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# The full power of the half-power

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**Abstract**

We use the complex square root to define a very simple homotopic invariant over the non-vanishing functions defined on the circle. As a consequence we provide easy proofs of the plane Brouwer fixed point theorem and the Fundamental Theorem of Algebra. The relation of this new invariant with the winding number and the Brouwer degree will be fully unveiled.


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## 1. Introduction

In [1] it was given a very simple proof of the no-retraction theorem in the plane based on the following elementary properties of the complex square root:

$(A_1)$ Any continuous $f : \overline{D_R} \to S_R$ admits a continuous square root in $\overline{D_R}$;
$(A_2)$ The identity $i : S_R \to S_R$ does not admit a continuous square root on $S_R$,

where $D_R \subset \mathbb{C}$ is the open disk of center 0 and radius $R > 0$ and $S_R$ its boundary.

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From the two previous claims it is clear that a retraction from $\overline{D_R}$ to $S_R$ (that is, a continuous function $r : \overline{D_R} \to S_R$ such that $r(z) = z$ for all $z \in S_R$) cannot exist. More generally, in this paper we will show that for $f : \overline{D_R} \to \mathbb{C}$ continuous such that $\gamma := f|_{S_R}$ does not vanish, whether or not $\gamma$ admits a continuous square root allows to distinguish between even and odd Brouwer degree of $f$.

Further developing this idea, by counting the exact number of times that a continuous square root for $\gamma \in C(S_R, \mathbb{C}\backslash\{0\})$ can be successively extracted over $S_R$, a more powerful tool is obtained. Remarkably, this value, ranging from 0 to $\infty$, shares some of the most important properties of the Brouwer degree (the existence property, homotopy invariance, etc.) while it is much easier to establish. In this way we shall obtain elementary proofs of many important theorems such as the invariance of domain, the Brouwer fixed point theorem or the Fundamental Theorem of Algebra.

The paper is organized as follows: in Section 2 we prove the main properties of the "square root counter" without any reference to the argument of a complex number neither to integration nor differentiation. As a consequence, in Section 3 we obtain a very simple proof of a common generalization of both the Brouwer fixed point theorem for the plane and the Fundamental Theorem of Algebra. In Section 4 we show how the square root can be used to define a continuous $n$th root over suitable subsets of the complex plane. Section 5 is more technical and contains a new characterization of the absolute value of the winding number in terms of the existence of continuous $n$th roots. In Section 6 we prove that, for any prime number $p$, the "$p$th root counter" provides actually the $p$-adic valuation of the winding number. Finally, we end the paper with further comments and relations in Section 7.

Once the paper was finished an anonymous referee pointed out to us the Ref. [3], where a similar approach to [1] using the complex square root was introduced by M. K. Fort, Jr., and applications were given to the fundamental theorem of algebra and Brouwer's fixed point theorem. After this we were also able to find an outline of Fort's method in K. R. Stromberg's classic book [10, p. 126, Exercise 20]. Roughly speaking, the proofs rely on the above property $(A_1)$ together with an extension of $(A_2)$ for arbitrary odd mappings, which coincides with Lemma 2.8. As a consequence, the key result [3, Theorem 3] is obtained, stating that if a continuous mapping $f : \overline{D_R} \to \mathbb{C}$ is odd or a nonzero integer power of the identity over $S_R$, then $f$ has at least one zero. The ideas in [3] certainly precoursed those in [1] but the main features of the present paper, namely, the homotopy invariance of the square root counter and its relation to the index, seem still unnoticed in the literature.

## 2. How many times does your function admit a continuous square root?

If $A := S_1\backslash\{-1\}$ then it is satisfied that $r_A(w) := \frac{w+1}{|w+1|}$ is a continuous square root over $A$. Indeed, for each $w \in A$ holds that

$$r_A(w)^2 = \frac{(w+1)^2}{|w+1|^2} = \frac{w+1}{\overline{w}+1} = \frac{w(1+\overline{w})}{\overline{w}+1} = w.$$

This allows to define a continuous square root over the set $A^+ := \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ given by $r_2(z) := \sqrt{|z|}r_A(z/|z|)$, or explicitly

$$r_2 : A^+ \to \mathbb{C}_{\Re(z)>0}, \quad z \mapsto \sqrt{|z|}\frac{z+|z|}{|z+|z||}. \tag{1}$$

We remark that only the real square root and the basic arithmetic operations were used in this construction. Furthermore, it is clear from the following result that $-r_2(z)$ defines the unique other continuous square root in $A^+$.

**Lemma 2.1.** *Let $X$ a connected topological space and $r, \tilde{r} : X \to \mathbb{C}\backslash\{0\}$ two continuous functions such that $r(x)^2 = \tilde{r}(x)^2$ for all $x \in X$. Then $r(x) = \tilde{r}(x)$ for all $x \in X$ or $r(x) = -\tilde{r}(x)$ for all $x \in X$.*

**Proof.** Since $\frac{r(x)^2}{\tilde{r}(x)^2} = \left(\frac{r(x)}{\tilde{r}(x)}\right)^2 = 1$ for all $x \in X$ it follows that the continuous function $\frac{r}{\tilde{r}}$ applies the connected set $X$ into the discrete one $\{-1, 1\}$, so it should be constant over $X$. □

Finally, setting $\pm ir_2(-z)$ we obtain the two unique continuous square roots over $A^- := \mathbb{C}\backslash\mathbb{R}_{\geq 0}$.

**Definition 2.1.** Let $\mathcal{A} := \{\gamma : S_R \to \mathbb{C}\backslash\{0\}\, continuous\}$ equipped with the uniform metric.

For $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ we say that $\gamma \in \mathcal{A}$ admits a continuous $2^m$-th root if there exists $\sigma \in \mathcal{A}$ such that $\sigma^{2^m} = \gamma$ and we define $v_2 : \mathcal{A} \to \mathbb{N}_0 \cup \{\infty\}$ by

$$v_2(\gamma) := \sup\{m \in \mathbb{N}_0 : \ \gamma \ admits\ a\ continuous\ 2^m - th\ root\},$$

where we understand $v_2(\gamma) = \infty$ if the previous set is unbounded.

If $f : \overline{D_R} \to \mathbb{C}$ is continuous and such that $f(z) \neq 0$ for all $z \in S_R$ we will write $v_2(f)$ with the meaning of $v_2(f|_{S_R})$.

It is clear that if $\gamma \in \mathcal{A}$ admits a continuous $2^m$-th root for some $m \in \mathbb{N}$ then it admits a continuous $2^{m'}$-th root for each $m' \in \mathbb{N}_0$ with $0 \leq m' \leq m$, so $v_2(\gamma) = \infty$ means that $\gamma$ admits a continuous $2^m$-th root for each $m \in \mathbb{N}$.

The following are very useful properties of $v_2$.

**Lemma 2.2.** *For $\gamma, \delta \in \mathcal{A}$, it holds that:*

1. $v_2(1/\gamma) = v_2(\gamma)$.
2. $v_2(\gamma \cdot \delta) \geq \min\{v_2(\gamma), v_2(\delta)\}$.
3. *If $v_2(\gamma) \neq v_2(\delta)$ then $v_2(\gamma \cdot \delta) = \min\{v_2(\gamma), v_2(\delta)\}$.*
4. *If $v_2(\gamma \cdot \delta) = \infty$ then $v_2(\gamma) = v_2(\delta)$.*
5. *If $\gamma \equiv c \in \mathbb{C} \setminus \{0\}$ then $v_2(\gamma) = \infty$.*
6. $v_2(\gamma^2) = v_2(\gamma) + 1$ *(here it is understood that $\infty + 1 = \infty$).*

**Proof.** The first three properties are straightforward consequences of the definition of $v_2$.

4. If $v_2(\gamma) = v_2(\delta) = \infty$, then we are done. Otherwise, $v_2(\gamma \cdot \delta) > \min\{v_2(\gamma), v_2(\delta)\}$ and the result follows from property 3.

5. Each $c \in \mathbb{C}\backslash\{0\}$ admits a square root that is again in $\mathbb{C}\backslash\{0\}$, so the process can be iterated.

6. We may assume that $m := v_2(\gamma) < \infty$ since, otherwise, the result is trivial. It is clear that $v_2(\gamma^2) \geq m+1$. Next, suppose that $v_2(\gamma^2) \geq m+2$, then there exists $\sigma \in \mathcal{A}$ such that

$$\gamma^2 = \sigma^{2^{m+2}} = \left(\sigma^{2^{m+1}}\right)^2.$$

From Lemma 2.1 it follows that $\gamma = \sigma^{2^{m+1}}$ or $\gamma = -\sigma^{2^{m+1}}$, and in both cases $v_2(\gamma) \geq m+1$, a contradiction. □

Also, it is easy to see that, under a suitable condition, the existence of a continuous square root can be iterated as many times as desired.

**Lemma 2.3.** *If $\gamma \in \mathcal{A}$ and $\gamma(S_R) \subset A^+$ or $\gamma(S_R) \subset A^-$, then $v_2(\gamma) = \infty$.*

**Proof.** For each $m \in \mathbb{N}$ the function $\sigma := \overbrace{r \circ r \circ \cdots \circ r}^{(m)} \circ \gamma$, where $r(z) = r_2(z)$ in case $\gamma(S_R) \subset A^+$ or $r(z) = i r_2(-z)$ in case $\gamma(S_R) \subset A^-$ and $r_2$ is given by (1), satisfies that $\sigma \in \mathcal{A}$ and $\sigma^{2^m} = \gamma$. □

**Corollary 2.1.** *The following properties hold:*

1. *If $\gamma \in \mathcal{A}$ and $\Re(\gamma) \geq 0$, then $v_2(\gamma) = \infty$.*
2. *If $f : \mathbb{C}\backslash\overline{D_{R_0}} \to \mathbb{C}$ is continuous for some $R_0 > 0$ and there exists $\lim_{|z|\to+\infty} f(z) \in \mathbb{C}\setminus\{0\}$, then $v_2\left(f_{|S_R}\right) = \infty$ for all large enough $R > 0$.*

**Proof.** The validity of 1 follows directly because in that case $\gamma(S_R) \subset A^+$. In case 2 it is clear that for all large enough $R > 0$ we have that $f_{|S_R} \in \mathcal{A}$ and $f(S_R)$ is contained in a closed ball that does not contain the origin, so $f(S_R) \subset A^+$ or $f(S_R) \subset A^-$. □

Next, we establish a basic version of Rouché's Theorem in our framework.

**Lemma 2.4.** *Let $\gamma, \delta \in \mathcal{A}$. If $|\delta - \gamma| < |\gamma|$, then $v_2(\gamma) = v_2(\delta)$.*

**Proof.** Because $\left|\frac{\delta}{\gamma} - 1\right| < 1$ and using Corollary 2.1 we have that $v_2\left(\frac{\delta}{\gamma}\right) = \infty$. Then the result follows from properties 4 and 1 in Lemma 2.2 □

Now, we are in a position to prove the main properties of $v_2$.

**Lemma 2.5** (*Continuity*). *$v_2 : \mathcal{A} \to \mathbb{N}_0 \cup \{\infty\}$ is continuous (i.e. locally constant).*

**Proof.** Given $\gamma \in \mathcal{A}$, let $\varepsilon := \inf|\gamma| > 0$. Then Lemma 2.4 implies that $v_2(\gamma) = v_2(\delta)$ for $|\delta - \gamma| < \varepsilon$. □

In other words, Lemma 2.5 says that $v_2$ is constant on each connected component of $\mathcal{A}$ which, in turn, implies the homotopy invariance, namely, if $h: S_R \times [0,1] \to \mathbb{C}\backslash\{0\}$ is continuous, then $v_2(h_0) = v_2(h_1)$, where $h_s(z) := h(z,s)$.

**Lemma 2.6.** *If $f: \overline{D_R} \to \mathbb{C}\backslash\{0\}$ is continuous, then $v_2(f) = \infty$.*

**Proof.** Consider the continuous homotopy $h: S_R \times [0,1] \to \mathbb{C}\backslash\{0\}$ between $f(0)$ and $f|_{S_R}$ given by

$$h(z,s) = h_s(z) := f(sz).$$

Then, from Lemma 2.5 and property 5 in Lemma 2.2 it follows that

$$v_2(f) = v_2(h_1) = v_2(h_0) = \infty. \quad \square$$

**Lemma 2.7** (*Existence*)**.** *Assume that $f: \overline{D_R} \to \mathbb{C}$ is continuous and $\varepsilon := \inf |f|_{|S_R} > 0$. If $v_2(f) \neq \infty$, then the equation $f(z) = w$ has a solution in $D_R$ for each $w$ such that $|w| < \varepsilon$.*

**Proof.** Let $g(z) := f(z) - w$, then for $z \in S_R$ it follows that $|g(z) - f(z)| = |w| < |f(z)|$. By Lemma 2.4, it follows that $v_2(g) \neq \infty$ and the conclusion is deduced from Lemma 2.6. $\square$

**Lemma 2.8** (*Borsuk*)**.** *If $\gamma \in \mathcal{A}$ is odd, then $v_2(\gamma) = 0$.*

**Proof.** Suppose $\delta^2 = \gamma$, then

$$\delta(-z)^2 = \gamma(-z) = -\gamma(z) = -\delta(z)^2 = (i\delta(z))^2.$$

From Lemma 2.1 it follows that $\delta(-z) \equiv i\delta(z)$ or $\delta(-z) \equiv -i\delta(z)$. In both cases, this implies

$$\delta(z) = \delta(-(-z)) = \pm i\delta(-z) = (\pm i)^2\delta(z) = -\delta(z),$$

a contradiction. $\square$

**Remark 2.2.** Notice that properties $(A_1)$ and $(A_2)$ stated in the Introduction trivially follow from Lemmas 2.6 and 2.8.

Moreover, Lemma 2.8 and property 6 in Lemma 2.2, imply that if $\gamma \in \mathcal{A}$ is odd, then for each $m \in \mathbb{N}_0$, $v_2\left(\gamma^{2^m}\right) = m$. In particular, for each $n \in \mathbb{Z}$ we have

$$v_2(z^n) = v_2(\overline{z}^n) = \nu_2(n),$$

where $\nu_2(\cdot)$ stands for the 2-adic valuation of the integers, that is

$$\nu_2(-n) = \nu_2(n) := \begin{cases} m & if\ n = 2^m \cdot k \geq 1\ and\ \gcd\{2,k\} = 1, \\ \infty & if\ n = 0. \end{cases}$$

In particular, the latter property, combined with Lemma 2.7, yields an elementary proof of the second part of [3, Theorem 3].

This property will be conveniently generalized for any $\gamma \in \mathcal{A}$ and any prime number $p$ in Section 6.

As a simple consequence of the previous properties we deduce a deep theorem: the invariance of domain.

**Theorem 2.3.** *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ continuous and injective. Then $f(U)$ is open and $f : U \to f(U)$ is a homeomorphism.*

**Proof.** Let $\overline{D_R(z_0)} \subset U$. We want to prove that $f(D_R(z_0))$ is a neighborhood of $f(z_0)$. Without loss of generality, we may assume that $z_0 = 0 = f(z_0)$. Consider in $S_R \times [0, 1]$ the continuous homotopy

$$h(z, s) = h_s(z) := f(z) - f(-sz),$$

which does not vanish since $f$ is injective. It follows from Lemma 2.5 that $v_2(h_s)$ is constant. Observe, moreover, that $h_0(z) = f(z)$ and $h_1(z) = f(z) - f(-z)$, which is odd and, consequently from Lemma 2.8

$$v_2(f) = v_2(h_0) = v_2(h_1) = 0.$$

To conclude, let $\varepsilon := \inf |f|_{|S_R}$, then $D_\varepsilon \subset f(D_R)$ from Lemma 2.7. □

## 3. A generalized version of the FTA

In [9] it was presented a version of Bolzano's theorem for an holomorphic function $f$ in $\Omega$, and continuous on $\overline{\Omega}$, assuming the condition

$$\Re(\overline{z} f(z)) > 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{C}$ is a bounded domain containing the origin.

As noticed by Mawhin in [5], when $\overline{\Omega} = D_R$ the previous condition is just the one used by Hadamard in his proof of the Brouwer fixed point theorem, namely,

$$\langle z, f(z)\rangle > 0 \quad \text{for all } z \in S_R,$$

where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $\mathbb{R}^2$.

We present here an extension to the continuous case of a result firstly stated in [6, Theorem 3.1] in the holomorphic case and proved using only the mean value property of holomorphic functions.

**Theorem 3.1** (**$n$-Hadamard**). *Let $f : \overline{D_R} \to \mathbb{C}$ be continuous and assume that for some $n \in \mathbb{N}$*

$$\Re(\overline{z}^n f(z)) \geq 0 \qquad \text{for all } z \in S_R. \tag{2}$$

*Then $f$ has a zero in $\overline{D_R}$.*

**Proof.** If $f$ vanishes in $S_R$ then we are done. If not, from Corollary 2.1 we have

$$v_2(\overline{z}^n f) = \infty,$$

and then by Lemma 2.2, property 4, and Remark 2.2 it follows that

$$v_2(f) = v_2(\overline{z}^n) = v_2(n) \neq \infty,$$

so $f$ vanishes in $D_R$ by Lemma 2.7. □

**Remark 3.2.** Notice that condition (2) can be replaced by

- $\Re(\overline{z}^n f(z))$ or $\Im(\overline{z}^n f(z))$ does not change sign on $S_R$,

because of the properties

$$\Re(\overline{z}(-w)) = -\Re(\overline{z}w) \quad and \quad \Re(\overline{z}(wi)) = -\Im(\overline{z}w) \quad for\ all\ z, w \in \mathbb{C}.$$

The same strategy used in the proof of Theorem 3.1, but applying instead the second property in Corollary 2.1, leads to the following variant of the sufficiency part of the main result in [4].

**Theorem 3.3.** *The continuous function $f:\mathbb{C} \to \mathbb{C}$ has a zero provided that there exists a continuous function $g:\mathbb{C} \to \mathbb{C}$ such that $\lim_{|z|\to+\infty} \frac{g(z)}{f(z)} \in \mathbb{C} \setminus \{0\}$ and $v_2(g) \neq \infty$ on $S_{R_k}$ for a sequence $R_k \to +\infty$.*

From $n$-Hadamard it is immediate to deduce the following generalization of the Brouwer fixed point theorem for the plane.

**Theorem 3.4** ($n$-**Rothe**). *If $f:\overline{D_R} \to \mathbb{C}$ is continuous and satisfies for some $n \in \mathbb{N}$*

$$|f(z)| \le |z^n| \quad for\ all\ z \in S_R, \tag{3}$$

*then there exists $z \in \overline{D_R}$ such that $z^n = f(z)$.*

**Proof.** Assume that $f:\overline{D_R} \to \mathbb{C}$ is continuous and satisfies condition (3). Define $g(z) := z^n - f(z)$ for all $z \in \overline{D_R}$, then for all $z \in S_R$

$$\Re(\overline{z}^n g(z)) = |z|^{2n} - \Re(\overline{z}^n f(z)) \ge |z|^{2n} - |\overline{z}^n f(z)| \ge R^{2n} - R^{2n} \ge 0.$$

Therefore the conclusion follows from Theorem 3.1 applied to $g$. $\square$

**Remark 3.5.** A version of Theorem 3.4 for holomorphic functions is given in [7, Corollary 5.3], where a value $z$ satisfying $z^n = f(z)$ is called an $n$-branch point.

Notice that Theorem 3.4, for $n = 1$, just asserts that any continuous function $f$ on $\overline{D_R}$ such that $f(S_R) \subset \overline{D_R}$ has a fixed point. This result, usually called Rothe's fixed point theorem, is equivalent to the Brouwer fixed point theorem, see [6].

In the same way as Hadamard's theorem and the Rothe fixed point theorem are equivalent, [6], we show now that so are $n$-Hadamard and $n$-Rothe.

**Theorem 3.6.** *Theorems 3.1 and 3.4 are equivalent.*

**Proof.** It only remains to prove that Theorem 3.4 implies Theorem 3.1: assume now that $f:\overline{D_R} \to \mathbb{C}$ is continuous and satisfies condition (2). Define $g(z) := r(z)^n - f(r(z))$ for all $z \in \mathbb{C}$ where the continuous retraction $r:\mathbb{C} \to \overline{D_R}$ is defined by $r(z) = z$ if $|z| \le R$ and $r(z) := \frac{R\cdot z}{|z|}$ if $|z| > R$. Then there exists $K > 0$ such that $|g(z)| \le K$ for all $z \in \mathbb{C}$ and from Theorem 3.4 it follows the existence of $z \in \mathbb{C}$ such that $z^n = g(z)$, that is

$$z^n = r(z)^n - f(r(z)). \tag{4}$$

Now suppose that $|z| > R$; then $r(z) = \frac{R}{|z|}z \in S_R$, $f(r(z)) = r(z)^n - z^n$ and

$$\Re\left(\overline{r(z)}^n f(r(z))\right) = R^{2n} - \frac{R^n}{|z|^n}|z|^{2n} = R^{2n} - R^n|z|^n < 0,$$

a contradiction to (2). So, $|z| \leq R$ and thus $r(z) = z$ and $f(z) = 0$ by (4). □

Clearly, Theorem 3.4 applied to $-f$ can be reformulated as the following generalization of the Fundamental Theorem of Algebra.

**Corollary 3.1.** *If $f : \overline{D_R} \to \mathbb{C}$ is continuous and satisfies (3) for some $n \in \mathbb{N}$ then $z^n + f(z)$ has a root in $\overline{D_R}$.*

**Remark 3.7.** Corollary 3.1 implies the Fundamental Theorem of Algebra because if $p$ is a complex polynomial of degree $n \geq 1$ (we may assume without loss of generality that its leading term is $z^n$) then $f(z) = p(z) - z^n$ satisfies (3) in $S_R$ for large enough $R > 0$.

Note that Corollary 3.1 also applies if $f(z) = q(z, \overline{z})$, where $q$ is any complex polynomial on two variables of degree at most $n - 1$, so it is more general than the FTA.

## 4. The square root implies all the others

Let $n \in \mathbb{N}$. Because the set of $n$th roots of 1 is finite, we may fix $\mu > 0$ such that $z^n \neq 1$ for all $z \in D_\mu(1)\backslash\{1\}$.

**Lemma 4.1.** *Assume that $\delta \in (0, 1)$ satisfies $\frac{2\delta}{1-\delta} < \mu$. Then the mapping $g : D_\delta(1) \to \mathbb{C}$ given by $g(z) = z^n$ is injective.*

**Proof.** Let $z, w \in D_\delta(1)$ satisfy $z^n = w^n$, then $\frac{z}{w}$ is an $n$th root of 1 and

$$\left|\frac{z}{w} - 1\right| = \frac{1}{|w|}|z - w| \leq \frac{2\delta}{1-\delta} < \mu.$$

We conclude that $\frac{z}{w} = 1$. □

Furthermore, a property analogous to the one established in Lemma 2.1 still holds true when one considers, instead of $z^2$, any multiplicative function $\varphi : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ such that $\varphi^{-1}(1)$ is a discrete set. In particular, setting $\varphi(z) = z^n$ we obtain the following extension of Lemma 2.1.

**Lemma 4.2.** *Let $X$ be a connected topological space and $r, \tilde{r} : X \to \mathbb{C} \setminus \{0\}$ two continuous functions such that $r(x)^n = \tilde{r}(x)^n$ for all $x \in X$. Then there exists $\xi$ an $n$th root of 1 such that*

$$r(x) = \xi\tilde{r}(x) \quad \textit{for all } x \in X.$$

*In particular, $\frac{r(x)}{r(y)} = \frac{\tilde{r}(x)}{\tilde{r}(y)}$ for all $x, y \in X$, and if $r(x_0) = \tilde{r}(x_0)$ for some $x_0 \in X$ then $r \equiv \tilde{r}$ in $X$.*

**Corollary 4.1.** *There exist $\varepsilon \in (0, 1)$ and a (unique) continuous mapping $r_n : D_\varepsilon(1) \to \mathbb{C}$ such that $r_n(z)^n = z$ and $r_n(1) = 1$.*

**Proof.** The existence of $r_n$ follows from Lemma 4.1 and the invariance of domain theorem. Uniqueness is a consequence of Lemma 4.2. □

**Corollary 4.2.** *Let $z_0 \neq 0$ and let $\tilde{\varepsilon} := \varepsilon|z_0|$, where $\varepsilon$ is defined as in the previous corollary. For each $w_0 \in \mathbb{C}$ such that $w_0^n = z_0$ there exists a unique continuous function $r : D_{\tilde{\varepsilon}}(z_0) \to \mathbb{C}$ such that $r(z)^n = z$ and $r(z_0) = w_0$.*

**Proof.** It suffices to consider $r(z) := w_0 r_n\left(\frac{z}{z_0}\right)$, with $r_n$ as in Corollary 4.1. □

Since in a real interval the intersection of two arbitrary connected subsets is again connected we obtain the following useful property.

**Lemma 4.3.** *Let $I \subset \mathbb{R}$ a nonempty interval, $\alpha : I \to \mathbb{C} \setminus \{0\}$ a continuous function and $w_0$ an nth root of $\alpha(x_0)$ for some $x_0 \in I$. Then there exists a unique continuous function $\rho : I \to \mathbb{C} \setminus \{0\}$ such that $\rho(x)^n = \alpha(x)$ for all $x \in I$ and $\rho(x_0) = w_0$.*

**Proof.** Due to Corollary 4.2, if $U$ and $V$ are open and connected subsets of $I$ containing $x_0$ in which are defined respective continuous $n$th roots of $\alpha$, $\rho_U$ and $\rho_V$, such that $\rho_U(x_0) = w_0 = \rho_V(x_0)$, then $\rho_V \equiv \rho_U$ on the connected set $U \cap V$. This allows to define the maximal open connected set $C$ as the union of all those $U$ and a unique continuous $n$th root of $\alpha$, $\rho : C \to \mathbb{C}$ such that $\rho(x_0) = w_0$. Using Corollary 4.2, it is readily seen that the open set $C$ is also nonempty and closed, then $C = I$ since $I$ is connected. □

Lemma 4.3 provides an easy way to extend the previous function $r_n$, defined in Corollary 4.1, uniquely to $A^+$.

**Corollary 4.3.** *There exists a unique continuous function $r_n : A^+ \to \mathbb{C}$ such that $r_n(1) = 1$ and $r_n(z)^n = z$ for all $z$.*

**Proof.** Consider the sets $A := S_1 \backslash \{-1\}$ and $B := \mathbb{R}_{>0}$. Since

$$h : \mathbb{R} \to S_1 \backslash \{-1\}, \quad t \mapsto h(t) := \frac{1 + it}{1 - it}$$

is a homeomorphism, the same conclusion of Lemma 4.3 holds true for $S_1 \backslash \{-1\}$. Then, there exist (unique) continuous $n$th roots of the identity function, $r_A$ and $r_B$, defined respectively on $A$ and $B$ such that $r_A(1) = 1 = r_B(1)$. Then, the function $r_n$ defined in Corollary 4.1 is extended to a (unique) continuous $n$th root in $A^+$ such that $r_n(1) = 1$ by

$$r_n(z) := r_B(|z|)\, r_A\left(\frac{z}{|z|}\right), \qquad z \in A^+. \quad \square$$

We end this section by noticing that, although our definition of the $n$th roots avoided completely the use of the polar form of a complex number, a geometric interpretation of $r_n$ can be straightforwardly given as follows.

**Lemma 4.4.** *Let $w \in S_1 \backslash \{\pm 1\}$ and set $1 = w_0, \ldots, w_n = w$ be the vertices of the shortest regular $n$-polygonal in $S_1$ joining $w$ and 1. Then $w_1 = r_n(w)$.*

**Proof.** By definition $|w_{j+1} - w_j| = |w_1 - 1|$ for all $j = 1, 2, \ldots, n-1$. We claim that $w_1^n = w$. Indeed, observe that $|w_1^2 - w_1| = |w_1 - 1|$ and, because $|w_1^2| = 1$ and $w_1^2 \neq 1$ we deduce that $w_1^2 = w_2$. The claim follows then by induction.

Furthermore the mapping $r(w) = w_1$ for $w \in S_1 \backslash \{\pm 1\}$ and $r(1) = 1$ is continuous. Then, from Lemma 4.2 we conclude that $r_n \equiv r$ on $S_1 \backslash \{-1\}$. □

**Remark 4.1.** The same reasoning as in the proof of Lemma 4.4 shows that if we set $1 = \xi_0, \xi_1, \ldots, \xi_n = 1$ be the vertices of the regular $n$-polygon in $S_1$ starting at 1 and according to its positive orientation, then $\xi_1^k = \xi_k$, for $k = 1, 2, \ldots, n$. Hence $\xi_1$ is a primitive $n$th root of 1, that is, $\xi_1^k \neq 1$ for $1 \leq k \leq n-1$ and, because there cannot be more than $n$ different roots, we conclude that the set of $n$th roots of 1 is exactly $\{\xi_1, \xi_1^2, \ldots, \xi_1^n = 1\}$.

**Remark 4.2.** From Lemma 4.2 and Remark 4.1 it is clear that the $n$ continuous $n$th roots in $A^+$ are given by $z \mapsto \xi r_n(z)$, where $\xi$ is an $n$th root of 1. In the same way, if $\xi$ is an $n$th root of $-1$, then the mapping $z \mapsto \xi r_n(-z)$ defines a continuous $n$th root over the set $A^-$.

## 5. Counting the number of turns

Now, let $c : [0, 1] \to S_R$ be a parametrization of $S_R$ with $c(0) = c(1)$ and $c|_{[0,1)}$ injective and define the set of closed continuous curves

$$\mathcal{C} = \{\tilde{\gamma} : [0, 1] \to \mathbb{C} \setminus \{0\} \text{ such that } \tilde{\gamma} \text{ is continuous and } \tilde{\gamma}(0) = \tilde{\gamma}(1)\}.$$

It is clear that if $\gamma \in \mathcal{A}$ then

$$\tilde{\gamma} = \gamma \circ c \in \mathcal{C}.$$

Conversely, given $\tilde{\gamma} \in \mathcal{C}$ we can set

$$\gamma(z) := \tilde{\gamma}(t) \quad \text{if } z = c(t) \text{ with } t \in [0, 1).$$

Despite the fact that $(c|_{[0,1)})^{-1} : S_R \to [0, 1)$ is discontinuous, the function $\gamma = \tilde{\gamma} \circ (c|_{[0,1)})^{-1}$ is continuous, since $c|_{[a,b]} \to c([a, b])$ is a homeomorphism when $[a, b] \neq [0, 1]$ and $\tilde{\gamma}(0) = \tilde{\gamma}(1)$. So, $\gamma \in \mathcal{A}$.

In fact, we have that

$$c^* : \mathcal{A} \to \mathcal{C}, \quad \gamma \mapsto \gamma \circ c,$$

is a homeomorphism.

For $\gamma \in \mathcal{A}$ let us consider $\eta(\gamma \circ c)$, the winding number of the closed curve $\gamma \circ c$ around 0, see [8]. Our main goal in this section is to give a very concise characterization of $|\eta(\gamma \circ c)|$ by means of the existence of continuous $n$th roots.

**Definition 5.1.** Let $\gamma \in \mathcal{A}$ and consider the set

$$R_\gamma := \{n \in \mathbb{N} : \ \gamma \ admits \ a \ continuous \ nth \ root\}.$$

Then we define $I : \mathcal{A} \to \mathbb{N}_0$ as

$$I(\gamma) := \begin{cases} \max R_\gamma & if \ R_\gamma \ is \ bounded, \\ 0 & otherwise. \end{cases}$$

The previous definition of $I$ allows to give a direct proof of the analogues of property 1, 4 and 5 in Lemma 2.2, that is:

**Lemma 5.1.** *For $\gamma, \delta \in \mathcal{A}$ we have:*

1. *$R_{1/\gamma} = R_\gamma$ and in particular $I(1/\gamma) = I(\gamma)$.*
2. *If $R_{\gamma\delta} = \mathbb{N}$ then $R_\gamma = R_\delta$ and in particular $I(\gamma) = I(\delta)$.*
3. *If $c \in \mathbb{C} \setminus \{0\}$ then $R_c = \mathbb{N}$ and in particular $I(c) = 0$.*

In turn, the two first properties are enough to prove the continuity of $I$ by adapting Lemma 2.4, with the aid of Corollary 4.3, and then proceeding as in Lemma 2.5.

**Lemma 5.2.** *$I : \mathcal{A} \to \mathbb{N}_0$ is continuous.*

The continuity of $I$ shall be crucial to prove the main result of this section since it implies its homotopy invariance. Notice that we are dealing with two different definitions of homotopies on the sets $\mathcal{A}$ and $\mathcal{C}$, namely:

- $\gamma, \delta \in \mathcal{A}$ are homotopic if there exists $h : S_R \times [0, 1] \to \mathbb{C}\backslash\{0\}$ continuous such that $h_0 = \gamma$ and $h_1 = \delta$, where $h_s(z) := h(z, s)$.
- $\tilde{\gamma}, \tilde{\delta} \in \mathcal{C}$ are homotopic if there exists $h : [0, 1] \times [0, 1] \to \mathbb{C}\backslash\{0\}$ continuous such that $h_0 = \tilde{\gamma}$, $h_1 = \tilde{\delta}$ and $h_s(0) = h_s(1)$ for all $s \in [0, 1]$, where $h_s(t) := h(t, s)$.

Clearly, both concepts are linked because if $h(z, s)$ is a homotopy between $\gamma, \delta \in \mathcal{A}$ then $\tilde{h}(t, s) := h(c(t), s)$ is a homotopy between $c^*(\gamma), c^*(\delta) \in \mathcal{C}$ and, conversely, a homotopy between elements of $\mathcal{C}$ induces a homotopy in $\mathcal{A}$. That is, both concepts are equivalent through the homeomorphism $c^*$.

**Theorem 5.2.** *For $\gamma \in \mathcal{A}$ holds that*

$$I(\gamma) = |\eta(\gamma \circ c)| .$$

For a proof, we shall proceed in a series of steps.

*Step 1*. **Root along a curve**. We shall define an $n$th root along the curve $\tilde{\gamma} \in \mathcal{C}$ as a continuous function $\rho_n : [0, 1] \to \mathbb{C}$ such that $\rho_n(t)^n = \tilde{\gamma}(t)$ for all $t \in [0, 1]$.

The following is a direct consequence of Lemma 4.3.

**Proposition 5.1.** *Let $\tilde{\gamma} \in \mathcal{C}$ and $w_n$ be an nth root of $\tilde{\gamma}(0)$. Then there exists a unique nth root along the curve $\tilde{\gamma}$, $\rho_n$, such that $\rho_n(0) = w_n$.*

*Step 2*. **Existence of $n$th roots and the root along a curve**. Now, for $\gamma \in \mathcal{A}$ fix $w_n$ an $n$th root of $\gamma(c(0))$ and set $\rho_n$ as the unique $n$th root along $\gamma \circ c$ such that $\rho_n(0) = w_n$.

**Proposition 5.2.** *For $\gamma \in \mathcal{A}$ and $n \in \mathbb{N}$ the following claims are equivalent:*

1. *$\gamma$ admits a continuous nth root, that is, $n \in R_\gamma$.*
2. *$\rho_n \in \mathcal{C}$, that is, $\rho_n(1) = \rho_n(0)$.*

**Proof.** If $\gamma(z) = \delta(z)^n$ with $\delta \in \mathcal{A}$, then

$$\rho_n(t)^n = \gamma(c(t)) = \delta(c(t))^n \quad \text{for all } t \in [0, 1]$$

and, from by Lemma 4.2,

$$\frac{\rho_n(1)}{\rho_n(0)} = \frac{\delta(c(1))}{\delta(c(0))} = 1.$$

Conversely, if $\rho_n \in \mathcal{C}$ then $\delta := (c^*)^{-1}(\rho_n) \in \mathcal{A}$ is a continuous $n$th root of $\gamma$. □

*Step 3*. **Reduction process**. In this step, for $\gamma \in \mathcal{A}$ we shall transform the closed curve $\gamma \circ c$ into an equivalent one $\tilde{\gamma}_1 \in \mathcal{C}$, for which the computation of $\eta(\tilde{\gamma}_1)$ and the ratio $\frac{\rho_n(1)}{\rho_n(0)}$ become easier. This will allow us to obtain the relation between the values $\eta(\gamma \circ c)$ and $I(\gamma)$.

Firstly, consider the continuous homotopy $h : S_R \times [0, 1] \to \mathbb{C} \setminus \{0\}$ given by

$$h(z, s) := l(s)\frac{\gamma(z)}{s(|\gamma(z)| - 1) + 1},$$

where $l : [0, 1] \to \mathbb{C} \setminus \{0\}$ is a continuous curve such that

$$l(0) = 1 \quad \text{and} \quad l(1) = \frac{|\gamma(c(0))|}{\gamma(c(0))} i.$$

This allows to assume that $\gamma : S_R \to S_1$ and $\gamma(c(0)) = \gamma(c(1)) = i$. Now, we define two "cancellation rules", according to the intuitive idea that the following two situations do not affect to the computation of $\eta$:

1. If the curve $\gamma \circ c$ passes through one of the poles $\pm i$ and returns to the same pole without passing through the opposite one, then this section may be deleted.
2. If the curve $\gamma \circ c$ moves from one pole to the opposite one and comes back through the same arc, then the whole cycle may be deleted.

With this in mind, we may formalize the situations described above by defining the following homotopies:

1. If $\gamma(c(t_a)) = \gamma(c(t_b))$ and $\gamma(c(t)) \neq -\gamma(c(t_a))$ for all $t \in (t_a, t_b)$, then take the homotopy
$$h(t, s) = (1 - s)\gamma(c(t)) + s\gamma(c(t_a)), \qquad \text{for } t \in [t_a, t_b]$$
and $h(t, s) = \gamma(c(t))$ otherwise. In other words, $\gamma \circ c$ is homotopic to a curve in $\mathcal{C}$ that remains unchanged for $t \notin [t_a, t_b]$ and keeps still between $t_a$ and $t_b$.
2. If $\gamma(c(t_a)) = -\gamma(c(t_b)) = \gamma(c(t_c))$ with $t_a < t_b < t_c$ and $\gamma(c([t_a, t_c]))$ is contained in $A^+$ or $A^-$, then fix two values $t_* \in (t_a, t_b)$ and $t^* \in (t_b, t_c)$ such that $\gamma(c(t_*)) = \gamma(c(t^*))$ and define the homotopy given by
$$h(t, s) = (1 - s)\gamma(c(t)) + s\gamma(c(t^*)), \qquad \text{for } t \in [t_*, t^*]$$

and $\gamma(c(t))$ otherwise. Again, when $s = 1$, after $t = t_*$ we remain at $\gamma(c(t^*))$ until $t = t^*$, so we do not pass through $\gamma(c(t_b))$. Next, combined with the previous rule, the whole interval $[t_a, t_c]$ may be eliminated.

Since $\gamma(c(0)) = \gamma(c(1)) = i$, then either $\gamma \circ c$ does not pass through $-i$ or else there exist finitely many points $\tilde{t}_0 = 0 < \tilde{t}_1 < \tilde{t}_2 < \cdots \tilde{t}_{2m-1} < \tilde{t}_{2m} \leq 1$ such that

- $\gamma(c(\tilde{t}_k)) = (-1)^k i$, for all $k = 0, 1, \ldots, 2\,m$,
- $\gamma(c(t)) \neq -\gamma(c(\tilde{t}_k))$ for all $t \in [\tilde{t}_k, \tilde{t}_{k+1})$ and $k = 0, 1, \ldots, 2m-1$,
- $\gamma(c(t)) \neq -i$ for all $t \in [\tilde{t}_{2m}, 1]$.

Now, the combination of the previous cancellation rules allows us to show that $\gamma \circ c$ is homotopic to a curve $\tilde{\gamma}_1 \in \mathcal{C}$ such that: either

$$\tilde{\gamma}_1 \equiv i \quad \text{(in which case we set } N = 0\text{)},$$

or there exist an even number $N > 0$, and points

$$t_0 := 0 < t_1 < t_2 < \cdots < t_{N-1} < 1 =: t_N$$

such that

- $\tilde{\gamma}_1(t_j) = (-1)^j i$ for all $j = 0, \ldots, N$,
- $\tilde{\gamma}_1([t_j, t_{j+1}]) \subset A_j$ for all $j = 0, \ldots, N-1$, where $A_j = A^{\pm}$, and
- the signs of $A_j = A^{\pm}$ alternate.

**Proposition 5.3.** *The previous reduction yields* $\eta(\gamma \circ c) = \pm\frac{N}{2}$.

**Proof.** It suffices to observe that $\eta(\gamma \circ c) = \eta(\tilde{\gamma}_1)$ and that every time the curve $\tilde{\gamma}_1$ moves from $\tilde{\gamma}_1(t_j)$ to $\tilde{\gamma}_1(t_{j+1})$, it performs half turn around the origin, always according to the same orientation of the circumference $S_1$. In consequence, the winding number of $\eta(\tilde{\gamma}_1)$ equals $\pm\frac{N}{2}$. $\square$

Regarding the computation of $I(\gamma)$, we have:

**Lemma 5.3.** *If $\rho_n$ is an nth root along $\tilde{\gamma}_1$, then $\rho_n(1) = \rho_n(0)$ if and only if $r_n(i)^{2N} = 1$.*

**Proof.** If $N = 0$ we have that $\tilde{\gamma}_1$ is constant so both conditions are obviously true. Then, suppose $N > 1$ and observe that

$$\frac{\rho_n(1)}{\rho_n(0)} = \prod_{j=0}^{N-1} \frac{\rho_n(t_{j+1})}{\rho_n(t_j)}.$$

Assume for example that $A_0 = A^-$ (namely, the positive orientation) and fix the $n$th root $r_n$ in $A^+$, given in Corollary 4.3, and $\tilde{r}_n(z) = \xi r_n(-z)$ in $A^-$ where $\xi$ is an arbitrary $n$th root of $-1$. From Lemma 4.2 it follows that

$$\frac{\rho_n(1)}{\rho_n(0)} = \underbrace{\frac{\tilde{r}_n(-i)}{\tilde{r}_n(i)}}_{j=0} \underbrace{\frac{r_n(i)}{r_n(-i)}}_{j=1} \cdots \underbrace{\frac{\tilde{r}_n(-i)}{\tilde{r}_n(i)}}_{j=N-2} \underbrace{\frac{r_n(i)}{r_n(-i)}}_{j=N-1} = \left(\frac{r_n(i)}{r_n(-i)}\right)^N.$$

To conclude, observe that because of Lemma 4.4 for $z \in S_1 \backslash \{-1\}$ we have that $r_n(\overline{z}) = \overline{r_n(z)}$. Hence

$$r_n(z) r_n(\overline{z}) = |r_n(z)|^2 = 1 \quad \text{for } z \in S_1 \backslash \{-1\},$$

and, in particular, $r_n(-i) = r_n(i)^{-1}$. $\square$

The condition $r_n(i)^{2N} = 1$ in the previous lemma becomes obviously true when $N = 0$ and observe, in this case, that $\gamma \circ c$ is homotopic to a constant. More generally, we have:

**Lemma 5.4.** *In the previous setting,*

$$r_n(i)^{2N} = 1 \iff n | \frac{N}{2}.$$

**Proof.** $\Longleftarrow$) follows immediately from the fact that $r_n(i)^{4n} = (r_n(i)^n)^4 = 1$.

$\Longrightarrow$) From Lemma 4.4 and Remark 4.1 we deduce that $r_n(i) \neq 1$ is the first $4n$-th root of 1 according to the positive orientation of $S^1$. Then it is primitive and, consequently, $4n$ divides $2N$. $\square$

Based on the preceding lemmas the following corollary is obtained which, combined with Proposition 5.3, completes the proof of Theorem 5.2.

**Corollary 5.1.** *In the previous situation, $I(\gamma) = \frac{N}{2}$.*

**Proof.** The conclusion is clear when $N = 0$ because in that situation $\gamma$ is homotopic to a constant. If, otherwise, $N > 1$ then $I(\gamma) = I(\gamma_1)$, where $\gamma_1 = (c^*)^{-1}(\tilde{\gamma}_1)$, and from Proposition 5.2 and the preceding lemmas we have that $I(\gamma_1) = n$, where $n$ is the maximum value that divides $\frac{N}{2}$, that is $n = \frac{N}{2}$. $\square$

As consequence of the reduction process and Corollary 5.1 we obtain the following characterization for the case $I(\gamma) = 0$.

**Proposition 5.4.** *In the previous setting the followings claims are equivalent:*

1. *$\gamma \in \mathcal{A}$ is homotopic to a constant in $\mathbb{C} \backslash \{0\}$,*
2. *$R_\gamma = \mathbb{N}$,*
3. *$I(\gamma) = 0$.*

Notice that a property derived from Proposition 5.4, which is not self-evident, is the following: if $\gamma \in \mathcal{A}$ has a continuous $n$th root for infinitely many values of $n$, then it has a continuous $n$th root for all $n \in \mathbb{N}$. Another property that can be derived is a characterization of simply connected regions without the use of complex integration theorems. To this end, let us firstly observe that if $\gamma : S_R \to \mathbb{C}$ is continuous, then the mapping

$$w \mapsto I(\gamma - w)$$

is constant over each connected component of $\mathbb{C} \backslash \mathrm{Im}(\gamma)$ and vanishes in the unbounded component. This is a well-known property of the winding number but, in the present context, follows trivially from the continuity of $I$. Moreover, a standard "paving" argument shows that if $U \subset \mathbb{C}$ is open and $K$ is an open subset of $U$, then there exists $\delta : S_R \to U \backslash K$ continuous such that $I(\delta - w) = 1$ for all $w \in K$.

**Proposition 5.5.** *Let $U \subset \mathbb{C}\backslash\{0\}$ be open and let $n > 1$. Then there exists a continuous nth root of the identity defined over U if and only if $I(\gamma) = 0$ for all $\gamma : S_R \to U$ continuous.*

**Proof.** Necessity: suppose $I(\gamma) \neq 0$ for some $\gamma \in C(S_R, U)$ and let $K := \overline{V}$, where $V$ is the connected component of $\mathbb{C}\backslash\text{Im}(\gamma)$ containing 0. Because $V$ is bounded, we deduce the existence of $\delta \in C(S_R, U)$ such that $I(\delta) = 1$. This implies that $\delta$ cannot have a continuous $m$th root for any $m > 1$ and, consequently, a continuous $n$th root of $z$ defined on $U$ cannot exist.

For the converse, we may assume w.l.o.g. that $U$ is connected and fix an arbitrary $z_0 \in U$. For $z \in U$, take a continuous arc $c : [0, 1] \to U$ joining $z_0$ and $z$ and $\rho : [0, 1] \to \mathbb{C}$ continuous such that $\rho(t)^n = c(t)$ for all $t$, that exists from Lemma 4.3. Next, define $r(z) := \rho(1)$, that is, $r(z)^n = c(1) = z$. The assumption $I(\gamma) = 0$ for all $\gamma \in C(S_R, U)$ implies that $r$ is well defined and continuous. □

**Corollary 5.2.** *Let $U \subset \mathbb{C}$ be open, then the following statements are equivalent:*

1. *$I(\gamma - w) = 0$ for all $\gamma \in C(S_R, U)$ and all $w \notin U$.*
2. *For each $w \notin U$ and all $n > 1$ there exists a continuous nth root of $z - w$ defined over $U$.*
3. *For each $w \notin U$ there exists $n > 1$ such that a continuous nth root of $z - w$ defined over $U$ exists.*

## 6. The p-adic valuation of the index

The results in Section 4 allow us to define, for each prime number $p$ and $\gamma \in \mathcal{A}$, the number $v_p(\gamma)$ exactly as in the case $p = 2$, and the analogues of Lemmas 2.2, 2.1, 2.4, 2.5, 2.6 and 2.7 are readily verified.

The main goal of this section is to prove that $v_p(\gamma)$ provides actually the $p$-adic valuation of $I(\gamma)$ and hence, in view of Theorem 5.2, the $p$-adic valuation of the winding number of $\gamma \circ c$ around 0.

We start by showing the following extension of Lemma 2.8 whose similar proof is omitted.

**Lemma 6.1** (*n*-**Borsuk**). *If $\gamma \in \mathcal{A}$ and there exist nth roots of $1$, $\xi$, $\tilde{\xi} \neq 1$, such that $\gamma(\xi z) = \tilde{\xi}\gamma(z)$ for all z, then $\gamma$ does not admit a continuous nth root (in particular, when $n = p$ is a prime number we have $v_p(\gamma) = 0$).*

Furthermore, we have the following extension of property 6 in Lemma 2.2:

**Lemma 6.2.** *For $\gamma \in \mathcal{A}$ and a prime number $p$ we have:*

1. *If $m \in \mathbb{N}_0$ then*

$$v_p\left(\gamma^{p^m}\right) = v_p(\gamma) + m.$$

2. *If $k \in \mathbb{N}$ such that $p$ does not divide $k$ then*

$$v_p\left(\gamma^k\right) = v_p(\gamma).$$

**Proof.** The proof of 1 is trivial for $m = 0$ and for $m = 1$ is analogous to property 6 in Lemma 2.2. Then 1 follows by induction.

For the proof of 2: since $\gcd(k, p) = 1$ then it is possible to write $1 = ak + bp$ for some integers $a, b$ and then

$$v_p(\gamma) = v_p(\gamma^{ak}\gamma^{bp}) \geq \min\{v_p(\gamma^{ak}), v_p(\gamma^{bp})\} = v_p(\gamma^{ak}),$$

since by 1 we have

$$v_p(\gamma^{bp}) = v_p((\gamma^b)^p) = v_p(\gamma^b) + 1 \geq v_p(\gamma) + 1 > v_p(\gamma).$$

Finally, 2 follows because $v_p(\gamma^{ak}) \geq v_p(\gamma^k) \geq v_p(\gamma)$. $\square$

As a straightforward consequence of Lemma 6.2, by writing $n = p^{\nu_p(n)} \cdot k$ where $\nu_p(\cdot)$ stands for the $p$-adic valuation of the integers, we obtain:

**Corollary 6.1.** *For each $n \in \mathbb{N}$ it holds that*

$$v_p\left(\gamma^n\right) = v_p(\gamma) + \nu_p(n).$$

**Remark 6.1.** From Lemma 6.1 it follows that $v_p(z) = v_p(1/z) = 0$ for all $p$ and then Corollary 6.1 implies that

$$v_p(z^n) = v_p(1/z^n) = \nu_p(n),$$

which is the analogous property to the one stated for $v_2$ in Remark 2.2. As $v_p(1) = \infty$ for all $p$ then we have that

$$v_p(z^n) = \nu_p(|n|) \quad for\ all\ n \in \mathbb{Z}. \tag{5}$$

In general, the following property holds for arbitrary $\gamma \in \mathcal{A}$.

**Theorem 6.2.** *If $\gamma \in \mathcal{A}$ then*

$$v_p(\gamma) = \nu_p(I(\gamma)) \quad for\ all\ p.$$

**Proof.** Assume firstly that $I(\gamma) = n \in \mathbb{N}$, that is, $\gamma = \delta^n$ with $n$ maximum. Then from Corollary 6.1 we have, for each prime $p$,

$$v_p(\gamma) = v_p(\delta^n) = v_p(\delta) + \nu_p(n).$$

Furthermore, if $\delta = \xi^p$, then $\gamma = \xi^{pn}$, which contradicts the maximality of $n$. Thus, $v_p(\delta) = 0$ and

$$v_p(\gamma) = \nu_p(n) = \nu_p(I(\gamma)) \quad \text{for all } p.$$

Next, assume that $I(\gamma) = 0$. From Proposition 5.4 it follows that $v_p(\gamma) = \infty$ for all $p$ and thus

$$v_p(\gamma) = \infty = \nu_p(0) = \nu_p(I(\gamma)) \quad \text{for all } p. \quad \square$$

**Remark 6.3.** From equality (5) and Theorem 6.2 it follows that

$$\nu_p(|n|) = v_p(z^n) = \nu_p(I(z^n)) \quad \textit{for all } p,$$

and then

$$I(z^n) = |n| \quad \textit{for all } n \in \mathbb{Z}.$$

## 7. Further comments

### 7.1. The reconstruction of I

In view of the results in the previous section, it is worthy to take a look at the properties of the index within the scope of the functions $v_p$ defined above. For this purpose notice, in the first place, that Theorem 6.2 shows that

$$I(\gamma) = I(\delta) \quad \text{if and only if } v_p(\gamma) = v_p(\delta) \text{ for all } p$$

and that

$$I(\gamma) = \begin{cases} \displaystyle\prod_{p \text{ prime}} p^{v_p(\gamma)} & \text{if } v_p(\gamma) < \infty \text{ for some } p, \\ 0 & \text{if } v_p(\gamma) = \infty \text{ for some } p. \end{cases}$$

Notice that the well definiteness of the previous formula relies on two nontrivial facts provided by Theorem 6.2:

- If $v_p(\gamma) < \infty$ for some $p$, then $v_p(\gamma) < \infty$ for all $p$.
- If $v_p(\gamma) < \infty$ for some $p$, then $v_q(\gamma) = 0$ for all $q$ except for a finite set.

### 7.2. The Hopf's Theorem

Observe, moreover, that our reduction process in Section 5, combined with the fact that the mapping $z \mapsto \overline{z}$ inverts the orientation of the plane, shows that:

- If $\gamma, \delta \in \mathcal{A}$ and $I(\gamma) = I(\delta)$, then either $\gamma$ and $\delta$ are homotopic or $\gamma$ and $\overline{\delta}$ are homotopic.

In particular, from Remark 6.3, this implies that each $\gamma \in \mathcal{A}$ is either homotopic to $z^n$ or $\overline{z}^n$ for some $n \in \mathbb{N}_0$ or, equivalently, that $\gamma$ is homotopic to $z^n$ for some $n \in \mathbb{Z}$. We may also notice that if $h : S_R \times [0, 1] \to \mathbb{C}\backslash\{0\}$ is a homotopy between $z^n$ and $z^k$, with $n, k \in \mathbb{Z}$, then $z^{-k}h(z, s)$ is a homotopy between $z^{n-k}$ and 1, that is, $n = k$. So, in each homotopy class in $\mathcal{A}$ there exists a unique representative of the form $z^n$ with $n \in \mathbb{Z}$. This yields an elementary version of the Hopf theorem that characterizes the homotopy classes in $\mathcal{A}$ by means of the winding number, see [8, Theorem 3.1].

On the other hand, we have seen in Lemma 5.2 and in Remark 6.3 that $I$ satisfies:

$(I_1)$ $I : \mathcal{A} \to \mathbb{N}_0$ is continuous.
$(I_2)$ $I(z^n) = |n| \quad$ for all$n \in \mathbb{Z}$.

It turns that these two properties characterize $I$: indeed, let $\tilde{I}$ any function satisfying $(I_1)$ and $(I_2)$. Then both $I$ and $\tilde{I}$ satisfy the homotopy invariance and since $\gamma \in \mathcal{A}$ is homotopic to $z^n$, for some $n \in \mathbb{Z}$, we have

$$\tilde{I}(\gamma) = \tilde{I}(z^n) = |n| = I(z^n) = I(\gamma) \quad \text{for all } \gamma \in \mathcal{A}.$$

### 7.3. A look at some properties of $I$

It is worthy noticing that, as mentioned in Section 5, the definition of $I$ allows to give a direct proof of those properties in Lemma 2.2 employed in the proof of the homotopy invariance. In fact, analogues of the subsequent lemmas in Section 2 are easily obtained, namely:

- If $\gamma \in \mathcal{A}$ and $\Re(\gamma) \geq 0$, then $R_\gamma = \mathbb{N}$ and in particular $I(\gamma) = 0$. Furthermore, due to the homotopy invariance the assumption may be replaced by $\mathrm{Im}(\gamma) \subset A$, where $A \subset \mathbb{C}\backslash\{0\}$ is an arbitrary simply connected set.
- If $|\gamma - \delta| < |\gamma|$, then $I(\gamma) = I(\delta)$.
- If $f : \overline{D_R} \to \mathbb{C}\backslash\{0\}$ is continuous, then $I(f|_{S_R}) = 0$.
- Let $f : \overline{D_R} \to \mathbb{C}$ be continuous with $\varepsilon := \inf |f|_{|S_R} > 0$. If $I(f|_{S_R}) \neq 0$, then $D_\varepsilon \subset \mathrm{Im}(f)$.

In this new context, Lemma 2.8 simply expresses the fact that:

- If $\gamma \in \mathcal{A}$ is odd, then $I(\gamma)$ is also odd.

The generalization given in Lemma 6.1 yields an analogous property and is left as an exercise for the reader.

It is also interesting to notice that Corollary 6.1 and Theorem 6.2 provide the identity:

- $I(\gamma^n) = nI(\gamma)$ for $\gamma \in \mathcal{A}$ and $n \in \mathbb{N}$.

However, the interpretation of properties 2 and 3 Lemma 2.2 in terms of $I$ is more subtle. It is intuitively clear the connection with the formula:

- $I(\gamma) + I(\delta) = \max\{I(\gamma\delta), I(\gamma\overline{\delta})\}$ for $\gamma, \delta \in \mathcal{A}$.

which can be immediately deduced using the fact that $\gamma \sim z^n$ and $\delta \sim z^k$ for some $n, k \in \mathbb{Z}$; however, it is not clear how this property can be obtained from the definition, i.e. without invoking the previous characterization.

### 7.4. The winding number and the Brouwer degree

With the notation introduced at Section 5 we can reformulate Theorems 5.2 and 6.2 just by saying that the following diagrams are commutative:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "c^*"] \arrow[d, "I"'] & \mathcal{C} \arrow[d, "\eta"] \\ \mathbb{N}_0 & \mathbb{Z} \arrow[l, "|\cdot|"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[d, "I"'] \arrow[dr, "v_p"] & \\ \mathbb{N}_0 \arrow[r, "v_p"'] & \mathbb{N}_0 \cup \{\infty\} \end{tikzcd}$$

Notice that all the previous properties of $I$ have a counterpart in terms of $\eta$; in particular, the first two properties in Lemma 5.1 correspond to

- $\eta(1/\tilde{\gamma}) = -\eta(\tilde{\gamma})$,
- $\eta(\tilde{\gamma}\tilde{\delta}) = 0 \implies \eta(\tilde{\gamma}) = -\eta(\tilde{\delta})$.

In fact both properties are a consequence of the more general one

- $\eta(\tilde{\gamma}\tilde{\delta}) = \eta(\tilde{\gamma}) + \eta(\tilde{\delta})$,

which, when $\tilde{\gamma}$ and $\tilde{\delta}$ are smooth, follows from the fact that

$$\frac{(\tilde{\gamma}\tilde{\delta})'}{\tilde{\gamma}\tilde{\delta}} = \frac{\tilde{\gamma}'}{\tilde{\gamma}} + \frac{\tilde{\delta}'}{\tilde{\delta}}.$$

Finally, we recall the relation between the winding number and the Brouwer degree, [2, Section 6.6]: if $\gamma \in \mathcal{A}$ and $f$ denotes any continuous extension of $\gamma$ to $\overline{D_R}$ then

$$\eta(\gamma \circ c) = d_B(f, D_R, 0),$$

where $d_B$ stands for the Brouwer degree. Notice that a continuous extension of $\gamma$ always exists by Tiezte's theorem and that the value of $d_B$ is independent of the particular extension $f$ because the Brouwer degree depends only on $f|_{S_R} = \gamma$.

Thus we also have

$$I(\gamma) = |d_B(f, D_R, 0)|.$$

This equality allows to prove that if $f : U \subset \mathbb{C} \to \mathbb{C}$ is an analytic function in the open set $U$, $\overline{D_R} \subset U$ and $f|_{S_r}$ does not vanish then $I(f|_{S_R})$ counts exactly the zeros of $f$ in $D_R$ with its multiplicity. Details are left to the reader: it is enough to use the additivity–excision property of the Brouwer degree and the relation between the index of an isolated zero and its multiplicity, see [2, Section 6.6].

In particular, if $h \not\equiv 0$ is an entire function (that is, $h$ is holomorphic in $\mathbb{C}$) then $h$ has a zero if and only if $I(h|_{S_R}) \neq 0$ for an unbounded set of values $R > 0$. This observation provides a natural candidate for the function $g$ in Theorem 3.3 that leads to the following simple asymptotic condition for the existence of a zero of a continuous function, see a related result in [7].

**Corollary 7.1.** *The continuous function $f : \mathbb{C} \to \mathbb{C}$ has a zero provided that there exists an entire function $h : \mathbb{C} \to \mathbb{C}$ such that $h(0) = 0$ and $\lim_{|z| \to +\infty} \frac{h(z)}{f(z)} \in \mathbb{C} \setminus \{0\}$.*

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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