Introduction to the theorico-practical classes of Strength of Materials 1

Bachelor in Civil Engineering

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9 December 2022

U.PORTO
FEUP FACULDADE DE ENGENHARIA
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The syllabus for the Civil Engineering Bachelor degree at FEUP includes 2.5 hours of theoretical lectures and 3 hours of theorico-practical lessons, per week, of Strength of Materials 1.

The working method recommended for the theorico-practical classes is centred on the resolution, by the students, of practical problems grouped in Worksheets. This document is used for a quick contextualization of the subjects covered in each theorico-practical lesson, so that the student can read the pages corresponding to each topic before the respective lesson. It does not replace in any way the systematic presentation of each topic and the underlying theory, made in the theoretical lecture classes.

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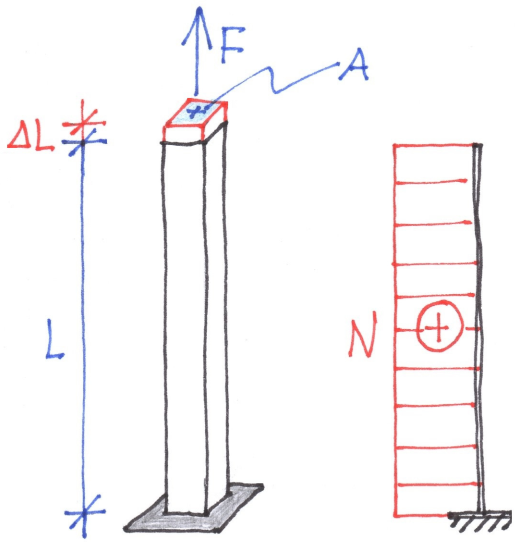
1. Axial force and normal stress

1.1. Stresses and deformations due to the axial force



The figure shows a bar subject to an axial force N.
The **normal stress** applied to the **bar's cross-section** is:

$$\sigma = \frac{N}{A}$$

where A is the cross-sectional area, and the stress direction is parallel to the bar axis (longitudinal direction). The **strain** ε in that same longitudinal direction is given by the **Hooke's law**, if the material has a **linear elastic behaviour** characterized by the **modulus of elasticity** E:

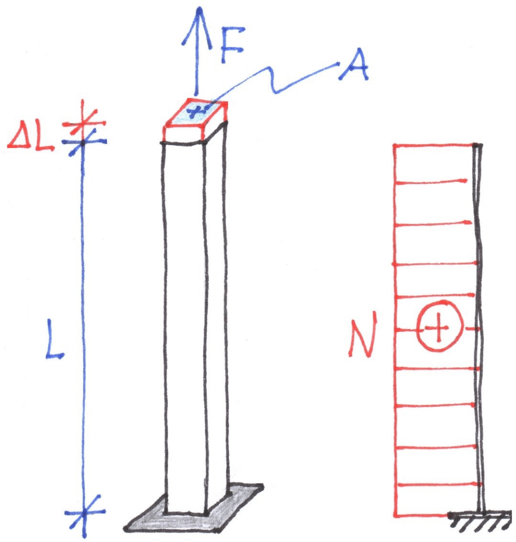
$$\sigma = E \cdot \varepsilon$$

The strain ε is the **relative length variation**:

$$\varepsilon = \frac{\Delta L}{L}$$

where L is the initial bar length and ΔL is the variation of length (**elongation**). By combining the three equations before, we conclude that the bar elongation can de directly calculated as:

$$\Delta L = \frac{N \cdot L}{E \cdot A}$$



The equations should be used with a coherent **system of units**, for example:

- N in kN
- A in m^2
- σ in kPa
- L and ΔL in m
- E in kPa
- ε is adimensional

In a correctly designed structure, the applied stress should not exceed the material resistance. To take into account the **uncertainties** associated to the quantification of the applied stresses and the material resistance, the applied stress is increased by a **partial safety coefficient** γ_S. In RM1, for simplification, we use γ_S = 1.5. The s**afety condition** is written as:

$$\sigma_{Sd} \leq \sigma_{Rd}$$

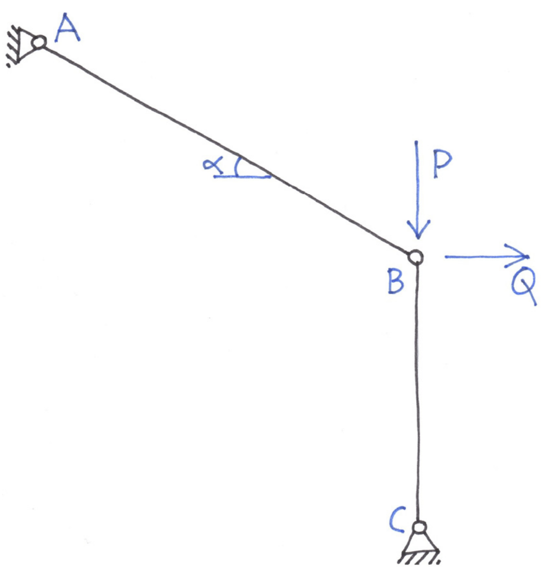
This is the so called **Ultimate Limit State** (ULS) condition, where σ_{Sd} is the ***design value*** of the **applied stress**:

$$\sigma_{Sd} = \gamma_S \cdot |\sigma_{\max}|$$

and σ_{Rd} is the *design value* of the **material resistance**. Additional explanations about the approaches used to check the safety can be found in Annex A.

1.2. Nodal displacement in plane structures with bars subject to axial force only

Example 1.1:



In plane structures, composed by a very small number of bars, the **nodal displacements** can be determined through a simple geometrical analysis.

The figure shows an example of an **articulated structure** in which only one of the nodes can move (node B). To calculate its displacement, we start by determining the axial forces in the bars, through the method of the **equilibrium of nodes**, for node B.

Then, the elongation of each bar, due to its axial force, is determined as:

$$\Delta L = \frac{N \cdot L}{E \cdot A}$$

Note that, if a bar is also subject to a **temperature variation** ΔT (ºC), its total elongation becomes:

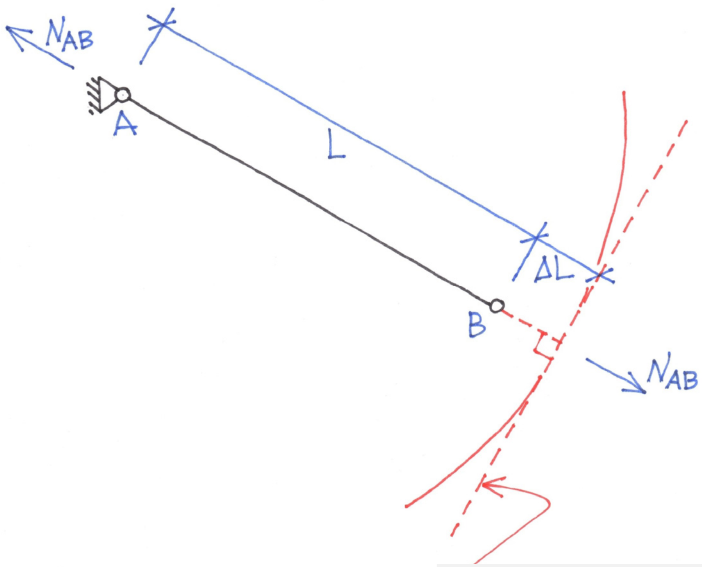
$$\Delta L = \frac{N \cdot L}{E \cdot A} + \alpha \cdot \Delta T \cdot L$$

where α is the **thermal dilation coefficient** (ºC^{-1}) of the material.

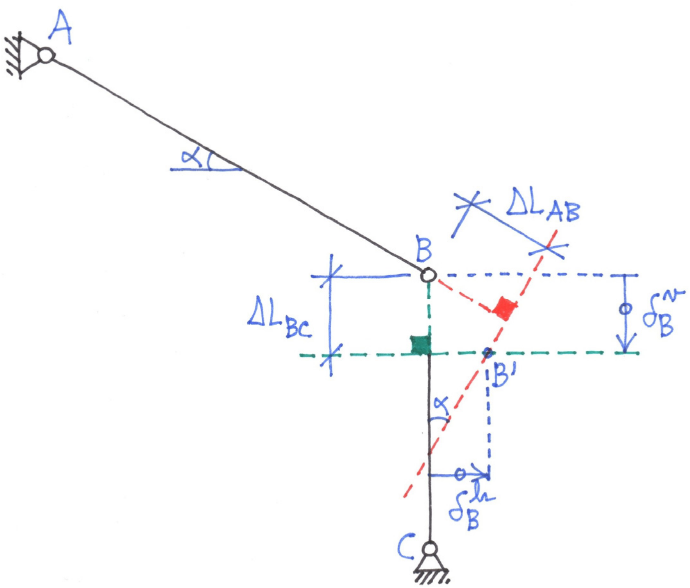
In a straight bar, with hinges at both extremities, without any load applied along its length, the **only effort that in not null** is the axial effort. This type of bar is called an **articulated bar**.

The displacement of node B is determined by analysing the deformations of the two bars connected to that node.

The figure shows the deformation of the bar AB. The position of node A is known. The bar elongation, ΔL, has been calculated. The bar can **rotate around the pinned support** A, thus describing the arc of circumference represented by the red continuous line. Because we are calculating **small displacements**, the arc of circumference can be approximated by its **tangent line**. This tangent to the circumference is perpendicular to the original bar direction.



The position of node B after the deformation lies on this dashed line, which is perpendicular to the straight line passing through the rotation centre.



The figure shows, in red dashed lines, the determination of a straight line which contains the position of node B after the deformation, imposed by the bar AB.

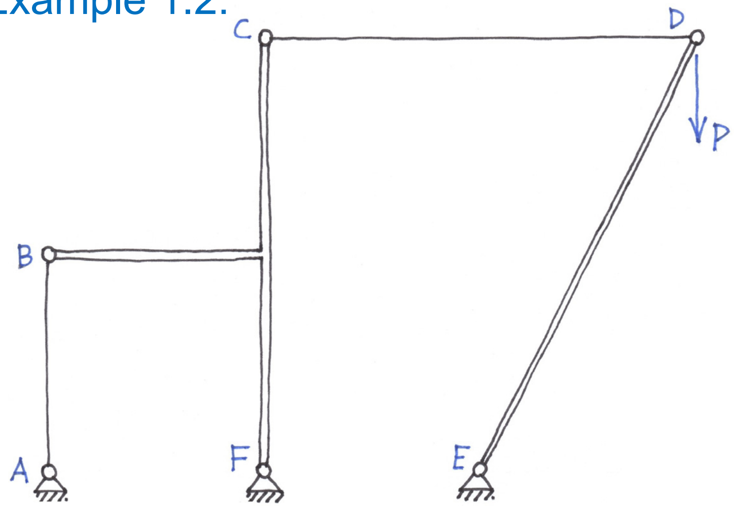
In green dashed lines, the figure shows the line which contains the final position of node B, imposed by the bar BC. The quantity ΔL_{BC} corresponds in this case to the decrease of the bar length, because the bar BC is subject to a compressive axial force.

Given that we have just determined two straight lines which contain the position of node B after the deformation, that position is on their intersection. We label the **position of node B after the deformation** as B'.

Then, we need to calculate the horizontal and the vertical **components of the node displacement**, δ_B^v and δ_B^h. In this example, we see graphically that the δ_B^v value is equal to ΔL_{BC}. The δ_B^h value is calculated using the trigonometric relationships, based on right triangles that have just been depicted in the figure.

In the graphical analysis of the node displacement, the deformations are **amplified**, so that the relationship between the various variables can be understood.

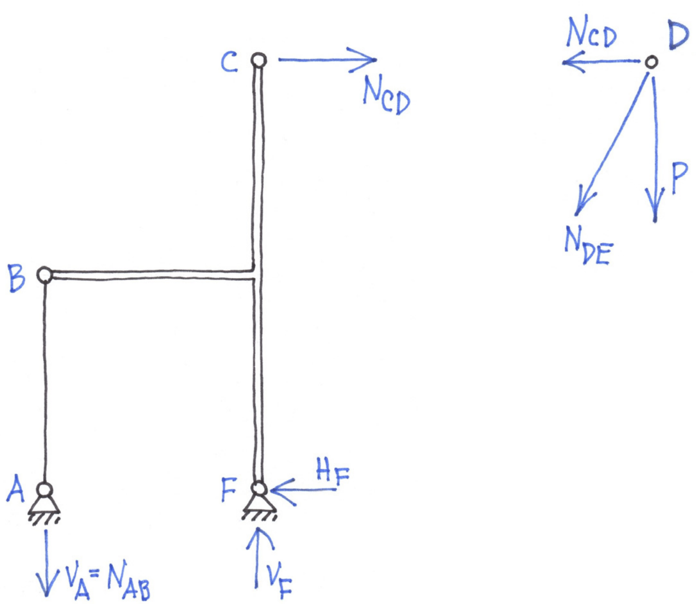
Example 1.2:



The figure shows a new example. This structure is composed by two **rigid bodies** (BCF and DE) and two deformable ties (articulated bars AB and CD) and is subject to a point load P. The objective consists in the calculation of the displacement of node D.

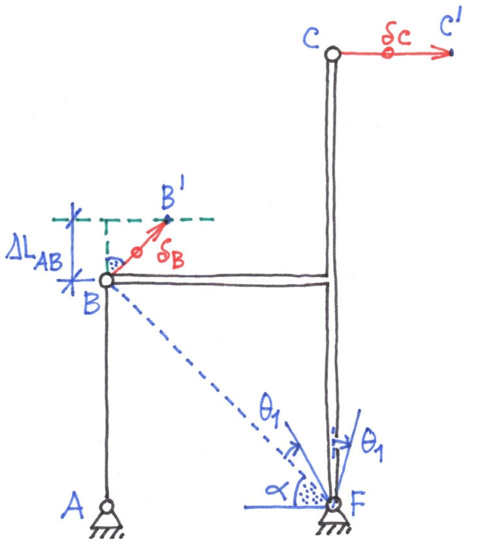
Before calculating the nodes' displacement, the axial forces in the deformable ties are needed.

Node D is connected to two articulated bars, therefore the axial forces in those bars can be determined through the method of the equilibrium of the nodes.



Once the N_{CD} value is known, we may **isolate the body** composed by the rigid member BCF and the tie AB, by representing all the actions applied to those body. Given that AB is an articulated bar, the horizontal reaction at the support A is null. The vertical reaction at the support A (equal to the axial force N_{AB}) can thus be calculated by equilibrium of this body.

Then, the **bar elongations**, ΔL_{AB} and ΔL_{CD} can be calculated. Both correspond to an increase of length, in this example.



In this example, three nodes can move (B, C and D). The only node whose displacement can be **calculated in the first place** is node C. Its position is determined by:

- the rotation of the rigid body BCD around the pinned support F (the position of B after the deformation, B', lies on a line perpendicular to BF);
- The elongation of the articulated bar AB, ΔL_{AB}, and the bar rotation around the pinned support A (B' lies on the green dashed line, perpendicular to the bar direction, represented in the figure).

The position B' is, therefore, on the intersection of the two straight lines just mentioned. From this graphical representation, we can see that the displacement of node B is:

$$\delta_B = \frac{\Delta L_{AB}}{\sin \alpha}$$

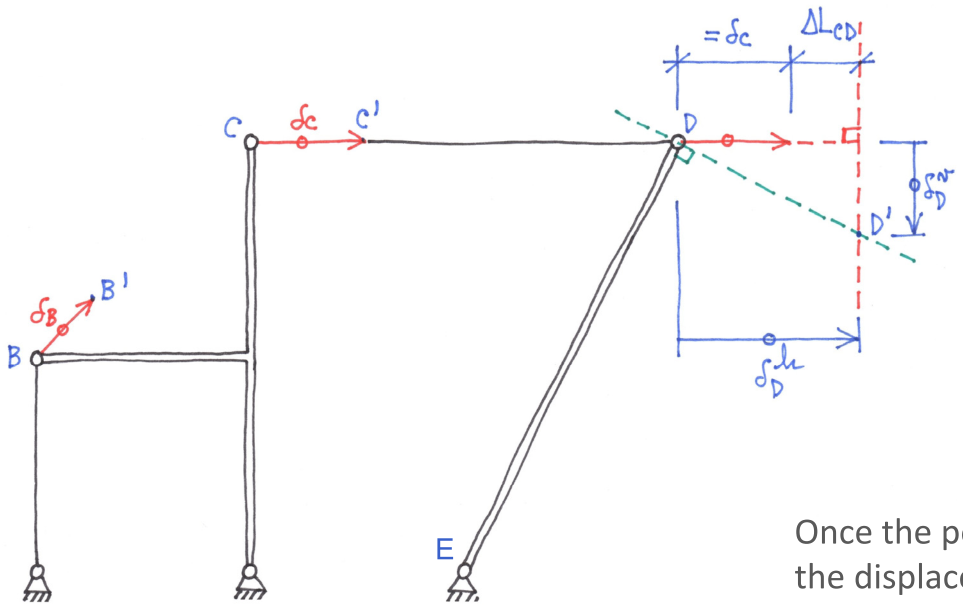
The angle α is known, because it can be calculated based on the structure dimensions.

Once δ_B is known, the rotation of the rigid body BCF can be calculated:

$$\tan \theta_1 = \frac{\delta_B}{\overline{BF}}$$

where $\overline{BF}$ is the distance between the nodes B and F. The angle θ_1 is the rotation of the rigid body BCF. It is a **very small angle**, and therefore: $\tan \theta_1 \approx \theta_1 \text{ [rad]}$. After the rotation of the rigid body is know, the displacement of any node on that body (namely node C) can be determined, because:

$$\tan \theta_1 = \frac{\delta_B}{\overline{BF}} = \frac{\delta_C}{\overline{CF}}$$



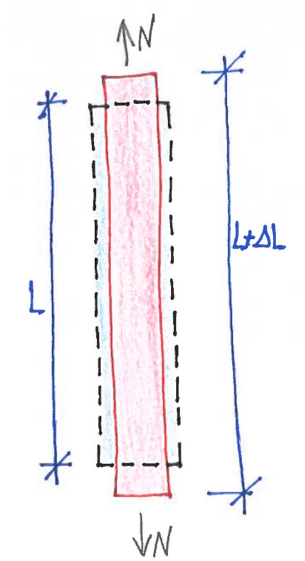
Once the position of node C after the deformation (C’) is known, the displacement of node D can de determined. Following the procedure learnt before, this position is on the intersection of:

- the green dashed line, which contains the possible positions for the extremity D of the bar ED after the deformation;
- the red dashed line which contains the possible positions for the extremity D of the bar CD after the deformation.

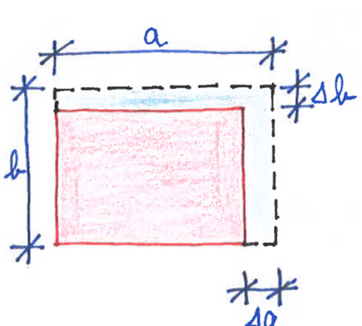
The position of node D after the deformation is thus D’ as shown in the figure.

1.3. Poisson's ratio (ν) in an elastic isotropic material

Side view of a bar subject to axial force:



Cross section of that same bar (at a different scale):



Legend:

- - - - Before the deformation

—— After the deformation

The figure shows a bar subject to an axial force N, made with an **isotropic** and **homogeneous** material (material with properties independent of the direction and the position), and represents the deformations due to that force (with a large amplification). In any position along the bar, the applied stress is just:

$$\sigma_L = \frac{N}{A}$$

where the subscript L denotes the stress direction: the longitudinal direction of the bar. The stress in the longitudinal direction is given by the Hooke's law:

$$\varepsilon_L = \frac{\sigma_L}{E}$$

In directions perpendicular to the load axis, the stress σ_L is responsible by deformations that depend on the **Poisson's ratio** of the material (ν):

$$\varepsilon_t = -\nu \frac{\sigma_L}{E}$$

If σ_L is a tensile stress, the strain ε_t is thus a negative quantity. The bar length after the deformation (i.e. after the load is applied) is $L + \Delta L$ and the cross-section sizes are $a + \Delta a$ and $b + \Delta b$, where:

$$\varepsilon_L = \frac{\Delta L}{L}$$

$$\varepsilon_t = \frac{\Delta a}{a} = \frac{\Delta b}{b}$$

1.4. Variation of volume and cross-sectional area

There are two alternatives to calculate the **variation of volume** (V) and the **variation of the cross-sectional area** (A) of a bar subject to an axial force.

Alternative 1:

Once the final bar dimensions (after the deformation) are know, the variation can be calculated as the difference between final and initial sizes:

$$\Delta V = V_{final} - V$$

$$\Delta A = A_{final} - A$$

Alternative 2:

The volume variation can be directly calculated knowing that the **volumetric strain** (ε_V), which gives the volume variation (ΔV) divided by the initial volume (V), is equal to the **sum of the linear strains in three orthogonal directions**. That is:

$$\varepsilon_V = \frac{\Delta V}{V}$$

$$\varepsilon_V = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

In a similar manner, the variation of the cross-sectional area can be calculated as:

$$\varepsilon_A = \frac{\Delta A}{A}$$

$$\varepsilon_A = \varepsilon_x + \varepsilon_y$$

NOTES:

- The strains ε_x and ε_y are in this case the strains in the transverse direction (ε_t) calculated before.
- The demonstration of the previous formulas for calculation of ε_V and ε_A was made in the theoretical lectures.

1.5. Deformations of a bar subject to axial force and temperature variation

Consider the bar shown in the image, subject simultaneously to:

- an axial force (in the direction of axis Z in the image);
- a temperature variation.

Which are the resulting strains ε_x, ε_y and ε_z?

The strains in directions X, Y and Z are calculated by **superposition of the effects** of the axial force and the temperature variation. Note that, in an isotropic material, the latter gives rise to a strain $\alpha \cdot \Delta T$ in any direction, where α is the thermal dilation coefficient of the material.

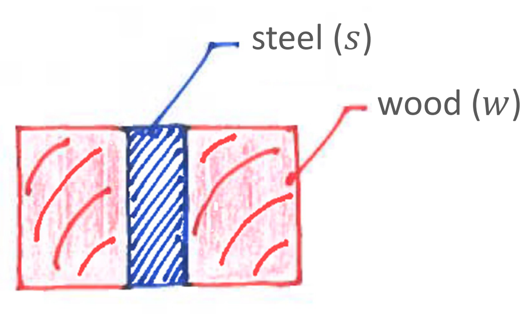
$$\varepsilon_z = \frac{\sigma_L}{E} + \alpha \cdot \Delta T$$

$$\varepsilon_x = \varepsilon_y = -\nu \frac{\sigma_L}{E} + \alpha \cdot \Delta T$$

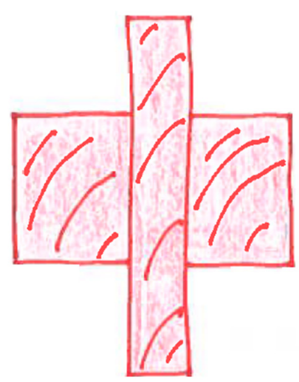
1.6. Sections made of two materials, subjected to axial force

Method of the cross-section homogenization

Real cross-section:



Cross-section homogenized in one of the materials (wood in this case):



Consider now a straight bar whose cross section is composed of two materials, the axial strain being equal in both ones. The bar is subject to an axial force N.

As expedite procedure for calculation the normal stress applied to each material is the **method of the cross-section homogenization**. It consists in transforming the real cross-section in an ***equivalent***, **homogenized cross-section**. The term *equivalent* means in this context that a bar with the homogenized cross-section has the same axial stiffness, the same elongation ΔL and the same stress in the homogenized material as the real bar made with the two materials.

The real cross-sectional areas of the two materials are named A_s and A_w. Their modulus of elasticity are named E_s and E_w. The **ratio of the modulus of elasticity** is named k_{hom}. If we choose, for the material of the homogenized cross-section, the material of lower modulus of elasticity, k_{hom} takes a value higher than 1:

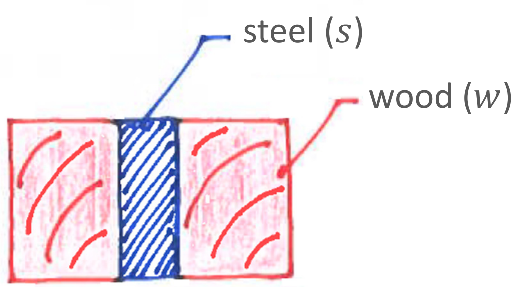
$$k_{hom} = \frac{E_s}{E_w}$$

The cross-sectional area of the homogenized cross-section is:

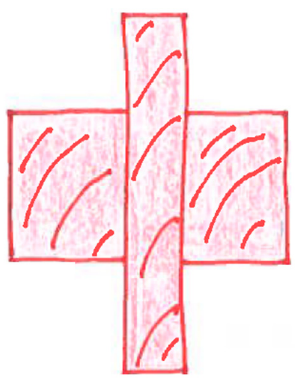
$$\bar{A} = A_w + A_s \cdot k_{hom}$$

The procedure used in the calculation of $\bar{A}$ means that, to transform the real steel cross-section part in a *equivalent* wood part with the same stiffness, the fact that the wood has a lower modulus of elasticity than steel was compensated by and increase of area in the homogenized cross section.

Real cross-section:



Cross-section homogenized in one of the materials (wood in this case):



A plane cross-section before the deformation is **kept plane after the deformation**. Therefore, the longitudinal strain due to the axial force is the same in both materials:

$$\varepsilon_s = \varepsilon_w$$

The normal stress in each material is given by the Hooke's law. Therefore, materials with different modulus of elasticity will have different stresses:

$$\sigma_w = \frac{N}{\bar{A}}$$

$$\sigma_s = \frac{N}{\bar{A}} \cdot k_{hom}$$

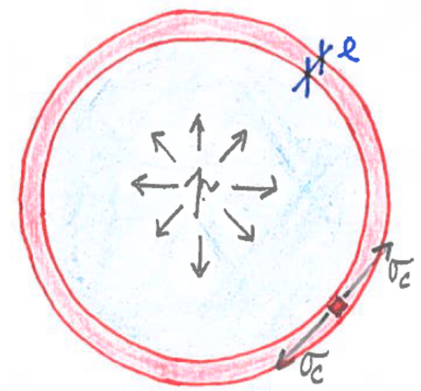
The demonstration of the equations for calculation of the homogenized area, and for calculation of the normal stresses, was made in the theoretical lectures.

Note that the homogenization method does not allow to calculate **stresses and strains due to temperature variations**, only the stresses due to axial forces. In a bar made with two materials, subject to temperature variations, the calculation of applied stresses is made by using a **condition of compatibility of deformations**. This issue is dealt with in Chapter 2, about hyperstatic structures.

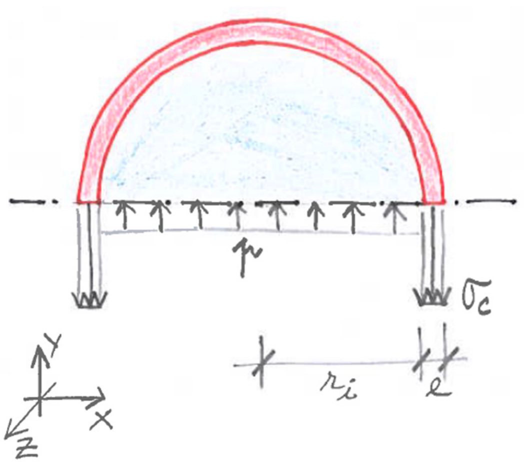
1.7 Stresses and deformations in thin-walled pressure vessels

Normal stress in the circumferential direction (σ_c)

Vessel's cross-section:



Equilibrium of forces:



Consider now a steel **thin-walled pressure vessel** (like a **closed pipe** or a **boiler**) containing a gas at a **pressure** p. The wall thickness is e and the internal radius is r_i. The internal pressure induces **normal stresses** on the steel structure, both in the **circumferential** and in the **longitudinal** directions.

The circumferential stress (σ_c), also called the **hoop stress**, can be calculated through the **equilibrium of forces** shown in the figure. A plane cut is made, so that the cut plane contains the longitudinal axis of the vessel. The stresses applied to the body under analysis are indicated in the image:

- the internal pressure p;
- the circumferential stress σ_c (if the thickness e is small with regard to the radius r_i, the stress can be assumed constant).

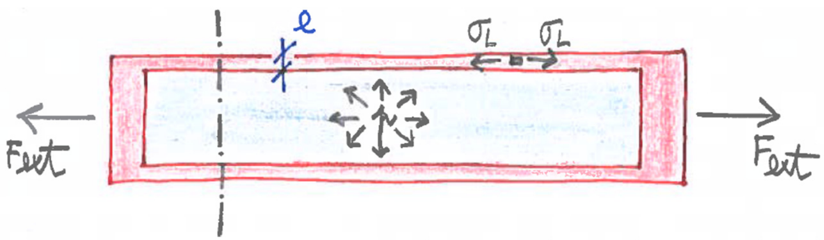
By writing the equilibrium of forces in the direction Y indicated in the figure:

$$\sum F_y = 0 \Leftrightarrow p \cdot 2r_i \cdot L = \sigma_c \cdot 2e \cdot L \Leftrightarrow \sigma_c = \frac{p \cdot r_i}{e}$$

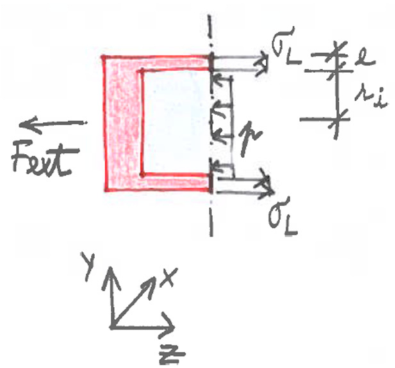
where L is the length of the pipe in the direction perpendicular to the image showing the equilibrium of forces (the pressure p is thus applied on a rectangular area $2r_i \cdot L$).

Normal stress in the longitudinal direction (σ_L)

Longitudinal section of a pipe:



Equilibrium of forces:



The first image represents the longitudinal section of a pipe. The applied actions are:

- the **internal pressure** p;
- An **external force**, in the longitudinal direction, F_{ext}.

To calculate the **longitudinal stress on the pipe walls**, σ_L, a plane cut is made as shown in the second image. The isolated body is be in equilibrium, and therefore the sum of forces in the direction Z has to be equal to zero:

$$\sum F_z = 0 \Leftrightarrow \sigma_L \cdot \pi \cdot ({r_e}^2 - {r_i}^2) - p \cdot \pi \cdot r_i^2 - F_{ext} = 0 \Leftrightarrow \sigma_L = \ldots$$

Area of the steel ring where the stress σ_L is applied

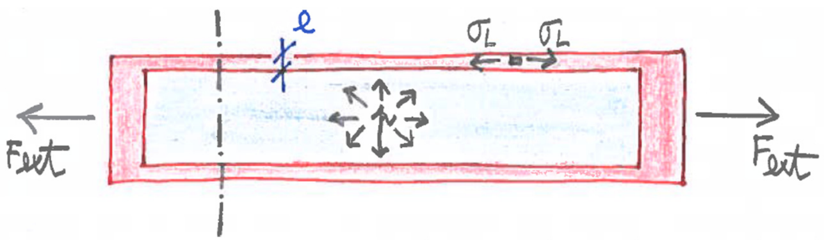
Internal circular area of the pipe

The variable r_e denotes the external radius: $r_e = r_i + e$.

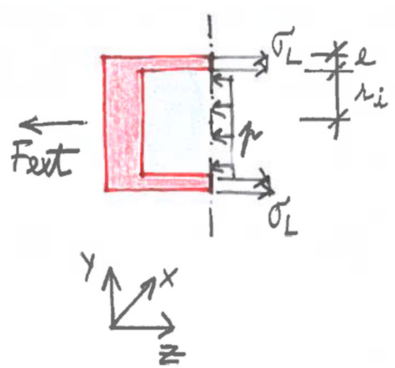
Note that the shape of the pipe end (spherical, disc-shaped, or other shape) does not have any influence on this σ_L value.

Normal stress in the longitudinal direction (σ_L)

Longitudinal section of a pipe:



Equilibrium of forces:



In the particular case of $F_{ext} = 0$ (i.e. when the pipe is not submitted to any external force in the longitudinal direction), a simplified formula for calculation of σ_L can be reached. To do so, note that the area of the steel ring where σ_L is applied can be written as the medium steel perimeter multiplied by he wall thickness:

$$2\pi \cdot r_m \cdot e$$

where r_m is the medium radius, $r_m = (r_i + r_e)/2$. Given that the thickness is small by comparison with the radius, the area where the pressure p is applied is approximately equal to:

$$\pi \cdot r_m^2$$

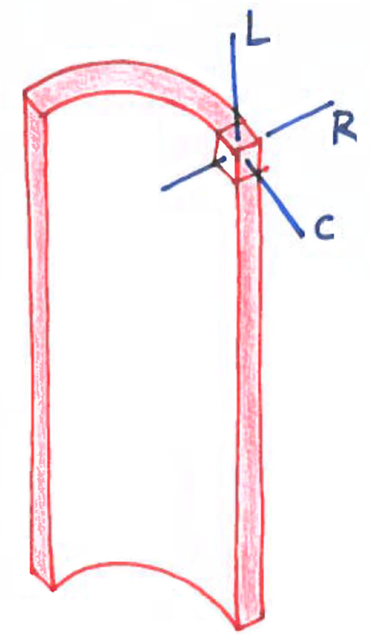
and the longitudinal stress becomes:

$$\sigma_L \approx \frac{p \cdot \pi \cdot r_m^2}{2\pi \cdot r_m \cdot e} = \frac{p \cdot r_m}{2e}$$

Therefore, if $F_{ext} = 0$, the longitudinal stress is:

$$\sigma_L \approx \frac{\sigma_c}{2}$$

Calculation of strains, through the superposition of effects



The calculation of the pipe (or vessel) deformation can be done by **superposition of the effects of the applied stresses**, σ_c and σ_L, calculated before. Three **orthogonal directions** have to be considered, the longitudinal (L), circumferential (C) and radial (R) directions represented in the image.

Taking into account the Hooke's law and the Poisson's ratio effect, the strains due to the circumferential stress are:

$$\varepsilon_c = \frac{\sigma_c}{E}$$

$$\varepsilon_L = \varepsilon_R = -\nu \frac{\sigma_c}{E}$$

Similarly, the strains due to the longitudinal stress are:

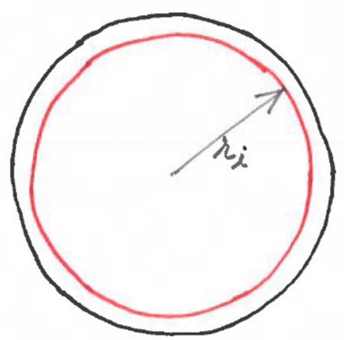
$$\varepsilon_L = \frac{\sigma_L}{E}$$

$$\varepsilon_c = \varepsilon_R = -\nu \frac{\sigma_L}{E}$$

The total strains are calculated by superposition (sum) of the strains due tot these two effects.

Note that the strain ε_R calculated in this page is the **strain in the steel structure**. It cannot be used to calculate the variation of the pipe's (or vessel's) internal radius. That issue is addressed in the next page.

Strain to calculate the variation of the vessel's internal radius



In some problems, we are interested in the calculation of the **variation of the internal volume** of a certain pipe or vessel. To do so, we need to determine the **variation of the internal radius** (r_i) of the pipe caused by the applied loading (internal pressure and eventually an external force). It can be demonstrated that the variation of the internal radius is calculated using the circumferential strain ε_c:

$$\varepsilon_c = \frac{\Delta r_i}{r_i}$$

Demonstration:

The initial **perimeter of the circumference of radius r_i** is $2\pi \cdot r_i$. Its variation after the deformation is naturally determined by the circumferential strain value, ε_c, because the strain ε_c is the deformation in the circumference direction:

$$\varepsilon_c = \frac{\Delta\text{Perimeter}}{\text{Perimeter}} = \frac{2\pi \cdot \Delta r_i}{2\pi \cdot r_i} = \frac{\Delta r_i}{r_i}$$

which was to be demonstrated.

2. Hyperstatic structures in which the deformable members are subjected to axial force only

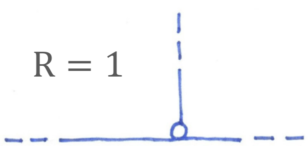
2.1. Degree of hyperstaticity in plane reticulated structures

$L = 1$

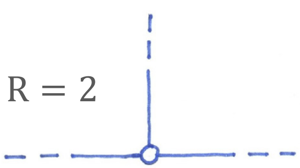
$L = 2$

$L = 3$

R = 1



R = 2



Reticulated structures may include bars not fully articulated (those which may have efforts N, V and M), fully articulated bars without applied loads in-between the nodes (those which do not have efforts V and M), supports and internal hinges.

Understanding whether the structure is **isostatic** or **hyperstatic** is paramount. The **degree of hyperstaticity** (H) of a **plane structure** can be determined as:

$$H = L - R + 3C - 3$$

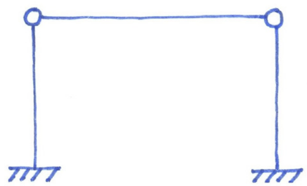
where:

L is the number of **links** to the exterior (support reactions);
R is the number of **rotation releases**, by hinges;
C is the number of **closed cells**;
H is the degree of hyperstaticity ($H = 0$ means the structure is isostatic).

The justification for the previous equation is as follows:

- A structure without closed cells ($C = 0$) and without internal hinges ($R = 0$) is isostatic if the number of support reactions (L) is equal to the number of equilibrium equations available to calculate such reactions (3).
- For each rotation release (R) added to the structure, the degree of hyperstaticity is decreased by 1, and a new equilibrium condition is added (the equilibrium of moments around the hinge).
- For each closed cell (C) added to the structure, the degree of hyperstaticity is increased by 3, because in order to transform that cell in an open structure, it would be necessary to cut (open) one bar cross section, thus releasing three efforts in the cross-section (N, V and M).

Example 2.1:



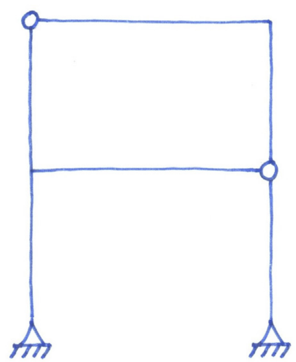
The **plane frame** (reticulated structure) in the example 2.1 is an hyperstatic structure, with a degree of hyperstaticity equal to 1, because:

$$L = 6$$
$$R = 2$$
$$C = 0$$

$$H = L - R + 3C - 3 = 1$$

The same conclusion could have been drawn directly by observation of the structure: by eliminating one internal connection, for example the articulated bar joining the tops of the two columns, two isostatic systems (the two vertical columns in this example) would be obtained.

Example 2.2:



The plane frame in example 2.2 is also a reticulated structure with a degree of hyperstaticity equal to 1, because:

$$L = 4$$
$$R = 3$$
$$C = 1$$

$$H = L - R + 3C - 3 = 1$$

2.2. Degree of hyperstaticity in plane articulated structures

Plane articulated structures, also known as **trusses**, are composed by articulated bars (bars connected to hinges in both extremities) only, subjected to point loads applied on the nodes only. Under these conditions, the only effort applied to each bar is the axial force. The **degree of hyperstaticity** (H) of a plane articulated structure can be determined as:

$$H = L + B - 2N$$

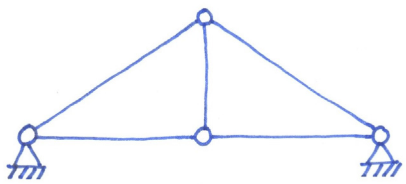
where:

L is the number of **links** to the exterior (support reactions);
B is the number of **bars**;
N is the number of **nodes** (including the ones directly connected to supports);
H is the degree of hyperstaticity ($H = 0$ means the structure is isostatic).

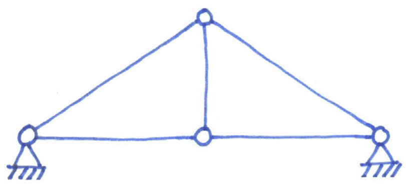
The justification for the previous equation is as follows:

- The number of unknows, which have to be calculated in order to fully characterize the forces in the structure, is equal to the sum of the number of support reactions (L) plus the number of bars (B), because one axial effort needs to be determined per bar.
- The number of available equilibrium equations is 2 per node, because each node has to be in equilibrium of forces in two orthogonal directions.
- If the number of unknowns is equal to the number of available equations, the structure is isostatic. Each additional unknown increases the degree of hyperstaticity by 1.

Example 2.3:



Example 2.3:



By applying the previous rule to the articulated structure in Example 2.3, one gets:

$$L = 4$$
$$B = 5$$
$$N = 4$$

$$H = L + B - 2N = 1$$

An articulated structure is a particular case of a reticulated structure, therefore the rule for the reticulated arrangement is also applicable:

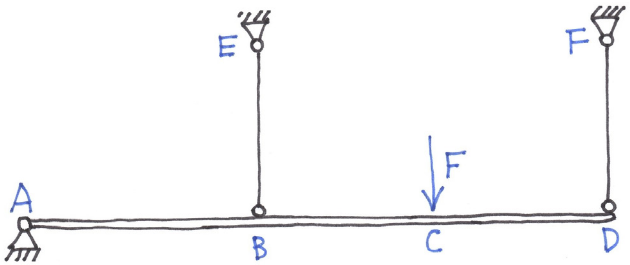
$$L = 4$$
$$R = 6$$
$$C = 2$$

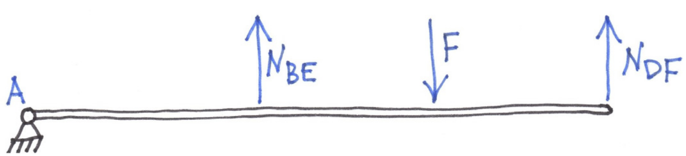
$$H = L - R + 3C - 3 = 1$$

The same conclusion could have been drawn knowing that a fully articulated plane structure, formed by adjacent triangles, in a simply supported condition ($L = 3$) is an isostatic structure. The truss in this example is formed by adjacent triangles. The number of support reactions is $L = 4$, therefore the degree of hyperstaticity is equal to 1.

2.3. Analysis of hyperstatic structures (H=1) with linear elastic material behaviour

Example 2.4:





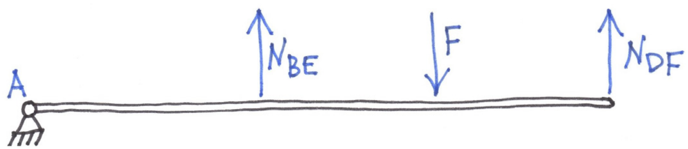
The purpose of this chapter is the calculation of **internal forces and displacements**, in structures with a degree of hyperstaticity equal to 1, in which the **deformable bars are articulated** (therefore the only internal effort is such bars is the axial effort). The figure shows one example of this type of structure. In this structure, the horizontal bar ABCD is **infinitely rigid** (i.e. its deformability due to applied forces is negligible), and it is subject to a point load F.

This is a reticulated structure, because the horizontal bar has internal efforts N, V and M. By applying the rule to check the degree of hyperstaticity, one gets:

$$L = 6$$
$$R = 2$$
$$C = 0$$

$$H = L - R + 3C - 3 = 1$$

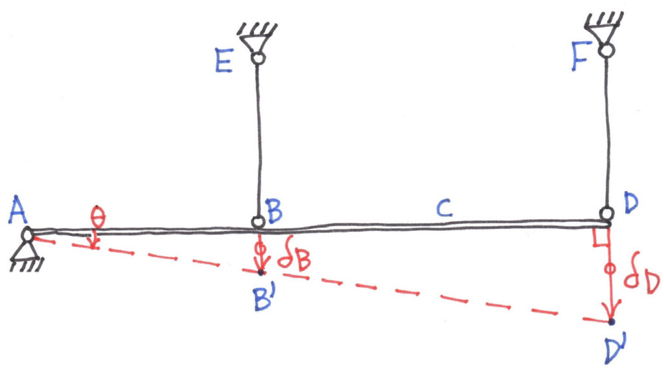
The same conclusion could have been drawn by analysing the equilibrium conditions of the body composed by the rigid bar only. To do so, the articulated bars can be replaced by the forces they apply on the rigid bar (the articulated ties have axial effort only). In this way, the body under analysis has 4 links to the exterior (the pinned support in A, plus the two forces induced by the ties) and no internal hinge, which means that it is **one time hyperstatic**.



To solve the structure, one needs to calculate the support reactions in A and the axial forces in the articulated ties. The equilibrium equations available are the 3 global equilibrium equations only:

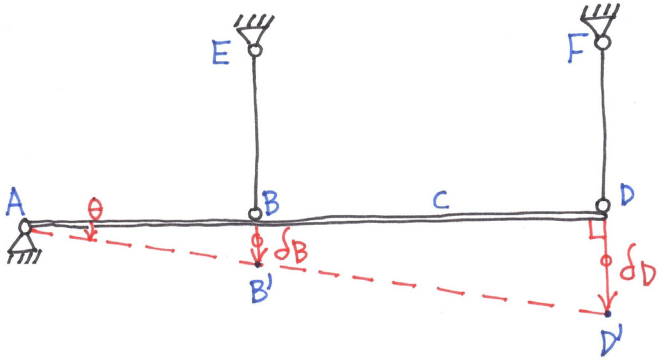
$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum m|_A = 0$$

Given that the number of unknowns is 4, **an additional equation is required** to solve the structure. This is the **equation of compatibility of deformations**.



The compatibility of deformations in this type of structure is ensured by the **rotation movement of the rigid body** around its pinned support (A in this example). Because we are dealing with small deformations, the arc of circumference described by any point of the rigid body, in this movement, can be approximated by its tangent (a straight line perpendicular to the straight line connecting the centre of rotation). Therefore, the equation of compatibility becomes:

$$\tan\theta = \frac{\delta_B}{\overline{AB}} = \frac{\delta_D}{\overline{AD}} \qquad \text{(form 1)}$$



The equation of compatibility will only be useful for solving the structure if it expresses a relationship between the axial forces on the ties. To get such a relationship, we start by replacing in the previous equation (form 1), the **node displacements** by their relationship with the **bars' elongation**. In this Example 2.4, the deformable bars are aligned with the node displacement, therefore:

$$\delta_B = \Delta L_{BE}$$
$$\delta_D = \Delta L_{DF}$$

and the equation of compatibility becomes:

$$\frac{\Delta L_{BE}}{\overline{AB}} = \frac{\Delta L_{DF}}{\overline{AD}} \qquad \text{(form 2)}$$

Finally, we need to introduce the relationship between the **elongation** and the **axial force**. Given that there is no temperature variation in this example, it becomes:

$$\frac{\left(\frac{NL}{EA}\right)_{BE}}{\overline{AB}} = \frac{\left(\frac{NL}{EA}\right)_{DF}}{\overline{AD}} \qquad \text{(form 3)}$$

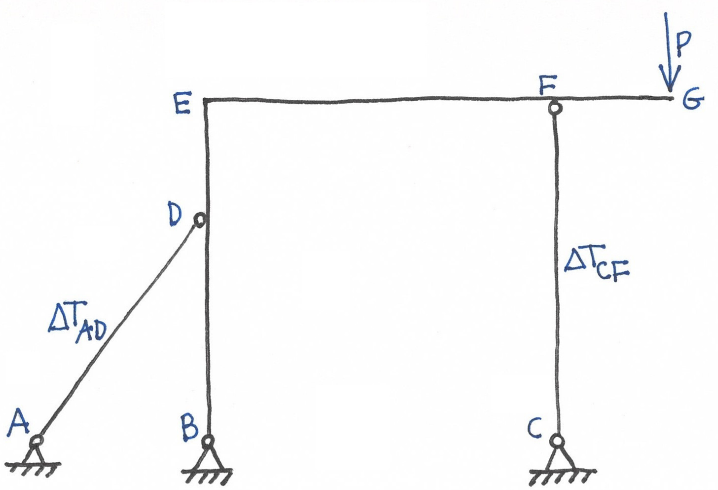
this being the fourth equation which, together with the equilibrium equations can be used to solve the structure.

The Example 2.4 before is a simple structure, where the node displacements are equal to the bars' elongation, and both ties are in tension.

In a more general case:

- the node displacement might be different from the bar elongation;
- there can be one bar in tension and another in compression;
- the deformable bars might be subjected to axial force and temperature variation.

Example 2.5:

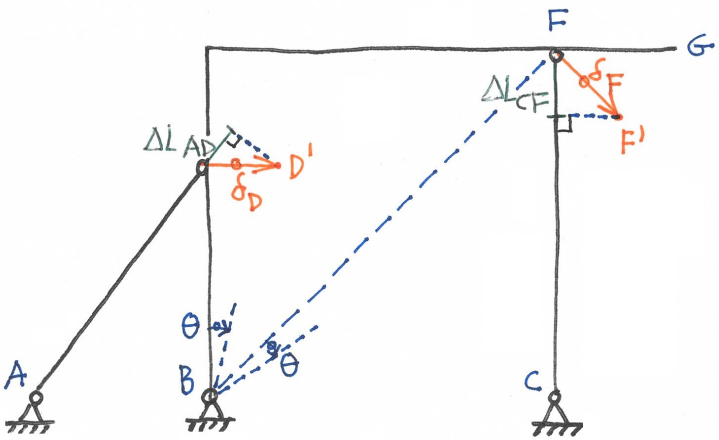


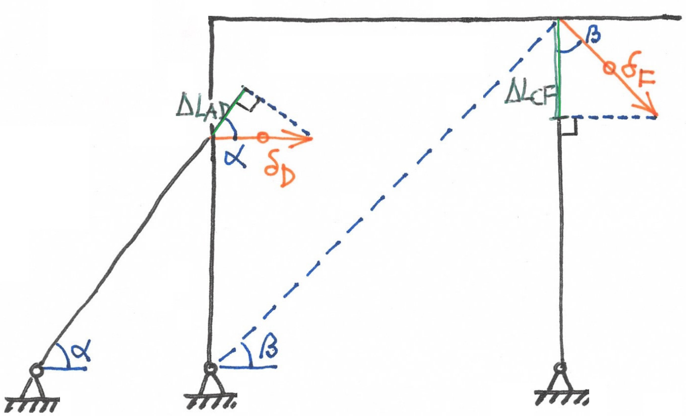
In this Example 2.5, the structure is composed by an infinitely rigid body BDEFG, and two deformable articulated bars. The **applied actions** consist in a point load P and temperature variations applied to the deformable bars.

As far as the **degree of hyperstaticity** is concerned, this structure is analogous the one in Example 2.4, and therefore it is also one time hyperstatic.

Regarding the temperature variations, it is important to realize that:

- **in an hyperstatic structure, temperature variations influence the value of the support reactions and the internal forces**, because the thermal deformations of a bar will be partially restrained by the remaining structure, giving rise to restraining forces.
- in an isostatic structure, temperature variations only affect the displacements, not the support reactions nor the internal efforts.





The compatibility of deformations in this Example 2.5 is governed by the rotation of the rigid body around the pinned support in B:

$$\tan\theta = \frac{\delta_D}{\overline{BD}} = \frac{\delta_F}{\overline{BF}} \qquad \text{(form 1)}$$

Then, we need to determine the **relationship between the node displacement and the bar's ΔL**. The displacement of point D, to its final position D', imposes that the bar AD elongates ΔL_{AD} as determined geometrically in the image. The relationship between δ_D and ΔL_{AD} is expressed based on the angle α, which is known because it depends on the structure geometry only.

$$\cos\alpha = \frac{\Delta L_{AD}}{\delta_D}$$

$$\cos\beta = \frac{\Delta L_{CF}}{\delta_F}$$

And therefore the compatibility condition becomes:

$$\frac{\Delta L_{AD}/\cos\alpha}{\overline{BD}} = \frac{\Delta L_{CF}/\cos\beta}{\overline{BF}} \qquad \text{(form 2)}$$

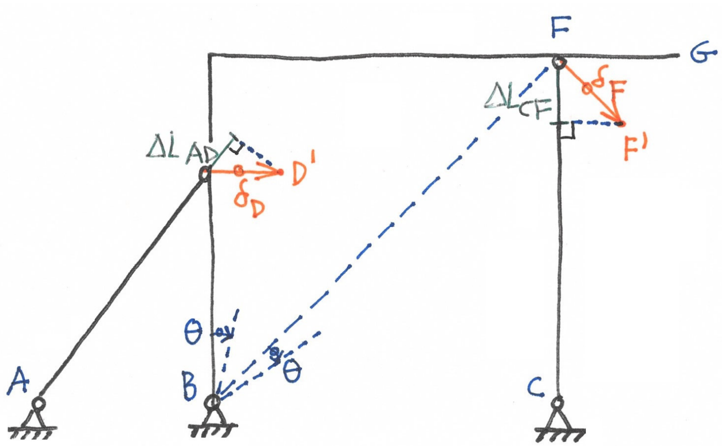
To get the equation of compatibility of deformations written in its final form, form 3, we need to replace the ΔL value by its relationship with the axial force and the temperature variation in the bar:

$$\Delta L = \frac{N\ L}{E\ A} + \alpha\ \Delta T\ L$$

It becomes:

$$\frac{\left[\frac{NL}{EA} + \alpha \Delta T L\right]_{AD}}{\overline{BD}\cos\alpha} = \frac{\left[\frac{NL}{EA} + \alpha \Delta T L\right]_{CF}}{\overline{BF}\ \cos\beta} \qquad \text{(form 3)}$$

Given that we are using in the calculation the relationship between ΔL, N and ΔT, there needs to exist **coherence** between these variables.

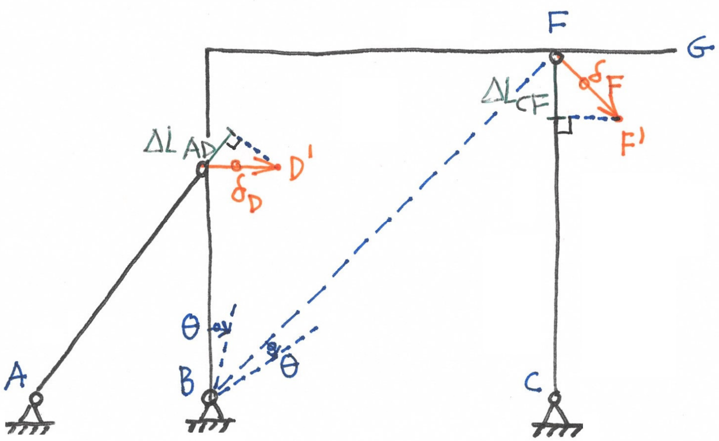


In this example, for **bar AD**, we see (in the representation of the compatibility of deformations) that **the variable ΔL_{AD} is the increase of the bar length**.

Consequently, the variable N_{AD} needs to be the tensile force in the bar, and the variable ΔT_{AD} represents the heating.

That means:

ΔL_{AD} is a positive quantity in case of increase of length;
N_{AD} is a positive quantity if it is a tensile force;
ΔT_{AD} is a positive quantity in the case of heating.

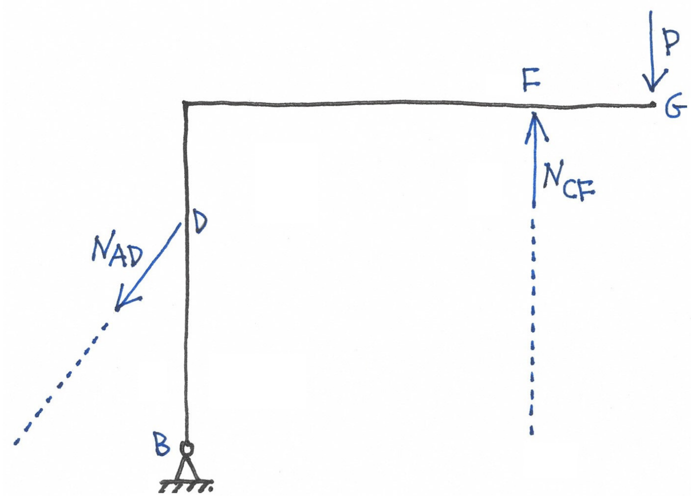


For **bar CF**, we see (in the representation of the compatibility of deformations) that **the variable ΔL_{CF} is the decrease of the bar length**.

Consequently, the variable N_{CF} needs to be the compression force in the bar, and the variable ΔT_{CF} is the bar cooling.

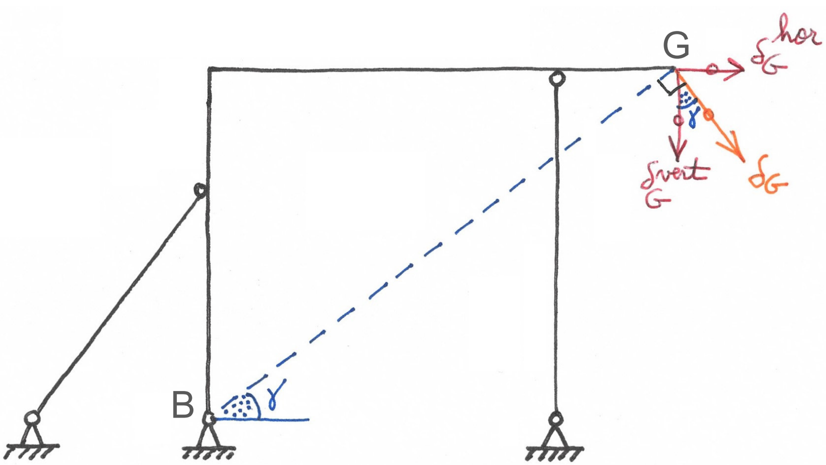
That means:

ΔL_{CF} is a positive number in case of decrease of length;
N_{CF} is a positive number if it is a compression force;
ΔT_{CF} is a positive number in the case of cooling.



Thus, the **equilibrium equations** will be written considering the forces applied to the rigid body as shown in the image.

There are other **alternatives** to deal with the sign conventions in the analysis of the hyperstatic structure. Whichever alternative is adopted, there needs to be coherence between ΔL, N and ΔT.



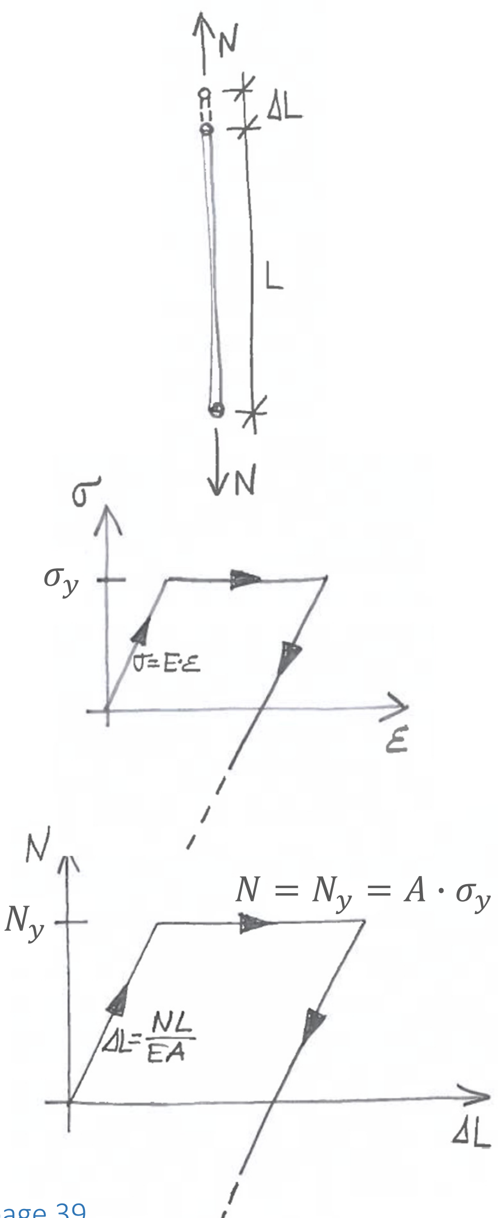
Once the axial forces in the deformable bars have been calculated, the rotation angle of the rigid body can be determined:

- by calculating the elongation of one of the deformable bars;
- and then, by using the rationale that was applied above to write the compatibility equation.

After that, the displacement of any other point belonging to that rigid body can be calculated. For point G:

$$\delta_G = \overline{BG} \tan \theta$$

2.4. Analysis of hyperstatic structures (H=1) with elasto-plastic material behaviour



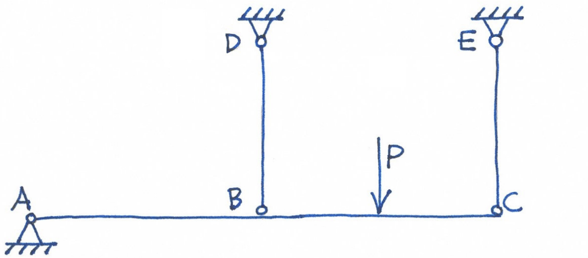
In many **ductile materials**, like for example steel, the stress-strain relationship is characterized by:

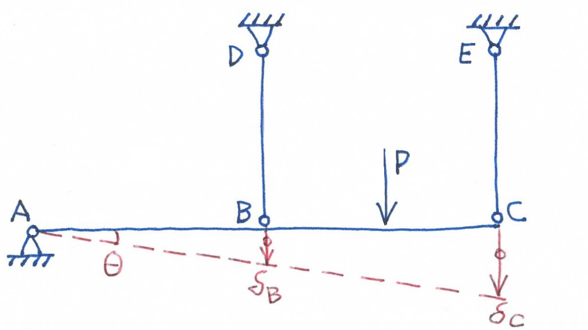
- a **linear elastic stage**, in which the stress-strain relationship is given by the Hooke's law;
- a **plastic stage**, once the **yield stress** σ_y is reached;
- if **unloading** occurs when the material point is on the plastic stage, the stress-strain trajectory is also characterized by the Hooke's law.

The images show the stress-strain relationship (σ–ε) and the force-elongation relationship (N–ΔL), for a bar subject to axial force, with **elasto-perfectly plastic behaviour**, characterized by a modulus of elasticity E and a yield stress σ_y. The **equations for the linear elastic stage** are also written in the images.

Note that, even though the real stress-strain relationship of many ductile materials, after σ_y is reached, is more complex, the structural analysis is often made assuming an elasto-perfectly plastic behaviour. This is the simplification adopted in RM too.

Example 2.6:





In this example, an hyperstatic structure is subjected to a point load P. The horizontal bar ABC is infinitely rigid. The articulated ties BD and CE have elasto-perfectly plastic behaviour, characterized by the modulus of elasticity E and the yield stress σ_y of the material.

Because the structure is hyperstatic, **while both ties are in the elastic stage**, the calculation of reactions and forces has to be made using:

- an equation of compatibility of deformations;
- equilibrium equations.

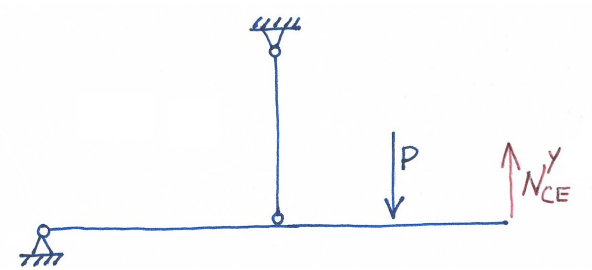
In this elastic stage, the **structure behaviour** (forces, displacements, …) is **proportional do the applied loading**.

To determine which is the **bar that enters the plastic stage in the first place**:

- calculate the structure for any (low) loading value;
- the bar where the stress is closest to the yield stress is the one which enters the plastic stage in first place (it would be the bar CE in this example).

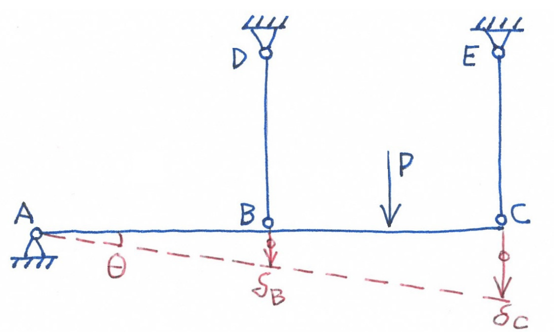
The l**oading value** (P in this example) **for which the structure enters the plastic stage** (let's call it P_y) can be determined by proportionality (based on the σ_{CE} result calculated for the aforementioned, low, P value):

$$\frac{\sigma_{CE}}{\sigma_{CE}^{y}} = \frac{P}{P_y}$$



For load values higher than P_y, one of the ties (CE in this example) is in the plastic stage. Thus, the force on this tie is known: it is the yield force. Consequently, the force on the other tie, and the support reactions, are calculated using the **equilibrium equations only**.

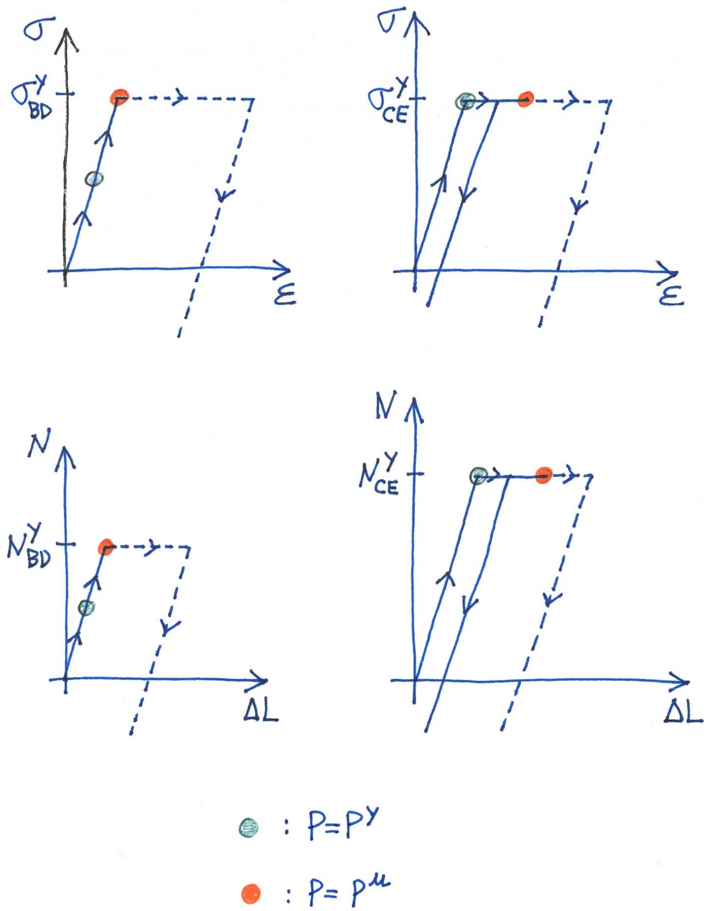
Note that, for $P > P_y$ the relationship $\Delta L = (NL)/(EA)$ is **valid only for** the tie which has not entered the plastic stage yet (BD in this case).



Once the elongation of bar BD is determined, the **displacement of any other point in the structure** can be calculated, because the rigid bar ensures the compatibility of displacements even after one of the bars is in the plastic stage.

The **ultimate load**, P_u, is the load for which both ties reach the respective yield force. This “ultimate load” is also called the “failure load”, the “rupture load”, the “collapse load”, among other synonyms.

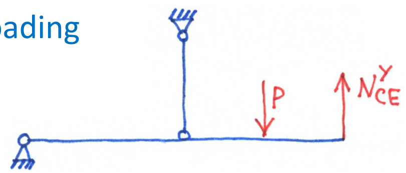
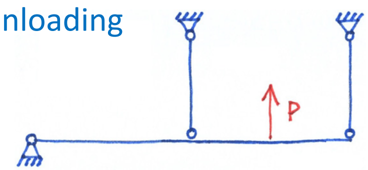
BD: CE:



The figures show the σ–ε and N–ΔL **trajectories** for bars BD and CE, in this example.

The following **points** are marked:

- the **end of the elastic stage** of the structure in this example ($P = P_y$);
- the point where the **ultimate load** of the structure is reached ($P = P_u$).

	N_{CE}	N_{BD}	δ_C
Loading 			
Unloading 			
Residual value = Loading + Unloading			

We often need to calculate the **residual displacements** and forces in the structure, when the structure is loaded with $P > P_y$ and then the load is completely removed (i.e., the structure is **unloaded**).

Because the structure **behaviour is different** in the **loading** and in the **unloading** stages, the residual values have to be determined by calculating separately the effects of loading and unloading, and then superposing those two effects. Organizing the calculation results in a table like the one in the image is a convenient procedure for this type of problem.

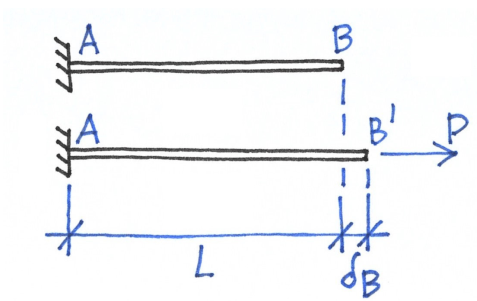
Note that the axial forces and the displacements **upon unloading** can be determined by **proportionality with** the ones calculated before, **for the elastic stage** of the structure behaviour (for $P \leq P_y$).

Unloading consists in applying a force in direction opposite to the direction of loading, so that the sum of loading and unloading effects gives, in the residual (final) state: P=0.

3. Fictitious Unit Load Method for calculation of deformations in plane articulated structures

3.1. Fundaments of the Fictitious Unit Load Method, and the use of energy methods in structural engineering

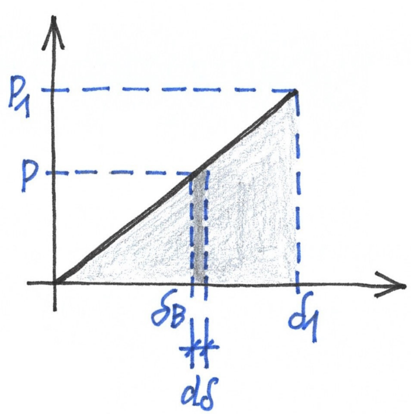
In the previous chapters, we dealt with the concepts of stress and strain. A third important concept, the concept of ***strain energy***, has now to be introduced.



The image shows a bar of length L, uniform cross-sectional area A and linear elastic behaviour characterized by the modulus of elasticity E. By applying, in a slowly increasing manner, a point load P, the extremity B undergoes a displacement δ_B.

The work W done by the load P, when it increases up to P_1, can be quantified as follows. The work dW done by the load P when the bar elongates by a small amount $d\delta$ is $dW = P \cdot d\delta$. Then, the total work W, as the load increases to P_1 is:

$$W = \int_0^{\delta_1} P \, d\delta$$



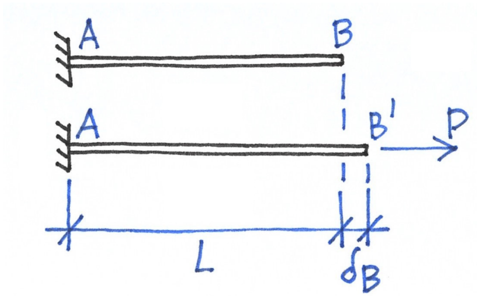
and is equal to the area under the load-displacement diagram:

$$W = \frac{1}{2} P_1 \, \delta_1$$

If the load and the displacement are expresses in N and m respectively, the work is given in joule (J), so 1 J = 1 N·m.

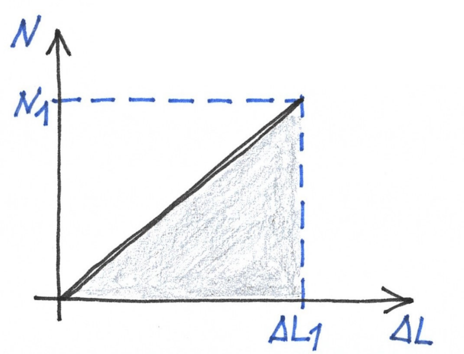
The physical ***law of energy conservation*** implies that the potential energy of the external force, which is consumed when the work is done, is stored as other energy form, or dissipated:

- part of the energy is stored in elastic strain energy, which is recoverable if the load is removed later;
- part can be stored as kinetic energy, if the body mass suffers a motion when the loads are not applied slowly;
- part can be dissipated by external friction, for example by rotation friction in unperfect pinned supports (some friction exists in movable supports, in the reality);
- part can be dissipated by plastic or viscous deformations of the material, if it is loaded beyond the elastic stage.



Energy methods are therefore very useful, for example, for the analysis of impact loading, or vibration problems. These type of problems are out of the scope of this theorico-practical classes. We will focus our attention on the application of energy methods to **quasi-static loading problems**, with **linear elastic material behaviour**, and in the theoretical scenario of frictionless supports. Under these conditions, the **work done by the external loading** is entirely stored as **elastic strain energy** U:

$$W = U$$



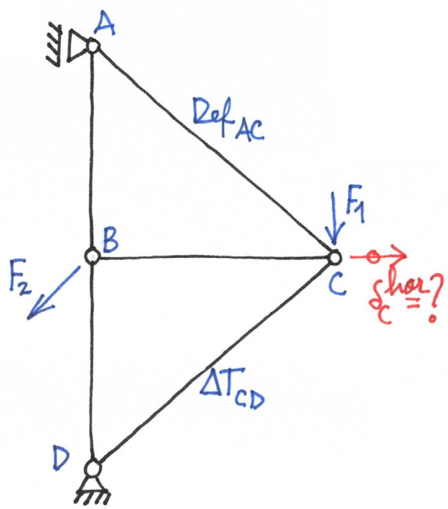
In the example considered herein, denoting as N_1 the axial force in the bar for $P = P_1$:

$$U = \frac{1}{2}\,P_1\,\delta_1 = \frac{1}{2}\,N_1\,\Delta L_1 = \frac{1}{2}\,\frac{N_1^2\,L}{E\,A}$$

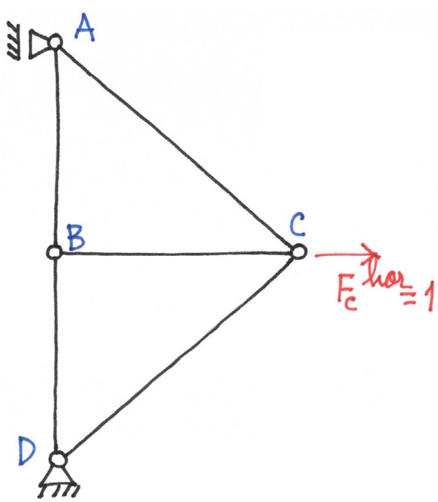
3.2. Calculation of a displacement in a plane articulated structure

Example 3.1:

Real loading:



Fictitious unit loading:



A common application of energy methods is the so-called **Fictitious Unit Load Method**, also known as the **Maxwell-Mohr Method**, for calculation of deformations in structures. According to the method, the procedure for calculation of a nodal displacement in an **articulated isostatic structure** is as follows:

- the axial forces N_i in all the structure's bars, due to the **real applied loading**, have to be calculated (using the method of the equilibrium of the nodes, or the method of the Ritter's sections);
- then, the bars' elongation ΔL_i, due to the applied loading and imposed deformations (temperature variations ΔT_i and fabrication defects Def_i) are determined as:

$$\Delta L_i = \left[\frac{N\,L}{E\,A} + \alpha\,\Delta T\,L + \mathrm{Def}\right]_i$$

- considering now the structure loaded only with a **fictitious unit load (in correspondence with the displacement to be determined)**, the axial forces in the bars have to be determined; the axial effort in bar i, due to this unit load, is denoted $\overline{N_i}$;

The objective in this example is the calculation of the horizontal displacement of node C, therefore the fictitious load is a horizontal unit load applied on node C.

- **considering real deformations and fictitious forces, the work of external forces is equal to the elastic strain energy**:

$$W = U$$

$$\Leftrightarrow 1 \times \delta_C^{\text{hor}} = \sum_{i=1}^{n} (\overline{N_i} \times \Delta L_i)$$

which allows to determine the requested displacement δ_C^{hor}. Coherent sign conventions have to be adopted for all the bars, namely: axial forces with positive value in case of tension; elongations ΔL_i with positive value in case of increase of length.

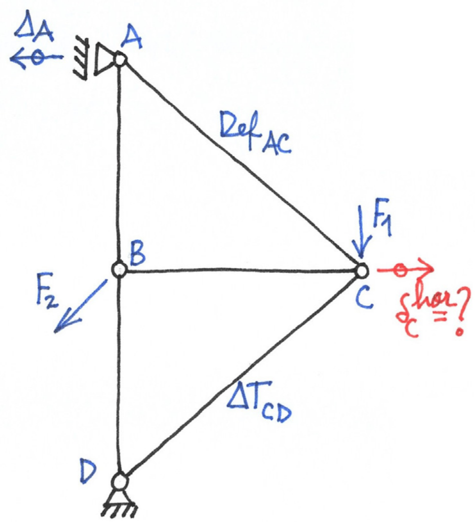
Note that in this Example 3.1 the only external force contributing to the work W is the unit fictitious load. The supports' reactions are also external forces, but they do not do any work in this example because there is no movement of any support.

The student interested in understanding the **demonstration of this method**, can find that explanation in the book *Mechanics of Materials*, by the authors Beer et al., in the chapters 11.11 to 11.13 (numbering in the 6^{th} edition). That demonstration is based on the Castigliano's Theorem.

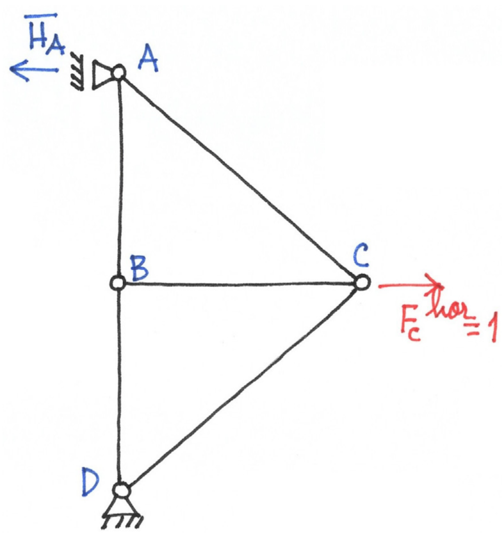
3.3. Influence of support settlements

Example 3.2:

Real loading:



Fictitious unit loading:



We often have to consider, in the calculation of the requested displacement, the influence of some support movement, such as the settlement of a foundation, or the deflection of some supporting structure. In this Example 3.2, there is a support settlement in support A, Δ_A. In this case, the equalization of the work of external forces and the elastic strain energy becomes:

$$W = U$$

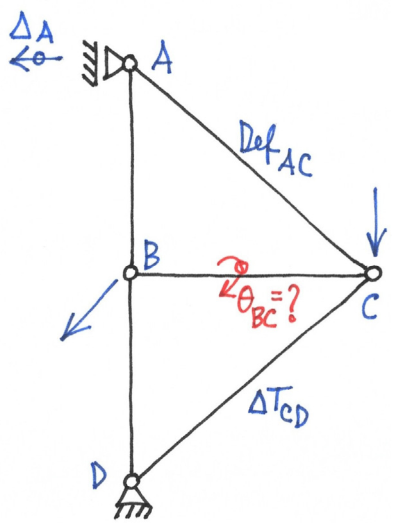
$$\Leftrightarrow 1 \times \delta_C^{\text{hor}} + \overline{H_A} \times \Delta_A = \sum_{i=1}^{n} \left(\overline{N_i} \times \Delta L_i \right)$$

where H_A is the support reaction due to the fictitious unit load. Once again, the previous equation gives the requested displacement if the settlement value (Δ_A) is known.

3.4. Calculation of a bar rotation in a plane articulated structure

Example 3.3:

Real loading:

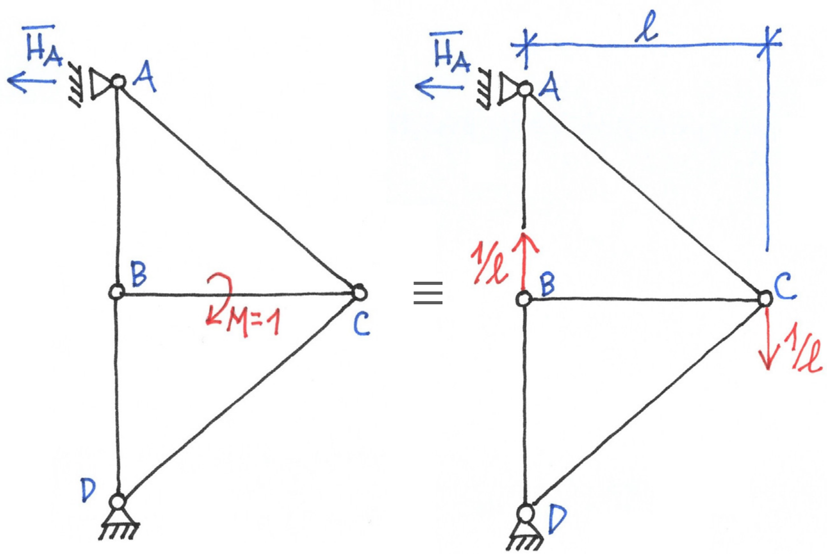


If the requested deformation is the **rotation** of a certain bar, the **fictitious unit load is a unit moment**. However, given that the loading in articulated structures should be composed of concentrated loads in the nodes only, that unit moment should be replaced by an equivalent binary of forces. The equalization of the work of external forces and the elastic strain energy becomes:

$$W = U$$

$$\Leftrightarrow 1 \times \theta_{BC} + \overline{H_A} \times \Delta_A = \sum_{i=1}^{n} (\overline{N_i} \times \Delta L_i)$$

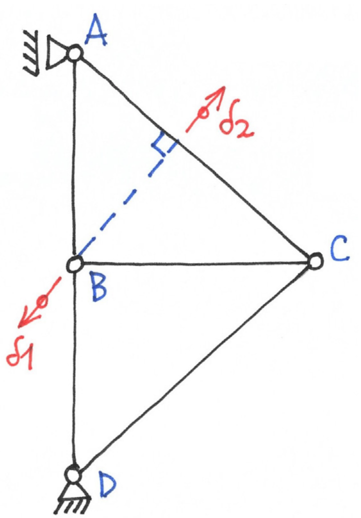
Fictitious unit loading:



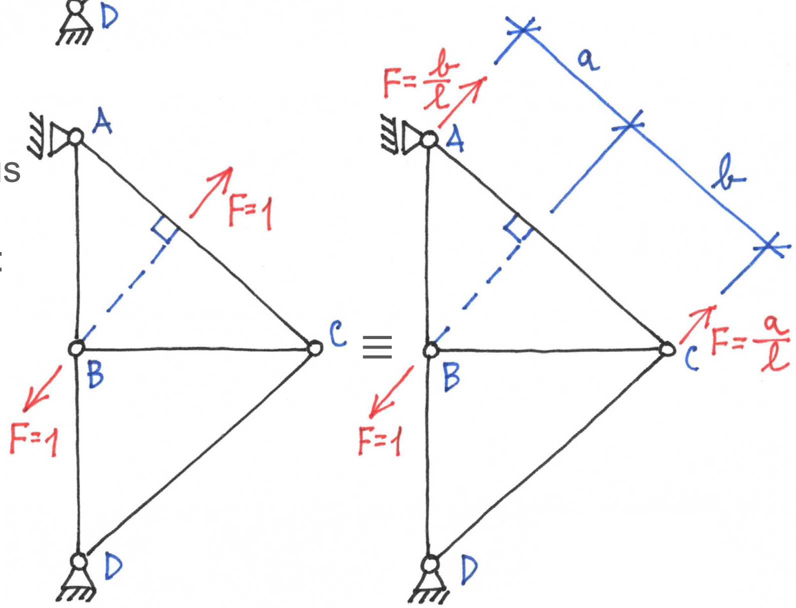
3.5. Calculation of a relative displacement between a bar and a node in a plane articulated structure

Example 3.4:

Real loading:



Fictitious unit loading:



In some problems, we are interested in the calculation of the **relative displacement** between a node and a bar. More specifically, that means the relative displacement between the node and the point on the bar closest to that node, as shown in the figure:

$$\delta_{\text{relative}} = \delta_1 + \delta_2$$

The correspondent **fictitious unit loading** consists of two **point loads in correspondence with these displacements**. However, because the loading in articulated structures should be composed of concentrated loads in the nodes only, the point load on the bar should be replaced by an equivalent pair of forces at the bar nodes. The equalization of the work and the elastic strain energy becomes:

$$W = U$$

$$\Leftrightarrow 1 \times \delta_1 + 1 \times \delta_2 = \sum_{i=1}^{n} (\overline{N_i} \times \Delta L_i)$$

3.6. Practical organization of the calculation results

For calculation of several deformation values for one structure submitted to a given real loading (like in Examples 3.1 to 3.4 shown before), the calculation steps can be conveniently organized in a table as follows:

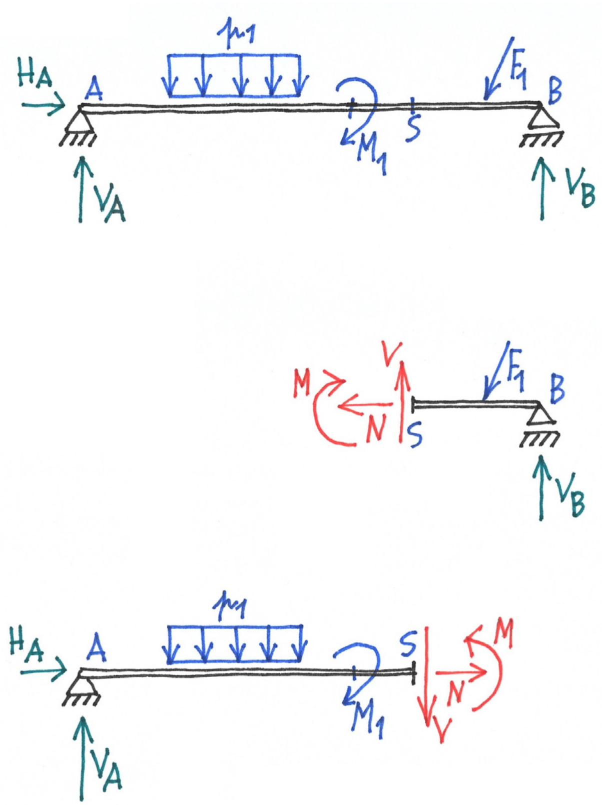
Bar	Real Loading		Calculation of δ_C^{hor}		Calculation of θ_{BC}		Calculation of $\delta_{\mathrm{relative}}$	
	N_i	ΔL_i	$\overline{N}_i$	$\overline{N}_i \cdot \Delta L_i$	$\overline{N}_i$	$\overline{N}_i \cdot \Delta L_i$	$\overline{N}_i$	$\overline{N}_i \cdot \Delta L_i$
AB								
AC								
BC								
BD								
CD								

If the real loading is unchanged, the ΔL_i values remain unchanged for the calculation of the several deformation values.

The summation of the values in this column gives the quantity $\sum(\overline{N}_i \cdot \Delta L_i)$ for calculation of δ_C^{hor}

4. Internal force diagrams in plane structures

4.1. Calculation of the diagrams by applying the equilibrium conditions



The letters N, V and M are universally adopted to identify these diagrams, where N stands for force ***N***ormal to the cross section, V stands for ***V***ertical, which is the direction of the shear force in horizontal beams (even though this is quite a peculiar attribution of letter, because many beams are not horizontal and, in those, the shear force is not vertical) and M stands for ***M***oment.

The internal force diagrams give the **resultant forces and moments** acting on any cross-section of a structure. In a plane structure, subjected to in-plane loading only, the internal forces can be characterized through the diagrams of:

- axial force (N);
- shear force (V);
- bending moment (M).

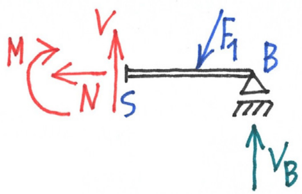
The image shows an example of a plane structure. To determine the internal forces in any cross-section S, we may start by isolating the part of the structure on one side of S and representing all the forces applied to it (applied loading, support reactions and internal forces on S). This is the so called **free-body diagram**. The free-body diagram has to be in equilibrium. Therefore, by writing the three equilibrium equations for that body, one gets:

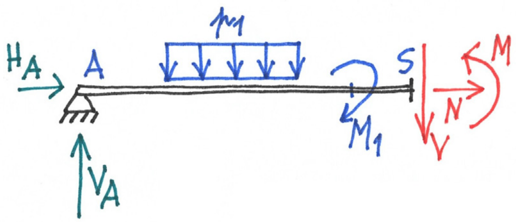
- $N = \sum F_i^{\parallel}$
- $V = \sum F_i^{\perp}$
- $M = \sum (F_i \cdot z_i + M_i)$

That is:

- N is the sum of all the force components parallel to the bar axis, applied **on one side of the cross section under analysis**;
- V is the sum of all the force components perpendicular to the bar axis, applied on one side of the cross section under analysis;
- M is the sum of concentrated moments M_i and the applied forces multiplied by the corresponding lever arms, applied on one side of the cross section under analysis.

4.2. Convention for positive internal forces, and for representation of the diagrams

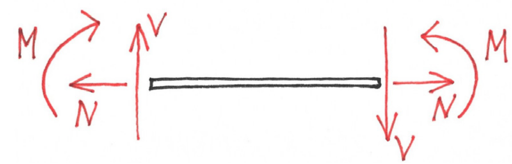


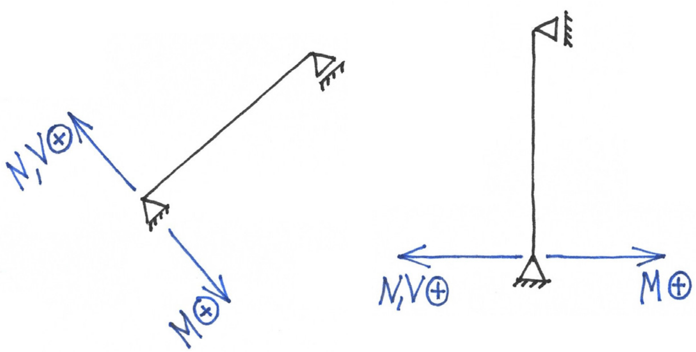


In these two free-body diagrams, the internal forces at section S are represented with their positive directions:

- in the first free-body diagram, we can see the **positive direction of forces acting on the** left **extremity of a bar segment**;
- in the second free-body diagram, we can see the positive direction of forces acting on the right extremity of a segment bar.

The convention for positive internal forces, applied to both the left and right extremities of a bar (or bar segment), can thus be summarized in an image as follows:

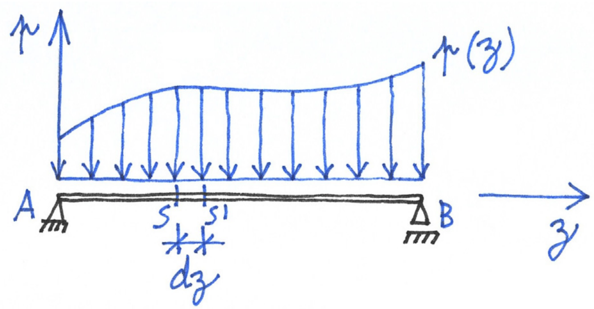


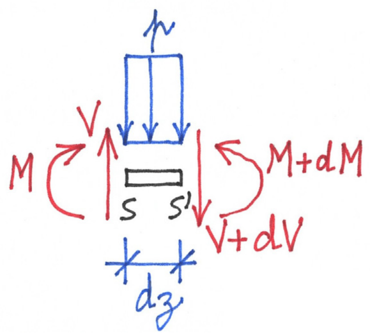


In the diagrams of internal forces, positive N and V values are represented on the upper side of the bar. **Positive M values** are represented **on the underside of the bar**. This convention for the bending moment diagram is (almost) universally adopted, so that the diagram is represented on the side of the bar **where the bar fibres are subjected to tensile stresses**.

In a **vertical bar**, the bottom extremity is usually assumed to be the “left” one, so that the positive internal force values are drawn on the sides indicated in the image.

4.3. Calculation of the diagrams by using the relationships between p, V and M





Considering that:

- $p(z)$ is the external **distributed loading** applied to a bar, in the direction **perpendicular** to the bar axis, in the position defined by the coordinate z; it has a positive sign when the direction is **downward**;
- $V(z)$ represents the equation of shear force in the bar;
- $M(z)$ represents the equation of bending moment in the bar.

The relationships between these variables are:

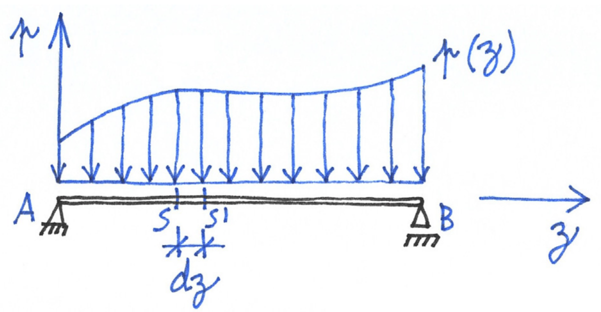
$$p(z) = -\frac{dV(z)}{dz}$$

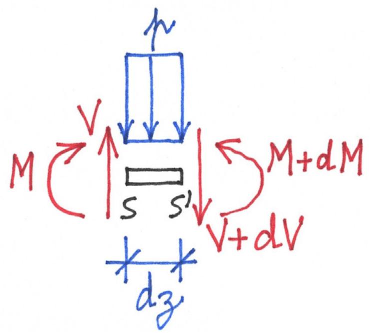
$$V(z) = \frac{dM(z)}{dz}$$

These two equations show that:

- if $p(z)$ is a polynomial function of **degree** n, $V(z)$ is a polynomial function of degree $n+1$ and $M(z)$ is of degree $n+2$;
- in the position where $V(z)=0$, $M(z)$ reaches a **local maximum or minimum**;
- if $p(z)$ is positive (i.e. it is a downward distributed load), $V(z)$ decreases for increasing z values, and $M(z)$ has an upward curvature;
- the diagram $V(z)$ has a higher **inclination** where $p(z)$ is higher; the diagram $M(z)$ has a higher slope where $V(z)$ is higher.

4.4. Demonstration of the relationships between p, V and M





Let us considerer a beam subjected to any distributed load $p(z)$, where $p(z)$ is the component of the distributed loading applied to the beam, in the direction perpendicular to the bar axis. If the distributed loading applied to the beam is not perpendicular to the beam axis, only the perpendicular component matters, for this purpose. The internal forces applied on sections S and S', separated by an infinitesimal distance dz, are shown in the second image. If the internal forces on the cross section S are V and M, they undergo variations dV and dM along the length dz.

By writing the **equilibrium conditions for the free-body diagram** represented in the second image, one gets:

$$\sum F_y = 0 \Leftrightarrow V - (V + dV) - p\,dz = 0$$

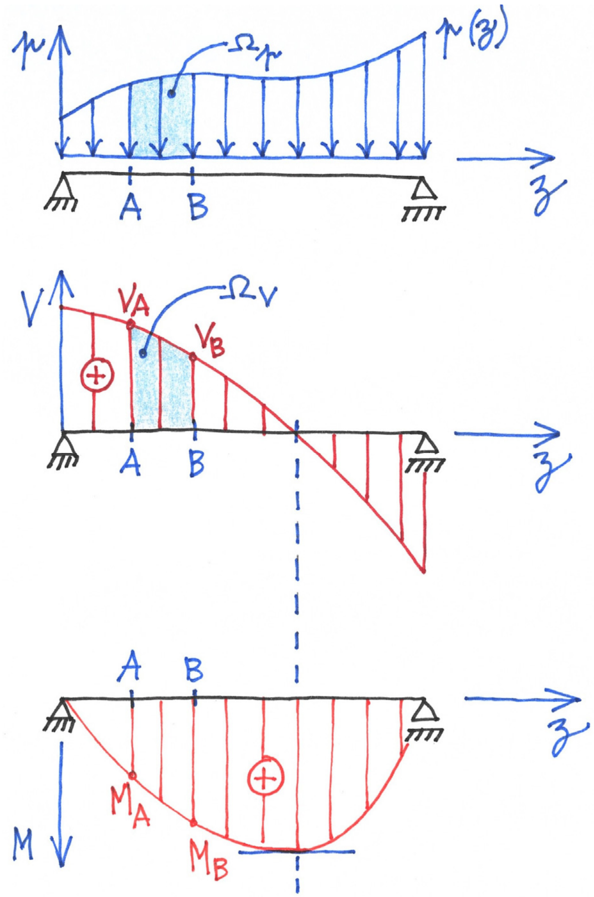
$$\Leftrightarrow p = -\frac{dV}{dz}$$

where y represents the direction perpendicular to the bar axis, and:

$$\sum m|_{S'} = 0 \Leftrightarrow M + V\,dz - p\frac{dz^2}{2} - (M + dM) = 0$$

$$\Leftrightarrow V = \frac{dM}{dz}$$

noting that $dz^2 \cong 0$ because it is a higher order infinitesimal.



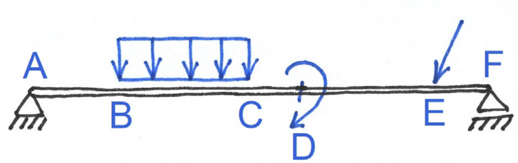
By integrating the equations $p = -dV/dz$, between two positions A and B, one concludes that:

$$\frac{dV}{dz} = -p \Leftrightarrow \int_A^B dV = -\int_A^B p\ dz \Leftrightarrow V_B - V_A = -\int_A^B p\ dz = -\Omega_p|_A^B$$

where $\Omega_p|_A^B$ is the **area enclosed by the diagram** $p(z)$ between the positions A and B. Similarly, by integrating $V = dM/dz$, between two positions A and B, one gets:

$$\frac{dM}{dz} = p \Leftrightarrow \int_A^B dM = \int_A^B V\ dz \Leftrightarrow M_B - M_A = \int_A^B V\ dz = -\Omega_V|_A^B$$

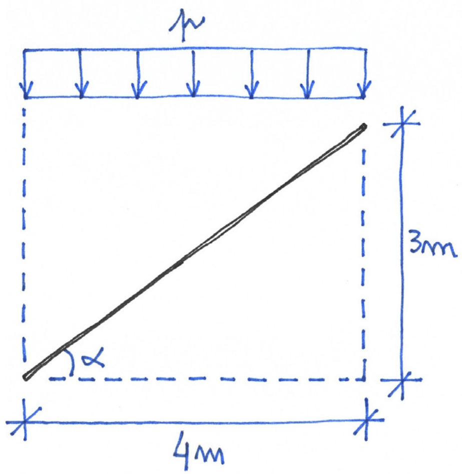
where $\Omega_V|_A^B$ is the area enclosed by the shear force diagram between the positions A and B.



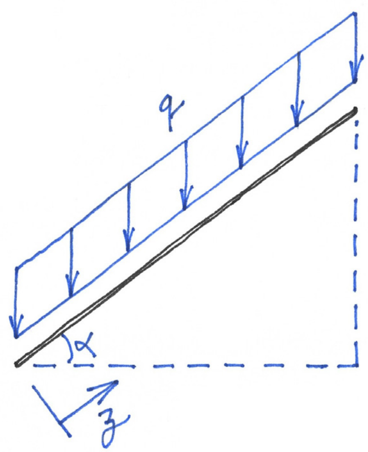
Note that:

- the **variable z** is used to characterize the position along the bar axis, because the variables x and y are reserved for characterization of the position of a point (a fibre) in the bar's cross section;
- in a certain bar, more that one equation of internal forces [$N(z)$, $V(z)$ and $M(z)$] might be needed, if there are **discontinuities** in the applied forces or concentrated moments. In the example shown in the image, separate equations would have to be written for the segments AB, BC, CD, DE and EF.

4.6. Bars inclined with respect to the uniform distributed loading



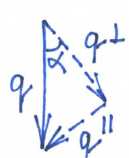
In some types of common structures, like roof beams and slabs, or stairways, **inclined beams** are subject to **vertical loading** (self-weight and other weights). In this example, an inclined beam segment is subjected to a uniformly distributed load p, which is a **quantity per unit metre measured in the horizontal direction**, i.e. its resultant is $p \cdot 4$.



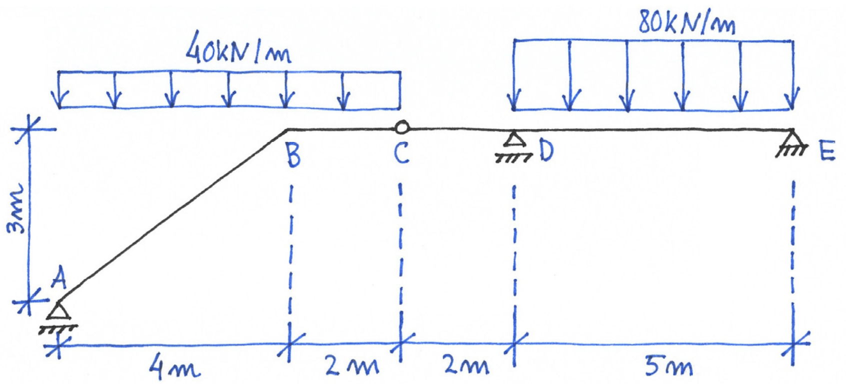
In order to determine the internal force diagrams for this inclined bar segment (which might be a part of a larger structure), it is necessary to calculate the value of the distributed load expressed in kN **per metre of bar length** (herein called load q). The determination of q is made by equalizing the resultant value of the distributed loads in the situations represented in the top and bottom images (the resultant has to be kept):

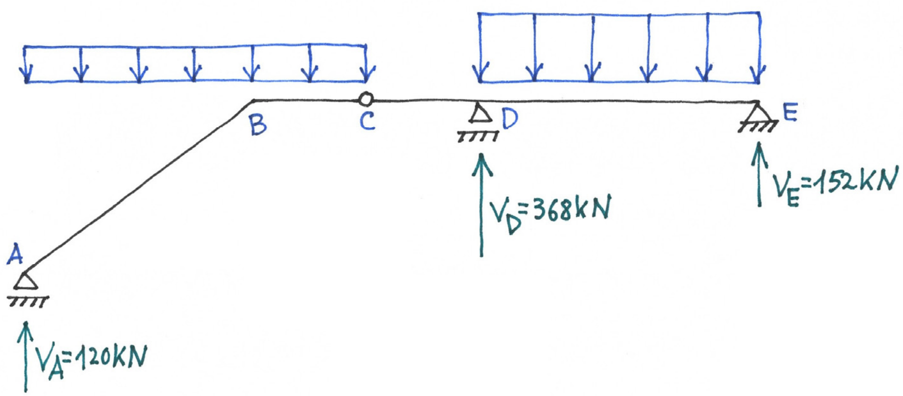
$$p \cdot 4 = q \cdot 5 \Leftrightarrow q = \frac{4}{5}\, p$$

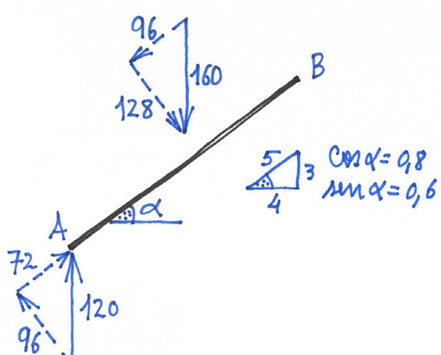
Once q is calculated, the components of this load q with the directions **parallel** and **perpendicular** to the bar axis ($q^{\parallel}$ and $q^{\perp}$) can be easily quantified through the trigonometric relations. The component $q^{\parallel}$ is to be used in the calculation of the axial force diagram. The component $q^{\perp}$ is to be used in the calculation of the shear force and bending moment diagrams of this bar.



4.7. Calculation example







Problem:

The image shows a **plane reticulated structure**, subjected to uniformly distributed loadings. The loading applied to bar AB is 40 kN per metre measured along the horizontal direction. Determine the internal force diagrams.

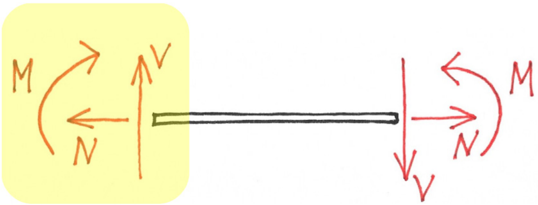
Different alternatives for calculation of the internal force diagrams will be explored.

Firstly, the **support reactions** have to be determined. This can be done by means of the three equations of global equilibrium, plus an equation of equilibrium of moments applied to the body ABC, around the hinge C. The results are shown in the image.

Bar AB – alternative 1:

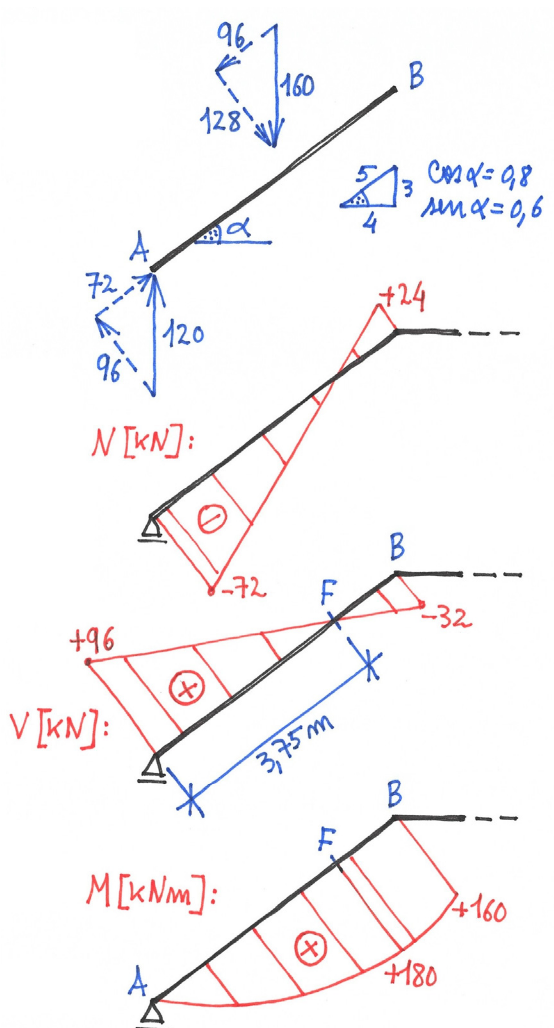
In this alternative, the diagrams will be drawn **without writing the equations** of internal forces.

The force applied on the support A, and the resultant of the distributed loading, are decomposed on their components parallel and perpendicular to the bar axis.



To determine the internal forces on the extremity A, we note that the only force applied on the left side of that point is the vertical force of 120kN (support reaction). The internal forces in this position would be positive if the direction of the applied normal and transverse forces had the directions highlighted in the image of the **convention** (with respect to the bar axis). The internal forces are thus:

- N_A = -72 kN
- V_A = +96 kN
- M_A = 0 (the resultant of moments on the left of A is zero)



To determine the internal forces on the extremity B, we **sum the resultants** of the forces applied on the left side of that position, shown in the image:

- N_A = -72 + 96 = +24 kN
- V_A = +96 – 128 = -32 kN

Noting that the shear force crosses zero between A and B, the exact position where V = 0 (herein called point F) has to be determined, because it is the position where M has a **local maximum**. Because $V(z)$ is a straight line in this bar, that position can be determined by proportionality (equivalence of triangles):

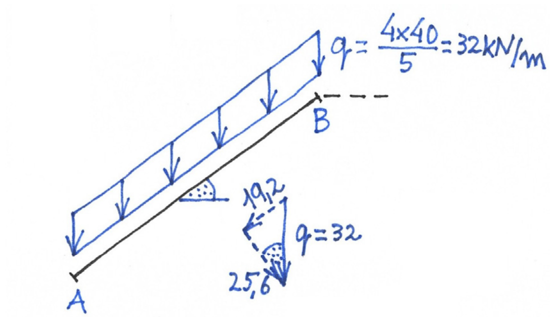
$$\frac{96+32}{96} = \frac{5}{x} \Leftrightarrow x = 3.75 \text{ m}$$

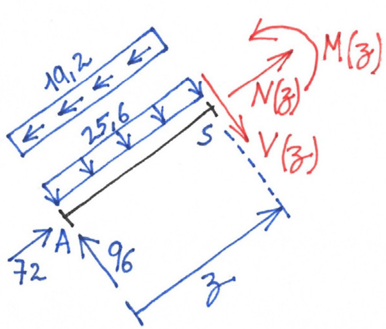
The bending moments can be calculated through the equalization of the **moment variation** and the **area enclosed by the shear force diagram**:

$$M_F - M_A = \Omega_V|_A^F \Leftrightarrow M_F - 0 = \frac{96 \times 3.75}{2} \Leftrightarrow M_F = 180 \text{ kNm}$$

The same procedure can be used to calculate M_B.

Bar AB – alternative 2:





In this alternative, the diagrams will be determined by using the **equilibrium equations** for the **free-body diagram**. The free-body diagram (shown in the second image) consists in all the forces applied to the bar segment. In this case, we are working with the bar segment starting from the leftmost bar extremity. The length of this segment is the **generic coordinate** $\boldsymbol{z}$.

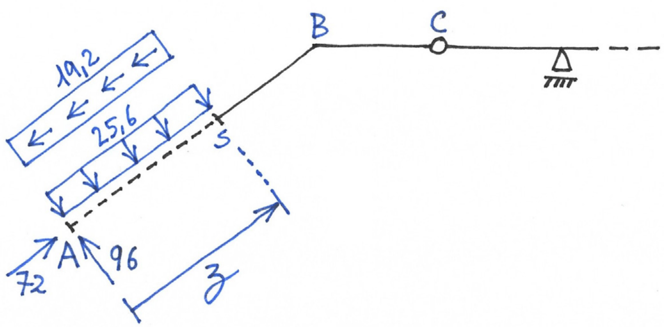
$$\sum F^{\parallel} = 0 \Leftrightarrow 72 + N(z) - 19.2\, z = 0$$
$$\Leftrightarrow N(z) = -72 + 19.2\, z$$

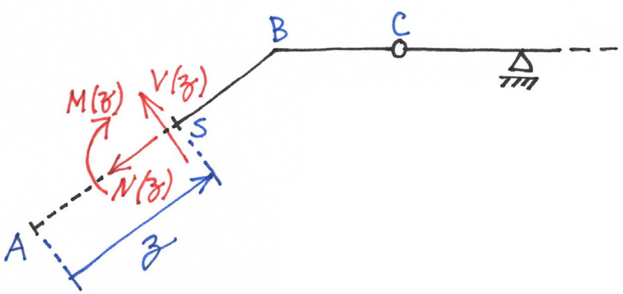
$$\sum F^{\perp} = 0 \Leftrightarrow 96 - V(z) - 25.6\, z = 0$$
$$\Leftrightarrow V(z) = 96 - 25.6\, z$$

$$\sum m|_S = 0 \Leftrightarrow 96\, z - 25.6\ z^2/2 - M(z) = 0$$
$$\Leftrightarrow M(z) = 96\, z - 25.6\ z^2/2$$

Once the expressions $N(z)$, $V(z)$ and $M(z)$ are known, the diagrams can be plotted.

Bar AB – alternative 3:





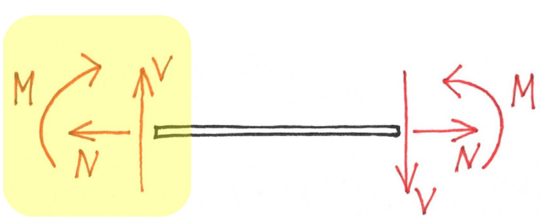
In this alternative, the diagrams will also be determined by writing the corresponding equations but, instead of writing the equilibrium equations for the free-body diagram, the rationale for calculation of the internal forces at point S (at a generic distance z from the leftmost bar extremity) will be as follows:

- the internal forces will be determined as the **resultants of the forces and moments** applied on the lefthand side of the point S;
- if the directions of the resultants, $N(z)$, $V(z)$ and $M(z)$ have directions as shown in the second image (i.e. if they correspond to the **convention** for internal forces, as shown in the third image), the internal forces are positive quantities (otherwise they are negative).

The internal forces at point S become:

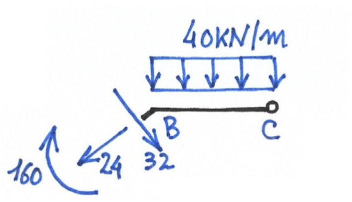
- $N(z) = -72 + 19.2\,z$
- $V(z) = 96 - 25.6\,z$

The bending moment equation can also be determined as the resultant of the forces on the lefthand side of S or, alternatively, by applying the relationship between the variation of moment and the **integral of the shear force equation** (remember that this integral is equivalent to the area enclosed by the shear force diagram):



$$M(z) - M_A = \int_0^z V(z)\,dz$$
$$\Leftrightarrow M(z) = \int_0^z (96 - 25.6\,z)\,dz$$
$$\Leftrightarrow M(z) = 96\,z - 25.6\,\,z^2/2$$

Bar BC – alternative 1

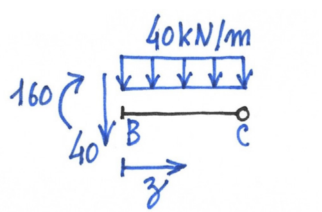


Before starting to determine the diagrams for the bar, it is useful to represent the **resultants** of **all the forces and moments applied on the lefthand side of the bar** BC. These resultants are the internal forces $N(z)$, $V(z)$ and $M(z)$ which have just been determined for the end of bar AB.

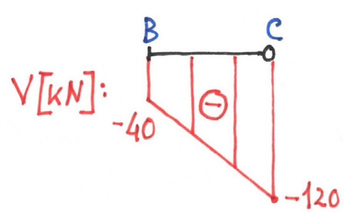
Given that the alignments of bars AB and BC are not coincident, in order to calculate $N(z)$ and $V(z)$ for bar BC, these forces of 24 kN and 32 kN shown in the image have to be decomposed into the axial and transverse directions of the bar BC.

But there is a more convenient alternative, which avoids the need to make such decomposition – see alternative 2.

Bar BC – alternative 2

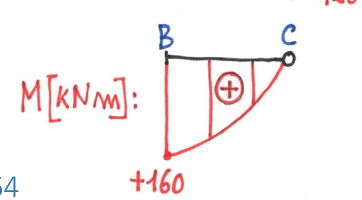


The resultants of all the forces and moments applied on the lefthand side of the bar BC can be directly quantified by **summing** the **resultants of applied loads and support reactions**. In this case, they are orthogonal to the bar segment under analysis, and therefore, by using this procedure, the force decompositions needed in alternative 1 can be avoided. These resultants correspond to a vertical downward force of 40 kN. The bending moment transmitted from bar AB to bar BC is 160 kNm (the bending moment it is not influenced by the different alignments of the two bars).

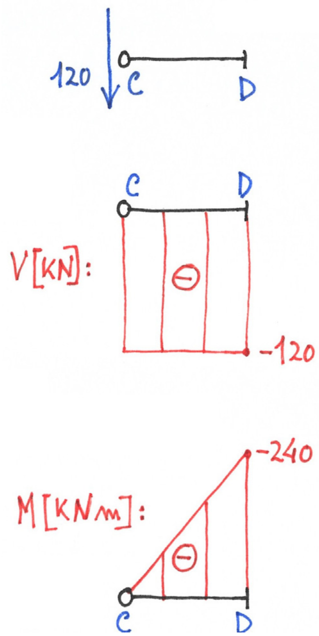


As soon as we know the resultants applied on B and the forces applied along BC (they are shown in the sketch), the diagrams can be drawn using one of the alternatives learnt before:

- $N(z) = 0$
- $V(z) = -40 - 40\,z$
- $M(z) = 160 - 40\,z - 20\,z^2$



Bar CD:

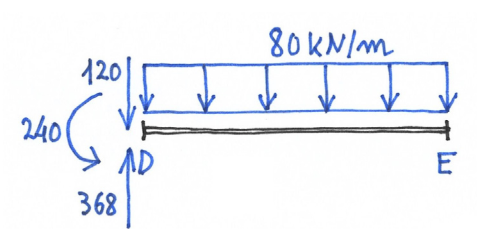


Once again, we should start by identifying the resultant internal forces applied to the leftmost bar extremity. The easiest way to achieve that is by just looking at the internal forces at the end of bar BC. Only shear force exists at this position ($V = -120$ kN), which means that the resultant on C is a vertical downward force.

We also need to identify any load applied along the bar (CD) length. In this case, no such load exists. Note that the force applied on point D (the vertical support reaction) is not needed. We are going to calculate the diagrams starting from the leftmost bar extremity and, therefore, **concentrated forces and moments applied** at the **end of the bar** do not influence the diagrams until that point is reached.

Then, starting from the leftmost bar extremity, the diagrams due to this vertical downward force can be plotted.

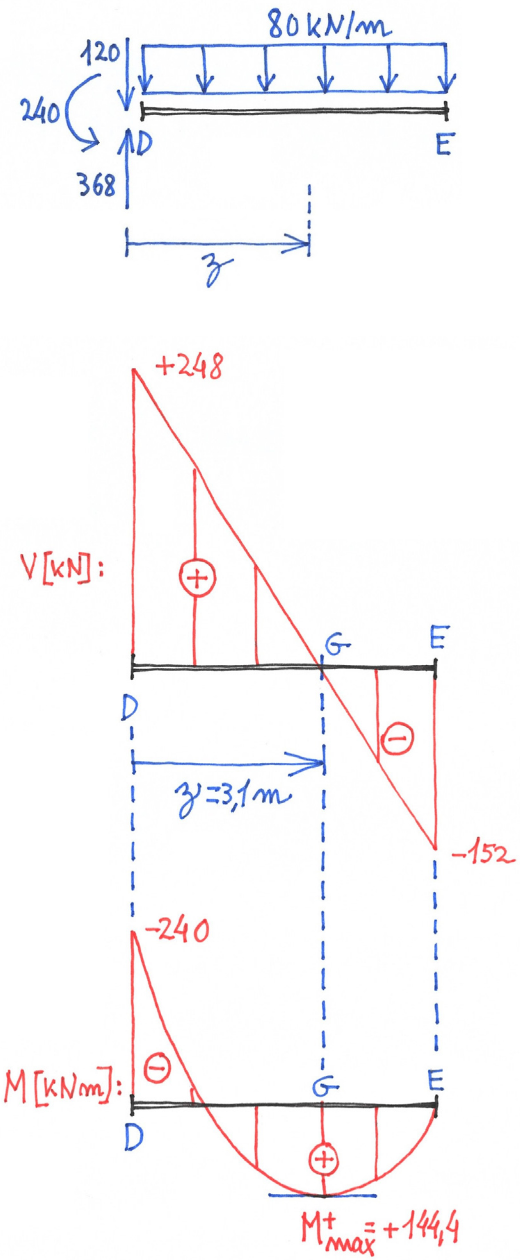
Bar DE – alternative 1:



Following the same procedure that has just been used for bars BC and CD, we should start by identifying the resultant internal forces applied to the leftmost bar extremity, which are equal to the internal forces at the end of bar CD:

- N = 0
- V = -120 kN
- M = -240 kNm

We also need the loads along the bar (DE) length, which in this case consist in the vertical reaction on support D and the distributed load of 80 kN/m. The concentrated force applied on E (the support reaction) is not used, for the same reason explained for bar CD.



After that, the expressions of $N(z)$, $V(z)$ and $M(z)$ can be written

- using the equilibrium conditions for the free-body diagram starting from the leftmost bar extremity,
- or by calculating the resultants of the loads applied to the left of the cross-section defined by the generic coordinate z:

$$N(z) = 0$$
$$V(z) = -120 + 368 - 80\,z = 248 - 80\,z$$

$$M(z) - M_D = \int_0^z V(z)\,dz$$
$$\Leftrightarrow M(z) - (-240) = \int_0^z (248 - 80\,z)\,dz$$
$$\Leftrightarrow M(z) = -240 + 248\,z - 40z^2$$

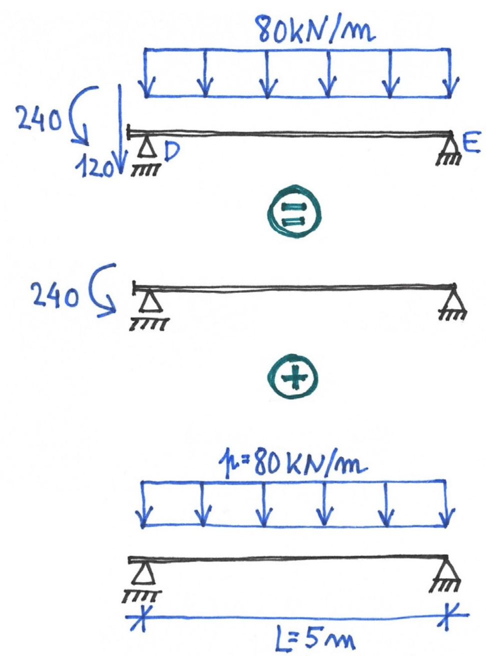
Then, the diagrams can be plotted. Given that the shear force diagram crosses zero, at the point herein called G, the coordinate of this point has to be determined:

$$V(z) = 0$$
$$\Leftrightarrow 248 - 80\,z = 0$$
$$\Leftrightarrow z = 3.1\text{ m}$$

And the corresponding local maximum of the bending moment diagram has to be calculated, too:

$$M(z = 3.1\text{m}) = +144.4\text{ kNm}$$

Bar DE – alternative 2:

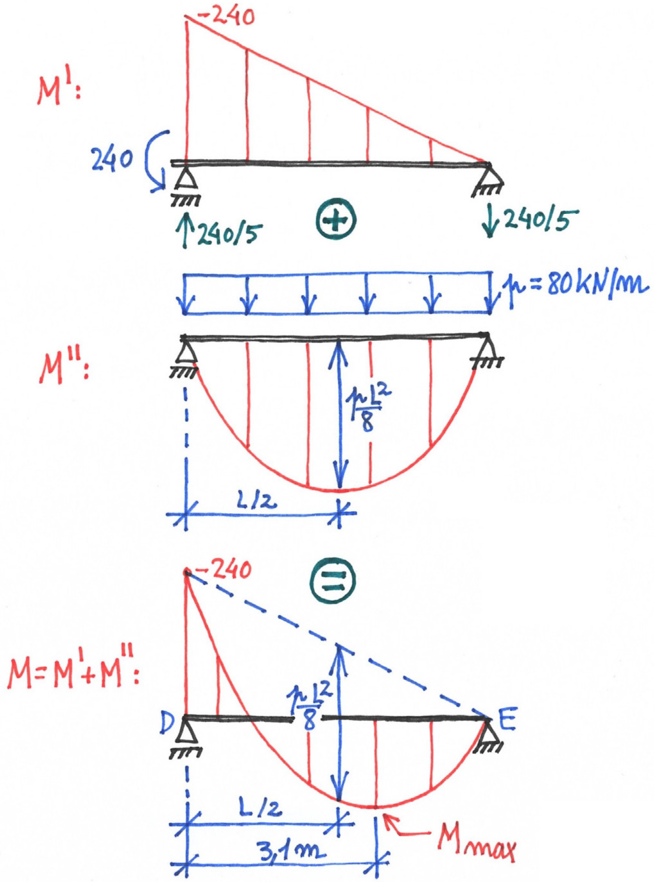


Bar DE is a bar segment, between two supports, subjected to a uniformly distributed loading perpendicular to the bar axis. This is a very common case in civil engineering structures. For this type of element, the diagrams of internal forces can be readily plotted using the following procedure:

- firstly, the structure is **cut** immediately on the left of the bar under analysis, and the internal forces on that cut are represented, so that the body shown in the first image can be isolated; these internal forces are the ones calculated before, at the end of the bar segment CD;

- secondly, one should realize that the vertical force on this cut (120 kN) does no play any influence on the diagrams of shear forces and bending moments, because the cut position coincides with a vertical support, and therefore this force is **directly transferred to** that **support**.

- finally, it is concluded that there are two loads applied to this simply supported bar DE: a bending moment on its leftmost extremity, and a uniformly distributed load of 80 kN/m; the internal forces diagrams can be determined by **superposition of the diagrams** for each of these individual loads.



The bending moment diagram due to the concentrated moment (herein called M') has to be a straight line, because the shear force diagram caused by this concentrated moment is constant along the bar DE. Given that the bending moments on sections D and E are known, the bending moment diagram M' can be depicted.

The bending moment diagram due to the **uniformly distributed loading**, in the supported beam, has the well known parabolic shape, with a maximum value at mid-span equal to $\boldsymbol{pL^2/8}$, where p is the distributed load value. This diagram is herein denoted by M''.

The total bending moment diagram is obtained as:

$$M = M' + M''$$

It can be represented by adding the parabolic line M'' to the straight line of the diagram M'. The parabolic shape of the diagram M can be represented by connecting the three known points:

- $M =$ -240 kNm at the leftmost extremity;
- $M = -240/2 + pL^2/8 =$ +130 kNm **at mid-span**;
- $M =$ 0 at the rightmost extremity.

Note that this point at mid-span ($z = L/2$) is useful to draw the diagram, but it does not coincide with the maximum positive bending moment. The latter was determined before, and amounts to:

$$M(z = 3.1\text{m}) = +144.4 \text{ kNm}$$

5. Simple plane bending

5.1. Types of bending

Even though this chapter 5 is about plane bending (pure or non-uniform), it is important to start by identifying the possible types of bending. The following **three designations** are usually adopted:

- Pure bending – when the only internal force in the bar is a **constant bending moment** (N = 0 and V = 0). It is also known as **circular bending** because the deformed axis of an initially straight bar is an arc of circumference.

- Non-uniform bending – when the **shear force V is not null**, and consequently the bending moment varies along the bar length. This designation is used only when the axial force N is null.

- Composed bending – when the **axial force N is not null**. If V is zero, the case is named pure composed bending, otherwise it is called non-uniform composed bending.

  The group formed by the two types of bending in which N = 0 can also be called the group of simple bending cases, by opposition with the case of composed bending.

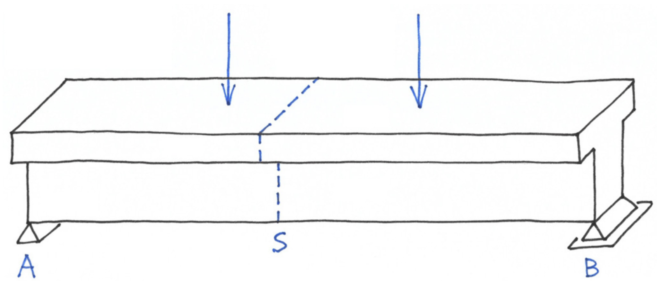
Each of the **aforementioned three types can be subdivided into**:

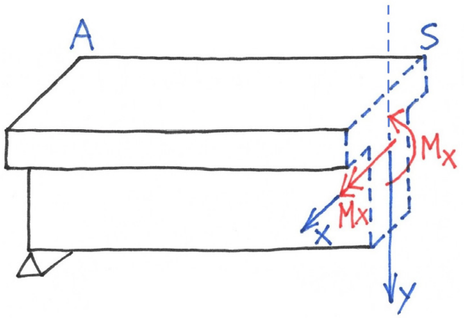
- Plane bending – when the **loading plane** contains one of the **principal axes of inertia** of the bar's cross-section. In this case, the deformed shape of the bar axis is also contained in that same plane.

- Inclined bending – when the loading plane is inclined with respect the principal axes of inertia of the bar's cross-section.

Torsion effects are not considered here, because they will be dealt with in RM2 only.

5.2. Calculation of stresses and strains due to simple plane bending





The example shown in the image is a simply supported beam, made with a **linear-elastic material**, having a cross-section with the shape of a T, symmetrical with respect to the axis Y shown in the second image. An **axis of symmetry** of a cross-section is always a **principal axis of inertia** of that cross-section.

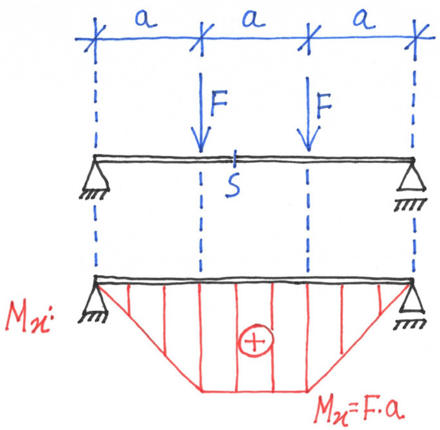
The beam is subjected to the two represented point loads, whose lines of action form a plane, the **loading plane**. This plane also contains the axis of symmetry Y, therefore this is an example of simple plane bending.

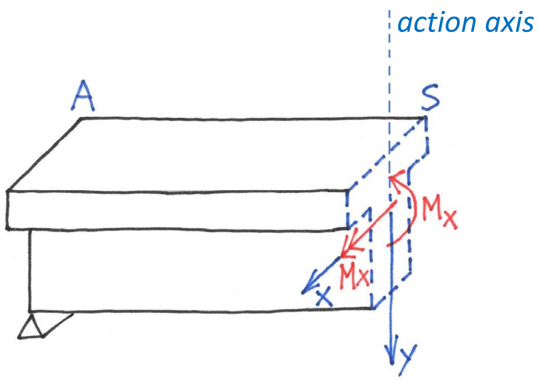
The beam cross sections are subjected to a bending moment, named $\boldsymbol{M_x}$ because it is a **moment around the principal axis $\boldsymbol{X}$** shown in the second image. It shows the moment acting on section S, **caused by loads and support reactions applied on the right side** of that section.

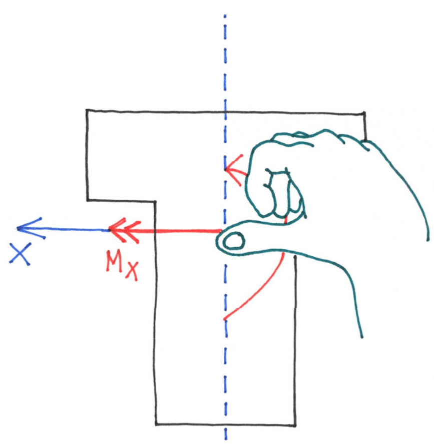
Two alternatives to represent the moment M_x are shown:

- the **curved arrow** around the axis X;
- the **moment vector $\boldsymbol{M_x}$**, with the direction of the axis X (moment vectors are represented by **double-headed straight arrows**)

The figure also shows the **action axis** in cross-section S. It is the intersection between the cross-section plane and the plane containing the applied loads. **The moment vector is perpendicular to the action axis**.



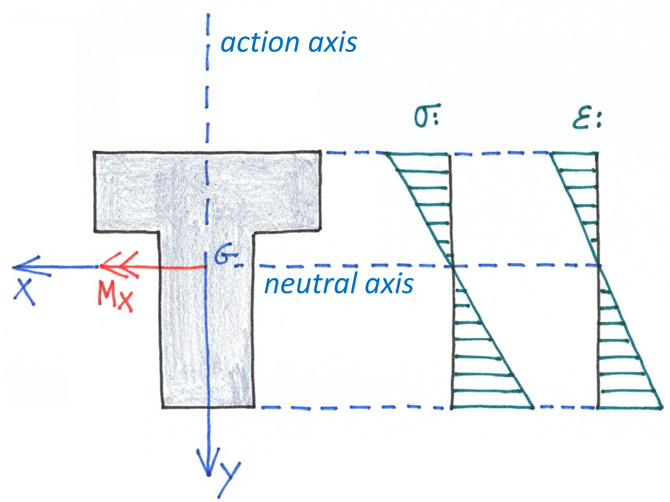




The beam in this example is subjected to a **positive bending moment** M_x, which, consequently, is responsible for tensile stresses on the bottom fibres of the cross-section, and compression on the top fibres. The representation of the curved arrow M_x is in accordance with these **tensile and compression effects**.

The direction of the moment vector is in accordance with the orientation of the curved arrow, and is determined by the **right-hand rule**:

> by curling the fingers of the right hand, so that they follow a rotation around the axis X and the fingers' tips press against the compressed region of the cross-section, the thumb of the right hand indicates the direction of the moment vector.



The image shows the cross section of this example, where the following entities are also represented: the geometrical centre G, the principal axes of inertia X and Y, the action axis and the moment vector M_x.

For calculation of the **normal stresses in the cross section**, σ_z, due to the applied bending moment M_x, an orthogonal system of axes, formed by the **principal axes of inertia of the cross section** (herein called X and Y), has to be used.

The intersection of the principal axes coincides with the **geometrical centre** of the cross section. An axis of symmetry is always a principal axis. In this example, Y is an axis of symmetry.

Plane bending is therefore the case in which:

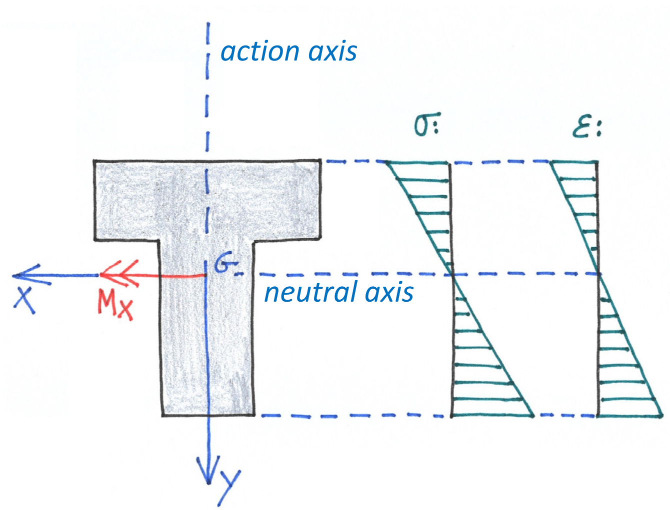
- the action axis is parallel to one of the principal axis of inertia of the cross section;
- the moment vector is parallel to the other principal axis.

The normal stress σ_z, in a fibre whose position is given by the coordinate y, is calculated as:

$$\sigma_z = \frac{M_x}{I_x} y$$

where I_x is the principal moment of inertia around X. The stress σ_z is therefore a **linear function of the coordinate y**. The diagram of stresses σ_z is represented as shown the image.

This equation is only applicable in the case of a **linear elastic material behaviour**.



The straight line containing the points where the stress is null is called the **neutral axis**. It is defined, in this case, by the equation:

$$y = 0$$

In the equation $\sigma_z = (M_x/I_x)\,y$, the moment $\boldsymbol{M_x}$ **is** a **positive** quantity when the **moment vector points in direction of the positive side of axis** $\boldsymbol{X}$ (i.e. when tensile stresses are applied to the bottom fibres of the cross section). The calculated stress σ_z becomes **positive** when it is a **tensile stress**.

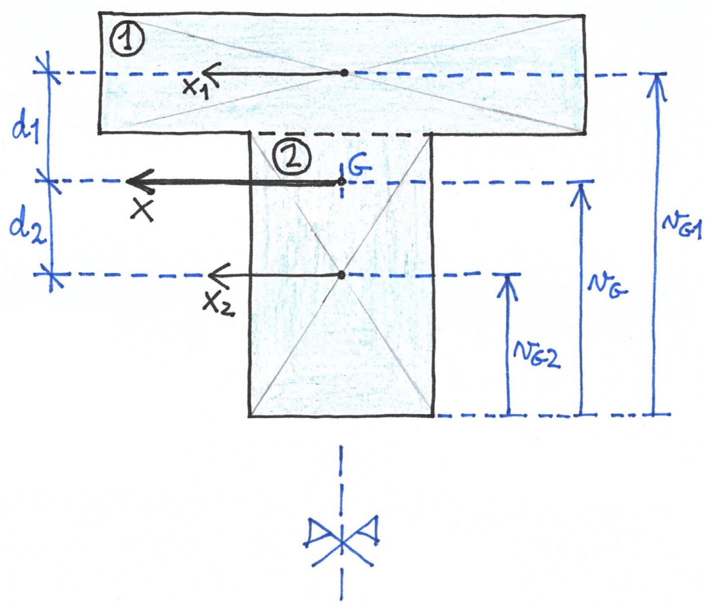
In an horizontal beam, the axes X and Y are usually used as shown in the image (X points **to the left**, Y points **downward**), so that the aforementioned sign conventions can be used: a positive moment M_x induces tensile stresses σ_z in fibres of positive coordinate y.

Given that the material is subjected to normal stress in the direction Z only, and the material has a linear elastic behaviour characterized by the modulus of elasticity E, the **strain** in that same direction (i.e. normal to the cross section) becomes:

$$\varepsilon_z = \frac{\sigma_z}{E}$$

For the sake of brevity, the stress σ_z and strain ε_z are frequently denoted by σ and ε only.

5.3. Calculation of the geometric properties of the cross-section



The image shows an example of a generic cross section composed by $\boldsymbol{n}$ **elementary parts**, in this case n = 2. An elementary part in this context can be either:

- a **simple plane shape**, like a rectangle, a triangle, a circle, half a circle, etc., for which their geometric properties (centre coordinates, area and moments of inertia) are given by known **closed formulas** [in this course unit, for elementary shapes other than the rectangle, their properties are give in the exam formulary];

- a standardized **metallic profile section** (like IPE and HEA beams, among others), whose geometric properties are given in **tables**.

If the **cross-section** is **symmetric** with respect to one axis (like the example in the image), it is known that:

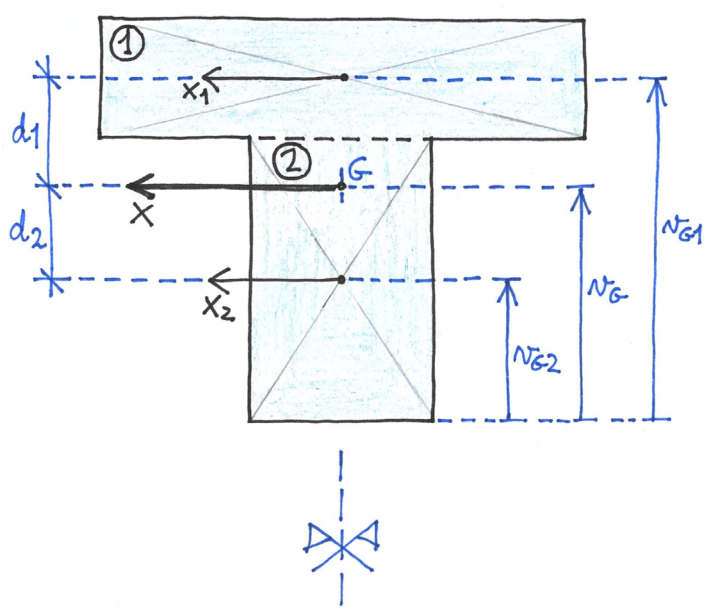
- its geometric centre, G, is contained in the axis of symmetry;

- the axis of symmetry is a principal axis of inertia, the other principal axis is perpendicular to the line of symmetry.

The **geometric centre** $\boldsymbol{G}$ is the average position of the areas A_i forming the cross-section.

The position of point G with respect to a certain **reference line** (the bottom line of the cross-section was selected as reference, in the image), herein named v_G, is given by:

$$v_G = \frac{\sum_{i=1}^{n}(A_i \cdot v_i)}{\sum_{i=1}^{n} A_i}$$



where A_i is the area of the elementary part i and v_i is the distance to the geometrical centre of the elementary part i, with respect to the refence line, as shown in the image.

The **moment of inertia** I_x around the principal axis X is given by the **Steiner's theorem**:

$$I_x = \sum_{i=1}^{n}(I_{xi} + A_i \cdot d_i^2)$$

where I_{xi} is the moment of inertia of the elementary part i, around the axis X_i, parallel to X, that crosses the geometric centre of the part i. The distance between the axes X_i and X is d_i.

If the elementary part i is a rectangle, it is well known that:

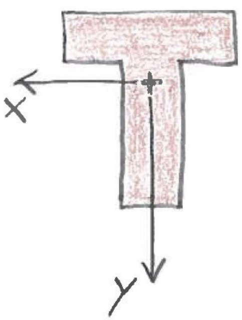
$$I_{xi} = \frac{B_i \cdot H_i^3}{12}$$

where B_i and H_i are the dimensions of the rectangle sides, parallel and perpendicular to X, respectively.

The moment of inertia around the principal axis Y, I_y, is calculated in an analogous manner.

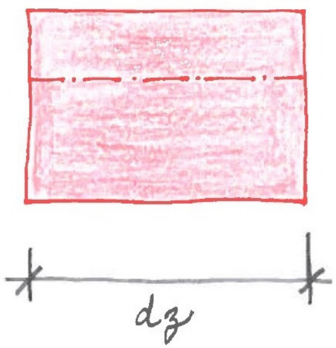
5.4. Curvature (χ) and radius of curvature (ρ) in plane bending

Generic cross section:

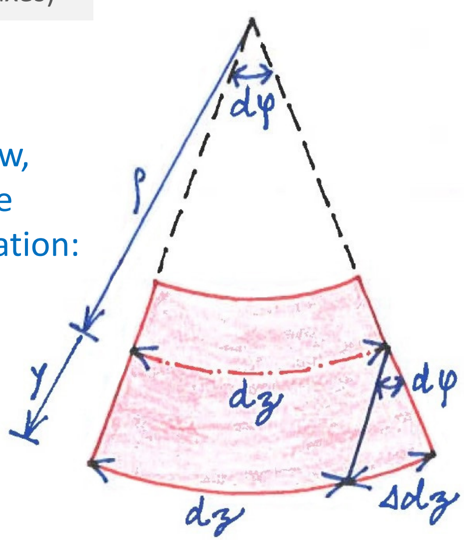


(X and Y are the principal axes)

Side view, before the deformation:



Side view, after the deformation:



Let us first consider the case of a prismatic bar segment, subjected to a constant bending moment (pure bending). The deformation of this type of member obeys to the **Bernoulli hypothesis** of **plane sections**:

> *In a prismatic bar under constant axial force and constant bending moment, the cross-sections remain plane and perpendicular to the axis of the bar during the deformation*.

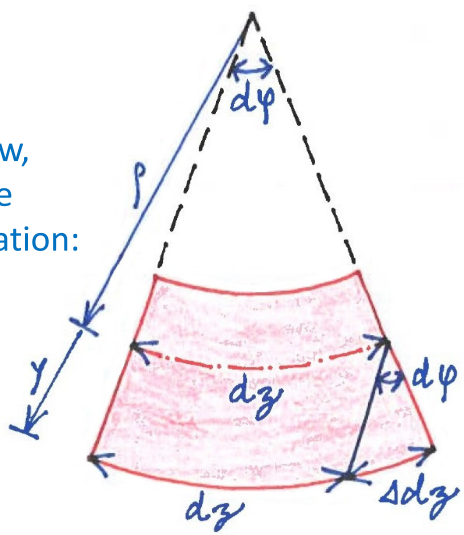
The image shows an example of plane pure bending. In fibres with coordinate y = 0, the normal stress σ_z is null, therefore the length dz of a bar segment before the deformation will be kept after the deformation in fibres with y = 0.

As the bending moment is constant, the deformed shape of the bar axis is an **arc of circumference**. If $d\varphi$ is the angle in radians defining the **relative rotation of the end cross-sections**, and ρ is the **radius of curvature** as shown in the image, it can be geometrically concluded that:

$$dz = \rho \cdot d\varphi$$

$$\Delta dz = y \cdot d\varphi$$

Side view, after the deformation:



The strain of a fibre of coordinate y can therefore be written as:

$$\varepsilon = \frac{\Delta dz}{dz} = \frac{y \cdot d\varphi}{\rho \cdot d\varphi} = \frac{1}{\rho} \cdot y$$

The **curvature** χ of a plane curve is, by definition, the inverse of the radius of curvature. Consequently:

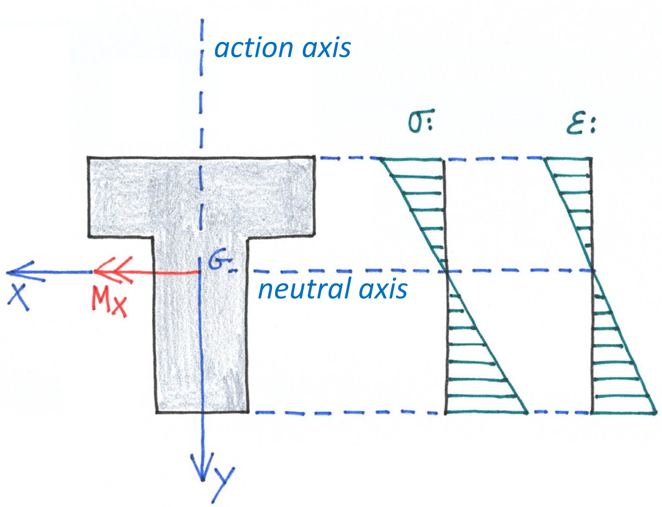
$$\varepsilon = \chi \cdot y$$

which means that the strain ε [adimensional] at a given fibre is the product of the cross-section curvature χ [m^{-1}] and the fibre coordinate y [m].

It is thus demonstrated that the **strain** ε **varies linearly** throughout the cross-section depth.

The explanation shown before in this sub-chapter 5.4 was made for the case of pure bending ($V = 0$). If, on the contrary, the shear force is not null, the relationships between ρ, χ, y and ε are the same. But the values of ρ and χ will vary for varying bending moment values.

5.5. Demonstration of the equation for calculation of the stresses due to simple plane bending



The internal forces applied to a cross section are the system of forces resulting from the stresses installed on that cross section.

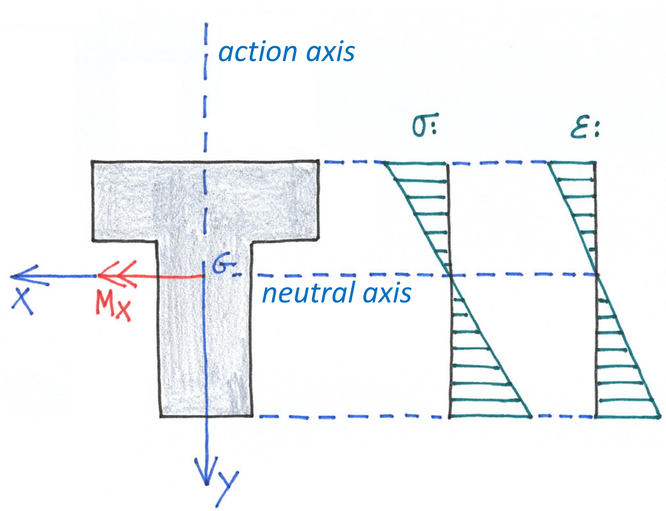
Therefore, if the principal axes of inertia are considered, the fact that the axial force is zero implies that the integral of the normal stresses is also zero, that is:

$$\int_A \sigma \, dA = 0 \Leftrightarrow \int_A \frac{E\,y}{\rho} dA = 0 \Leftrightarrow \int_A y \, dA = 0$$

The integral $\int y \, dA$ represents the first moment of area of the cross section with respect to the axis X. Therefore, the previous equation simply states that this integral is zero because the axis X crosses the centroid of the cross-section.

The resulting moment of the normal stresses in the cross-section has to be equal to the applied bending moment M_x, that is:

$$\int_A \sigma \, y \, dA = M_x \Leftrightarrow \int_A \frac{E\,y}{\rho} y \, dA = M_x \Leftrightarrow \frac{E}{\rho} \int_A y^2 \, dA = M_x$$



The integral $\int y^2 dA$ represents the moment of inertia I_x (also named the second moment of area of the cross section). Therefore, the previous equation becomes:

$$\frac{E}{\rho} I_x = M_x \Leftrightarrow \frac{1}{\rho} = \frac{M_x}{E I_x}$$

this being a very useful relationship between the curvature radius ρ, the bending moment M_x, and the term EI_x (which is named the **bending stiffness**).

By introducing, in the previous equation, the already known relationships:

$$\varepsilon = \frac{1}{\rho} \cdot y$$

and

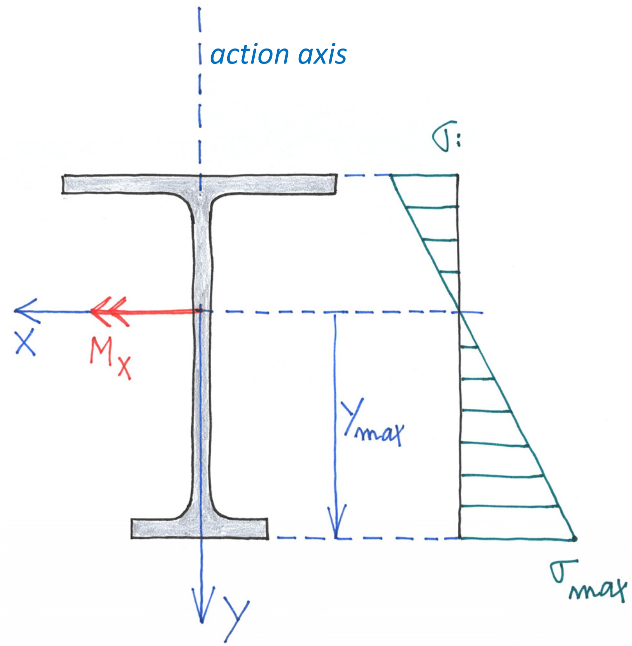
$$\sigma = E\,\varepsilon$$

it can be concluded that it is equivalent to:

$$\sigma = \frac{M_x}{I_x} y$$

which was to be demonstrated.

5.6. Bending modulus



The **bending modulus** around the axis X, W_x, is defined as:

$$W_x = \frac{I_x}{y_{max}}$$

where y_{max} is the absolute value of the coordinate of the cross-section fibre **farthest from the axis**. The maximum normal stress due to bending can thus be expressed as:

$$|\sigma_{\max}| = \frac{|M_x|}{I_x} y_{max} = \frac{|M_x|}{W_x}$$

It is already known that the condition of safety in **Ultimate Limit State** is:

$$\sigma_{Sd} \leq \sigma_{Rd}$$

$$\Leftrightarrow 1.5 \cdot |\sigma_{\max}| \leq \sigma_{Rd}$$

Therefore, in the case of materials with **equal tensile and compressive strengths** (e.g. steel), it becomes:

$$1.5 \cdot \frac{|M_x|}{W_x} \leq \sigma_{Rd}$$

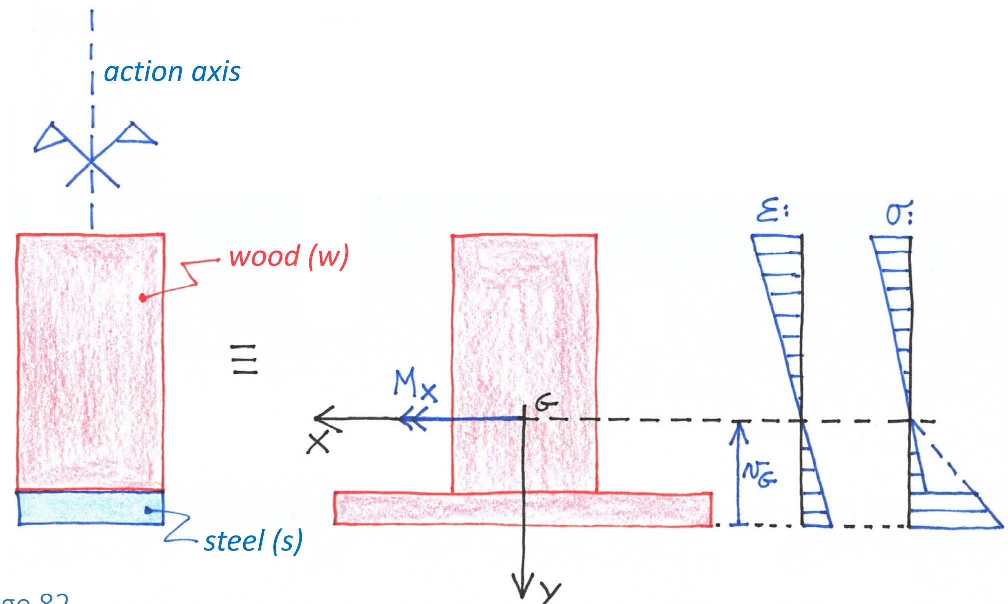
The previous equation is useful in **design problems**, in which M_x and σ_{Rd} are known, and the objective is the determination of the required cross-section size, so that the Ultimate Limit State condition is met. This equation allows to calculate the **required W_x value**, and then it is possible to find (e.g. in a table of steel profiles) a convenient cross-section geometry.

The bending modulus determined in this page is used in calculations considering a **linear-elastic material behaviour** (governed by the Hooke's law). It can thus be named $W_{x,\mathrm{el}}$.

5.7. Sections made of two materials, subjected to simple plane bending

The calculation of stresses and strains, in a cross section made of two materials, with linear elastic behaviour, subjected to bending, can be made using the **method of the cross-section homogenization**, in an analogous manner to the procedure presented in subchapter 1.6 for bars subjected to axial force. The image illustrates this procedure for a cross section formed by steel (s) and wood (w) parts, perfectly bonded, subject to a bending moment M_x around the principal axis X.

The method consists in transforming the real cross-section in an ***equivalent***, **homogenized cross-section**. The term *equivalent* means in this context that a bar with the homogenized cross-section has the same bending stiffness, the same curvature χ and the same stress in the homogenized material as the real bar made with the two materials.



If we choose, for the material of the homogenized cross-section, the material of lower modulus of elasticity, the **ratio of the modulus of elasticity**, k_{hom}, takes a value higher than 1, that is

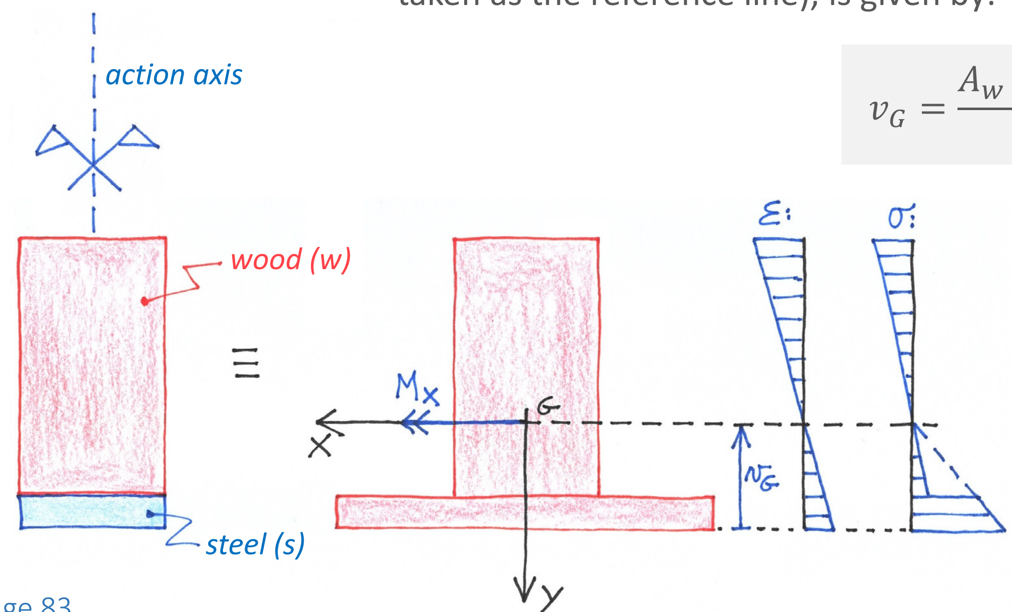
$$k_{hom} = \frac{E_s}{E_w}$$

where E_s and E_w are the modulus of elasticity of the steel and wood parts, respectively. The cross-sectional area of the homogenized cross-section is:

$$\bar{A} = A_w + A_s \cdot k_{hom}$$

where A_w and A_s are the real cross-sectional areas of the wood and steel parts. The point G in the image denotes the **centre of rigidity**. Whilst a geometrical centre is, by definition, the average position of the areas A_i, the centre of rigidity is the **average position of the stiffnesses** $E_i\ A_i$. The axis of symmetry contains the point G. Its position along the vertical axis, measured with respect to a certain reference line (in the image, the **bottom layer** of the cross-section was taken as the reference line), is given by:

$$v_G = \frac{A_w \cdot v_{G,w} + A_s \cdot v_{G,s} \cdot k_{hom}}{\bar{A}}$$



where $v_{G,w}$ and $v_{G,s}$ are the distances from the reference line to the centroids of the wood and steel parts, respectively. And the moment of inertia of the homogenized cross section, around the principal axis X, is:

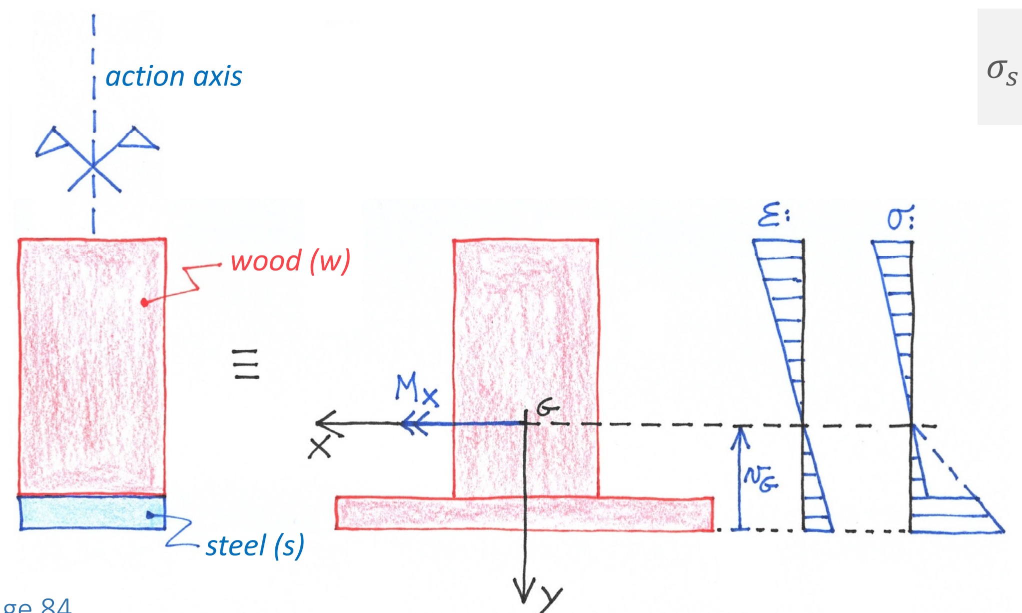
$$\overline{I_x} = I_{x,w} + I_{x,s} \cdot k_{hom}$$

where $I_{x,w}$ and $I_{x,s}$ are the moments of inertia, with respect to X, of the elementary areas of wood and steel. They are therefore determined through the Steiner's theorem.

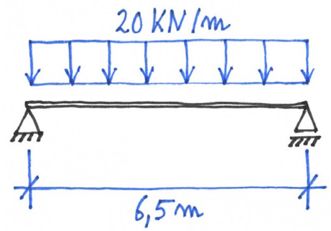
Owing to the Bernoulli's hypothesis of plane sections, the **strain** diagram **varies linearly over the cross-section depth**, as in the image. The stress diagrams in both materials are given by:

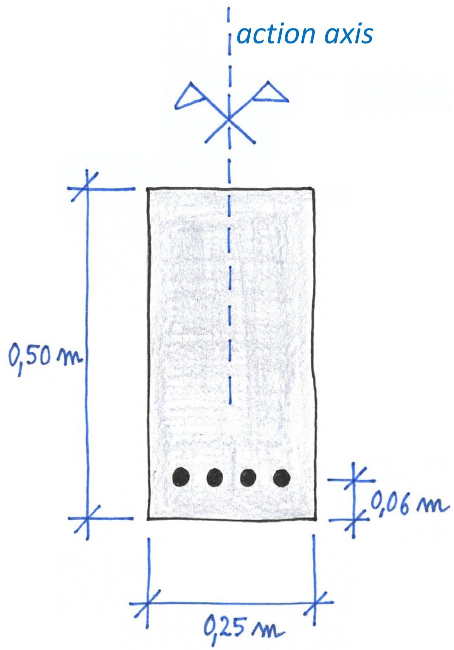
$$\sigma_w = \frac{M_x}{\overline{I_x}}\, y$$

$$\sigma_s = \frac{M_x}{\overline{I_x}}\, y \cdot k_{hom}$$



5.8. Reinforced concrete beams, subjected to simple plane bending, considering that the tensile strength of concrete is null





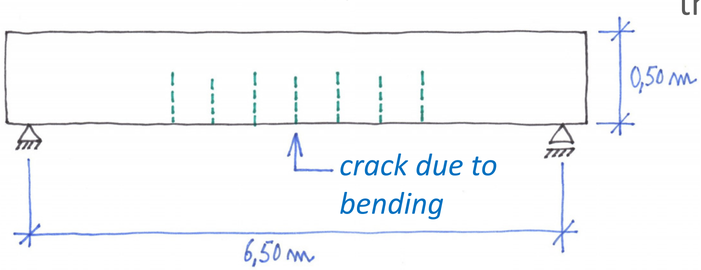
Example 5.1:

A typical case of a cross section made of two materials is the **reinforced concrete beam**. The image shows one example of this type of beam, in which a rectangular concrete cross section is reinforced with 4 bars with a diameter of 20 mm. The tensile strength of concrete is very small (by comparison with its compressive strength) and consequently cracks are formed in the tensile region, as shown in the schematic image at the bottom of the page.

Concrete cracking due to applied actions is normal, if a sufficient amount of steel reinforcement is provided to resist the tensile forces, and also to keep the **crack width** under acceptable limits.

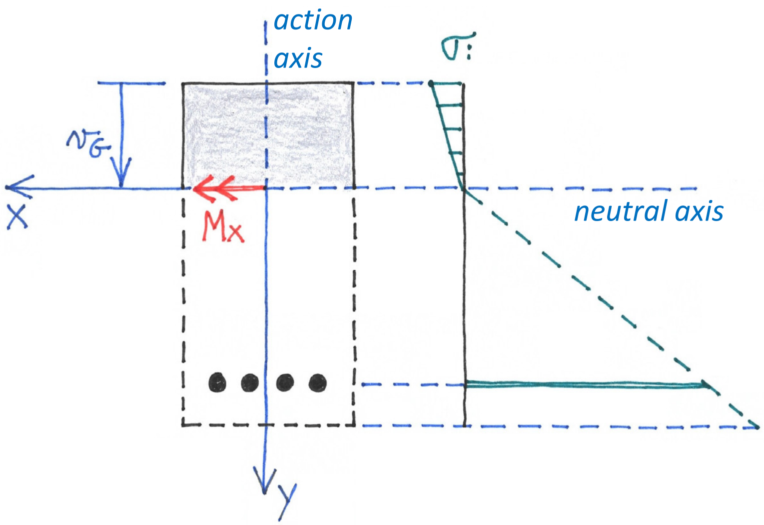
The control of cracking and the calculation of crack widths is out of the scope of this course unit. But, the calculation of stresses at the cross section of a crack is a topic that is dealt with here.

Once a crack is formed, it gives rise to a material discontinuity, which does not allow the transmission of tensile stresses. Consequently, the calculation of normal stresses is made by **assuming that tensile strength of the concrete is zero**, the tensile stresses being resisted by the steel reinforcement only. And the concrete resists to the compression stresses only.



Additional material and geometrical data for this calculation example:

- steel: E_s = 200 GPa
- concrete: E_c = 25 GPa
- cross-sectional area of steel: ϕ = 20mm $\Longrightarrow A_s$ = 3.142 cm^2 × 4 bars



That means the **effective cross section** will be formed by the concrete region above the neutral axis only (shaded region, in the image), which will be in compression, plus the steel bars, which will be in tension.

If we choose concrete as the material of the homogenized cross-section:

$$k_{hom} = E_s/E_c = 200/25 = 8$$

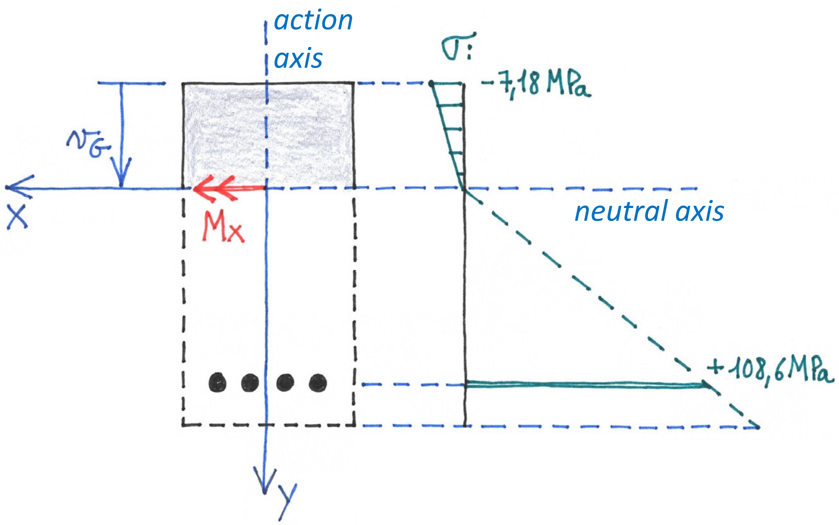
Before the calculations are made, the **depth of the compressed concrete region** is not known. But that depth (v_G in the image) can be determined, through the formula for calculation of the centre of rigidity, because it will be the only unknown in the problem of determination of the position of that centre:

$$v_G = \frac{0.25 \cdot \frac{v_G^2}{2} + A_s \cdot 0.44 \cdot k_{hom}}{0.25 \cdot v_G + A_s \cdot k_{hom}}$$

$$\Leftrightarrow v_G = \frac{0.25 \cdot \frac{v_G^2}{2} + 4 \cdot 0.0003142 \cdot 0.44 \cdot 8}{0.25 \cdot v_G + 4 \cdot 0.0003142 \cdot 8}$$

By solving this second order polynomial equation, we get:

$$v_G = 0.1522 \text{ m}$$



Once the value of v_G has been determined, the effective cross section, which supports the applied bending moment, becomes fully characterized.

The moment of inertia of this homogenised, effective, cross section can thus be calculated. The result is:

$$I_x = 0.002239 \text{ m}^4$$

The **maximum stresses** in the beam under analysis will occur at the (cracked) mid-span cross section, where the applied bending moment is highest:

$$M_{x,max} = \frac{p\,L^2}{8} = 105.6 \text{ kNm}$$

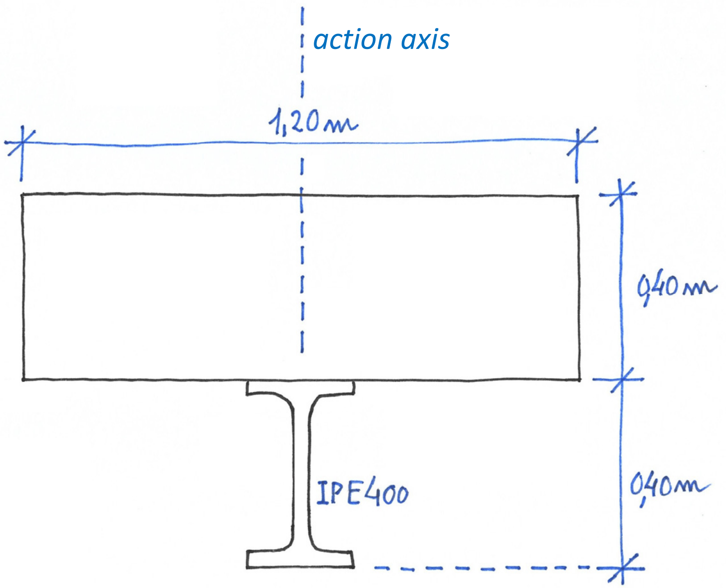
The highest concrete compression stress becomes:

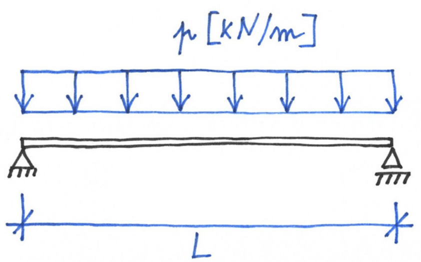
$$\sigma_c = \frac{M_x}{\overline{I_x}}\, y = \frac{105.6}{0.002239} \cdot (-0.1522) = -7.18 \cdot 10^3 \text{ kPa}$$

The highest steel stress, **calculated at the centre of the steel area**, becomes:

$$\sigma_c = \frac{M_x}{\overline{I_x}}\, y\, k_{hom} = \frac{105.6}{0.002239} \cdot (0.44 - 0.1522) \cdot 8 = +108.6 \cdot 10^3 \text{ kPa}$$

5.9. Steel-concrete composite beams, subjected to simple plane bending, considering that the tensile strength of concrete is null





Additional data:

- steel: E_s = 206 GPa
- concrete: E_c = 20.6 GPa; γ = 25 kN/m^3
- L = 8 m
- p = 60 kN/m plus the beam self-weight

Example 5.2:

Another typical case of a cross section made of two materials is the beam composed by a steel profile, rigidly bonded to a concrete element, as shown in the image.

Once again, the **tensile strength of concrete is disregarded**, i.e., it is taken as zero. By doing so, we are considering that, if tensile stresses are applied to concrete, it will crack, and therefore the **effective cross-section** (the one which effectively resists the applied bending moment) is composed by the region of concrete under compression, and the steel member.

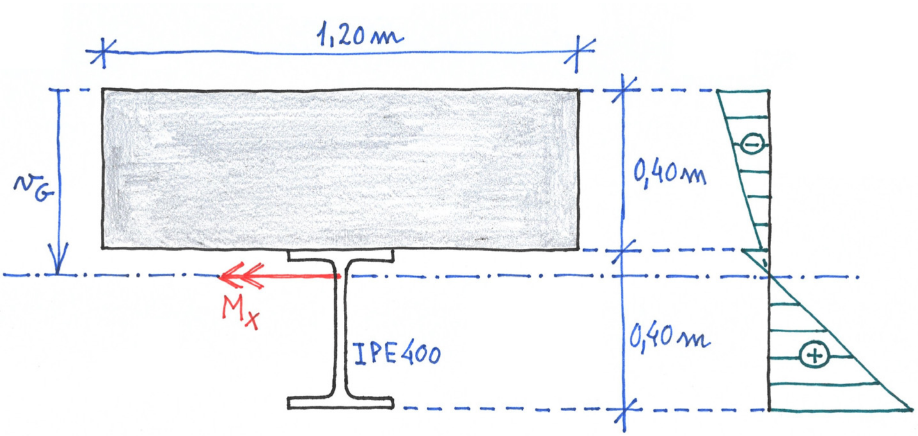
However, unlike Example 5.1, in this Example 5.2 we do not know whether:

i. all the concrete member is effective (this is the case if all the concrete is under compression);
ii. or, on the contrary, cracks are formed and only a certain part of the concrete member is effective.

Consequently, we need to check, by calculations, **which of these hypotheses** i. **or** ii. **is the true one**. That depends on the geometry and material properties of the concrete and steel members.

We will choose concrete as the material of the homogenized cross-section.

Hypothesis i:

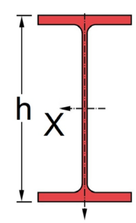


Given that the beam is subjected to a positive bending moment M_x, **all** the concrete member will be under compression if the neutral axis crosses the steel part, i.e., if the coordinate v_G shown in the image is not lower than 0.40 m. If the hypothesis was true, then the normal stress diagram would have the shape represented in the image.

Assuming, by hypothesis, that the entire concrete area s effective, the calculation of the coordinate v_G gives:

$$v_G = \frac{A_c \cdot v_{G,c} + A_s \cdot v_{G,s} \cdot k_{hom}}{\bar{A}}$$

where the subscripts c and s denote concrete and steel, respectively. The geometrical properties of the IPE400 cross-section can be found in a table of steel profiles:



- A = 84.5 cm^2
- I_x = 23130 cm^4
- h = 40 cm

It becomes:

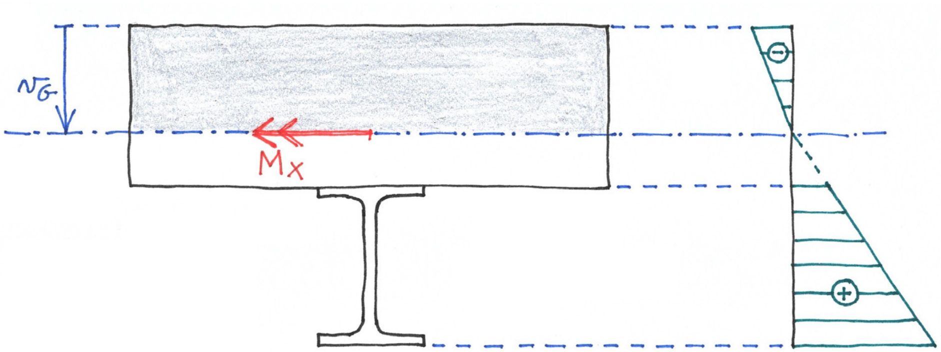
$$\bar{A} = A_c + A_s \cdot k_{hom} = 120 \cdot 40 + 84.5 \cdot 10 = 5645 \text{ cm}^2$$

$$v_G = \frac{120 \cdot 40 \cdot 20 + 84.5 \cdot 60 \cdot 10}{5645} = 25.99 \text{ cm}$$

It is lower than 40 cm, which **means that** the **hypothesis is not the true one**. The depth of concrete in compression will be lower than 40 cm.

Hypothesis ii:

The correct v_G value should therefore be calculated considering that **only a portion of concrete**, the one above the neutral axis, **will be effective**. It will be different from the previously determined v_G = 25.99 cm under the hypothesis (which turned out to be false) of entire concrete in compression. The true v_G value will therefore de determined as:



$$v_G = \frac{120 \cdot v_G \cdot v_G/2 + 84.5 \cdot 60 \cdot 10}{120 \cdot v_G + 84.5 \cdot 10}$$

$$\Leftrightarrow 60 \cdot v_G^2 + 845 v_G - 50700 = 0$$

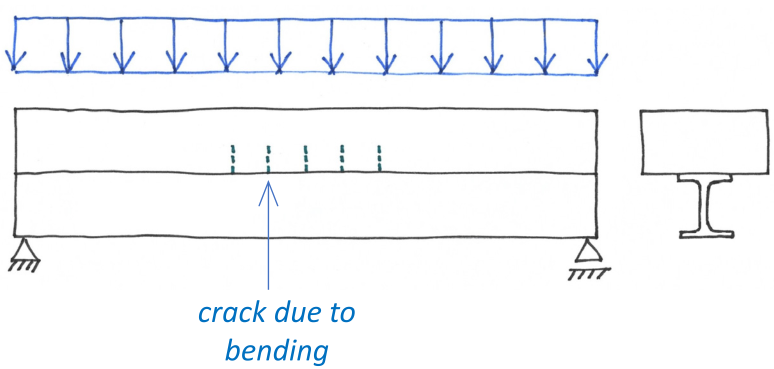
$$\Leftrightarrow v_G = 22.87\ cm$$

That means that bending cracks will occur, as shown schematically in the side view of the structure, at the bottom of the page.

The moment of inertia, of the homogenized section, around the principal horizontal axis, can now be calculated, considering only the effective portion of concrete:

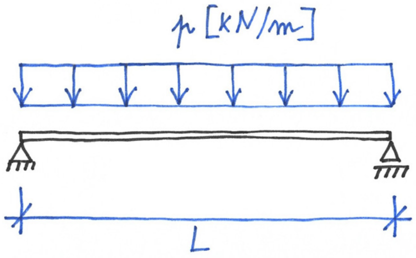
$$\bar{I_x} = \frac{120 \cdot 22.87^3}{12} + 120 \cdot 22.87 \cdot \left(\frac{22.87}{2}\right)^2 +$$

$$+23130 + 84.5 \cdot (60 - 22.87)^2 = 1.875 \cdot 10^6\ cm^4$$



Note that, in both of these Examples 5.1 and 5.2, the calculation procedure assumes a **linear elastic behaviour of steel**, and a **linear elastic behaviour of concrete in compression**. The consideration of yielding effects in steel, as well as the non-linear behaviour of concrete in compression, is out of the scope of this part of the course unit.

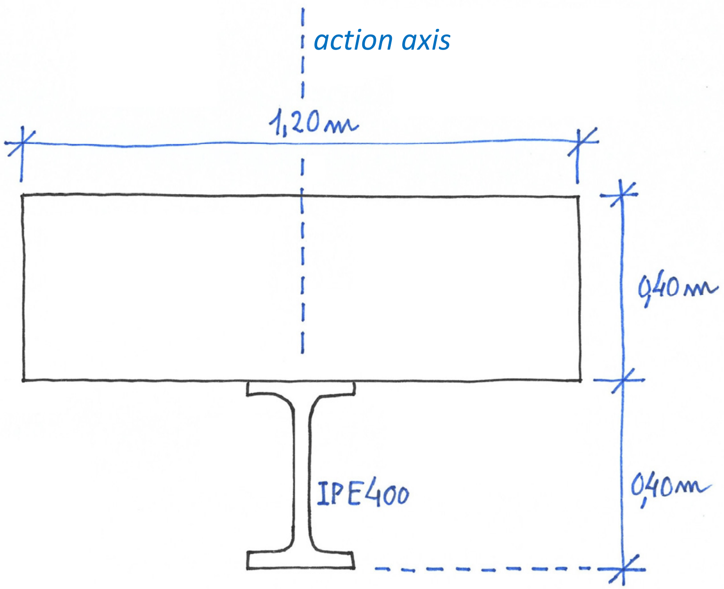
Calculation of $M_{x,\max}$ at the mid-span cross-section:



The beam self-weight, in kN/m, has to be quantified, based on the **specific weight** (γ) of the materials which, by definition, is the weight per unit volume:

$$\gamma = \frac{\text{weight [kN]}}{\text{volume [m}^3\text{]}}$$

The example data indicates that, for concrete, γ = 25.0 kN/m³. The specific weight of steel is γ = 77.0 kN/m³.



The **self-weight load per unit length**, p_{sw} [kN/m], of a linear member (beam, column, …) is calculated as:

$$p_{sw} = \frac{\gamma \cdot A \cdot L}{L} = \gamma \cdot A$$

where A is the cross-section area. It becomes:

- for concrete: $p_{sw} = \gamma \cdot A = 25 \cdot 1.2 \cdot 0.4 = 12\ \text{kN/m}$
- for steel: $p_{sw} = 77 \cdot 0.00845 = 0.65\ \text{kN/m}$

Alternatively, the steel self weight might have been quantified based on the **mass** value given in the **table of steel profiles**. For the IPE400, the mass is 66.3 kg/m. Therefore:

$$\frac{1\ \text{kgf}}{66.3\ \text{kgf}} = \frac{9.81\ \text{N}}{x} \Leftrightarrow x = 0.65\ \text{kN/m}$$

The total applied loading is therefore p = 60 + 12 + 0.65 = 72.65 kN/m, and the maximum bending moment, at mid-span, becomes:

$$M_{x,\max} = \frac{p\,L^2}{8} = \frac{72.65 \cdot 8^2}{8} = 581.2\ \text{kNm}$$

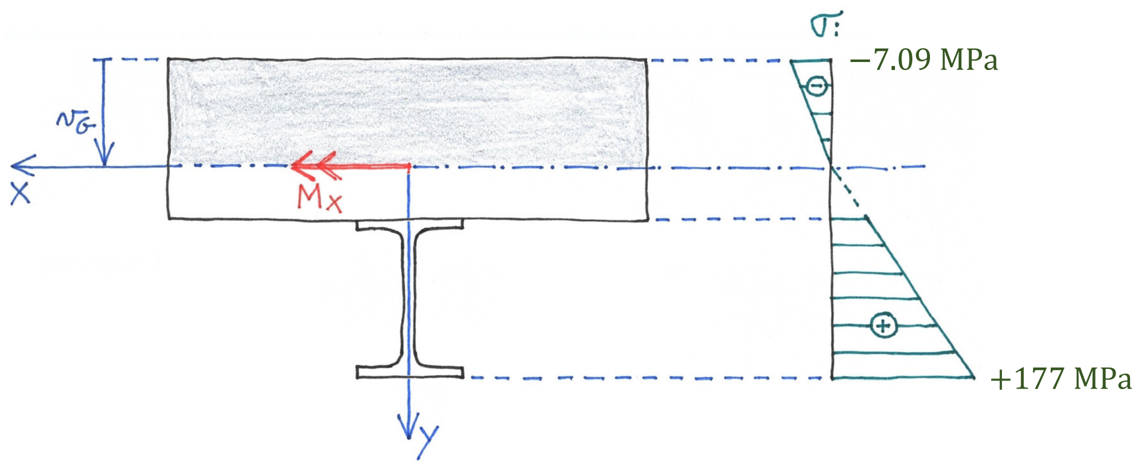
Calculation of the normal stress diagram at the mid-span (cracked) cross-section:

The highest compression stress in the concrete member becomes:

$$\sigma_c = \frac{M_x}{\bar{I}_x} y = \frac{581.2}{0.01875} \cdot (-0.2287) = -7.09 \cdot 10^3 \text{ kPa}$$

The highest steel stress occurs at the bottom fibre of the cross-section:

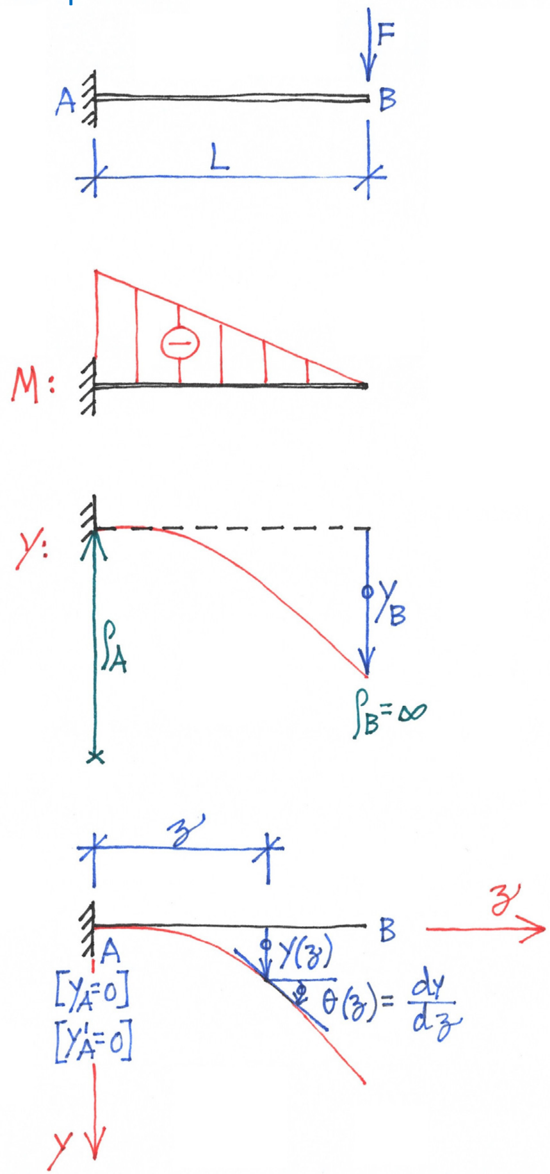
$$\sigma_s = \frac{M_x}{\bar{I}_x} y \, k_{hom} = \frac{581.2}{0.01875} \cdot (0.8 - 0.2287) \cdot 10 = +177.1 \cdot 10^3 \text{ kPa}$$



6. Calculation of displacements, rotations and curvatures in isostatic structures subject to plane bending

6.1. Method of Integration of the Curvature Equation
6.1.1. Fundaments of the method

Example 6.1:



We saw in subsection 5.5 that, in a bar with linear elastic behaviour, the relationship between the bending moment and the curvature radius (ρ) of the deformed shape of the bar axis is given by:

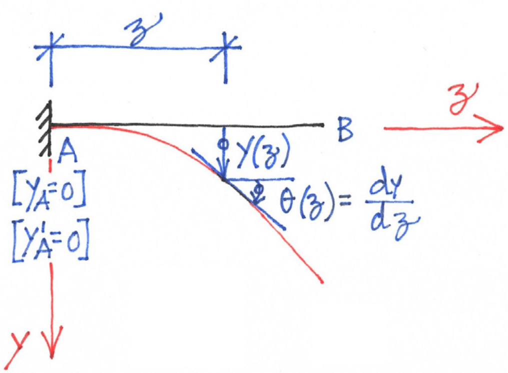
$$\frac{1}{\rho} = \frac{M(z)}{EI}$$

where $M(z)$ is the bending moment, which can be variable along the development of the bar, E is the modulus of elasticity of the material, and I is the moment of inertia of the cross section.

Let us consider that **the deformed shape of the bar axis is an equation $y(z)$**, where y is the value of the displacement (in the direction perpendicular to the undeformed bar axis) at a position of abscissa z. Since we will be working with small deformations, the **rotation θ of the bar (in radians)**, at a position of abscissa z, is practically equal to the first derivative of the function $y(z)$, that is:

$$\frac{dy}{dz} = \tan\theta \approx \theta \text{ [rad]}$$

Example 6.1:



Regarding the concavity of the curve $y(z)$, the relationship

$$\frac{d^2y}{dz^2} = -\frac{M(z)}{EI}$$

means that, when the bending moment is negative, as in this Example 6.1, the second derivative of the function $y(z)$ becomes positive, i.e. the concavity of the function $y(z)$ is turned to the positive side of axis y.

Furthermore, it is known, from elementary calculus, that the radius of curvature of a plane curve with equation $y(z)$ [see example in the image] can be expressed as:

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dz^2}}{\left[1 + \left(\frac{dy}{dz}\right)^2\right]^{3/2}}$$

In the case of small deformations, the denominator $[1 + (dy/dz)^2]^{3/2}$ is practically equal to 1 (for θ = 0.01 radians, for example, that denominator would amount to 1.0002), and therefore the previous expression can be written as:

$$\frac{1}{\rho} \approx \frac{d^2y}{dz^2}$$

which means:

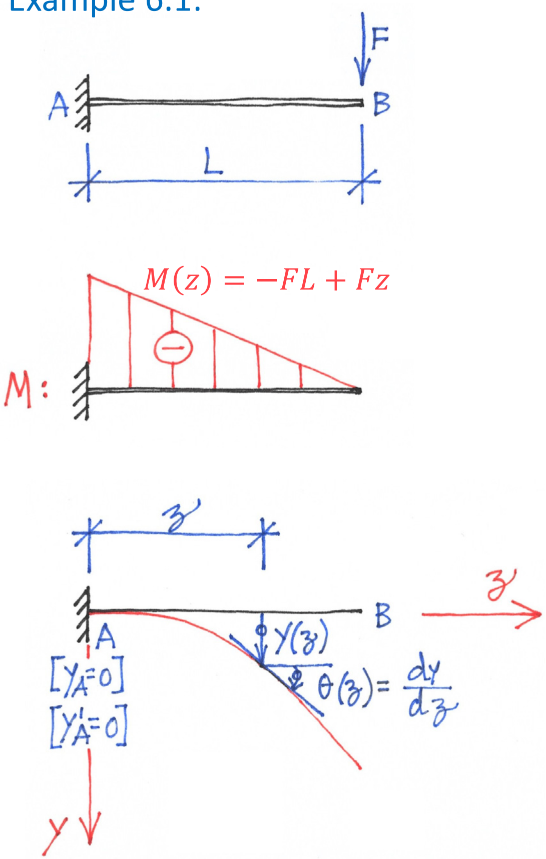
$$\frac{d^2y}{dz^2} \approx -\frac{M(z)}{EI}$$

this being the basic relation used in the displacement calculation method named the **Method of Integration of the Curvature Equation**. By simplification, the sign "$=$" will be used from now on instead of "$\approx$".

Note that a **minus sign** was introduced in the previous equation, for consistency of the sign conventions. Thus, when using the usual sign convention of the bending moment diagram (positive moment is the one that causes tension in the bottom fibre of the bar), the positive displacement is the downward one.

6.1.2. Practical usage of the method

Example 6.1:



Using an alternative notation to present the second derivative of the function $y(z)$, one can also write:

$$y''(z) = -\frac{M(z)}{EI}$$

$M(z)$ being the equation that gives the bending moment at each position of abscissa z. The equation for **the rotation $\boldsymbol{\theta}$** at each position [recall that $y'(z) = \theta(z)$ in radians] **can be obtained by integration of the previous expression**. In the case of prismatic bars of constant cross section, the bending stiffness EI is a constant, and therefore:

$$y''(z) = -\frac{M(z)}{EI}$$

$$\Leftrightarrow EIy''(z) = -M(z)$$

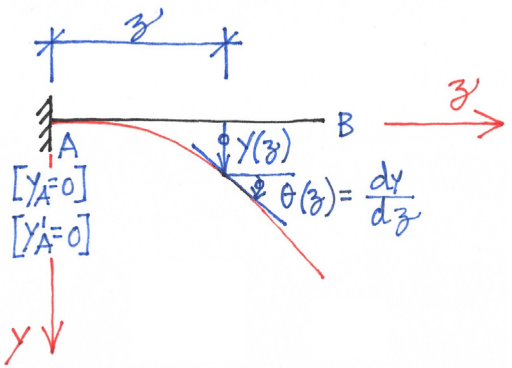
$$\Leftrightarrow EIy'(z) = \int_0^z -M(z)dz + C_1$$

where C_1 is an integration constant. **Integrating the expression once more, gives the equation $\boldsymbol{y(z)}$**, which expresses the displacement at each position of abscissa z :

$$EIy(z) = \int_0^z \left[\int_0^z -M(z)dz + C_1\right] dz + C_2$$

$$\Leftrightarrow EIy(z) = \int_0^z \left[\int_0^z -M(z)dz\right] dz + C_1 z + C_2$$

where C_2 is another integration constant. To complete the determination of the equations $y'(z)$ and $y(z)$, it is necessary to determine the **integration constants**. For this, two **boundary conditions** are used, imposed by the **nature of the supports**.



In the example shown in the image, the fixed support (bracket) imposes that the displacement and rotation are zero at the left end of the bar:

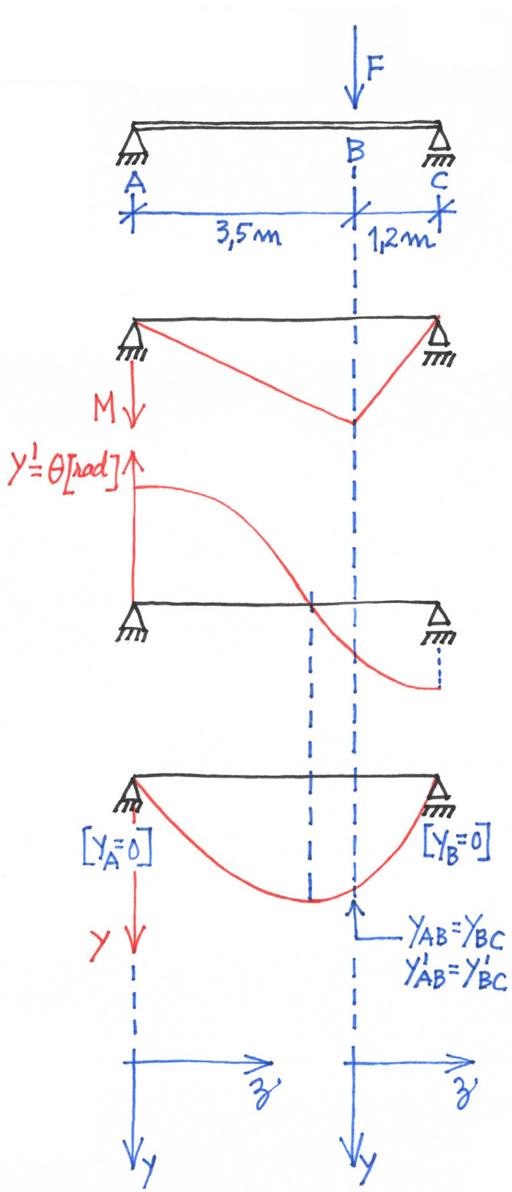
$$y(z = 0) = 0$$

$$y'(z = 0) = 0$$

By introducing these two conditions in the expressions of $y'(z)$ and $y(z)$ shown in the page before, the value of the constants C_1 and C_2 is determined.

Note that in this chapter the variable y represents the displacement in the direction perpendicular to the undeformed bar axis, while in the previous chapter it represented the coordinate of a point on the cross section, measured relative to the neutral axis of the section.

Example 6.2:



For bar segment AB:
$0 \leq z \leq 3.5$ m

For bar segment BC:
$0 \leq z \leq 1.2$ m

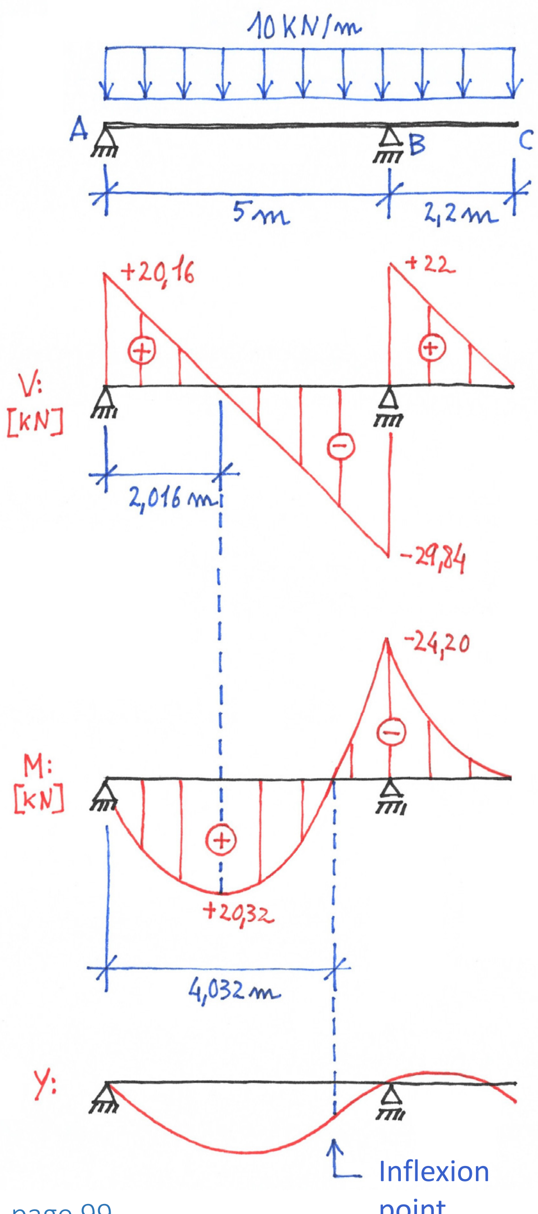
In Example 6.1, a single equation was sufficient to characterize the bending moment diagram for the entire structure.

On the contrary, when the loading or the structure have discontinuities, more than one bending moment equation is required. In Example 6.2, two equations will be needed. Each of these equations will have to be integrated twice, resulting in four integration constants (C_1 to C_4) whose determination is possible using four **boundary conditions**. In this problem, two of these conditions are imposed by the fact that the displacement y is zero at the **supports** A and C. The remaining two, are **conditions of continuity between the bar segments** AB and BC: the value of y at point B, given by the equation of the bar AB, must be equal to the value of y at this point in the equation of the bar BC. The same equality applies to the values of y'. The rotations immediately to the left and to the right of point B would not be equal only if there were a hinge at that position.

As we know, the equation $y(z)$ has a **local maximum** at the position where its derivative is zero. Therefore, to determine the maximum displacement in a bar, it is necessary to search for the position where the function $y'(z)$ is zero (see example in the image). That position may be different from the position of maximum bending moment.

6.1.3. Calculation example

Example 6.3:



Problem:

Consider the steel beam ABC (E = 206 GPa; INP180), subjected to a uniformly distributed loading of 10 kN/m (Example 6.3).

i. Make schematic representation (sketch) of the deformed bar axis, before calculating the corresponding equation $y(z)$.
ii. Calculate the maximum displacement and rotation of the bar axis, through the Method of Integration of the Curvature Equation.

In order to make the sketch of the deformed bar axis, one should start by determining the bending moment diagram, because the bending moment indicates the curvature:

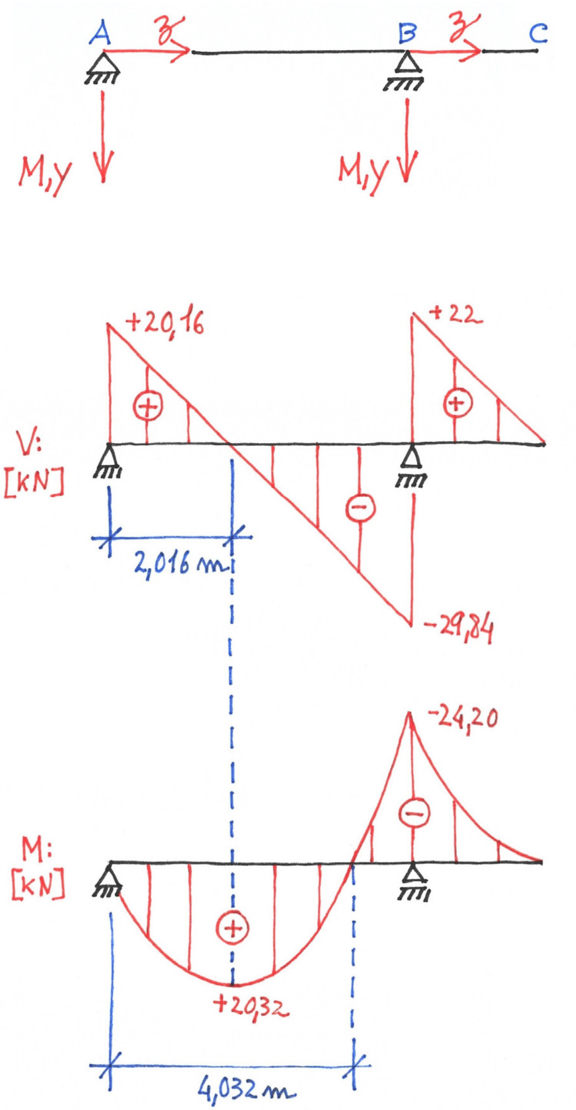
- the relationship $y''(z) = -M(z)/(EI)$ implies that, if the bending moment is positive, the concavity of the curve $y(z)$ is turned upwards;
- if the bending moment is negative, the concavity is turned downwards.

After calculating the reactions at the supports A and B, the diagram of shear forces and bending moments can be determined. The results are shown in the image.

Then, the deformed shape of the bar axis, y, can be sketched, knowing that:

- the bending moment value indicates the curvature (and consequently there will be an inflexion point at the position where $M = 0$);
- the supports on positions A and B prevent any vertical displacement on those positions.

The result is shown in the image. Note that we do not know yet whether the vertical displacement of point C is upward or downward.



In order to calculate the maximum displacement and rotation, let’s now apply the Method of Integration of the Curvature Equation.

We need to start by writing the equations of bending moments. In this Example 6.3, two equations are needed, because the equation for the bar segment AB is different from the one for BC.

The equations $M(z)$ may be written by determining the equation $V(z)$ for the bar segment, and then using the relationship between $M(z)$ and the integral of $V(z)$. The axes to be used for each bar segment are shown in the image.

For bar AB, it becomes:

$$0 \leq z \leq 5 \text{ m}$$

$$V(z) = 20.16 - 10z$$

$$M(z) - M_A = \int_0^z V(z)\, dz \Leftrightarrow M(z) = 20.16z - 5z^2$$

and for bar BC it becomes:

$$0 \leq z \leq 2.2 \text{ m}$$

$$V(z) = 22 - 10z$$

$$M(z) - M_B = \int_0^z V(z)\, dz \Leftrightarrow M(z) = -24.20 + 22z - 5z^2$$

Integration of the curvature equation for bar AB:

The limits for the abscissa of this bar segment are: $0 \leq z \leq 5$ m.

We start from the fundamental relationship:

$$EIy''(z) = -M(z)$$

$$\Leftrightarrow EIy''(z) = -20.16z + 5z^2$$

By integrating it, we get the equation for the rotation (θ in radians = y'):

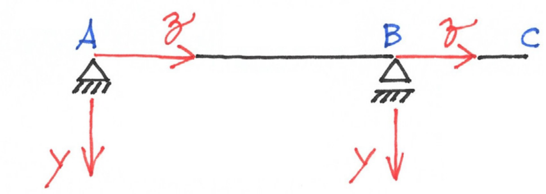
$$EIy'(z) = \int_0^z -20.16z + 5z^2 dz + C_1$$

$$\Leftrightarrow EIy'(z) = -\frac{20.16}{2}z^2 + \frac{5}{3}z^3 + C_1$$

And, by integrating it once again, we get the equation for the deflection y:

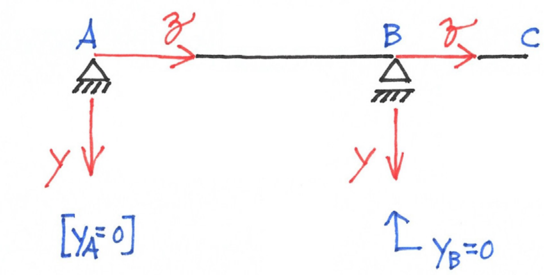
$$EIy(z) = \int_0^z -\frac{20.16}{2}z^2 + \frac{5}{3}z^3 dz + C_1 z + C_2$$

$$\Leftrightarrow EIy(z) = -\frac{20.16}{6}z^3 + \frac{5}{12}z^4 + C_1 z + C_2$$



The moment of inertia of the cross section INP180 can be found in a table of steel profiles, and is I = 1450 cm^4 (assuming that the cross section is correctly oriented), and therefore the bending stiffness EI is:

$$EI = 206 \times 10^6 \times 1450 \times 10^{-8} = 2987 \text{ kNm}^2$$



To determine the integration constants C_1 and C_2, two boundary conditions are needed. The boundary conditions are values known in advance, for y or y', usually at the extremities of the bar segment under analysis. For the segment AB, we know that:

$$y_A = y(z = 0) = 0$$

$$y_B = y(z = 5\text{ m}) = 0$$

By replacing these conditions on the equation for $y(z)$ determined on the previous page, we get:

$$y(z = 0) = 0$$

$$\Leftrightarrow EI \cdot 0 = -\frac{20.16}{6} \cdot 0^3 + \frac{5}{12} \cdot 0^4 + C_1 \cdot 0 + C_2$$

$$\Leftrightarrow C_2 = 0$$

and:

$$y(z = 5\text{ m}) = 0$$

$$\Leftrightarrow EI \cdot 0 = -\frac{20.16}{6} \cdot 5^3 + \frac{5}{12} \cdot 5^4 + C_1 \cdot 5 + 0$$

$$\Leftrightarrow C_1 = 31.91(6)$$

The equations $y(z)$ and $y'(z)$ for bar AB are now completely determined:

$$y'_{AB}(z) = \frac{1}{2987}\left[-\frac{20.16}{2} z^2 + \frac{5}{3} z^3 + 31.91(6)\right]$$

$$y_{AB}(z) = \frac{1}{2987}\left[-\frac{20.16}{6} z^3 + \frac{5}{12} z^4 + 31.91(6) z\right]$$

Integration of the curvature equation for bar BC:

By applying the same procedure to bar BC:

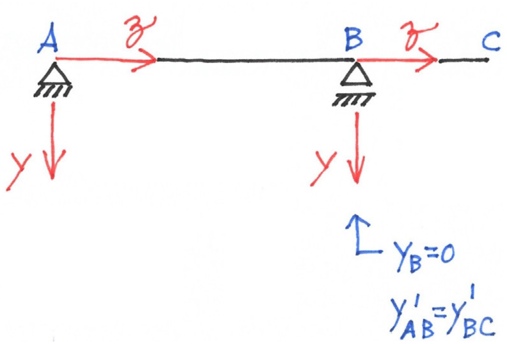
$$0 \leq z \leq 2.2 \text{ m}$$

$$M(z) = -24.2 + 22z - 5z^2$$

$$EIy''(z) = -M(z) = 24.2 - 22z + 5z^2$$

$$EIy'(z) = 24.2z - \frac{22}{2}z^2 + \frac{5}{3}z^3 + C_3$$

$$EIy(z) = \frac{24.2}{2}z^2 - \frac{22}{6}z^3 + \frac{5}{12}z^4 + C_3 z + C_4$$



The boundary conditions for determination of the integration constants C_3 and C_4 are found at the left extremity of the bar. At that position, we know in advance that the deflection is zero, and the rotation y' is also known because it can be calculated using the equation previously determined for the rotation of bar AB:

$$y_B = y(z = 0) = 0$$

$$y'_{BC}(z = 0) = y'_{AB}(z = 5\text{m}) = \frac{1}{2987}\left[-\frac{20.16}{2} \cdot 5^2 + \frac{5}{3} \cdot 5^3 + 31.91(6)\right] = \frac{-11.75}{2987}$$

The integration constants can now be determined:

$$EI \cdot y'_{BC}(z = 0) = 24.2 \cdot 0 - \frac{22}{2} \cdot 0^2 + \frac{5}{3} \cdot 0^3 + C_3$$

$$\Leftrightarrow 2987 \cdot \frac{-11.75}{2987} = 0 + C_3 \Leftrightarrow C_3 = -11.75$$

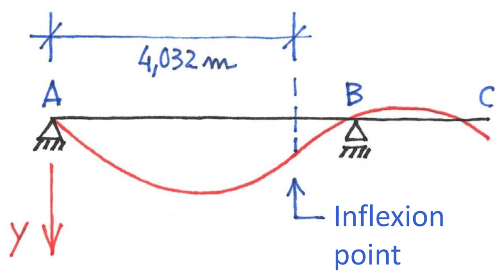
$$EIy_{BC}(z=0)=\frac{24.2}{2}\cdot 0^2-\frac{22}{6}\cdot 0^3+\frac{5}{12}\cdot 0^4+C_3\cdot 0+C_4$$

$$\Leftrightarrow EI\cdot 0=0+C_4 \Leftrightarrow C_4=0$$

The equations $y(z)$ and $y'(z)$ for bar BC are now completely determined:

$$y'_{BC}(z)=\frac{1}{2987}\left[24.2z-\frac{22}{2}z^2+\frac{5}{3}z^3-11.75\right]$$

$$y_{BC}\,(z)=\frac{1}{2987}\left[\frac{24.2}{2}z^2-\frac{22}{6}z^3+\frac{5}{12}z^4-11.75z\right]$$



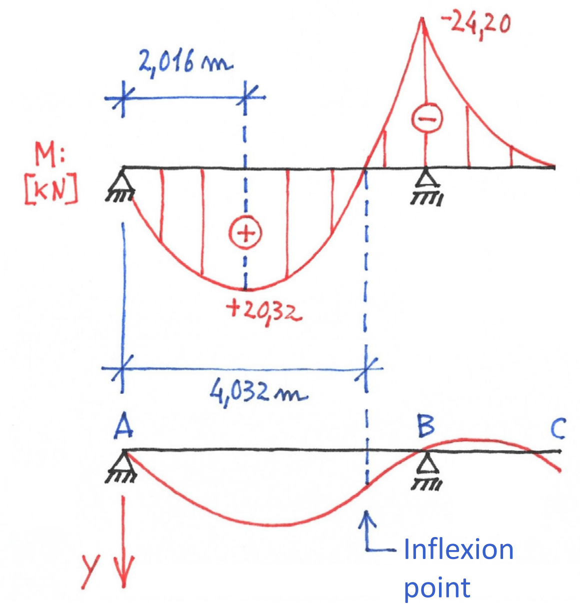
Once the equations of $y(z)$ and $y'(z)$ have been obtained for both bar segments (AB and BC), we can finally calculate the required maximum displacement and rotation values. The sketch of the deformed shape of the bar axis (which has been drawn before, and is replicated in the image on the left) is useful for the identification of the positions where the calculations have to be made.

We need to search for the maximum downward displacement by calculating $y(z)$:

- in the position of the local maximum reached in bar AB, which occurs in the abscissa z where $y'_{AB}(z)=0$;
- in point C, because the maximum displacement may occur not only in the points where $y'(z)=0$, but also in (free) bar extremities.

We need to calculate the maximum upward displacement:

- in the abscissa z where a local maximum is reached in bar BC, i.e. in the abscissa z where $y'_{BC}(z)=0$.



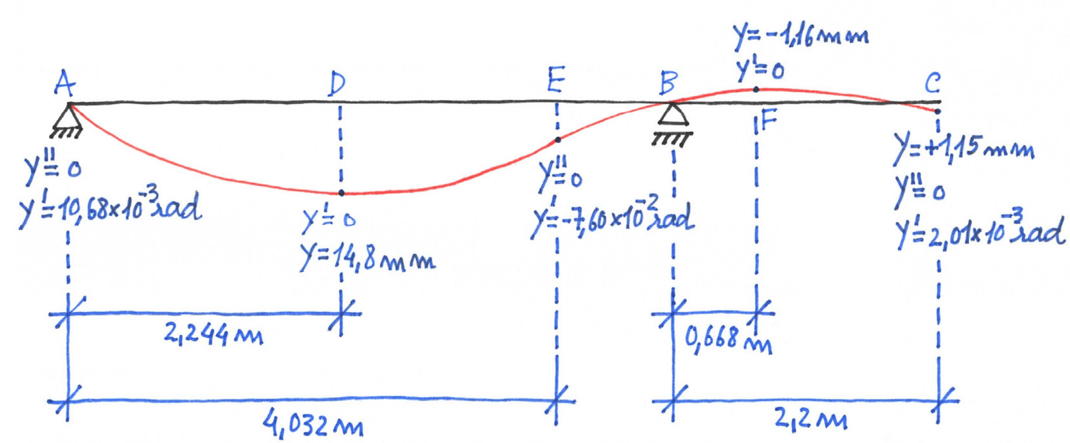
We need to search for the maximum rotation by calculating $y'(z)$ in all the positions where the function $y'(z)$ reaches a local maximum, i.e. in the abscissas where $y''(z) = 0$. Given that

$$y''(z) = -\frac{M(z)}{EI}$$

they coincide with the abscissas where $M(z) = 0$:

- point A;
- inflexion point;
- point C.

The results, in terms of maximum displacements and rotations, are shown in the image below:



6.2. Fictitious Unit Load Method
6.2.1. Fundaments of the method

The **Fictitious Unit Load Method** is also known as the **Maxwell-Mohr Method**. It is used for calculation of displacements or rotations in structures, and it is derived from the physical law of energy conservation, which implies that the work done by the external loading is entirely stored as elastic strain energy.

The calculation of displacements or rotations in isostatic structures, using this method, is made in the same way as in subsection 3.2, but:

- in subsection 3.2, the deformable bars were subjected to axial force only;
- now we are considering not only the **deformation due to the axial force**, but **also due to the bending moment** and, for that reason, we will have to calculate the elastic strain energy due to bending.

It is well known that, in plane structures, the resultant forces in the bars' cross sections are:

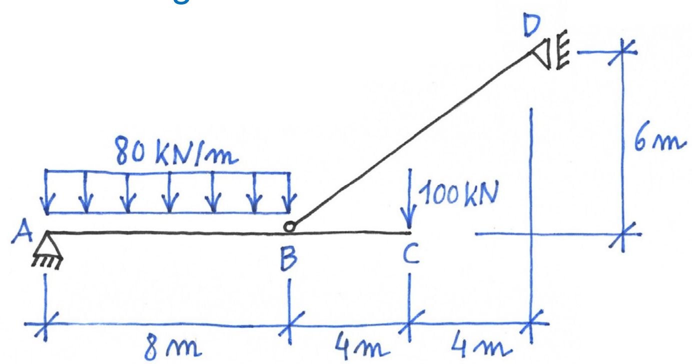
- the axial force N;
- the bending moment M;
- and the shear force V.

We are not considering the **deformations due to shear force** in these practical classes of the course unit RM1 because these deformations are, in current structures, negligible when compared with the deformations due to bending moments and axial forces.

6.2.2. Practical usage of the method

Example 6.4:

Real loading:



In an isostatic structure, the calculation of the displacement of a certain point, in a certain direction (δ_C^{vert} in this Example 6.4), is made through the procedure described below.

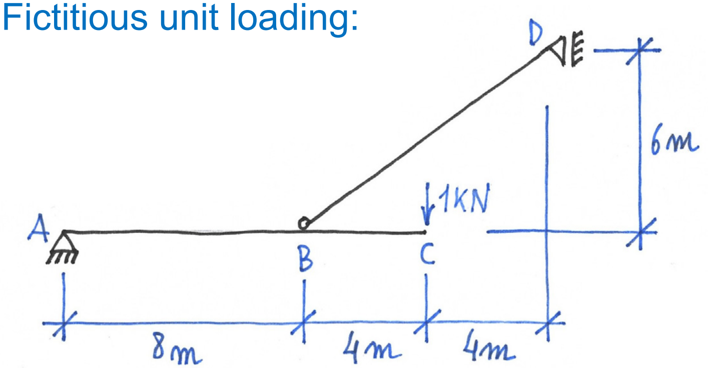
Considering the **real loading**, calculate:

- the axial force, N_i, for every bar i in the structure;
- the elongation, ΔL_i, for every bar, due to applied loading, temperature variations ΔT_i and fabrication defects Def_i;

$$\Delta L_i = \left[\frac{N\ L}{E\ A} + \alpha\ \Delta T\ L + \mathrm{Def}\right]_i$$

- the bending moment diagram, $M_i(z)$, for every bar.

Fictitious unit loading:



Considering the structure loaded only with a **fictitious unit load** (in correspondence with the displacement to be determined), calculate:

- the axial force in every bar i, denoted $\overline{N_i}$;
- the bending moment diagram in every bar i, denoted $\overline{M_i}(z)$.

According to this method, the work of external forces, W, is equal to the elastic strain energy, U, considering fictitious forces and real deformations:

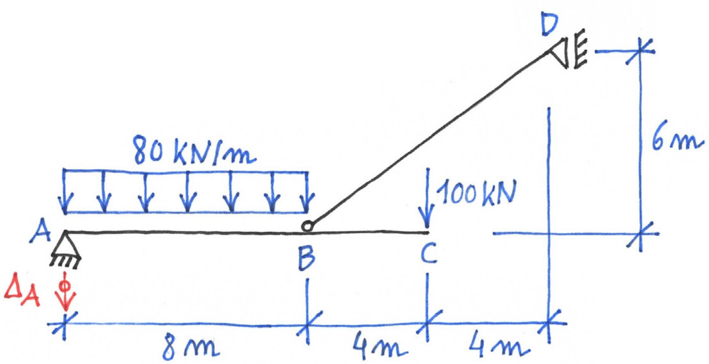
$$W = U$$

$$\Leftrightarrow 1 \times \delta_C^{\text{vert}} = \underbrace{\sum_{i=1}^{n} (\overline{N_i} \times \Delta L_i)}_{\text{effect of the axial bar deformations}} + \underbrace{\sum_{i=1}^{n} \left[\int_0^{L_i} \frac{M_i(z) \times \overline{M_i}(z)}{EI} \, dz \right]}_{\text{effect of bending deformations}}$$

where L_i denotes the length of bar i and n is the total number of bars in the structure. This expression allows to determine the displacement δ_C^{vert}. Coherent **sign conventions** must be adopted for all the bars, for example: axial forces with positive value in case of tension; elongations ΔL_i with positive value in case of increase of length; the usual convention for positive bending moments in plane structures.

In this Example 6.4, the only external force contributing to the work W is the fictitious unit load, because there is no movement of any support.

Example 6.5:



If there is some support movement, like in Example 6.5, the equalization of the work of external forces and the elastic strain energy becomes:

$$W = U$$

$$\Leftrightarrow 1 \times \delta_C^{\text{vert}} + \overline{H_A} \times \Delta_A = \sum_{i=1}^{n} (\overline{N_i} \times \Delta L_i) + \sum_{i=1}^{n} \left[\int_0^{L_i} \frac{M_i(z) \times \overline{M_i}(z)}{EI} dz \right]$$

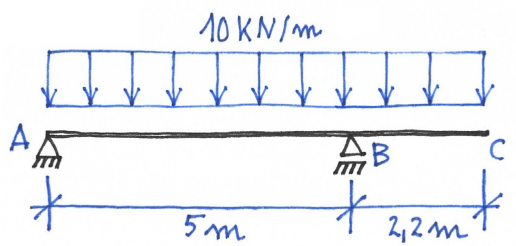
where $\overline{H_A}$ is the support reaction due to the fictitious unit load. Once again, this equation gives the displacement δ_C^{vert}, if the support movement value, Δ_A, is known.

Remember that the calculation of the internal force diagrams due to the real loading (N_i and M_i) is not influenced by temperature variations, fabrication defects or support movements. **In isostatic structures**, these **imposed deformations** do not give rise to any internal force in the structure.

If the purpose of the analysis is the calculation of the rotation at a certain position, the fictitious unit load will be a unit moment at that position.

6.2.3. Calculation example

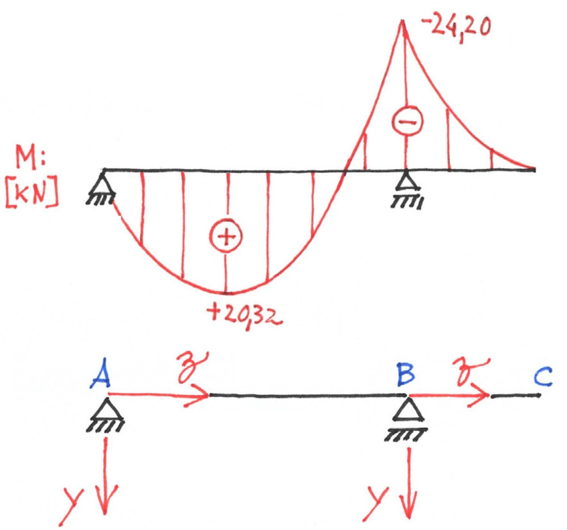
Example 6.6:



Problem:

Consider the steel beam ABC (E = 206 GPa; INP180), subjected to a uniformly distributed loading of 10 kN/m (the same structure and loading as in Example 6.3). Calculate the vertical displacement of point C, through the Fictitious Unit Load Method.

Moment diagram due to real loading:



We start by calculating the internal force diagrams and the bar elongations, due to the real loading:

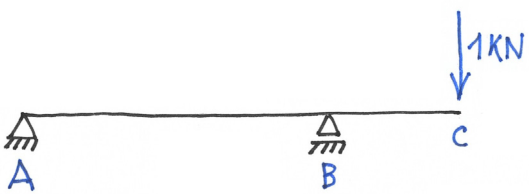
- the axial force is zero in every bar;
- the elongation is zero in every bar;
- the bending moment diagram and the corresponding equations were determined before in Example 6.3.

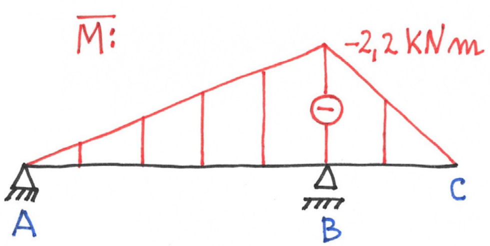
The bending moment equations, using the systems of axes shown in the image at the bottom, are:

$$M_{AB}(z) = 20.16z - 5z^2$$

$$M_{BC}(z) = -24.2 + 22z - 5z^2$$

Fictitious unit load, and corresponding diagram of bending moments:





The fictitious unit load, for calculation of the vertical downward displacement of point C, is a vertical downward unit force in C.

The bending moment diagram, due to this unit load, takes the shape shown in the image, and the corresponding equations are:

$$\overline{M_{AB}}(z) = -0{,}44z$$

$$\overline{M_{BC}}(z) = -2.2 + z$$

According to the method, the work of external forces, W, is equal to the elastic strain energy, U, considering fictitious forces and real deformations. In the structure under analysis, the effect of axial bar deformations is null, in the calculation of U:

$$W = U$$

$$\Leftrightarrow 1 \times \delta_C^{\text{vert}} = \sum_{i=1}^{n}(\overline{N_i} \times \Delta L_i) + \sum_{i=1}^{n}\left[\int_0^{L_i} \frac{M_i(z) \times \overline{M_i}(z)}{EI} dz\right]$$

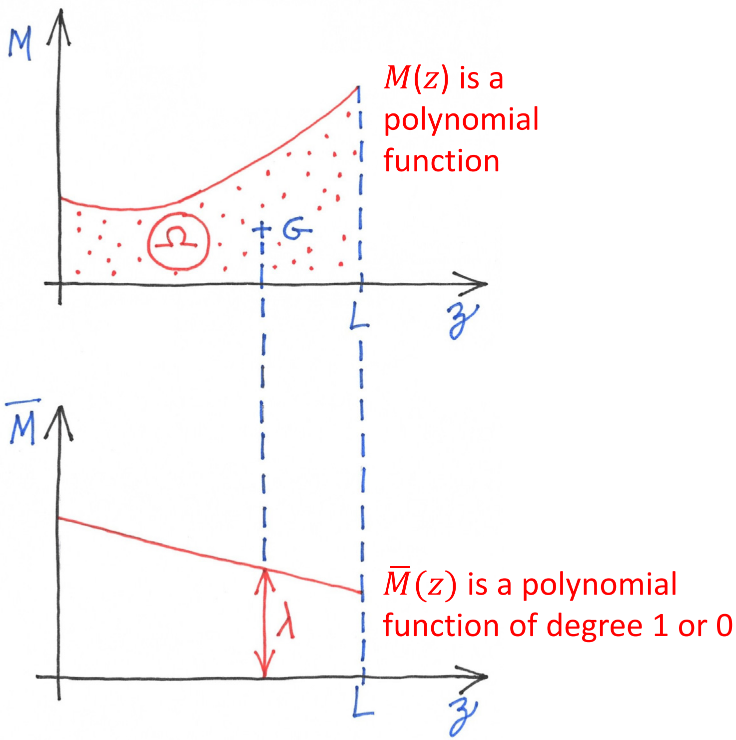
$$\Leftrightarrow \delta_C^{\text{vert}} = \frac{1}{EI}\int_0^5 M_{AB}(z) \times \overline{M_{AB}}(z)\, dz + \frac{1}{EI}\int_0^{2.2} M_{BC}(z) \times \overline{M_{BC}}(z)\, dz$$

$$\Leftrightarrow \delta_C^{\text{vert}} = \frac{1}{2987}\int_0^5 (20.16z - 5z^2) \times (-0{,}44z)\, dz + \\ + \frac{1}{2987}\int_0^{2.2}(-24.2 + 22z - 5z^2) \times (-2.2 + z) dz$$

$$\Leftrightarrow \delta_C^{\text{vert}} = -0.008654 + 0.009803$$

$$\Leftrightarrow \delta_C^{\text{vert}} = +0.00115 \text{ m}$$

6.2.4. Bonfim Barreiros Method, for calculation of the integral of the product of two polynomial functions



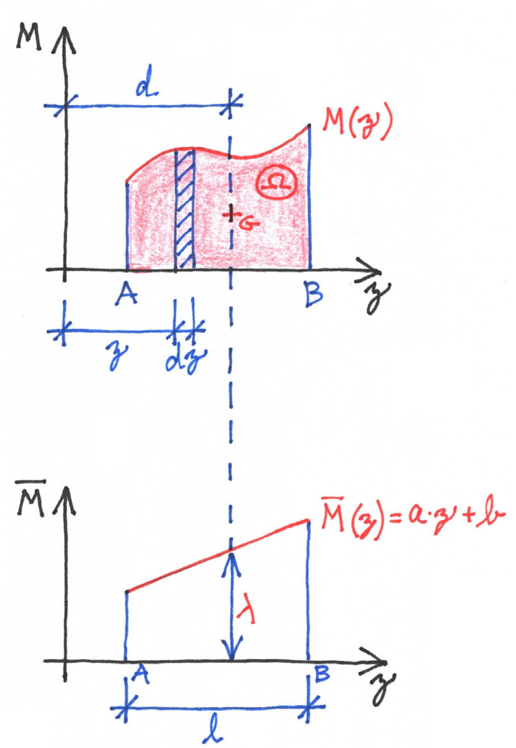
The diagrams of bending moments are usually simple geometric shapes, like triangles, rectangles or 2nd order parabolas. In these cases, the integral $\int_0^{L_i} M_i(z) \times \overline{M_i}(z)dz$ needed in the application of the Fictitious Unit Load Method can be calculated in an alternative manner, by using the Bonfim Barreiros Method.

The Bonfim Barreiros Method, also known as the Vereshchagin Method, is used for calculation of the integral of the product of two polynomial functions, one being of degree 1 (or degree 0). The method states that:

$$\int_0^L M(z) \times \overline{M}(z)dz = \Omega \times \lambda$$

where:

- Ω is the area enclosed by the function $M(z)$
- λ is the $\overline{M}(z)$ value for an abscissa equal to the abscissa of the geometric centre G of the area Ω



Demonstration:

If the diagram $\overline{M}(z)$ is a straight line, whose equation we will denote herein as $\overline{M}(z) = a \cdot z + b$, then:

$$\int_A^B M(z) \cdot \overline{M}(z) dz$$

$$= \int_A^B M(z) \cdot (a\, z + b) dz$$

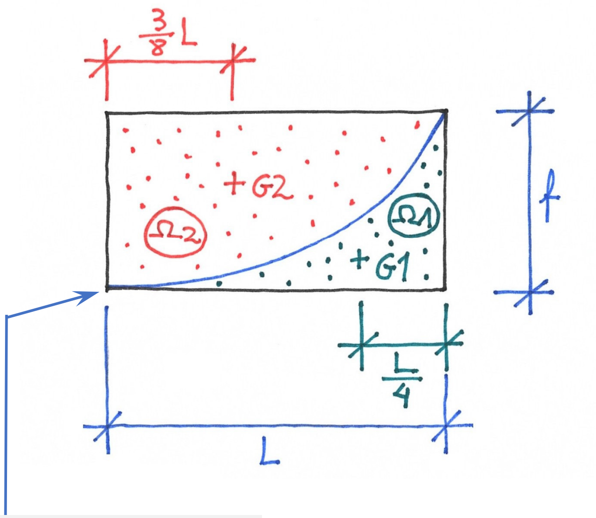
$$= \text{a} \int_A^B M(z) \cdot z\, dz + \text{b} \int_A^B M(z)\, dz$$

$$= a \cdot d \cdot \Omega + b \cdot \Omega$$

$$= \Omega \cdot (a \cdot d + b)$$

$$= \Omega \cdot \lambda \quad \text{(which was to be demonstrated)}$$

6.2.5. Area and position of the centre of 2nd degree parabolas of the type $y = k\,x^2$



Position where the first derivative of the function $M(z)$ needs to be zero, i.e., the tangent to the curve needs to be horizontal

If the diagram of bending moments $M(z)$ is a 2nd degree parabola, with the shape shown in the image, the area enclosed by the diagram is

$\Omega_1 = \frac{1}{3}\,f\,L$ in the case of the smaller parabola (in green in the image)

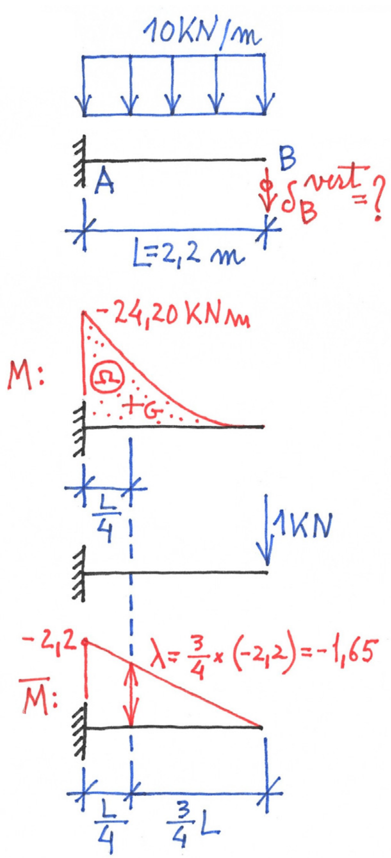
$\Omega_2 = \frac{2}{3}\,f\,L$ in the case of the larger parabola (in red in the image)

The position of the centre of these parabolas is shown in the image.

These formulas are only valid if, at the point indicated in the image, the first derivative of the function $M(z)$ is zero, i.e., if the tangent to the curve is horizontal as shown in the image.

6.2.6. Calculation example

Example 6.7:



Problem:

Consider the cantilever beam shown in the image, with bending stiffness EI = 2987 kNm^2, subjected to a uniformly distributed loading of 10 kN/m^2. Determine the vertical displacement of the free extremity B, using the Fictitious Unit Load Method and the Bonfim Barreiros Method.

Bending moment diagram due to the real loading:

Given that the shear force is zero at the point B, the tangent to the bending moment diagram is horizontal in this position.

The maximum bending moment occurs in A, and can be calculated as the resultant of the moments due to loading on the right side of this cross section A:

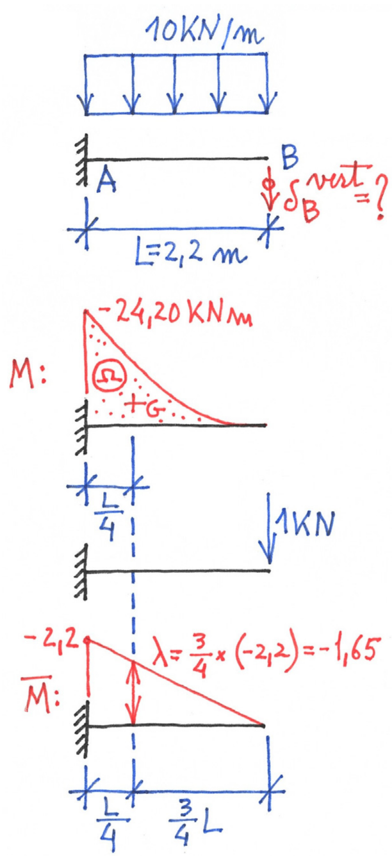
$$M_A = -10 \times 2.2 \times \frac{2.2}{2}\ \text{kNm}$$

The shape of the diagram is of the same type as the parabola 1 in subchapter 6.2.5.

Bending moment diagram due to the fictitious unit loading:

The fictitious unit load must be an unit vertical force in B. The corresponding diagram is therefore a straight line, with a maximum value of:

$$\overline{M_A} = -1 \times 2.2\ \text{kNm}$$



Displacement calculation:

According to the Fictitious Unit Load Method, the vertical displacement of point B is given by:

$$W = U$$

$$\Leftrightarrow 1 \times \delta_B^{\text{vert}} = \sum_{i=1}^{n} (\overline{N_i} \times \Delta L_i) + \sum_{i=1}^{n} \left[\int_0^{L_i} \frac{M_i(z) \times \overline{M_i}(z)}{EI} dz \right]$$

$$\Leftrightarrow \delta_B^{\text{vert}} = \frac{1}{EI} \int_0^{2.2} M_{AB}(z) \times \overline{M_{AB}}(z)\, dz$$

Using the Bonfim Barreiros Method:

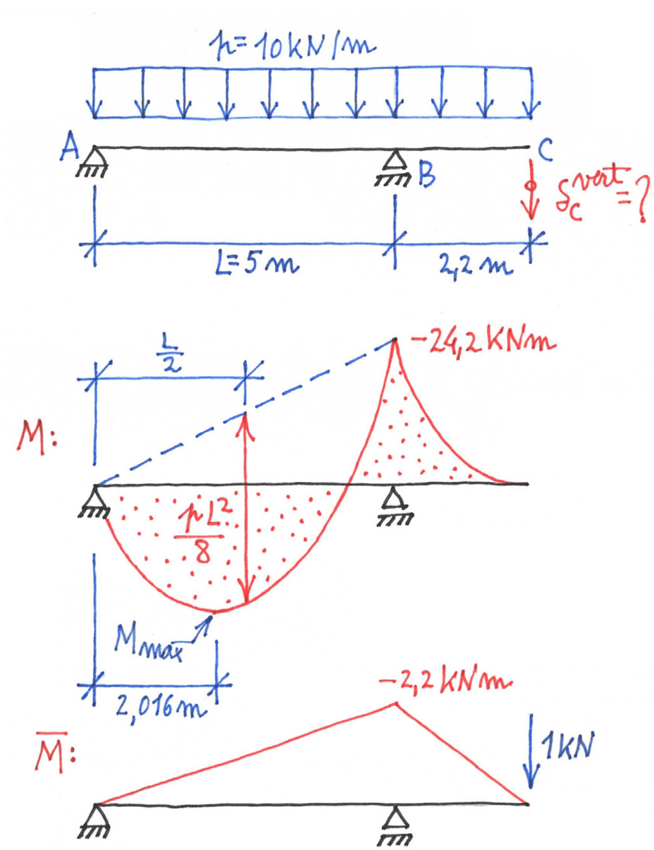
$$\int_0^{2.2} M_{AB}(z) \times \overline{M_{AB}}(z)\, dz = \Omega \times \lambda = \left[\frac{-24.2 \times 2.2}{3} \right] \times \left[\frac{3}{4} \times (-2.2) \right] = 29.282$$

and therefore:

$$\delta_B^{\text{vert}} = \frac{1}{EI} \times \Omega \times \lambda = \frac{29.282}{2987} = 0.009803 \text{ m}$$

6.2.7. Application of the Bonfim Barreiros Method, on bars subject to uniformly distributed loading, decomposing the $M(z)$ diagram into simple geometric shapes

Example 6.8:



Problem:
Consider the steel beam ABC (E = 206 GPa; INP180), subjected to a uniformly distributed loading of 10 kN/m (the same structure and loading as in Examples 6.3 and 6.6). Calculate the vertical displacement of point C.

In this example, we will apply the Fictitious Unit Load Method, but unlike we did in Example 6.6, now the Bonfim Barreiros Method will be used for calculation of the integrals $\int M(z)\,\overline{M}(z)\,dz$ for the two bars.

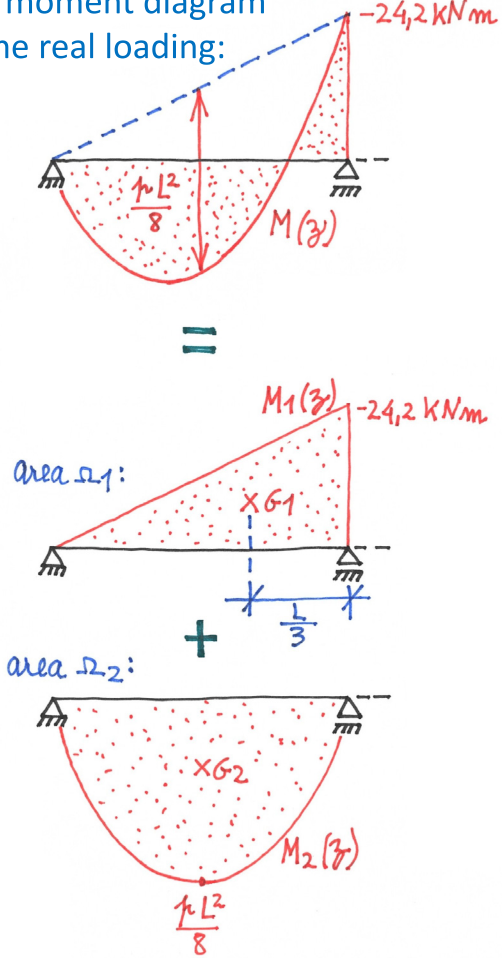
Bending moment diagram due to the real loading:

The bending moment diagram due to real loading has already been determined in Example 6.3. As shown in the image on the left, in bar AB, the area enclosed by the diagram $M(z)$ is not a simple geometric shape, and therefore that area Ω and its geometric centre cannot be calculated directly. Let's see how to solve this issue in the next page.

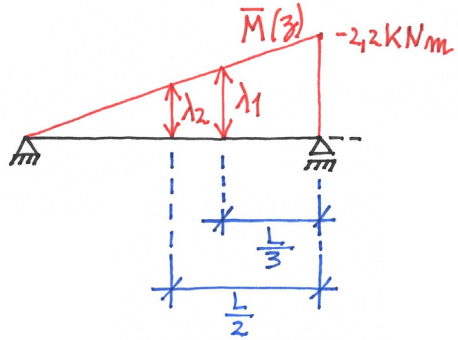
Bending moment diagram due to the fictitious loading:

The fictitious unit load for calculation of the desired displacement is a concentrated, vertical, unit load in C. For this loading, it can be easily seen that the shear force is constant in AB, and is also constant in BC. Consequently, the bending moment diagram will be a linear function in AB, and also in BC. Given that no sudden change of bending moment can occur in B, the diagram becomes fully known once $\overline{M_B}$ is calculated.

Bending moment diagram
due to the real loading:



Bending moment diagram
due to the fictitious loading:



Application of the Bonfim Barreiros Method, by decomposing the $M(z)$ diagram into simple geometric shapes:

The area Ω, enclosed by the diagram $M(z)$ in bar AB, and its geometric centre, cannot be calculated directly. However, this type of bending moment diagram, in a bar segment subjected to a uniformly distributed load p, can be decomposed into a triangle and a parabola, as explained in subchapter 4.7.

$$\Omega = \Omega_1 + \Omega_2$$

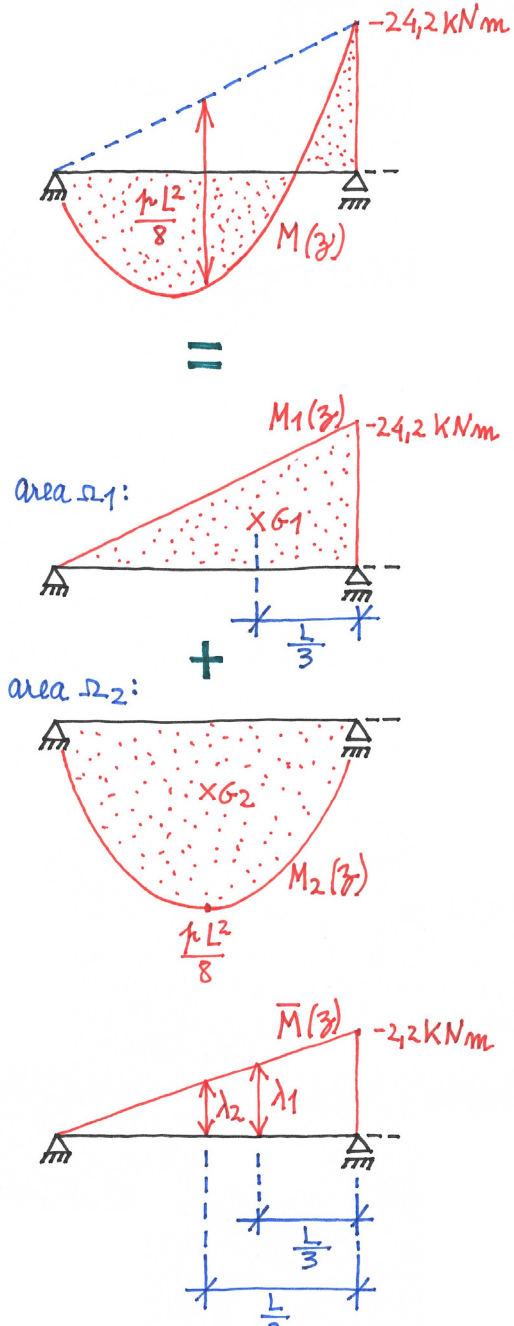
Therefore, for bar AB:

$$\int_0^L M(z) \times \bar{M}(z)\, dz =$$

$$= \int_0^L [M_1(z) + M_2(z)] \times \bar{M}(z)\, dz =$$

$$= \int_0^L M_1(z) \times \bar{M}(z)\, dz + \int_0^L M_2(z) \times \bar{M}(z)\, dz =$$

$$= \Omega_1 \times \lambda_1 + \Omega_2 \times \lambda_2$$



Calculation of the vertical displacement of point C:

The bending moment diagrams, for both the real loading and the fictitious unit load, have already been determined. In this Example 6.8, there is no axial bar elongation. Therefore, according to the Fictitious Unit Load Method:

$$W = U$$

$$\Leftrightarrow 1 \times \delta_C^{\text{vert}} = \frac{1}{EI}\int_0^5 M_{AB}(z) \times \overline{M_{AB}}(z)\, dz + \frac{1}{EI}\int_0^{2.2} M_{BC}(z) \times \overline{M_{BC}}(z)\, dz$$

As shown in the page before, the integral expression for bar AB can be calculated, using the Bonfim Barreiros method, as :

$$\int_0^5 M_{AB}(z) \times \overline{M_{AB}}(z)\, dz = \Omega_1 \times \lambda_1 + \Omega_2 \times \lambda_2 =$$

As far as the area Ω_2 is concerned, its maximum bending moment is $pL^2/8 =$ 31.25, and the area is (as explained in subchapter 6.2.5):

$$\Omega_2 = \frac{2}{3}\, f\, L = \frac{2}{3} \times 31.25 \times 5 = 104.1(6)$$

Thus:

$$\Omega_1 \times \lambda_1 + \Omega_2 \times \lambda_2 = \left[\frac{5 \times (-24.2)}{2}\right] \times \left[\frac{2}{3} \times (-2.2)\right] + [104.1(6)] \times \left[\frac{-2.2}{2}\right] = -25.850$$

The calculation of $\Omega \times \lambda$ for bar BC is equal to the one made in Example 6.7. Therefore:

$$\delta_C^{\text{vert}} = \frac{1}{EI}\, [-25.850 + 29.282] = \frac{1}{2987} \times 3.432 = 0.00115 \text{ m}$$

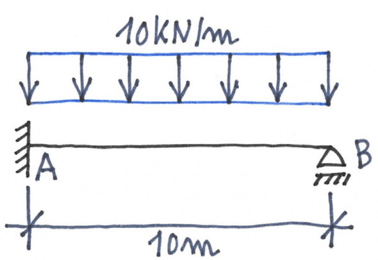
7. Calculation of support reactions and force diagrams in hyperstatic structures (H = 1) with bars subjected to bending

7.1. Calculation procedure

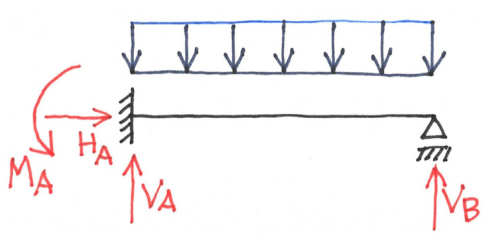
In subchapter 2.3, we have seen the procedure for calculation of the support reactions and the internal forces, in structures one time hyperstatic. It was done therein for structures in which the deformable bars were subjected to axial force only.

Let's now see a generic procedure for analysis of plane structures, one time hyperstatic, in which there can be deformations due to bending moment also.

Example 7.1:



Loading and support reactions:



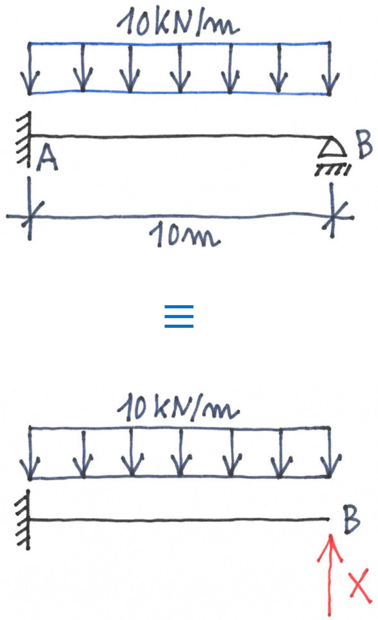
The Example 7.1 is an hyperstatic structure, with degree of hyperstaticity H = 1. It is subjected to a uniformly distributed load, and the bar's bending stiffness is EI = 8×10^4 kNm^2.

For calculation of the support reactions in this structure, there are:

- **4 unknows (support reactions)**;
- **3 equations of global equilibrium**.

The additional equation, needed for calculation of the 4 support reactions, is:

- **1 condition of compatibility of deformations**.



The calculation procedure is based on the following fact:

> *The structure behaviour is not changed if one of the support reactions is replaced by a force X, if the displacement of the point is the same as in the real structure.*

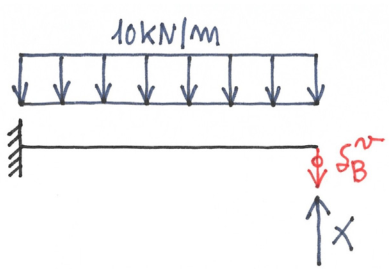
The force X is named the **hyperstatic unknown**. The two structures in the image are equivalent if the vertical displacement of point B, δ_B^{V}, is:

$$\delta_B^{\mathrm{V}} = 0$$

This is the **condition of compatibility of deformations** for this structure.

If the hyperstatic unknown is calculated, then the remaining support reactions can be determined through the equilibrium equations, and the internal force diagrams can also be obtained.

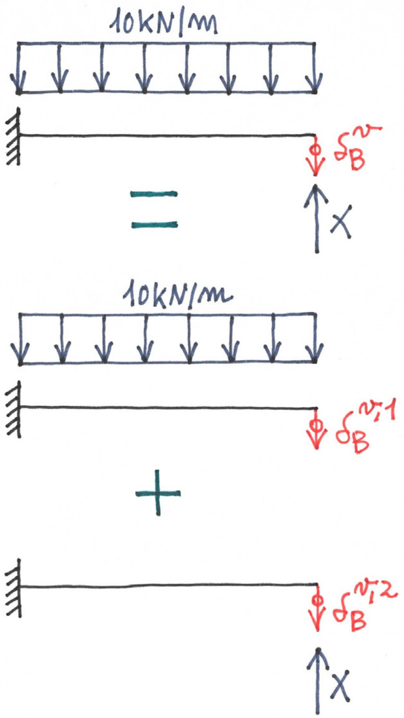
Procedure for calculation of the hyperstatic unknown X:



1) Determine the internal force diagrams, treating the hyperstatic unknown X as if it were an external action.

2) **Calculate the displacement** that appears in the compatibility equation (δ_B^{V} in this example), using the Method of Integration of the Curvature Equation, or the Fictitious Unit Load Method. This displacement will be expressed as a function of X.

3) By **introducing the displacement** calculated in step 2 **in the compatibility condition equation**, the value of X is determined.



Problem:

Determine the support reactions and the internal force diagrams for the structure shown in Example 7.1, subjected to a uniformly distributed loading of 10 kN/m.

We need to start by calculating the displacement δ_B^{v} due to the applied loading and the hyperstatic unknown X.

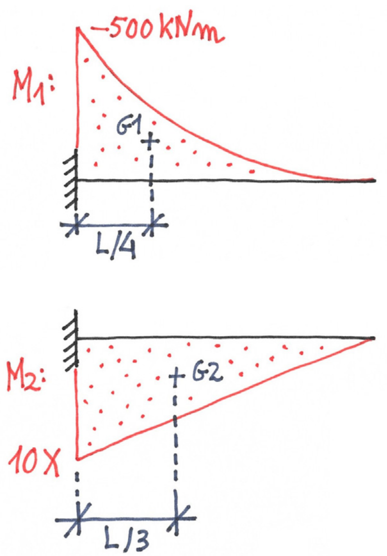
In order to get simple bending moment diagrams, it is convenient to calculate δ_B^{v} through the superposition of the effects of:

- the applied loading of 10 kN/m (displacement called $\delta_B^{\mathrm{v},1}$);
- the hyperstatic unknown (displacement called $\delta_B^{\mathrm{v},2}$);

so that $\delta_B^{\mathrm{v}} = \delta_B^{\mathrm{v},1} + \delta_B^{\mathrm{v},2}$, and the equation of compatibility of deformations becomes:

$$\delta_B^{\mathrm{v},1} + \delta_B^{\mathrm{v},2} = 0$$

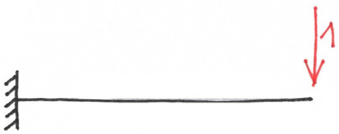
Bending moment diagrams due to real loading



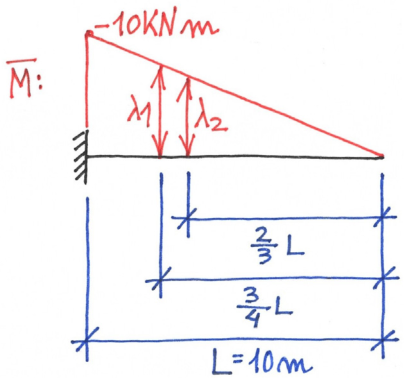
The resulting diagrams of bending moments due to the applied loading (M_1) and the hyperstatic unknown (M_2) are shown in the image.

The diagram M_1 is a second order parabola, and the first derivative of M_1 in point B is zero. Therefore, the area enclosed by this diagram and its geometric centre can be calculated as shown in subchapter 6.2.5.

Let's calculate the displacements $\delta_B^{v,1}$ and $\delta_B^{v,2}$ using the Fictitious Unit Load Method, and the Bonfim Barreiros method.



The image shows the fictious unit load, and the corresponding bending moment diagram, $\overline{M}$.



The displacements become:

$$\delta_B^{v,1} = \frac{1}{EI} \cdot \Omega_1 \cdot \lambda_1 = \frac{1}{8 \cdot 10^4} \left[\frac{-500 \cdot 10}{3} \right] \cdot \left[\frac{3}{4} \cdot (-10) \right] = +0.15625 \text{ m}$$

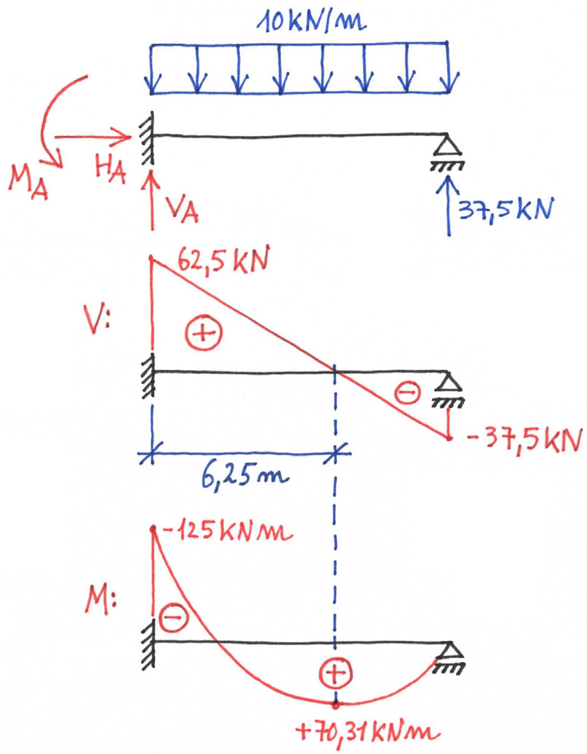
$$\delta_B^{v,2} = \frac{1}{EI} \cdot \Omega_2 \cdot \lambda_2 = \frac{1}{8 \cdot 10^4} \left[\frac{10 \cdot 10 \cdot X}{2} \right] \cdot \left[\frac{2}{3} \cdot (-10) \right] = -0.0041(6) \cdot X$$

By introducing these displacements in the equation of compatibility of deformations, the hyperstatic unknow is determined.

$$\delta_B^{v,1} + \delta_B^{v,2} = 0$$

$$\Leftrightarrow +0.15625 \text{ m} - 0.0041(6) \cdot X = 0$$

$$\Leftrightarrow X = 37.5 \text{ kN}$$

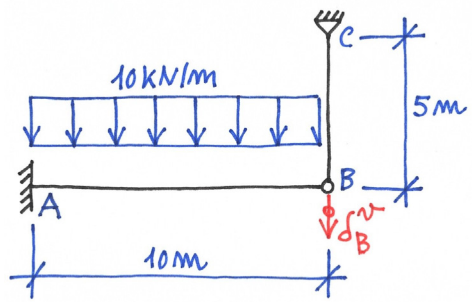


Once the hyperstatic unknown has been quantified, the remaining support reactions can be determined by using the equilibrium equations.

Then, the internal force diagrams can be calculated. The results are shown in the image.

7.2. Example of a structure supported by an articulated bar

Example 7.2:



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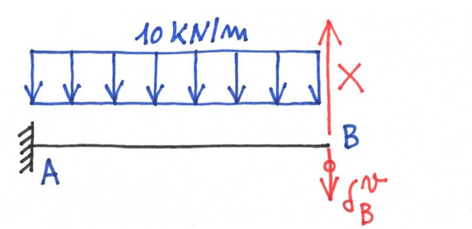


Problem:

Consider the plane structure, one time hyperstatic, shown in the image, subjected to a uniformly distributed loading of 10 kN/m. The bending stiffness of bar AB is EI = 8×10^4 kNm^2. The cross-sectional area of bar BC is A = 2 cm^2 and its modulus of elasticity is E = 200 GPa. Determine the support reactions.

Bar BC is articulated, and therefore the only internal force in this bar that is not zero is the axial force. Therefore, the hyperstatic structure can be solved by replacing this articulated bar by the axial force it applies on the remaining structure, this being the hyperstatic unknown X.

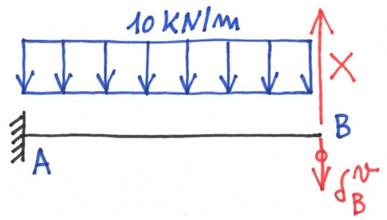
The structure in the image below is equivalent to the real one (in the image above), if the displacement δ_B^{v} is the same in both structures.

This means that the displacement calculated in the structure shown in the image below should be:

$$\delta_B^{\mathrm{v}} = \Delta L_{BC}$$

$$\Leftrightarrow \delta_B^{\mathrm{v}} = \frac{X \cdot L_{BC}}{E_{BC} \cdot A_{BC}}$$

this being the equation of compatibility of deformations.



The expression that gives the displacement δ_B^{v} due to the applied loading and the hyperstatic unknown X is the same as the expression calculated in Example 7.1:

$$\delta_B^{\mathrm{v}} = +0.15625 \text{ m} - 0.0041(6) \cdot X$$

By introducing this expression into the equation of compatibility of deformations, the hyperstatic unknown X is obtained:

$$\delta_B^{\mathrm{v}} = \frac{X \cdot L_{BC}}{E_{BC} \cdot A_{BC}}$$

$$\Leftrightarrow 0.15625 \text{ m} - 0.0041(6) \cdot X = \frac{X \cdot 5}{200 \cdot 10^6 \cdot 2 \cdot 10^{-4}}$$

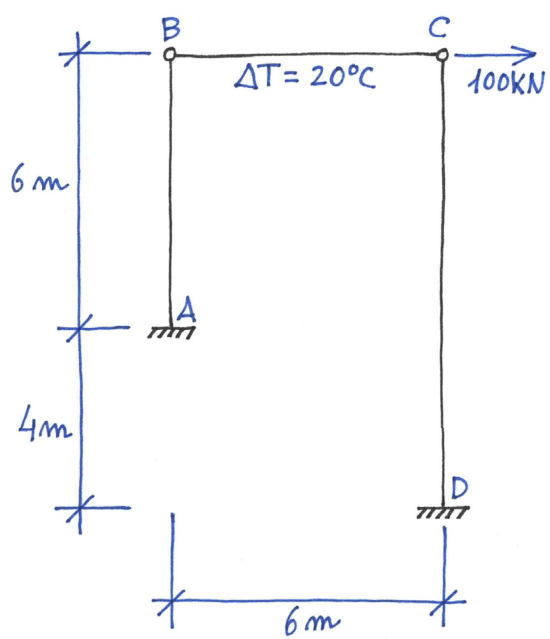
$$\Leftrightarrow X = 36.4 \text{ kN}$$

The structures in Examples 7.1 and 7.2 are very similar. The difference is that the vertical (rigid) support in 7.1 is replaced by a flexible support (tie BC) in example 7.2. Because of the support flexibility, the vertical force in point B is smaller in Example 7.2.

Once the hyperstatic unknown X has been determined, the remaining support reactions can be easily calculated, by applying the equilibrium conditions.

7.3. Example of a structure with an internal articulated bar

Example 7.3:



Problem:

The structure in the image is composed by two columns (AB and CD) of different height, with bending stiffness EI = 1.83·10^5 kNm^2, fixed at the base.

The top extremity of the columns is linked by an articulated bar (E = 200 GPa, A = 8.04 cm^2, α = 1.2·10^{-5} ºC^{-1}).

An horizontal concentrated force is applied to node C, and bar BC undergoes a temperature increase of 20ºC.

The purpose of this problem is the calculation of the axial force in the articulated bar, N_{BC}.

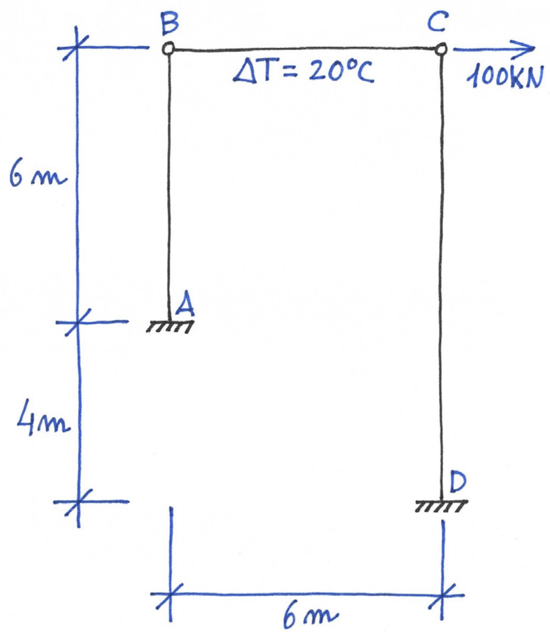
The structure is one time hyperstatic.

Therefore, one equation of compatibility of deformations is needed for calculation of the support reactions.

The structure has an internal articulated bar, in which the only internal force that is not zero is the axial force. Thus, it is convenient to make the calculations considering that the hyperstatic unknown is this axial force:

$$X = N_{BC}$$

Real structure:



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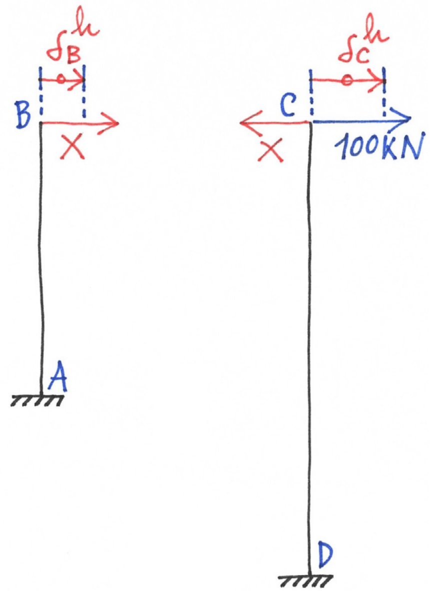
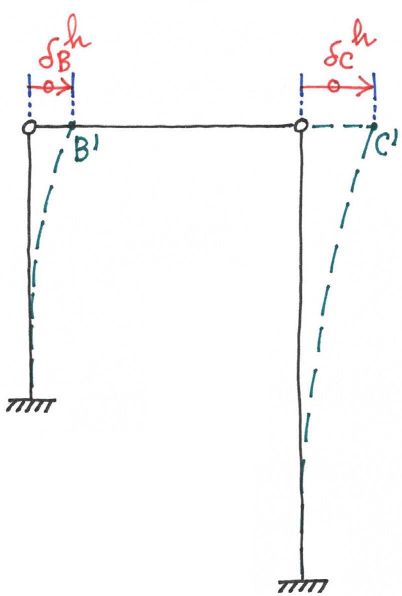
Structure after replacing one of the links by the hyperstatic unknown X:





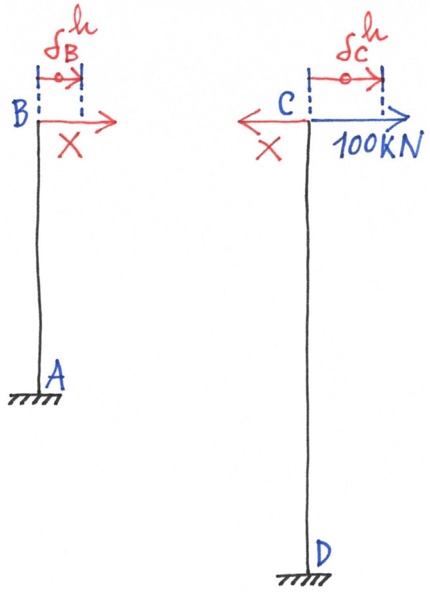
The structure in which the bar BC was replaced by the hyperstatic unknown X will be equivalent to the real one if the displacements of the points where the link was removed are equal to the real ones, i.e.:

$$\delta_C^h = \delta_B^h + \Delta L_{BC}$$

$$\Leftrightarrow \delta_C^h = \delta_B^h + \left(\frac{X \cdot L}{E \cdot A}\right)_{BC} + (\alpha \cdot \Delta T \cdot L)_{BC}$$

this being the equation of compatibility of deformations.

Note that consistent sign conventions have to be used for the axial force X and for the bar elongation ΔL. In this case, the variable X represents the **tensile force** in the bar and, consistently, ΔL represents the increase of the bar length, and ΔT represents the temperature increase.

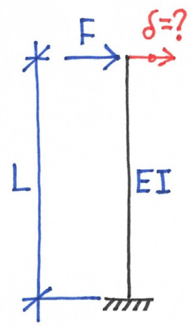


We need to determine now the expressions that give the displacements δ_B^h and δ_C^h as a function of the variable X.

Note that we need to make more than one calculation for the same type of structure and loading, i.e.:

> we need to calculate the horizontal displacement at the top of a column fixed at the base, due to an horizontal force applied at the top.

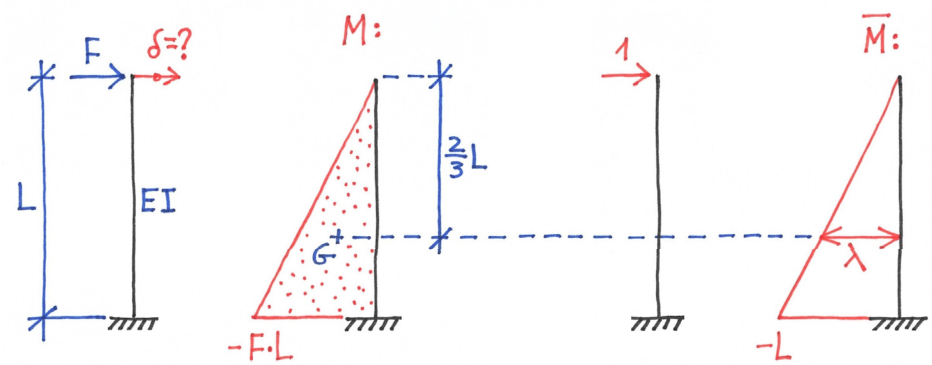
Therefore, we may start by finding a generic expression for this simple case:



Let's calculate such a generic expression, by employing the Fictitious Unit Load Method, in combination with the Bonfim Barreiros method.

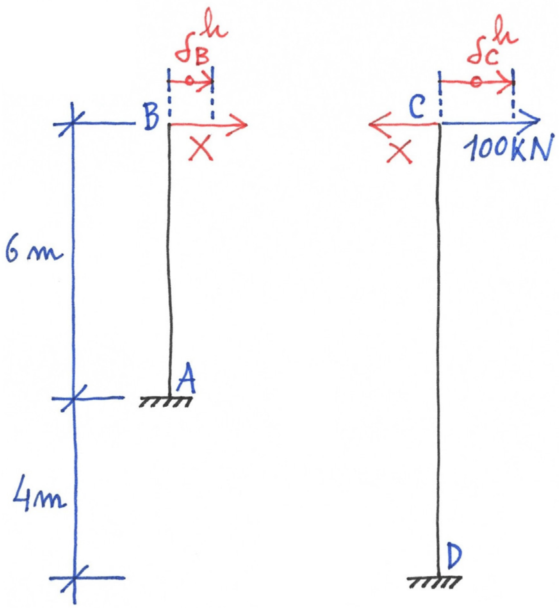
Real loading:

Fictitious unit load :



It becomes:

$$\delta = \frac{1}{EI} \cdot \Omega \cdot \lambda = \frac{1}{EI} \cdot \left[\frac{-F \cdot L \cdot L}{2}\right] \cdot \left[\frac{2}{3} \cdot (-L)\right] = \frac{FL^3}{3EI}$$



Therefore:

$$\delta_B^h = \frac{FL^3}{3EI} = \frac{X \cdot 6^3}{3 \cdot 1.83 \cdot 10^5} = \frac{3}{7625}X$$

$$\delta_C^h = \frac{FL^3}{3EI} = \frac{(100 - X) \cdot 10^3}{3 \cdot 1.83 \cdot 10^5} = \frac{100 - X}{549}$$

and, by introducing these expressions in the condition of compatibility deformations:

$$\delta_C^h = \delta_B^h + \left(\frac{X \cdot L}{E \cdot A}\right)_{BC} + (\alpha \cdot \Delta T \cdot L)_{BC}$$

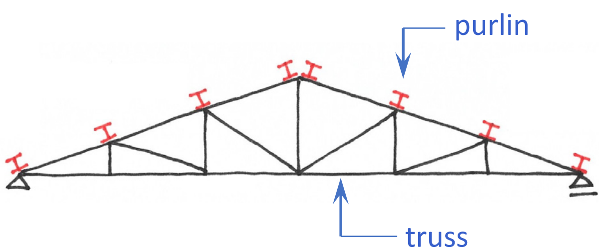
$$\Leftrightarrow \frac{100 - X}{549} = \frac{3 \cdot X}{7625} + \frac{X \cdot 6}{200 \cdot 10^6 \cdot 8.04 \cdot 10^{-4}} + 1.2 \cdot 10^{-5} \cdot 20 \cdot 6$$

$$\Leftrightarrow X = 80.24 \text{ kN}$$

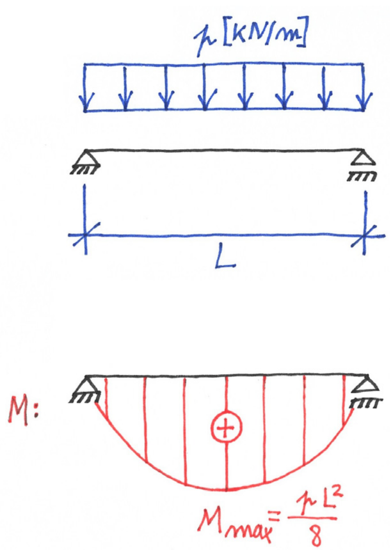
8. Simple inclined bending

8.1. Simple inclined bending in bars with a single loading plane

Example 8.1:



Simply supported *purlin* structure, subjected to a gravitational load p :



According to the **definition**, presented at the beginning of chapter 5, inclined bending is the case when the loading plane is inclined with respect the principal axes of inertia of the bar's cross-section.

If the axial force is zero, then the case is named **simple inclined bending**. Otherwise ($N \neq 0$), it is named **composed inclined bending**. In this chapter 8, only simple inclined bending is covered. The case of composed inclined bending will be dealt with in Strength of Materials 2.

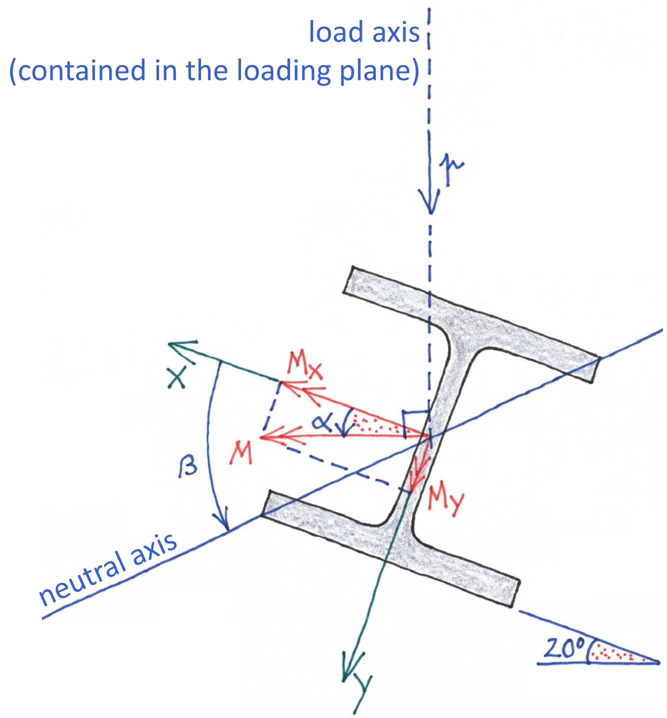
A typical example of a bar subjected to inclined bending is the secondary beam (called purlin), in a roof structure, supported by the inclined upper chord of the main roof structures (the articulated plane structures, also called trusses).

The gravitational loadings (self-weight and remaining weights) act along a vertical plane. None of the principal axes of the purlin's cross section is parallel to the loading plane, therefore this is a case of inclined bending.

In order to calculate the normal stresses on the purlin's cross section, we need to start by determining the applied bending moments. If the purlin is simply supported on the trusses, the purlin span will be equal to the separation between trusses, L.

The bending moment diagram in the purlin is shown in the image.

P*urlin* cross section, load axis, bending moment vector and diagram of normal stresses:



The **load axis**, usually represented in the cross section picture, is the intersection between the loading plane and the cross-section plane. It is a vertical line in this Example 8.1.

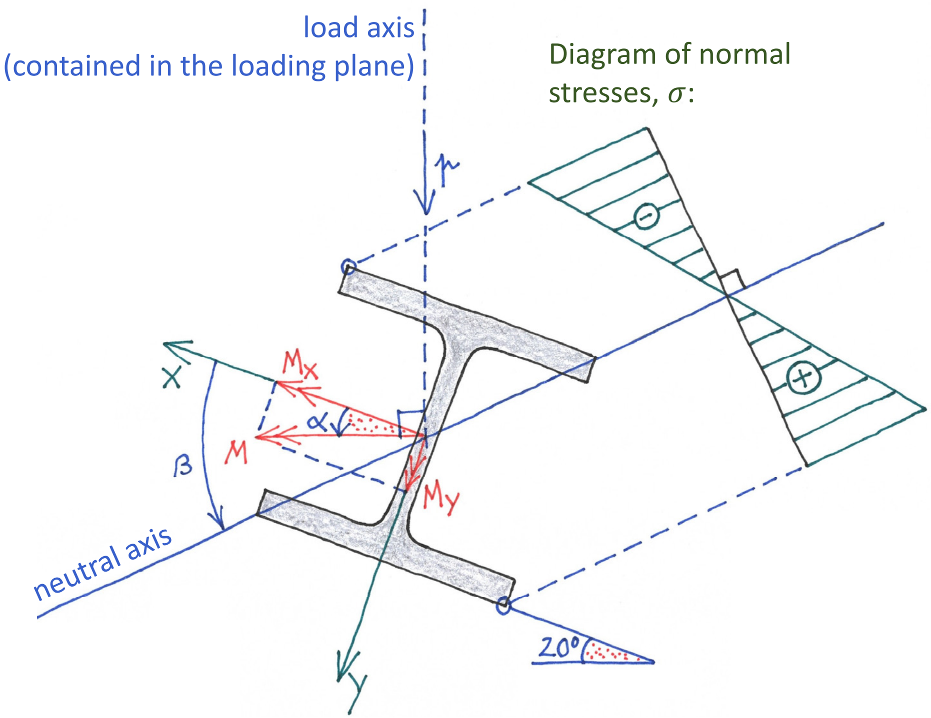
It was explained in chapter 5 that the **bending moment vector**, M, is perpendicular to the load axis. Its direction is given by the **right-hand rule**, explained in that same chapter. In this example, the vector M points to the left, because it is a bending moment which induces tensile stresses at the bottom cross-section fibres, and compression at the upper ones.

The principal axes of the cross section, X and Y, are also shown in the image.

The moment vector M can be decomposed into its **components along the principal axes** directions: M_x and M_y. Note that a simple definition of inclined bending can also be written as:

> *A cross section is subjected to inclined bending if both M_x and M_y (the components of the applied bending moment vector in the direction of the principal axes) are not zero.*

In our **convention**, M_x and M_y will be positive quantities if they point in the positive axis direction. Consequently, in this example 8.1, both M_x and M_y are positive quantities.



In inclined bending problems, the angle α denotes the **inclination of the bending moment vector**. It is measured as shown in the image:

> *the angle α is measured starting from the positive side of axis X, in the anti-clockwise direction, until the vector M.*

With these conventions:

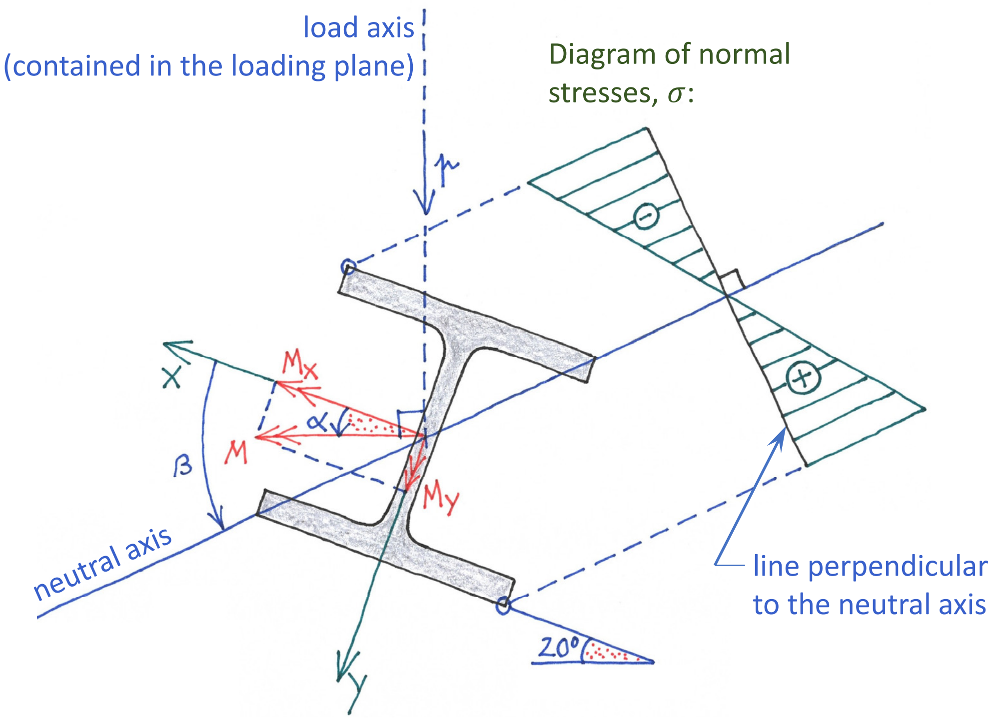
$$M_x = |M| \cdot \cos\alpha$$

$$M_y = |M| \cdot \sin\alpha$$

Each of the vectors M_x and M_y represents a plane bending vector. Therefore, the **normal stress** σ, in a point of the cross section of coordinates x and y (measured in the referential of the principal axes), can be calculated through the superposition of the effects of M_x and M_y:

$$\sigma = \frac{M_x}{I_x} \cdot y - \frac{M_y}{I_y} \cdot x$$

Note the **minus sign before M_y**, which is needed for consistency of the sign conventions. A positive quantity σ in the previous expression means that it is a tensile stress.



It was seen in chapter 5 that the straight line containing the points of the cross section where σ is zero is named the **neutral axis**.

In an inclined bending problem, the neutral axis is a straight line, whose **inclination**, denoted as β, can be determined through the expression:

$$\tan\beta = \tan\alpha \cdot \frac{I_x}{I_y}$$

where the angle β is measured using the same convention as angle α. Besides that, if the axial force in the cross section is zero, then the neutral axis crosses the cross-section's centre.

The **points of highest stresses** will be those farthest from the neutral axis. Therefore, before plotting the normal stress diagram, the neutral axis position has to be calculated and represented.

The **normal stresses diagram** is plotted over a line orthogonal to the neutral axis.

Demonstration of the equation for calculation of the neutral axis slope:

The neutral axis is, by definition, the geometric place of the points (points of coordinates x and y) where the stress σ is zero. Therefore, the equation $y(x)$ of the neutral axis is obtained by setting the normal stress function equal to zero:

$$\sigma = \frac{M_x}{I_x} \cdot y - \frac{M_y}{I_y} \cdot x = 0$$

$$\Leftrightarrow \frac{y}{x} = \frac{M_y}{M_x} \cdot \frac{I_x}{I_y} \qquad \text{[eq. 8.a]}$$

The ratio M_y/M_x is:

$$\frac{M_y}{M_x} = \frac{|M| \cdot \sin\alpha}{|M| \cdot \cos\alpha} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

and the ratio y/x is the slope of the straight line, equal to the tangent of its inclination:

$$\frac{y}{x} = \tan\beta$$

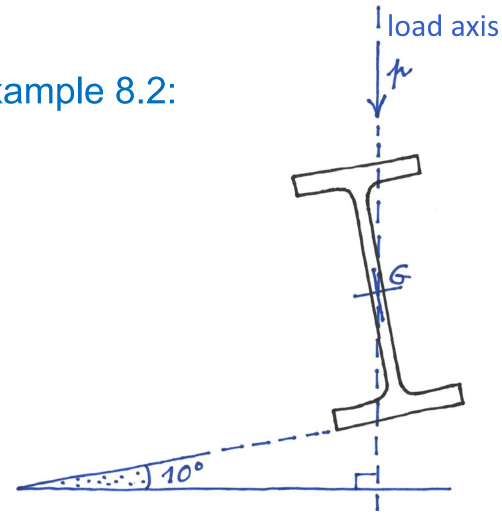
Thus, [eq. 8.a] can be written as:

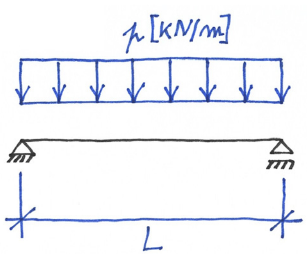
$$\tan\beta = \tan\alpha \cdot \frac{I_x}{I_y}$$

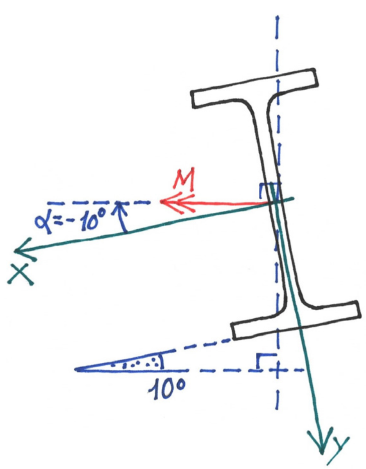
(which was to be demonstrated)

8.2. Calculation example

Example 8.2:







Problem:

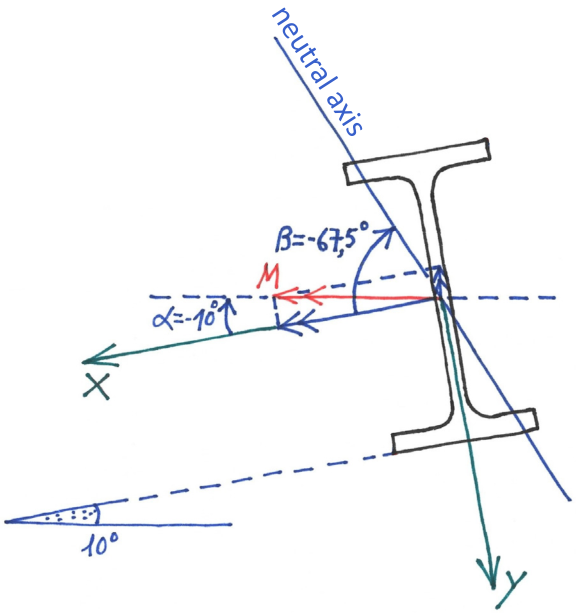
The image shows the cross section and the load axis of a simply supported purlin, in the roof of a warehouse distribution centre. The purlin's span is L = 6.5 m, and it is made with a steel profile IPE 240. The roof inclination is 10˚. Determine the neutral axis and the diagram of normal stresses, at the cross section of highest bending moment, for a uniformly distributed vertical loading p = 5 kN/m.

The maximum bending moment occurs at the mid span cross section, and is equal to:

$$M = \frac{p\,L^2}{8} = \frac{5 \cdot 6.5^2}{8} = 26.41\ \text{kNm}$$

This is a positive bending moment, therefore it causes tensile stresses at the bottom part of the cross section and, consequently, the moment vector points to the left (according to the right-hand rule).

It is known that an axis of symmetry is a principal axis, therefore the two principal axes are as shown in the image.



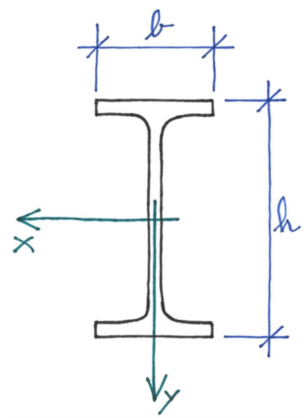
The angle α that characterizes the inclination of the bending moment vector is α = -10°.

The components of the moment vector in the directions of the principal axes are:

$$M_x = |M| \cdot \cos\alpha = 26.41 \cdot \cos(-10°) = 26.01\ \mathrm{kNm}$$

$$M_y = |M| \cdot \sin\alpha = 26.41 \cdot \sin(-10°) = -4.586\ \mathrm{kNm}$$

The cross section characteristics can be found in a table of steel profiles:



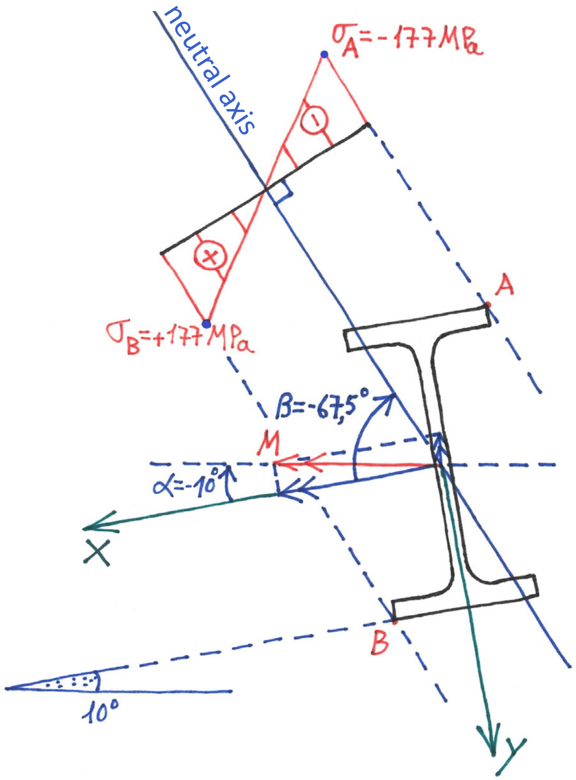
h = 24cm
b = 12cm
A = 39.1 cm^2
I_x = 3892 cm^4
I_y = 284 cm^4

The inclination of the neutral axis, β, is calculated as follows:

$$\tan\beta = \tan\alpha \cdot \frac{I_x}{I_y}$$

$$\Leftrightarrow \tan\beta = \tan(-10°) \cdot \frac{3892}{284}$$

$$\Leftrightarrow \beta = -67.52°$$



The points farthest from the neutral axis are the positions A e B shown in the image.

The normal stress in these points become:

$$\sigma_A = \frac{M_x}{I_x} \cdot y - \frac{M_y}{I_y} \cdot x =$$

$$= \frac{26.01}{3892 \cdot 10^{-8}} \cdot (-0.12) - \frac{-4.586}{284 \cdot 10^{-8}} \cdot (-0.06) \text{ kPa}$$

$$= -177 \text{ MPa}$$

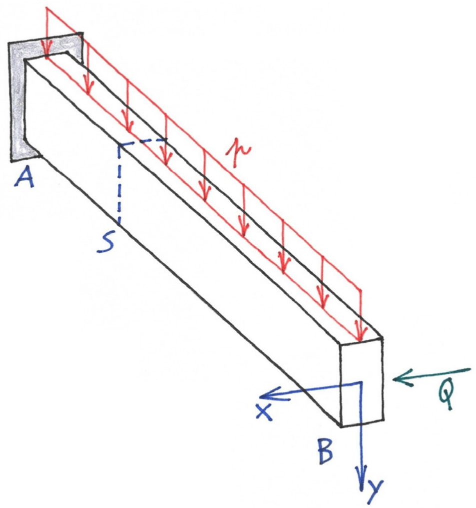
$$\sigma_B = \frac{26.01}{3892 \cdot 10^{-8}} \cdot 0.12 - \frac{-4.586}{284 \cdot 10^{-8}} \cdot 0.06 \text{ kPa}$$

$$= 177 \text{ MPa}$$

The diagram of normal stresses is represented in the image.

8.3. Simple inclined bending in bars with more than one loading plane

Example 8.3:



In the Examples 8.1 and 8.2 before, the applied loading and the support reactions were all contained in a single loading plane (a vertical plane, in those cases).

Real structures are often subjected to loads in multiple directions. The image shows, in perspective, a cantilever bar, subjected to a vertical distributed load p and a horizontal concentrated force Q.

In this example, these loads are aligned with the principal axes of the cross section. Therefore, we can calculate directly:

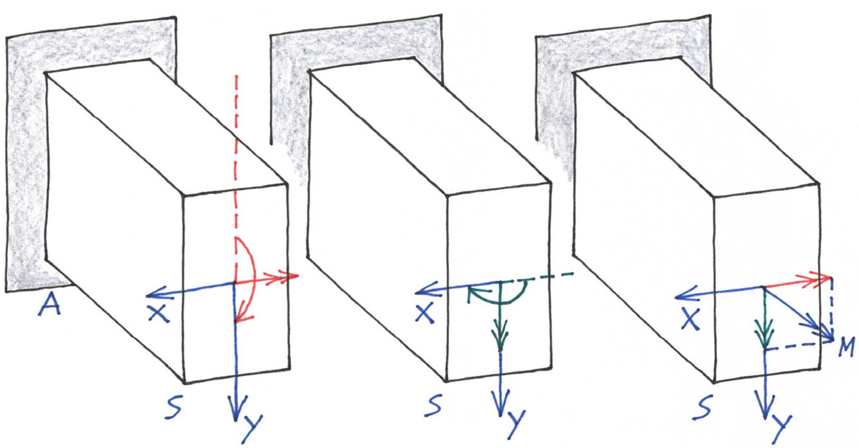
- the bending moment, in any cross-section, due to the effect of load p;
- the bending moment, in any cross-section, due to force Q.

Calculation of the moment vector M in cross section S:

Effect of load p:

Effect of load Q:

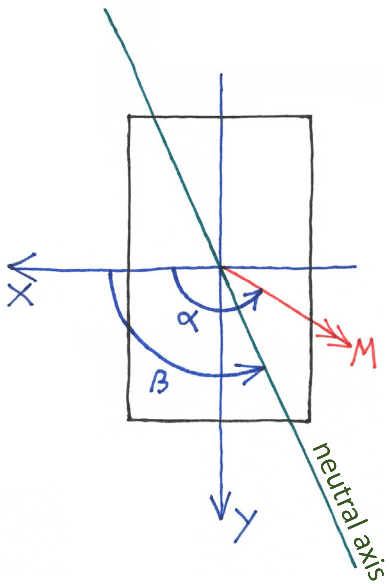
Resultant bending moment, M:



The image shows the calculation of the bending moments in section S, calculated as the resultants of the actions applied on the right side of that cross section.

The direction in which the moment vector points is determined through the right-hand rule, both for the effects of p and Q. The resultant bending moment, M, is the vectorial sum of vectors M_x and M_y.

Planification of the cross section S and calculation of the neutral axis' inclination, β:



The image shows the cross-section S and the applied resultant bending moment, M. If we want to represent the diagram of normal stresses, the inclination of the neutral axis, β, has to be determined. To do so, we need to calculate the angle α that characterizes the inclination of the resultant moment vector with respect to the positive side of axis X.

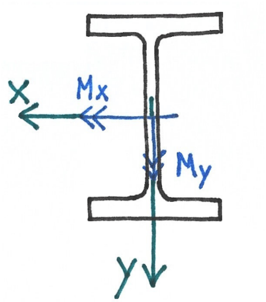
In this case, bearing in mind the orientation of the components M_x and M_y determined in the page before, it would be:

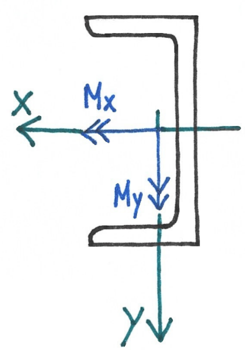
$$\alpha = 90° + \operatorname{atan}\frac{|M_x|}{|M_y|}$$

and the angle β would be calculated through the already known expression:

$$\tan\beta = \tan\alpha \cdot \frac{I_x}{I_y}$$

8.4. Design of sections with boundary embedded in a rectangle and $\sigma_{Rd,t} = \sigma_{Rd,c}$





Structural designers often face the following problem:

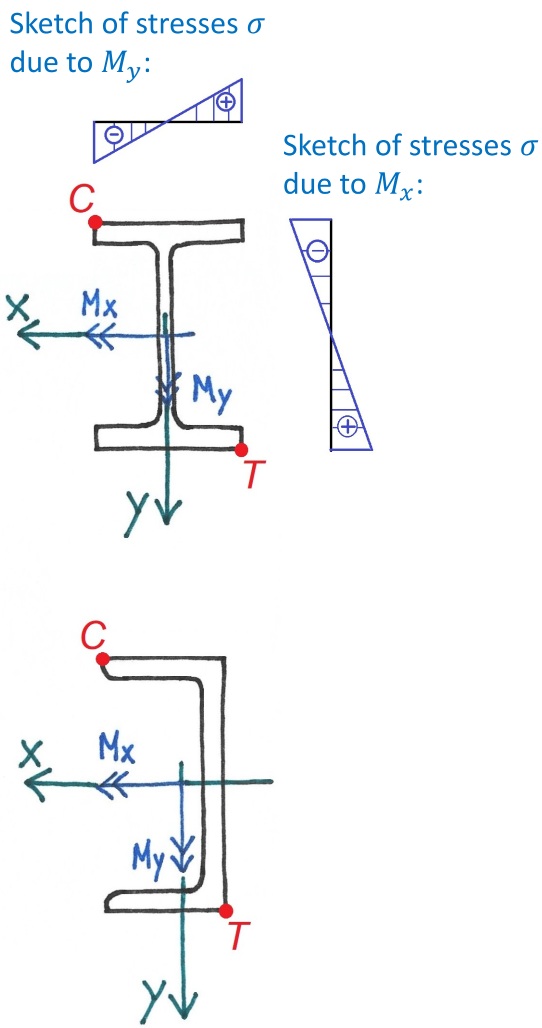
> The bending moments M_x and M_y applied to a certain steel bar are known. The material strength is known. The required cross-section size, from a table of steel profiles, has to be determined, so that the ultimate limit state condition is met, as far as the normal stresses are concerned.

If the cross-section's outer corners are embedded in a rectangle (like in the two examples shown in this page), the maximum stress in the cross section, is given by:

$$|\sigma_{\max}| = \frac{|M_x|}{W_x} + \frac{|M_y|}{W_y}$$

no matter which is the sign of the applied bending moments. In the expression before, W_x and W_y are the bending modulus around axes X and Y respectively.

For this design problem, we just need the stress of highest absolute value. We do not need to know whether it is a tensile or a compression stress, because the material strength of steel is the same for tension and for compression. And we do not need to calculate the inclination of the neutral axis either.



Justification for the fact the, in this type of problem, the expression $|\sigma_{\max}| = |M_x|/W_x + |M_y|/W_y$ can be used without thinking about which is the point of highest stress:

In the sections shown in the image, subjected to the applied bending moments indicated therein, one can see that the points of highest tension and compression are the points T and C indicated.

This conclusion is drawn just by analysing mentally the superposition of the diagrams due to M_x and M_y, as shown in the image.

We can conclude that, in the "I" section, the absolute value of the highest compression is equal to the absolute value of the highest tension. This is because the section is symmetric around both X and Y.

On the contrary, in the "Ɔ" section, the stress of highest absolute value occurs in point C. Note that, no matter which is the sign of M_x and M_y, and no matter which is the orientation of the cross section, there would be a point where the stress of highest absolute value due to M_x, i.e. $|M_x|/W_x$ would be summed to the stress of highest absolute value due to M_y, i.e. $|M_y|/W_y$.

Procedure for determination of the required cross-section size, so that the ultimate limit state condition is met:

The ultimate limit condition is:

$$\sigma_{Sd} \leq \sigma_{Rd}$$

$$\Leftrightarrow 1.5\ |\sigma_{\max}| \leq \sigma_{Rd}$$

$$\Leftrightarrow 1.5\left(\frac{|M_x|}{W_x} + \frac{|M_y|}{W_y}\right) \leq \sigma_{Rd} \qquad \text{[eq. 8.b]}$$

We have this single condition, but two unknows, W_x and W_y. Therefore, the unknows cannot be calculated directly through this equation. Instead, the determination of the required cross section size has to be made iteratively. In each iteration:

1) pick a section size from the table of steel profiles, and read the corresponding bending modulus values, W_x and W_y;
2) apply the ultimate limit condition [eq. 8.b];
3) if the condition is respected (i.e. $\sigma_{Sd} \leq \sigma_{Rd}$), a smaller section size has to be picked in the following iteration;
4) on the contrary, if the condition is not respected ($\sigma_{Sd} > \sigma_{Rd}$), then a larger section size has to be picked in the following iteration.

Steps 1) to 4) are repeated until the most economical (i.e., the smallest) cross section size, for which $\sigma_{Sd} \leq \sigma_{Rd}$, is found.

Hint: one may start the iterations by selecting the required section size for the isolated action of M_x and M_y, i.e.:

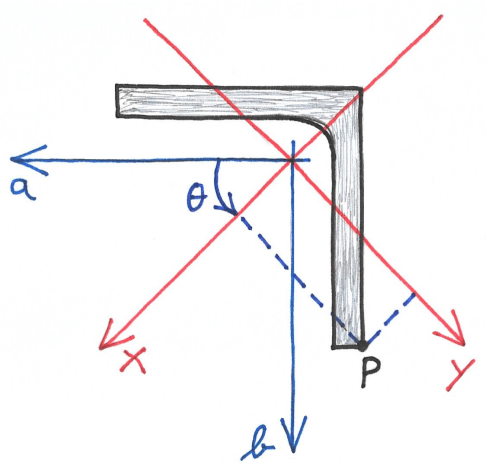
$$W_x \geq \frac{1.5 \cdot |M_x|}{\sigma_{Rd}}$$

and:

$$W_y \geq \frac{1.5 \cdot |M_y|}{\sigma_{Rd}}$$

These two conditions provide a lower bound for the required cross-section size.

8.5. Calculation of coordinates of a point after rotation of the axes referential



To calculate the stresses at a point (a section's vertex, for example), its coordinates are needed, expressed in the central principal axes of inertia (CPAI) referential.

In cross-sections of the type shown in the image (angle of equal flanges), the CPAI are the axes X and Y, because X is an axis of symmetry. The determination the coordinates of a point P is straightforward in a system of axes parallel to the section sides (axes a, b in the image, for example). However, it is not so straightforward in the case of a rotated system of axes (axes X, Y for example).

However, based on the coordinates (a, b) of point P in the referential a, b and on the angle of rotation of the referential (angle θ, measured as shown in the image), the coordinates of P in the referential X, Y can be easily determined as:

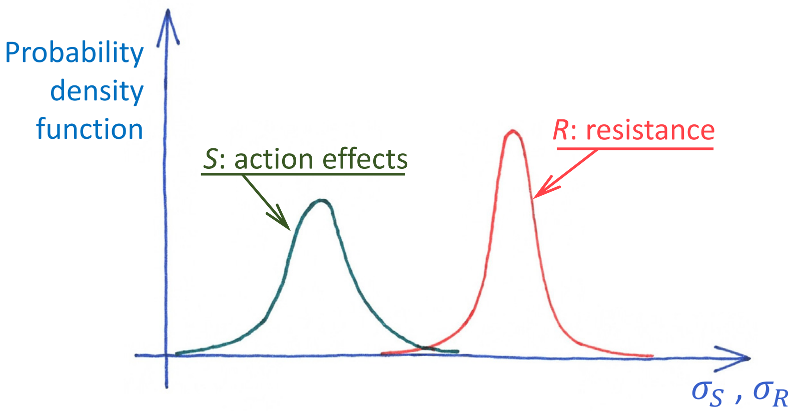
$$x = a \cdot \cos\theta + b \cdot \sin\theta$$

$$y = -a \cdot \sin\theta + b \cdot \cos\theta$$

Annex A

Basic notions of structural safety and reliability

A.1. Probabilistic approach for safety assessment in ultimate limit state (ULS)



One of the conditions which has to be satisfied, for a structure to be safe, consists in ensuring that the resistance is higher than the effect of the applied actions.

The actions considered in this course unit are:

- applied forces (concentrated or distributed) and concentrated moments;
- temperature variations;
- fabrication defects.

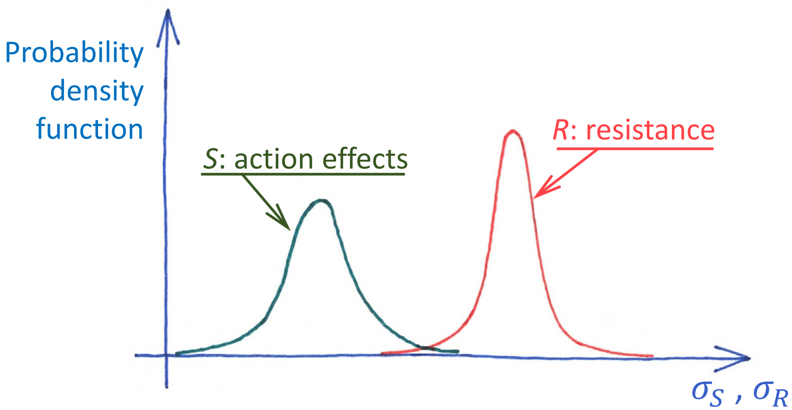
In this course unit, the safety assessment is made by analysing the normal stress value, only. So, the safety condition is:

$$\sigma_S \leq \sigma_R$$

where:

σ_S is the applied stress (it is an "action effect")
σ_R is the resistant stress (material strength)

In the reality, the shear stresses have also to be taken into account, as well as the combination of stresses applied, in each material point, in multiple directions. These issues are dealt with in the course unit of Strength of Materials 2.



In structural engineering problems, the applied stress σ_S cannot be considered a determinist parameter (i.e. a parameter with a fixed, known, value). Instead, it is a statistical variable (a variable having values that vary, due to random causes). It means that there is some uncertainty about the exact value of the stress.

The same can be said about the resistant stress σ_R.

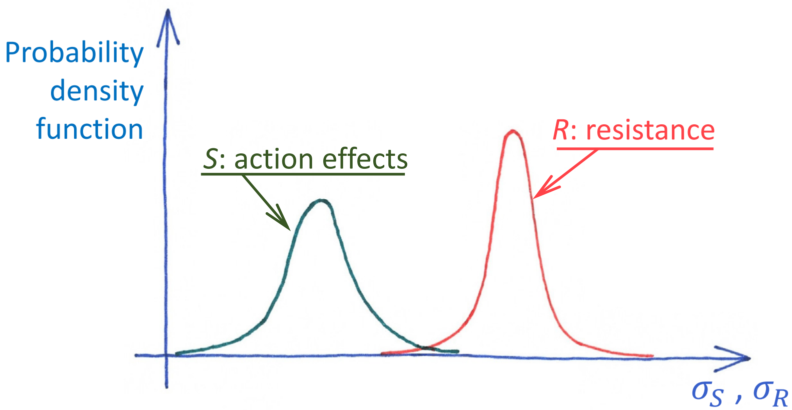
Exemplification of the sources of uncertainty:

A structural designer (civil engineer) is designing a simply supported steel beam, which is part of a residential building. To ensure that the beam is safe, in terms of normal stresses, the maximum applied stress σ_S at the cross-section of highest bending moment should not exceed the material strength σ_R.

In order to calculate the applied stress σ_S, the effect of "live loading" (loading due to furniture, occupancy of people, etc.) is represented by a uniformly distributed loading, in kN/m^2. Let it be called the load q. The value of q, for typical residential buildings, may be set in design codes, based on past observations of the loads applied in existing buildings of the same type. Based on such observations, the variable q can be characterized through:

- the estimated mean value;
- the estimated standard deviation;
- the estimated type of statistical distributions (the Gaussian distribution is usually assumed in this type of study).

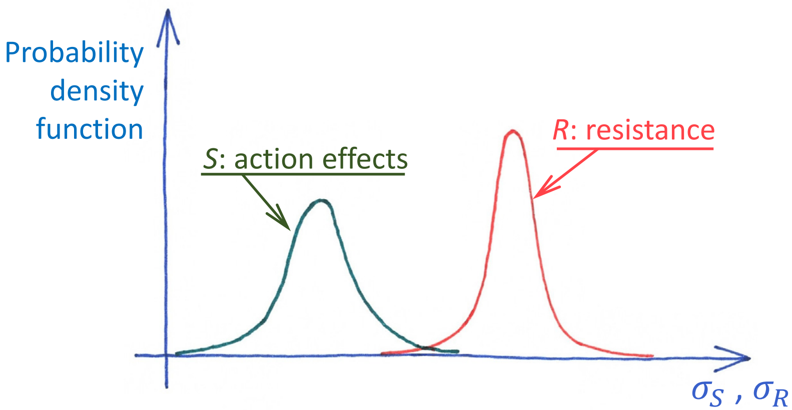
Based on these three data, the **probability density function** for the variable q can be plotted.



The applied stress also depends on the structure geometry (span, cross-section dimensions, etc.). Even though the geometry is fixed in the design drawings, the geometry as built will not be exactly equal to the prescribed one (in concrete constructions this variability is larger than in steel structures). Therefore, the geometry variables (span length, width and depth of the cross-section, etc.) can also be characterized through probability density functions. The variability in the geometry data is usually much less important than the variability of the loading value, q.

Taking into account the loading and the geometry data, the applied stress σ_S can also be characterized through a probability density function.

The resistant stress σ_R is also a statistical variable, whose value is characterized based on the information obtained in tests of material specimens, conducted in the past. The material strength value obtained in each test varies, depending on random factors that affect the material production (raw materials, factory conditions, etc.). With the aid of these test results, the material strength of each steel class can be characterized by means of a certain statistical distribution shape (usually the Gaussian distribution), and its estimated mean value and standard deviation value.



If this safety assessment procedure is followed, the average value of σ_S will be significantly lower than the average value of σ_R.

The applied stress would only reach the resistant stress in a very improbable situation, which is called the **ultimate limit state**.

Probabilistic approach for safety assessment:

The designer has to ensure that the **probability of failure** is extremely small, taking into account the uncertainty associated to the applied stress variable σ_S, and to the resistant stress variable σ_R. The admitted probability of failure is obviously not the same for every type of structure (an hospital or a critical bridge structure, for example, are not designed for the same probability of failure as an agricultural storage facility).

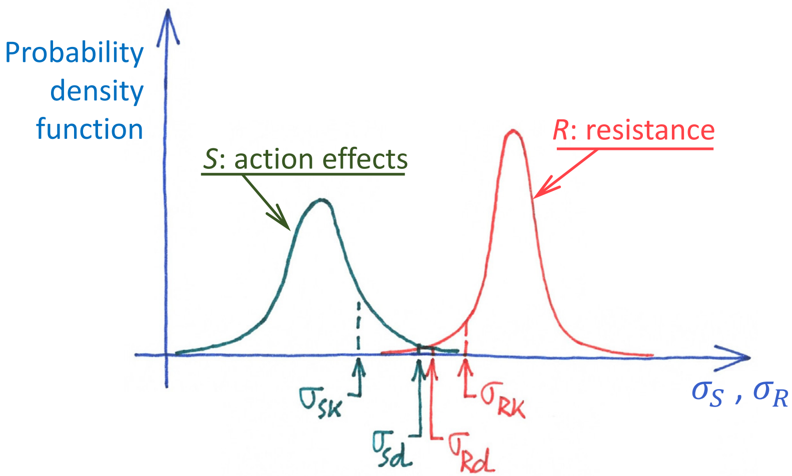
That condition can be written as:

$$P_f = \text{Prob}\,\{R \leq S\} \leq 10^{-5}$$

- P_f: Probability of failure
- $\text{Prob}\,\{R \leq S\}$: Probability of the resistance being lower than the applied stress
- 10^{-5}: Reference value for the admitted probability of failure, generally adopted for current structures

The calculation of the probability of failure, in a structural analysis problem, can be done by employing probabilistic analysis methodologies such as the Monte Carlo simulation method. This is a very time consuming procedure, and therefore it is rarely applied in practice. It is used only in special studies.

A.2. Method of the Partial Safety Coefficients (semi-probabilistic approach) for safety assessment in ULS



Given that the probabilistic analysis approach is not feasible for current applications, an alternative approach has been developed (and prescribed in the design codes). It is the so called Method of the Partial Safety Coefficients. It is considered a semi-probabilistic method, because the safety coefficients were calibrated so that this method leads to a designed structure with a geometry similar to the result of the direct application of the probabilistic approach.

In the Method of the Partial Safety Coefficients, the ultimate limit state (ULS) condition is expressed as:

$$\sigma_{Sd} \leq \sigma_{Rd}$$

Design value of the applied stress (action effects) — σ_{Sd}

Design value of the material strength (stress resisted by the material) — σ_{Rd}

$$\Leftrightarrow \gamma_S \cdot \sigma_{Sk} \leq \frac{\sigma_{Rk}}{\gamma_M}$$

Representative value (nominal or characteristic) of the applied stress (action effect) — σ_{Sk}

Representative value (nominal or characteristic) of the material strength (usually it is the characteristic value corresponding to the 5% quantile) — σ_{Rk}

Partial Safety Coefficient for actions (in RM we consider, by simplification, γ_S = 1.5) — γ_S

Partial safety factor to take into account the uncertainty in the determination of the material strength and the uncertainty of the calculation model (in RM, we use directly the σ_{Rd} value) — γ_M

NOTE: Currently, the letter "E" is often used instead of "S" to denote action **e**ffects.

In bending problems, the normal stress σ is directly related to the bending moment value, M. Therefore, the safety assessment can also be made by checking the bending moment values:

$$M_{Sd} \leq M_{Rd}$$

Design value of the applied bending moment (action effects) → M_{Sd}

Design value of the resistant bending moment (it is, therefore, determined using the design value of the stress resisted by the material) → M_{Rd}

Representative value (nominal or characteristic) of the applied bending moment (action effect) → M_{Sk}

$$\Leftrightarrow \gamma_S \cdot M_{Sk} \leq M_{Rd}$$

Partial safety coefficient for actions (in RM we consider, by simplification, γ_S = 1.5) → γ_S

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