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Semilinear problems at resonance with nonlinearities depending only on the derivative

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We deal with second-order semilinear equations with a nonlinear term depending only on the derivative. Our main results extend or complement some previous work by J. Mawhin.

KEYWORDS

Leray–Schauder continuation theorem, Neumann boundary value problem, periodic boundary value problem, semilinear problems at resonance

MSC CLASSIFICATION

34B15, 34C25

1 | INTRODUCTION

In the paper[1] Cañada and Drábek studied, in particular, the scalar problem

$$u'' + g(u') = \tilde{f} + s, \; t \in [a, b], \tag{1}$$

with Neumann

$$u'(a) = u'(b) = 0$$

or periodic

$$u(a) - u(b) = u'(a) - u'(b) = 0$$

boundary conditions.

Their main goal was to describe for each given $\tilde{f}$ with mean value zero its *solvability set*, that is, the set of $s \in \mathbb{R}$ for which problem (1) has a solution. Assuming

(CD) $g : \mathbb{R} \to \mathbb{R}$ is of class C^1 and bounded,

they prove that the solvability set of (1) with Neumann or periodic conditions is a singleton (see Cañada and Drábek[1, Theorems 3.3 and 3.4]). This contrasts with the periodic problem for the pendulum equation

$$u'' + \sin(u) = \tilde{f} + s, \; t \in [a, b],$$

Dedicated to Jean Mawhin, on the occasion of his 80th birthday.

where the solvability set is an interval $[s_-, s_+]$ and the reduction of the interval to a single point, the so-called *degeneracy* problem, is still open.[2]

Mawhin[3] extended the previous work to systems of the form

$$u'' = g(t, u') + \widetilde{f} + s, \ t \in [a, b], \tag{2}$$

where $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, $\widetilde{f} : [a, b] \to \mathbb{R}^n$ has mean value zero and $s \in \mathbb{R}^n$. He proved that if condition

(M) $\lim_{\|u\| \to \infty} \frac{g(t,u)}{\|u\|} = 0$ uniformly in $t \in [a, b]$,

holds, then the sets

$$\mathcal{J}^{(\mathcal{N})}_{\widetilde{f},n} = \{s \in \mathbb{R}^n : \text{the Neumann problem for (2) is solvable}\}, \tag{3}$$

and

$$\mathcal{J}^{(\mathcal{P})}_{\widetilde{f},n} = \{s \in \mathbb{R}^n : \text{the periodic problem for (2) is solvable}\}, \tag{4}$$

are both nonempty (see Mawhin[3, Theorems 1 and 4]). Moreover, in the scalar case ($n = 1$), he also generalized the uniqueness results in Cañada and Drábek[1] by proving that $\mathcal{J}^{(\mathcal{N})}_{\widetilde{f},1}$ and $\mathcal{J}^{(\mathcal{P})}_{\widetilde{f},1}$ are both singletons.

In case $n \geq 2$ the question of the uniqueness was solved in the negative sense for the Neumann problem in a previous work,[4] where Almira and Del Toro showed a counterexample satisfying (M) but for which $\mathcal{J}^{(\mathcal{N})}_{\widetilde{f},n}$ contains a continuum. On the other hand, the uniqueness for the periodic problem whenever $n \geq 2$ was stated there as an open problem. Some other related results and extensions can be found in previous works.[5–7]

The aim of this note is to improve and complement some of the results in previous works.[3,4] The paper is organized as follows: in Section 2 we present, by means of the Leray–Schauder continuation theorem, a slightly more general condition than in Mawhin[3] ensuring that both $\mathcal{J}^{(\mathcal{P})}_{\widetilde{f},n}$ and $\mathcal{J}^{(\mathcal{N})}_{\widetilde{f},n}$ are nonempty. In Section 3 we use the theory of the differential inequalities for proposing a partial extension of a uniqueness result in Mawhin.[3] Finally, in Section 3 we settle in the negative sense the open problem stated in Almira and Del Toro[4] with respect to the uniqueness for the periodic problem in case $n \geq 2$.

Throughout the paper, we are going to use the following assumptions and notations: by simplicity, we assume that $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $\widetilde{f} \in \tilde{C}([a, b], \mathbb{R}^n) := \left\{ f \in C([a, b], \mathbb{R}^n) : \int_a^b f(x)dx = 0 \right\}$ and $s \in \mathbb{R}^n$.

For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ we consider the maximum norm, that is,

$$\|x\| = \max_{i=1,2,\dots,n} |x_i|,$$

and for $h \in C([a, b], \mathbb{R}^n)$ we will denote

$$\|h\|_\infty = \max_{t \in [a,b]} \|h(t)\| \text{ and } \bar{h} = \frac{1}{b-a} \int_a^b h(s)\mathrm{d}s \in \mathbb{R}^n.$$

Notice that $\|\bar{h}\|_\infty \leq \|h\|_\infty$ (where with a slight abuse of notation we consider $\bar{h}$ as a constant function)

Finally, for $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ we consider the usual partial ordering induced by the cone $\mathbb{R}^n_+$, defining

$$x \leq y \Longleftrightarrow x_i \leq y_i \ \text{ for all } \ i = 1, 2, \dots, n,$$

$$x < y \Longleftrightarrow x_i < y_i \ \text{ for all } \ i = 1, 2, \dots, n.$$

We will say that x and y are *comparable* when $x \leq y$ or $y \leq x$ and *strongly comparable* when $x < y$ or $y < x$.

2 | EXISTENCE RESULTS

2.1 | The periodic problem

Let us consider the periodic problem

$$u'' = g(t, u') + \widetilde{f} + s,\ t \in [a, b],\ u(a) - u(b) = u'(a) - u'(b) = 0. \tag{5}$$

The proof of the following auxiliary result is straightforward and so it is omitted.

Lemma 2.1. *If y is a solution of the problem*

$$y' = g(t, y(t) - \overline{y}) - \overline{g(t, y(t) - \overline{y})} + \widetilde{f},\ t \in [a, b],\ y(a) = 0 = y(b), \tag{6}$$

then

$$u(t) = c + \int_a^t (y(s) - \overline{y})\mathrm{d}s$$

is a family of solutions for problem (5) with $s = -\overline{g(t, y(t) - \overline{y})} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}^n$ *arbitrary.*

Based upon the previous lemma, we present an extension of Mawhin[3, Theorem 4.]

Theorem 2.2. *If*

$$\limsup_{\|u\|\to\infty} \frac{\|g(t, u)\|}{\|u\|} < \frac{1}{4(b-a)} \text{ uniformly in } t \in [a, b]. \tag{7}$$

then there exists a value of $s \in \mathbb{R}^n$ *for which there is a family of solutions for problem (5).*

Proof. In view of Lemma 2.1 it is enough to show the solvability of problem (6), which is equivalent to the fixed point equation

$$y = Ty \tag{8}$$

where $T : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$ is given by

$$Ty(t) := \int_a^t \left[g(s, y(s) - \overline{y}) - \overline{g(s, y(s) - \overline{y})} + \widetilde{f}(s)\right] \mathrm{d}s.$$

It is a standard exercise to show that operator T is completely continuous. So, if we prove that the possible solutions of the operator equation $y = \lambda Ty$ are bounded independently of $\lambda \in (0, 1)$ then the Leray–Schauder continuation theorem, Mawhin[8, Theorem 7.1,] will provide the desired fixed point for (8).

On the other hand, note that condition (7) implies that there exist $\nu < \frac{1}{4(b-a)}$ and $M > 0$ such that

$$\|g(t, u)\| \le \nu\|u\| + M, \text{ for all } (t, u) \in [a, b] \times \mathbb{R}^n. \tag{9}$$

Then, from (9) it follows that $y = \lambda Ty$ for some $\lambda \in (0, 1)$ implies

$$\begin{aligned}
\|y\|_\infty = \lambda\|Ty\|_\infty &\le \int_a^b \|g(\cdot, y(\cdot) - \overline{y}) - g(\cdot, \bar{y(\cdot)} - \overline{y}) + \widetilde{f}\|_\infty \mathrm{d}s \\
&\le (b-a)\left(\|g(\cdot, y(\cdot) - \overline{y}) - g(\cdot, \bar{y(\cdot)} - \overline{y})\|_\infty + \|\widetilde{f}\|_\infty\right) \\
&\le (b-a)\left(2\|g(\cdot, y(\cdot) - \overline{y})\|_\infty + \|\widetilde{f}\|_\infty\right) \\
&\le (b-a)\left(2[\nu\|y - \overline{y}\|_\infty + M] + \|\widetilde{f}\|_\infty\right) \\
&\le (b-a)\left(4\nu\|y\|_\infty + 2M + \|\widetilde{f}\|_\infty\right).
\end{aligned}$$

Therefore,

$$\|y\|_\infty \leq \frac{(b-a)(2M+\|\widetilde{f}\|_\infty)}{1-4(b-a)\nu}$$

and the proof is complete. □

2.2 The Neumann problem

Let us consider now the Neumann problem

$$u'' = g(t,u') + \widetilde{f} + s, \ t \in [a,b], \ u'(a) = 0 = u'(b). \tag{10}$$

An analogous of Lemma 2.1, that we state also without proof, is the following.

Lemma 2.3. *If y is a solution of the problem*

$$y' = g(t,y(t)) - \overline{g(t,y(t))} + \widetilde{f}, \ t \in [a,b], \ y(a) = 0 = y(b), \tag{11}$$

then

$$u(t) = c + \int_a^t y(s)\mathrm{d}s$$

is a family of solutions for problem (10) with $s = -\overline{g(t,y(t))} \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ arbitrary.

Theorem 2.4. *If*

$$\limsup_{\|u\|\to\infty} \frac{\|g(t,u)\|}{\|u\|} < \frac{1}{2(b-a)} \ \textit{uniformly in } t \in [a,b]. \tag{12}$$

then there exists a value of $s \in \mathbb{R}^n$ for which there is a family of solutions for problem (10).

Proof. The proof follows the same lines as that of Theorem 2.2: from Lemma 2.3 it is enough to show the solvability of problem (11), that turns out to be equivalent to the fixed point equation $y = Ty$, where now $T : C(I,\mathbb{R}^n) \to C(I,\mathbb{R}^n)$ is given by

$$Ty(t) := \int_a^t \left[g(s,y(s)) - \overline{g(s,y(s))} + \widetilde{f}(s)\right] \mathrm{d}s.$$

Again, the operator T is completely continuous, condition (12) implies that there exist $\nu < \frac{1}{2(b-a)}$ and $M > 0$ such that

$$\|g(t,u)\| \leq \nu\|u\| + M, \ \text{for all}\,(t,u) \in [a,b] \times \mathbb{R}^n, \tag{13}$$

and if $y = \lambda Ty$ for some $\lambda \in (0,1)$ then

$$\begin{aligned}
\|y\|_\infty = \lambda\|Ty\|_\infty &\leq \int_a^b \|g(\cdot,y(\cdot)) - \overline{g(\cdot,y(\cdot))} + \widetilde{f}\|_\infty \mathrm{d}s \\
&\leq (b-a)\left(\|g(\cdot,y(\cdot)) - \overline{g(\cdot,y(\cdot))}\|_\infty + \|\widetilde{f}\|_\infty\right) \\
&\leq (b-a)\left(2\|g(\cdot,y(\cdot))\|_\infty + \|\widetilde{f}\|_\infty\right) \\
&\leq (b-a)\left(2\nu\|y\|_\infty + 2M + \|\widetilde{f}\|_\infty\right).
\end{aligned}$$

So,

$$\|y\|_\infty \leq \frac{(b-a)(2M+\|\widetilde{f}\|)}{1-2(b-a)\nu}$$

and the proof follows from the Leray–Schauder continuation theorem. □

3 | A STRUCTURAL PROPERTY FOR $\mathcal{J}_{\widetilde{f},n}^{(\mathcal{N})}$

Next we present a partial extension of Mawhin[3, Theorem 3] to the case $n \geq 2$. The key concept that allows such extension is quasi-monotonicity, a central concept in carrying over properties from scalar differential equations to systems,[9,10]: a function $g : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$, $g := (g_1, g_2, \dots, g_n)$, is called *quasi-monotone increasing*, or just *quasi-monotone*, if for each $t \in [a,b]$ and each $i = 1, 2, \dots, n$ we have

$$g_i(t, u_1, u_2, \dots, u_n) \leq g_i(t, v_1, v_2, \dots, v_n) \text{ if } u_i = v_i \text{ and } u_j \leq v_j \text{ for all } j \neq i.$$

Theorem 3.1. *Assume that $g(t,u)$ is quasi-monotone. Then, for each $\widetilde{f}$ the set $\mathcal{J}_{\widetilde{f},n}^{(\mathcal{N})}$ can not contain two strongly comparable elements.*

Proof. To the contrary, suppose that for some $\widetilde{f}$ there exist $s_1 < s_2 \in \mathbb{R}^n$ and x, y solutions, respectively, of problems

$$x' = g(t,x) + \widetilde{f} + s_1, \ t \in [a,b], \ x(a) = 0 = x(b)$$

and

$$y' = g(t,y) + \widetilde{f} + s_2, \ t \in [a,b], \ y(a) = 0 = y(b).$$

Then,

$$x(a) = y(a), \ x'(a) = g(t, x(a)) + \widetilde{f}(a) + s_1 < g(t, y(a)) + \widetilde{f}(a) + s_2 = y'(a),$$

and therefore, there exists $\varepsilon > 0$ such that

$$x(t) < y(t) \text{ for all } t \in (a, a+\varepsilon).$$

Furthermore,

$$x'(t) - g(t, x(t)) < y'(t) - g(t, y(t)) \text{ for all } t \in [a,b],$$

and then a well established result in the theory of the differential inequalities (see Walter[10, Theorem 12.V]) implies that

$$x(t) < y(t) \text{ for all } t \in (a,b],$$

attaining a contradiction with the fact that $x(b) = 0 = y(b)$. □

Remark 3.1. Theorem 3.1 is, in fact, an extension of Mawhin[3, Theorem 3] since any scalar function is quasi-monotone and for $n = 1$ the set $\mathcal{J}_{\widetilde{f},1}^{(\mathcal{N})}$ does not contain two strongly comparable elements if and only if it has at most one element.

On the other hand, Theorem 3.1 is not true, in general: indeed, consider the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g(u_1, u_2) := (u_2 - u_1, -2u_1 + u_2) \text{ for all } (u_1, u_2) \in \mathbb{R}^2.$$

Then, it is easy to verify that for each $\alpha \in \mathbb{R}$ the function

$$u(t) := (-\alpha\cos(t), -\alpha t + \alpha\sin(t) - \alpha\cos(t)) \text{ for all } t \in [0, 2\pi],$$

is a solution of the problem

$$u'' = g(u') + (\alpha, \alpha), \ u'(0) = 0 = u'(2\pi),$$

which means that

$$\{(\alpha,\alpha) \in \mathbb{R}^2 : \alpha \in \mathbb{R}\} \subseteq \mathcal{J}_{0,2}^{(\mathcal{N})}.$$

Therefore, $\mathcal{J}_{0,2}^{(\mathcal{N})}$ admits infinitely many strongly comparable elements. Of course, the nonlinearity $g(u_1, u_2)$ is not quasi-monotone.

Finally, we point out that the proof of Theorem 3.1 seems not work for the periodic boundary conditions. We do not know whether the result is true or not in the periodic setting.

4 | A NON-UNIQUENESS RESULT FOR THE PERIODIC PROBLEM WITH $n \geq 2$

In Almira and Del Toro[4] it was left as an open problem to determine if $\mathcal{J}_{\tilde{f},n}^{(\mathcal{P})}$ is a singleton for $n \geq 2$. In this section we present an example answering this question in the negative sense: consider $h : \mathbb{R} \to \mathbb{R}$ a bounded and C^∞ function such that $h(x) = -x$ for all $x \in [-2, 2]$ and define

$$g(t, u_1, u_2, u_3, \dots , u_n) = (h(u_2 - u_1 \sin(t) + u_2 \cos(t)), h(-u_1), 0, \dots , 0).$$

Then $g : [0, 2\pi] \times \mathbb{R}^n \to \mathbb{R}^n$ belongs to $C^\infty([0, 2\pi] \times \mathbb{R}^n, \mathbb{R}^n)$ and it is bounded, so g obviously satisfies condition (M). Let us now consider the problem

$$\begin{aligned} u''(t) &= g(t, u'(t)) + s, \ \ t \in [0, 2\pi], \\ u(0) &= u(2\pi), \ \ u'(0) = u'(2\pi). \end{aligned} \tag{14}$$

For each $\alpha \in [-1, 1]$ the periodic function $u_\alpha(t) = (\alpha \cos(t), \alpha \sin(t), 0, \dots , 0)$ clearly satisfies

$$u'_\alpha(t) = (-\alpha \sin(t), \alpha \cos(t), 0, \dots , 0),$$

$$u''_\alpha(t) = (-\alpha \cos(t), -\alpha \sin(t), 0, \dots , 0),$$

and

$$\begin{aligned} u''_\alpha(t) - g(t, u'_\alpha(t)) &= (-\alpha \cos(t), -\alpha \sin(t), 0, \dots , 0) \\ &\quad - (h(\alpha \cos(t) + \alpha \sin^2(t) + \alpha \cos^2(t)), h(\alpha \sin(t)), 0, \dots , 0) \\ &= (\alpha, 0, 0, \dots , 0). \end{aligned}$$

Therefore, u_α solves the periodic problem (14) with $s_\alpha = (\alpha, 0, 0, \dots , 0)$ and $n \geq 2$. Then for $\tilde{f} = 0$ we have

$$\{s_\alpha : \alpha \in [-1, 1]\} \subseteq \mathcal{J}_{0,n}^{(\mathcal{P})}$$

and in consequence the uniqueness part of Cañada and Drábek[1, Theorem 3.4] cannot be added to Mawhin.[3, Theorem 4]

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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