictly increasing or bimodal in k $\forall n$ values, and sufficient conditions can be given. We recall this result.

PROPOSITION 6. *Define $A = \frac{p}{\beta^{\frac{p-1}{p}}}(s_w - p\Delta s) < 0$, $B = \frac{p}{\beta^{\frac{p-1}{p}}}(s_w + p\Delta s) > 0$, and*

$$M_m = \frac{-[(2p-1)A-(2p+1)B] + \sqrt{[(2p-1)A-(2p+1)B]^2 + 4(p^2-1)AB}}{2(-1-p)A} > 1.$$

(i) *If $s_w - p\Delta s \geq 0$, then map $g_n(k)$ is strictly increasing.*

(ii) *Let $s_w - p\Delta s < 0$ and define $H(M) = \frac{AM^2+BM}{M^{1/p}(1+M)^3}$.*

  (a) *If $H(M_m) \geq \frac{\delta-1}{\alpha}$, then $g_n(k)$ is strictly increasing.*

  (b) *If $H(M_m) < \frac{\delta-1}{\alpha}$, then $g_n(k)$ admits a maximum point k_M and a minimum point k_m such that $1 < k_M < k_m$.*

From Proposition 6 a sufficient condition for $\lambda_2(E_{0k_B}) \in (0,1)$ or $\lambda_2(E_{hk_2}) \in (0,1)$ can be obtained. In fact, if $g_n(k)$ is strictly increasing, then, taking into account Proposition 5, it must hold that ${g'}_0(k_B) \in (0,1)$ and ${g'}_h(k_2) \in (0,1)$, and consequently E_{0k_B} is a saddle point, while E_{hk_2} is a stable node.

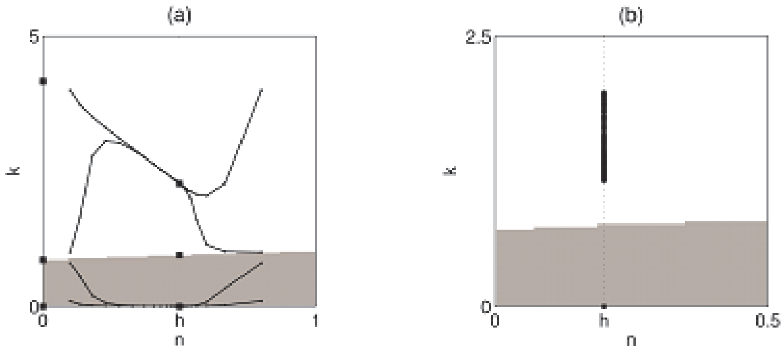

FIG. 1. (a) *Fixed points of T and sets B_1 (gray region) and B_2 (white region) if α is sufficiently high (i.e., six fixed points are owned) and Δs is low enough (i.e., $g_n(k)$ is strictly increasing). Some trajectories are depicted. Parameter values: $\delta = 0.6$, $\alpha = 10$, $\beta = 0.8$, $s_w = 0.2$, $s_r = 0.21$, $p = 10$, $r = 1.5$, and $h = 0.5$.* (b) *Attractors of system (T, D) and their own basins: if p is great enough (i.e., $g_n(k)$ is bimodal). The attractor Λ_2 belonging to the invariant line $n = h$ is strange. Parameter values: $\delta = 0.2$, $\alpha = 1$, $\beta = 0.9$, $s_w = 0.1$, $s_r = 0.9$, $p = 12$, $r = 1.1$, and $h = 0.2$.*

Observe that if Δs is low enough, then condition (i) of Proposition 6 holds (a Δs does exist such that $g_n(k)$ is strictly increasing $\forall \Delta s < \Delta s$). In this case, the dynamics of T is quite simple: both the structure of the attractors (fixed points) and that of their basins (connected sets) are simple. More precisely, since the fixed points E_{hk_2} and E_{h0} are both locally stable, the economic system converges to a steady state characterized by a zero (poverty trap) or a positive capital per capita growth rate, while the population growth rate converges to h for all initial conditions $n_0 > 0$. In such a case two different sets exist, namely, $B_1 \subset R_+^2$ and $B_2 \subset R_+^2$ (such that $B_1 \cap B_2 = \emptyset$), as well as a curve C which separates such sets, passing through the saddle points E_{0k_A} and E_{hk_1}, such that trajectories starting from set B_1 will approach E_{h0}, while trajectories starting from set B_2 will approach E_{k_2} (see Figure 1(a)). Consequently, featuring the economic system could be ambiguous with respect to initial condition close to C and perturbations on it. To summarize, no cycles or complex features are observed if the difference between the two propensities to save is low enough, confirming the results proved in previous works in which concave production functions were taken into account.

Consider now case (ii) of Proposition 6. Condition $p > s_w/\Delta s$ is necessary for $g_n(k)$ being bimodal. Furthermore, $\lim_{p\to+\infty} H(M_m) = +\infty$ so that a $p_1 > 0$ does exist such that $g_n(k)$ is bimodal $\forall p > p_1$. Let $\bar{p} = \max\{2, s_w/\Delta s, p_1\}$. The previous arguments prove the following proposition.

PROPOSITION 7. *A $\bar{p} > 0$ does exists such that $g_n(k)$ is bimodal $\forall p > \bar{p}$.*

In order to assess the possibility of complex dynamics arising, we focus on the case in which Proposition 7 holds, as it states a sufficient condition for system T restricted to the invariant line N_h (i.e., the one-dimensional map $g_h(k)$) being bimodal (having two critical points k_M and k_m). In fact, if $g_h(k)$ is bimodal, then system T may produce complex dynamics occurring on the invariant set N_h.

As previously proved, set N_0 is repelling; hence, in what follows, we focus on the dynamics of T restricted to the set $D = (0, +\infty) \times [0, +\infty)$, i.e., the system (T, D). Notice that set D is positively invariant; in fact, for any initial condition $(n_0, k_0) \in D$, all the images $T^t(n_0, k_0)$ of any rank t belong to the set D. System (T, D) always admits a locally stable fixed point, given by E_{0h}, belonging to the attracting line K_0,

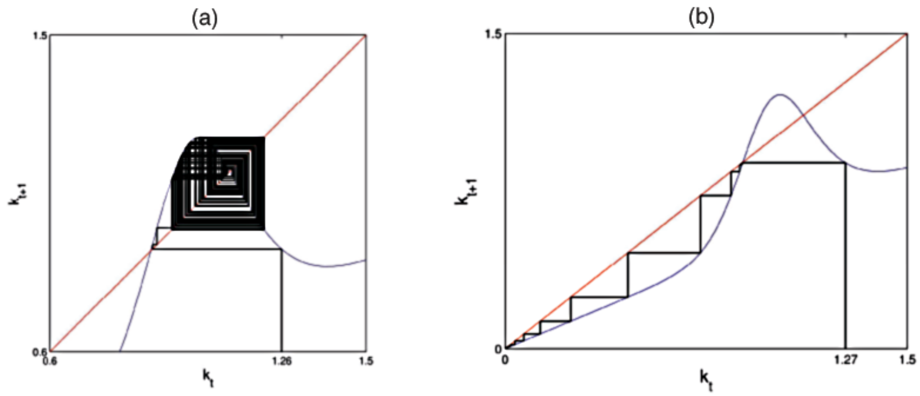

FIG. 2. *K-L staircase diagram for map $g_h(k)$ and the following parameter values: $\delta = 0.2$, $\alpha = 0.4$, $\beta = 0.9$, $h = 0.5$, $s_w = 0.1$, $s_r = 0.9$, and $p = 11$.* (a) $k_0 = 1.26$ *generates a trajectory converging to a complex set;* (b) $k_0 = 1.27$ *generates a trajectory converging to the poverty trap.*

for all parameter values. Furthermore, this attractor may coexist with another one belonging to $N_h - \{(0, h)\}$. Since in Brianzoni, Mammana, and Michetti (2012a) the map $g_n(k)$ was completely studied $\forall n > 0$, all the results proved in that work can be used to describe the dynamics of (T, D) for a given $n = h$. As a consequence, the features of (T, D) are completely known for all initial conditions on the invariant set N_h. We briefly recall these properties.

A trajectory starting from (h, k_0), $k_0 \geq 0$, may converge to a steady state (for instance, the poverty trap) or to a more complex attractor, which may be periodic (an m-period cycle) or chaotic. Anyway multistability may occur due to the coexistence of two attractors, and consequently their basins of attraction have to be studied (as they may have a complex structure). As an example, in Figure 2 we present two trajectories converging to different attractors for close initial conditions taken on the line N_h: in panel (a) the initial condition $(h, 1.26)$ produces a trajectory converging to a very high period cycle or to a chaotic set; in panel (b) the initial condition is $(h, 1.27)$ and the long term dynamics converges to the origin. It is very important to underline that, even if the invariant line N_h is globally attracting for (T, D), the dynamical study of the one-dimensional restriction of system T to this line cannot completely describe the dynamics of the two-dimensional system (T, D) when coexisting attractors are present. In fact, a trajectory starting from an initial condition $(n_0, k_0) \in D$ will converge to an attractor belonging to the line N_h. Anyway, in the case of coexistence of attractors, which is typical for bimodal maps, one has to determine to which of the two coexisting attractors the system will converge, depending on the initial condition. This kind of study requires an analysis of the global dynamical properties of the two-dimensional system, that is, an analysis which is not based on the linear approximation of the map (see, among others, Bischi, Gardini, and Kopel (2000) and Sushko, Agliari, and Gardini (2005)).

Let $\Lambda_1 = E_{00}$ and $\Lambda_2 \subset N_h - \{E_{00}\}$ be the two coexisting attractors of (T, D). Then system (T, D) always admits trajectories converging to Λ_1 so that we define $B_1 \subset D$ as the set of points generating trajectories converging to Λ_1. Furthermore, for certain parameter values, (T, D) admits an attractor Λ_2, and let B_2 be the basin of attraction of Λ_2 (i.e., the set of initial conditions generating trajectories converging to Λ_2). Then $B_2 = \text{Int}(D/B_1)$, where $\text{Int}(M)$ denotes the interior points of set M.

In Figure 1(a) the attractors of T (consisting of fixed points) are presented and their own basins B_1 and B_2 are, respectively, depicted in gray and white. A different case is presented in Figure 1(b), as the attractor Λ_2 is a complex set. In both cases the basins of attraction have a simple structure, as they consist of connected sets.

In order to discuss the global bifurcations that are responsible for a change in the structure of such basins, we have to analyze the properties of critical curves of system (T, D) (see also Bischi and Gardini (2000), Bischi, Mammana, and Gardini (2000), and Dieci, Bischi, and Gardini (2003)).

More formally, the two-dimensional map $(n', k') = T(n, k)$ is noninvertible since the rank-1 preimages $(n, k) = T^{-1}(n', k')$ may not exist or may be more than one. In this case the plane can be subdivided into regions Z_j, $j \geq 0$, whose points have j distinct rank-1 preimages. Generally, as the point (n', k') varies, pairs of preimages appear or disappear as it crosses the boundaries separating the different regions; hence such boundaries are characterized by the presence of at least two coincident (merging) preimages. Following the notation of Mira et al. (1996) and Abraham, Gardini, and Mira (1997), the critical curve of rank-1, denoted by LC, is defined as the locus of points having two or more coincident rank-1 preimages, located on a set denoted by LC_{-1} called the curve of merging preimages. LC is the two-dimensional generalization of the notion of critical value of a one-dimensional map. Arcs of LC separate the plane into regions characterized by a different number of real preimages.

For the two-dimensional map (T, D), which is an endomorphism, with f and g continuously differentiable, the locus LC_{-1} is given by the set of points such that

$$|DT(n,k)| = \frac{rh^2}{(h+(r-1)n)^2}\left(\frac{1}{1+n}[1-\delta+\alpha(G(k)+kG'(k))]\right) = 0,$$

where $DT(n, k)$ is the Jacobian matrix of the map T given in (10). The following proposition trivially holds.

Proposition 8. *The locus LC_{-1} of the phase plane is made up of two curves representing the set of points (n, k) belonging to the horizontal lines $k = k_m$ and $k = k_M$, where k_m and k_M are the minimum and the maximum points of $g_n(k)$, or $LC_{-1} = \{\emptyset\}$.*

In fact the locus

$$\frac{\partial g}{\partial k}(n,k) = 0 \tag{11}$$

is given by the set of points (n, k) such that $n > 0$ and k solves (11). Recall that function g can be bimodal or strictly monotonic in k. In the first case LC_{-1} is composed of two horizontal lines (LC^a_{-1} and LC^b_{-1}) of equations $k = k_m$ and $k = k_M$, respectively. In the second case $LC_{-1} = \{\emptyset\}$—the map is invertible.

In the following we assume $p > \bar{p}$, so that map $g_n(k)$ is bimodal in k; the set of points for which the determinant of the Jacobian matrix vanishes is presented in Figure 3(a). With LC being the rank-1 image of LC_{-1}, i.e., $LC = T(LC_{-1})$, the following proposition can be easily proved.

Proposition 9. *The rank-1 image LC is the union of two branches, $LC^a = T(LC^a_{-1})$ and $LC^b = T(LC^b_{-1})$.*

The two branches of the rank-1 image LC are shown in Figure 3(b). Since $g(n, k)$ is bimodal in k, the one-dimensional map $= g_n(k)$ is of the kind $Z_1 - Z_3 - Z_1$, where the three different sets are separated by $g(n, k_m) = LC^a$ and $g(n, k_M) = LC^b$. In fact points k with $k < LC^a$ or $k > LC^b$ have a unique preimage; points satisfying

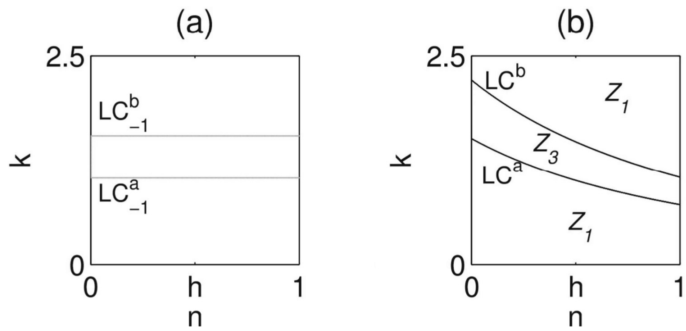

FIG. 3. (a) *Critical curves of rank*-0, LC_{-1} *for system* (T, D) *and the following parameter values:* $\delta = 0.2$, $\alpha = 1$, $\beta = 0.9$, $s_w = 0.1$, $s_r = 0.7$, $p = 8$, $r = 1.1$, *and* $h = 0.5$. (b) *Critical curves of rank*-1, $LC = T(LC_{-1})$, *for the same parameter values as in panel* (a). *These curves separate the plane into regions* Z_1 *and* Z_3, *whose points have different numbers of pre-images.*

$LC^a < k < LC^b$ have three distinct preimages; and each of the points $k = LC^a$ and $k = LC^b$ has two preimages which merge in a critical point together with a second distinct preimage called the *extra-preimage*. As a consequence, system (T, D) is noninvertible and of $(Z_1 - Z_3 - Z_1)$-type. In other words the plane is divided into several unbounded open regions: two regions Z_1 whose points have one preimage, and a region Z_3 whose points generate three real rank-1 preimages.

In order to describe the global dynamics of (T, D), we will perform a mainly numerical analysis focusing on the set of initial conditions $D' = (0, \bar{n}] \times [0, \bar{k}]$; that is, we consider initial conditions starting from initial population growth rates and initial capital levels less than a fixed value. Using numerical simulations it can be shown that the curve LC^a is strictly decreasing in $(0, \bar{n}]$ and that it moves downward as parameter p increases so that, given the other parameter values, a threshold value $\tilde{p}$ does exist such that a contact bifurcation (i.e., a contact between a critical curve and the basin boundary) occurs (regarding this kind of bifurcation, see, among others, Abraham, Gardini, and Mira (1997)). At this parameter value LC^a collides with the basin boundary (see Figure 4(a)) and a global bifurcation occurs causing the transformation of B_1 from connected to nonconnected; i.e., it is given by an infinite sequence of nonconnected regions (or holes) inside B_2. This bifurcation is due to the fact that a portion of the basin B_1 enters in a region characterized by a higher number of preimages (and hence the preimages of any rank of such a portion also belong to B_1; see Bischi, Gardini, and Mira (2011)).

Obviously a subset $B_0 \subset B_1$, with $\Lambda_1 \subset B_0$, exists such that trajectories starting from B_0 converge to Λ_1 (immediate basin); hence if the economy starts from a *low* level of economic growth, it will fall in the poverty trap. On the other hand, after the contact bifurcation, B_0 admits new preimages given by $B_{-1} = \{(n, k) : T(n, k) = B_0\}$, and consequently initial conditions belonging to B_{-1} also generate trajectories converging to Λ_1, as B_{-1} is mapped into set B_0 after one iteration. The previous procedure can be repeated while considering the preimages of rank-2 of the set B_0, namely, B_{-2}. Again initial conditions belonging to the set B_{-2} generate trajectories converging to B_0 after two iterations. The story repeats and a set of nonconnected portions is created, so that the contact between the critical set and the basin boundary marks the transition from simple connected to nonconnected basins. Finally the basin

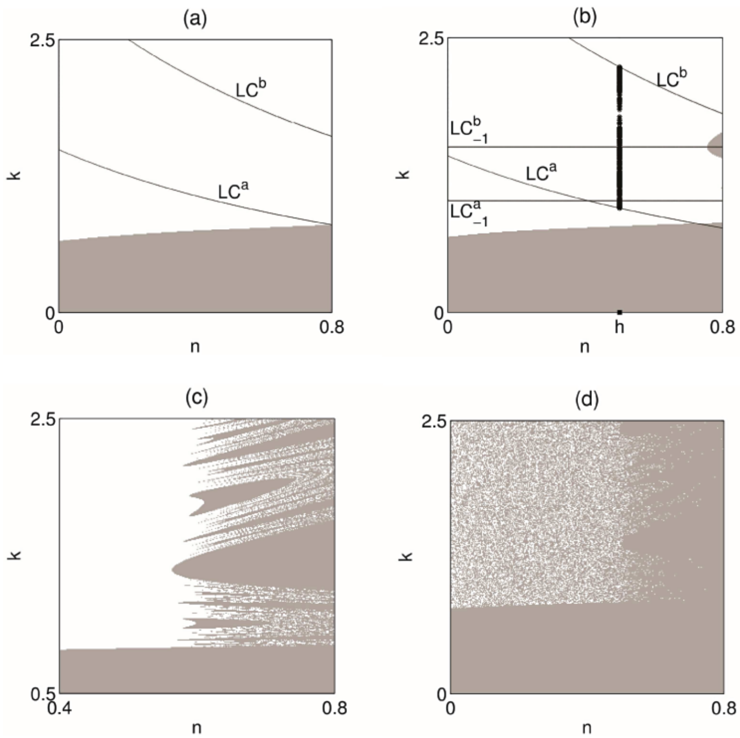

FIG. 4. (a) *Immediately before the contact bifurcation occurring at* $p = 9.51$*, the collision between the critical curve and the basin boundary is shown.* (b) *Basins of attraction for* $p = 11.2$ *of* Λ_1 *(the white region) and* Λ_2 *(the gray region) after the contact bifurcation: a gray hole is depicted. The two attractors are also presented.* (c) *Basins of attraction for* $p = 16$. (d) *Basins fractalization for* $p = 18.35$.

of attraction of the poverty trap is given by

$$B_1 = B_0 \cup_{i\geq 1} B_{-i}.$$

In such a case an economic policy trying to push up the investment does not guarantee an escape from the poverty trap. By using numerical computations and fixing $\delta = 0.2$, $\alpha = 1$, $\beta = 0.9$, $s_w = 0.1$, $s_r = 0.9$, $r = 1.1$, and $h = 0.5$, the bifurcation value $\tilde{p} = 9.51$ is obtained, and in Figure 4(b) the situation occurring immediately after this global bifurcation is shown (a hole has appeared, shown by the gray region inside the white region).

Observe that the attractor Λ_2 does not disappear after this bifurcation. This depends on the fact that the portion of curve LC involved in the contact bifurcation does not belong to the boundary of the absorbing area containing the attractor Λ_2. The basin structure increases in complexity if p further increases, as shown in Figure 4(c), and the gray area increases too (i.e., the set of initial conditions generating sequences converging to the poverty trap).

As parameter p is further increased, the basin boundary has a contact with Λ_2 and a final bifurcation occurs: the attractor Λ_2 disappears and almost all trajectories converge to the poverty trap. In Figure 4(d) the situation before the final bifurcation occurring at $p = 20.51$ is represented, and the distribution of white and gray points appears quite complicated.

Bifurcations concerning both the structure of the attractors Λ_1 and Λ_2 and the structure of their basins are strictly related to the values of the key parameter p, which gives information about the elasticity of substitution between production factors (which decreases as p increases). The analysis of the dynamics we proposed shows that the elasticity of substitution in the nonconcave production function affects the final long run dynamics of the growth model, i.e., it increases in complexity, when shareholders save more than workers.

4. Conclusions. In this paper we investigated the dynamics of the Solow growth model with differential saving and Beverton–Holt population growth rate in the case of nonconcave production function. Fixed points and other invariant sets of T were determined, and the local stability analysis was conducted. About the global properties of the system, we first proved that the model admits two coexisting attractors and then described their structure as parameters of the system vary showing that complex features emerge as p is increased (so that the elasticity of substitution between production factors is low enough). This evidence confirms the results obtained in previous works with concave production functions.

Anyway, in contrast to other studied cases, with a nonconcave production function the invariant set characterized by zero capital per capita is an attracting set for all parameter values so that the system may converge to the poverty trap.

As the structure of the basins of attraction may also be very complicated, the economy may converge to a zero growth rate also starting from a situation with high initial level of capital. Furthermore, since the model may admit coexisting attractors (consisting of fixed points, cycles, or more complex sets), a final bifurcation occurs at which complicated dynamics is ruled out for very low values of the elasticity of substitution.

A further interesting question is what happens when population dynamics with an economic feedback is introduced, i.e., where f is a function of both n and k, but we leave such an approach to future research.

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