Journal of Computational and Applied Mathematics 414 (2022) 114454

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Numerical analysis of a thermoelastic dielectric problem arising in the Moore–Gibson–Thompson theory

N. Bazarra [a], J.R. Fernández [a,*], R. Quintanilla [b]

[a] Universidade de Vigo, Departamento de Matemática Aplicada I, 36310 Vigo, Spain
[b] Departamento de Matemáticas, E.S.E.I.A.A.T.-U.P.C., Colom 11, 08222 Terrassa, Barcelona, Spain

ARTICLE INFO

Article history:
Received 1 April 2021
Received in revised form 25 March 2022

Keywords:
Moore–Gibson–Thompson thermoelasticity
Dielectric material
Finite elements
A priori error estimates
Numerical simulations

ABSTRACT

In this paper, we numerically study a thermoelastic problem arising in the Moore–Gibson–Thompson theory. Dielectrics effects are also included within the model. The corresponding problem is written in terms of the displacement field, the temperature and the electric potential. A viscous term is added in the heat equation to provide the numerical analysis of the corresponding variational problem. Then, by using the finite element method and the implicit Euler scheme fully discrete approximations are introduced. A discrete stability property and a priori error estimates are obtained. Finally, one- and two-dimensional numerical simulations are shown to demonstrate the accuracy of the approximation and the behavior of the solution.

1. Introduction

Thermoelastic dielectrics have deserved much attention in the recent time and many authors have dedicated theirs works to this theory. At the same time, in the last years a big interest has been devoted to the so-called Moore–Gibson–Thompson thermoelasticity. This theory can be obtained after the introduction of a relaxation parameter to the so-called type III heat equation,[1] and it has received a big attention over the last two years (see, for instance, [1–15]).

It has been shown that the Moore–Gibson–Thompson thermoelasticity can be obtained after assuming suitable constitutive functions, for the theory of thermoviscoelasticity, when the invariance of the infinitesimal entropy production is invariant under time reversal (proposed by Gurtin [16]). It was also studied by Borghesiani and Morro [17,18], but Ciarletta and Ieşan [19] considered the theory when elastic deformations were included too.

By using this last approach and considering suitable constitutive functions, Fernández and Quintanilla [20] proposed the Moore–Gibson–Thompson theory for thermoelastic dielectrics. In fact, they obtained the existence and uniqueness of solutions as well as the exponential decay for the case of a rigid solid. Exponential decay for the one-dimensional case could be also proved by using the usual arguments joined with the specific ones proposed in that paper.

In this work, we want to continue the study of the Moore–Gibson–Thompson theory for thermoelastic dielectrics, but here we aim to provide a numerical study. However, the direct study of the problem introduced in [20] has revealed to be very difficult and, for this reason, we have considered a small perturbation of the problem. We have added a viscous term of the form "$\epsilon \Delta\dot{\theta}$" in the heat equation, and this new problem has allowed a good development. As we will see, the

* Corresponding author.
E-mail addresses: nbazarra@uvigo.es (N. Bazarra), jose.fernandez@uvigo.es (J.R. Fernández), ramon.quintanilla@upc.edu (R. Quintanilla).

1 The introduction of the relaxation parameter is motivated to overcome the unbounded speed of propagation of thermal waves phenomena in the case of type III theory. Therefore, this process is similar to the one used to obtain the Maxwell–Cattaneo heat equation from the Fourier heat equation.

https://doi.org/10.1016/j.cam.2022.114454
0377-0427/

solutions to both problems (the original and perturbed ones) are very similar and we will provide the numerical analysis of the perturbed problem. We will also conclude a new knowledge of the original problem from the numerical point of view.

The structure of the paper is as follows. In Section 2 we present the system and conditions determining the Moore–Gibson–Thompson thermoelastic dielectric problem and, at the same time, we propose a small perturbation of it. We finish this section recalling an existence and uniqueness result corresponding to the perturbed problem. In Section 3, we compare the solutions to both problems and we show that, for a certain measure, the difference is of type $O(\epsilon)$. Then, in Section 3 we describe the fully discrete approximations of the corresponding variational problem by using the finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are shown. Finally, some one- and two-dimensional numerical simulations are presented in Section 4 to demonstrate the accuracy of the approximation and the behavior of the solution.

2. The problem: existence and uniqueness

In this section, we introduce the model and we recall an existence and uniqueness result (details can be found in [20]).

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the domain and denote by $[0, T]$, $T > 0$, the time interval of interest. The boundary of the body $\partial\Omega$ is assumed to be Lipschitz. Moreover, let $\boldsymbol{x} \in \overline{\Omega}$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x} = (x_j)_{j=1}^d$ and t, and a subscript after a comma, under a variable, represents its spatial derivative with respect to the prescribed variable, that is $f_{i,j} = \dfrac{\partial f_i}{\partial x_j}$. The time derivatives are represented as a dot for the first order and two dots for the second order. Finally, as usual, the repeated index notation is used for the summation.

We focus our attention to the isotropic and homogeneous case, although we emphasize that the analysis of the anisotropic case could be done in a similar way. In this situation, our system of equations can be written in $\Omega \times (0, T)$ as follows (see [20]):

$$\rho \ddot{u}_i = \mu^* u_{i,jj} + (\lambda + \mu^*) u_{j,ji} - \beta^*(\theta_{,i} + \tau \dot{\theta}_{,i}), \tag{1}$$

$$A^*(\dot{\theta} + \tau \ddot{\theta}) = -\beta^* \dot{u}_{i,i} + \kappa^* \Delta\alpha + \kappa \Delta\theta - Q^* \Delta\phi - Q \Delta\dot{\phi}, \tag{2}$$

$$-\gamma \Delta\dot{\phi} - \gamma^* \Delta\phi = -Q^* \Delta\alpha - Q \Delta\theta. \tag{3}$$

Here, $\boldsymbol{u} = (u_i)_{i=1}^d$ denotes the displacement field, α and $\theta = \dot{\alpha}$ are the thermal displacement and the temperature, respectively, and ϕ represents the electric potential. Moreover, parameters $\mu^*, \lambda^*, \beta^*, A^*, \kappa^*, \kappa, Q^*, Q, \gamma$ and γ^* represent constitutive coefficients, ρ is the material density and $\tau > 0$ denotes a relaxation parameter related to the Moore–Gibson–Thompson theory.

In this work, in order to provide the numerical analysis of the corresponding problem later, we need to modify the field equation for the temperature field, adding an additional term of the form $\epsilon \Delta\dot{\theta}$. The resulting field equations are the following:

$$\rho \ddot{u}_i = \mu^* u_{i,jj} + (\lambda + \mu^*) u_{j,ji} - \beta^*(\theta_{,i} + \tau \dot{\theta}_{,i}), \tag{4}$$

$$A^*(\dot{\theta} + \tau \ddot{\theta}) = -\beta^* \dot{u}_{i,i} + \kappa^* \Delta\alpha + \kappa \Delta\theta + \epsilon \Delta\dot{\theta} - Q^* \Delta\phi - Q \Delta\dot{\phi}, \tag{5}$$

$$-\gamma \Delta\dot{\phi} - \gamma^* \Delta\phi = -Q^* \Delta\alpha - Q \Delta\theta, \tag{6}$$

where ϵ is a positive parameter which is assumed small. Obviously, this equation for the temperature field now becomes parabolic. We note that this new term has not thermomechanical interpretation and it is only introduced to study this problem from the numerical point of view.

In order to complete the description of the problem, we impose the boundary conditions, for a.e. $\boldsymbol{x} \in \partial\Omega$ and $t \in [0, T]$,

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0}, \quad \alpha(\boldsymbol{x}, t) = \phi(x, t) = 0, \tag{7}$$

and the initial conditions, for a.e. $\boldsymbol{x} \in \overline{\Omega}$,

$$\begin{aligned} &\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}^0(\boldsymbol{x}), \quad \dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{v}^0(\boldsymbol{x}), \quad \alpha(\boldsymbol{x}, 0) = \alpha^0(\boldsymbol{x}), \\ &\theta(\boldsymbol{x}, 0) = \theta^0(\boldsymbol{x}), \quad \dot{\theta}(\boldsymbol{x}, 0) = \zeta^0(\boldsymbol{x}), \quad \phi(\boldsymbol{x}, 0) = \phi^0(\boldsymbol{x}). \end{aligned} \tag{8}$$

Proceeding as in [20] we could prove the following theorem which states the existence of a unique solution to problem (4)–(8).

Theorem 1. *Assume the following conditions on the constitutive coefficients:*

$$\begin{aligned} &\rho > 0, \quad \mu^* > 0, \quad \lambda^* + \mu^* > 0, \quad \kappa - \tau\kappa^* > 0, \quad \gamma - \tau\gamma^* > 0, \\ &\gamma > 0, \quad \kappa^*\gamma^* > (Q^*)^2, \quad \kappa^* > 0, \quad \gamma^* > 0, \quad \kappa > 0, \quad \epsilon > 0, \\ &(\kappa - \tau\kappa^*)(\gamma - \tau\gamma^*) > (Q - \tau Q^*)^2. \end{aligned}$$

Therefore, problem (4)–(8) has a unique solution with the following regularity:

$$\begin{aligned}&\boldsymbol{u} \in C([0,T];[H^1(\Omega)]^d)\cap C^2([0,T];[L^2(\Omega)]^d),\\&\theta \in C^1([0,T];H^1(\Omega))\cap C^2([0,T];L^2(\Omega)),\\&\phi \in C([0,T];H^1(\Omega))\cap C^1([0,T];L^2(\Omega)).\end{aligned} \tag{9}$$

We also note that in [20] it is suggested that the energy of the original problem (1)–(3), (7) and (8) decays exponentially for the one-dimensional case.

3. Convergence of the solution when parameter ϵ tends to zero

In this section, we consider the convergence of the solution to problem (4)–(8) to the solution to problem (1)–(3), (7) and (8) when parameter ϵ tends to zero.

Let us denote by $(\boldsymbol{u},\alpha,\phi)$ the solution to problem (1)–(3), (7) and (8) and by $(\boldsymbol{u}^\epsilon,\alpha^\epsilon,\phi^\epsilon)$ the solution to problem (4)–(8). Let us define functions $\boldsymbol{w}=\boldsymbol{u}-\boldsymbol{u}^\epsilon$, $\eta=\alpha-\alpha^\epsilon$ and $\psi=\phi-\phi^\epsilon$.

It is easy to check that $(\boldsymbol{w},\eta,\psi)$ satisfies the following system:[2]

$$\begin{aligned}&\rho\ddot{\boldsymbol{w}}=\mu^* w_{i,jj}+(\lambda^*+\mu^*)w_{j,ji}-\beta^*(\dot\eta_{,i}+\tau\ddot\eta_{,i}),\\&A^*(\ddot\eta+\tau\dddot\eta)=-\beta^*\dot w_{i,i}+\kappa^*\Delta\eta+\kappa\Delta\dot\eta+\epsilon\Delta\ddot\eta-Q^*\Delta\psi-Q\Delta\dot\psi-\epsilon\Delta\ddot\alpha,\\&\dot\psi=\gamma^{-1}(Q^*\eta+Q\dot\eta-\gamma^*\psi),\end{aligned}$$

with homogeneous and null initial and boundary conditions.

We define now the functional:

$$\begin{aligned}E(t)=&\frac{1}{2}\int_\Omega\Big(\rho\dot w_i\dot w_i+\mu^* w_{i,j}w_{i,j}+(\lambda^*+\mu^*)w_{i,i}w_{i,i}\Big)\,d\sigma\\&+\frac{1}{2}\int_\Omega\Big(A^*(\dot\eta+\tau\ddot\eta)^2+\kappa^*|\nabla(\eta+\tau\dot\eta)|^2+(\epsilon+\tau\bar\kappa)|\nabla\dot\eta|^2\\&+\gamma^*|\nabla(\psi+G)|^2+\tau\bar\gamma|\nabla G|^2\Big)\,d\sigma\\&-\int_\Omega\Big(Q^*\nabla(\eta+\tau\dot\eta)\cdot\nabla(\psi+\tau G)+\tau\bar Q\nabla\dot\eta\cdot\nabla G\Big)\,d\sigma,\end{aligned}$$

where $G=\gamma^{-1}(Q^*\eta+Q\dot\eta-\gamma^*\psi)$, and we used the notation $\bar\kappa=\kappa-\tau\kappa^*$, $\bar\gamma=\gamma-\tau\gamma^*$ and $\bar Q=Q-\tau Q^*$.

It follows that

$$\begin{aligned}\dot E(t)=&-\int_\Omega\Big(\bar\kappa|\nabla\dot\eta|^2+\bar\gamma|\nabla G|^2-2\bar Q\nabla\dot\eta\cdot\nabla G\Big)\,d\sigma\\&+\epsilon\int_\Omega\nabla\ddot\alpha\cdot\nabla(\dot\eta+\tau\ddot\eta)\,d\sigma-\tau\epsilon\int_\Omega|\nabla\ddot\eta|^2\,d\sigma.\end{aligned}$$

Therefore, it leads that we have, for every $\delta>0$,

$$\begin{aligned}\dot E(t)\le&\frac{\epsilon^2}{2\delta}\int_\Omega|\nabla\ddot\alpha|\,d\sigma+\frac{\delta}{2}\int_\Omega|\nabla(\dot\eta+\tau\ddot\eta)|^2\,d\sigma\\&-\int_\Omega\Big(\bar\kappa|\nabla\dot\eta|^2+\tau\epsilon|\nabla\ddot\eta|^2+\bar\gamma|\nabla G|^2-2\bar Q\nabla\dot\eta\cdot\nabla G\Big)\,d\sigma.\end{aligned}$$

Now, we can assume that ϵ is small enough to guarantee that

$$\bar\kappa|\nabla\dot\eta|^2+\bar\gamma|\nabla G|^2-2\bar Q\nabla\dot\eta\cdot\nabla G\ge\frac{\epsilon}{\tau}\Big(|\nabla\dot\eta|^2+|\nabla G|^2\Big),$$

and so, we have

$$\begin{aligned}\dot E(t)\le&\frac{\epsilon^2}{2\delta}\int_\Omega|\nabla\ddot\alpha|^2\,d\sigma+\frac{\delta}{2}\int_\Omega|\nabla(\dot\eta+\tau\ddot\eta)|^2\,d\sigma\\&-\frac{\epsilon}{\tau}\int_\Omega\Big(|\nabla\dot\eta|^2+\tau^2|\nabla\ddot\eta|^2\Big)\,d\sigma.\end{aligned}$$

We observe that

$$|\nabla(\dot\eta+\tau\ddot\eta)|^2\le 2(|\nabla\dot\eta|^2+\tau^2|\nabla\ddot\eta|^2).$$

[2] In view of the boundary conditions the last equation of system (4)–(6) is equivalent to the last equation we propose here.

Therefore, choosing $\delta=\epsilon/\tau$ it follows that

$$\dot{E}(t)\leq\frac{\tau\epsilon}{2}\int_{\Omega}|\nabla\ddot{\alpha}|^2\,d\sigma.$$

Next, we shall bound the integral $\int_{\Omega}|\nabla\ddot{\alpha}|^2\,d\sigma$.

We note that the energy related to the solution to problem (1)–(3), (7) and (8), defined as

$$\begin{aligned}H(t)&=\frac{1}{2}\int_{\Omega}\Big(\rho\ddot{u}_i\ddot{u}_i+\mu^*\dot{u}_{i,j}\dot{u}_{i,j}+(\lambda^*+\mu^*)\dot{u}_{i,i}\dot{u}_{j,j}\Big)\,d\sigma\\&+\frac{1}{2}\int_{\Omega}\Big(A^*(\ddot{\alpha}+\tau\dddot{\alpha})^2+\kappa^*|\nabla(\dot{\alpha}+\tau\ddot{\alpha})|^2+\tau\bar{\kappa}|\nabla\ddot{\alpha}|^2\\&+\gamma^*|\nabla(\dot{\phi}+\tau\dot{F})|^2+\tau\bar{\gamma}|\nabla\dot{F}|^2\Big)\,d\sigma\\&-\int_{\Omega}\Big(Q^*\nabla(\dot{\alpha}+\tau\ddot{\alpha})\cdot\nabla(\dot{\phi}+\tau\dot{F})+\tau\bar{Q}\nabla\ddot{\alpha}\cdot\nabla\dot{F}\Big)\,d\sigma,\end{aligned}$$

where $F=\gamma^{-1}(Q^*\alpha+Q\dot{\alpha}-\gamma^*\phi)$, it satisfies $\frac{d}{dt}H(t)\leq 0$ and so, $H(t)\leq H(0)$.

Moreover, we can see that there exists a positive constant C, depending only on the constitutive parameters, such that

$$\int_{\Omega}|\nabla\ddot{\alpha}|^2\,d\sigma\leq CH(t).$$

Thus, we have

$$\dot{E}(t)\leq\frac{\tau\epsilon C}{2}H(0),$$

and, by integration, it leads

$$E(t)\leq\frac{\tau C}{2}t\epsilon H(0).$$

We note that value $H(0)$ is obtained from values $\ddot{\boldsymbol{u}}(0)$, $\dddot{\alpha}(0)$, $\dot{\phi}(0)$ and $\dot{F}(0)$, which are well defined thanks to regularities (9).

Hence, we may conclude that, for a finite final time, $\boldsymbol{w}$, η and ψ tend to zero when ϵ also converges to zero.

4. Fully discrete approximations: an a priori error analysis

In order to numerically analyze problem (4)–(8), we introduce its variational formulation. Let $Y=L^2(\Omega)$, $H=[L^2(\Omega)]^d$ and $Q=[L^2(\Omega)]^{d\times d}$ and denote by $(\cdot,\cdot)_Y$, $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$, $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let us define the variational spaces $V=[H_0^1(\Omega)]^d$ and $E=H_0^1(\Omega)$ with respective scalar products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_E$, and norms $\|\cdot\|_V$ and $\|\cdot\|_E$.

By using Green's formula and boundary conditions (7), we write the variational formulation of problem (4)–(8) in terms of the velocity field $\boldsymbol{v}=\dot{\boldsymbol{u}}$, the temperature speed $\zeta=\dot{\theta}$ and the electric potential speed $\psi=\dot{\phi}$.

Problem VP. *Find the velocity field $\boldsymbol{v}:[0,T]\to V$, the temperature speed $\zeta:[0,T]\to E$ and the electric potential speed $\psi:[0,T]\to E$ such that $\boldsymbol{v}(0)=\boldsymbol{v}^0$, $\zeta(0)=\zeta^0$, and, for a.e. $t\in(0,T)$ and for all $\boldsymbol{w}\in V$ and $r,l\in E$,*

$$\begin{aligned}&\rho(\dot{\boldsymbol{v}}(t),\boldsymbol{w})_H+\mu^*(\nabla\boldsymbol{u}(t),\nabla\boldsymbol{w})_Q+(\lambda^*+\mu^*)(\operatorname{div}\boldsymbol{u}(t),\operatorname{div}\boldsymbol{w})_Y\\&\quad+\beta^*(\nabla(\theta(t)+\tau\zeta(t)),\boldsymbol{w})_H=0,\qquad(10)\\&A^*(\zeta(t)+\tau\dot{\zeta}(t),l)_Y+\epsilon(\nabla\zeta(t),\nabla l)_H+\kappa(\nabla\theta(t),\nabla l)_H=-\beta^*(\operatorname{div}\boldsymbol{v}(t),l)_Y\\&\quad-\kappa^*(\nabla\alpha(t),\nabla l)_H+Q^*(\nabla\phi(t),\nabla l)_H+Q(\nabla\psi(t),\nabla l)_H,\qquad(11)\\&\gamma(\nabla\psi(t),\nabla r)_H+\gamma^*(\nabla\phi(t),\nabla r)_H=Q^*(\nabla\alpha(t),\nabla r)_H+Q(\nabla\theta(t),\nabla r)_H,\qquad(12)\end{aligned}$$

where the displacement field $\boldsymbol{u}$, the temperature θ, the thermal displacement α and the electric potential ϕ are then recovered from the relations:

$$\begin{aligned}&\boldsymbol{u}(t)=\int_0^t\boldsymbol{v}(s)\,ds+\boldsymbol{u}^0,\quad\theta(t)=\int_0^t\zeta(s)\,ds+\theta^0,\\&\alpha(t)=\int_0^t\theta(s)\,ds+\alpha^0,\quad\phi(t)=\int_0^t\psi(s)\,ds+\phi^0.\end{aligned}\qquad(13)$$

Now, we consider a fully discrete approximation of Problem *VP*. This is done in two steps. First, we assume that the domain $\overline{\Omega}$ is polyhedral and we denote by $\mathcal{T}^h$ a regular triangulation in the sense of [21]. Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$V^h = \{\boldsymbol{z}^h \in [C(\overline{\Omega})]^d \cap V \;;\; \boldsymbol{z}^h_{|Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h\}, \tag{14}$$

$$E^h = \{r^h \in C(\overline{\Omega}) \cap E \;;\; r^h_{|Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h\}, \tag{15}$$

where $P_1(Tr)$ represents the space of polynomials of degree less than or equal to one in the element Tr, i.e. the finite element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}^{0h}$, $\boldsymbol{v}^{0h}$, ζ^{0h}, θ^{0h}, α^{0h} and ϕ^{0h}, are given by

$$\begin{aligned} &\boldsymbol{u}^{0h} = \mathcal{P}^h_1 \boldsymbol{u}^0, \quad \boldsymbol{v}^{0h} = \mathcal{P}^h_1 \boldsymbol{v}^0, \quad \zeta^{0h} = \mathcal{P}^h_2 \zeta^0, \quad \theta^{0h} = \mathcal{P}^h_2 \theta^0, \\ &\alpha^{0h} = \mathcal{P}^h_2 \alpha^0, \quad \phi^{0h} = \mathcal{P}^h_2 \phi^0, \end{aligned} \tag{16}$$

where $\mathcal{P}^h_1$ and $\mathcal{P}^h_2$ are the classical finite element interpolation operators over V^h and E^h, respectively (see, e.g., [21]).

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = nk$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z^n = z(t_n)$ and, for the sequence $\{z^n\}_{n=0}^N$, we denote by $\delta z^n = (z^n - z^{n-1})/k$ its corresponding divided differences.

Therefore, using the implicit Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk}. *Find the discrete velocity field $\boldsymbol{v}^{hk} = \{\boldsymbol{v}^{hk}_n\}_{n=0}^N \subset V^h$, the discrete temperature speed $\zeta^{hk} = \{\zeta^{hk}_n\}_{n=0}^N \subset E^h$ and the discrete electric potential speed $\psi^{hk} = \{\psi^{hk}_n\}_{n=0}^N \subset E^h$ such that $\boldsymbol{v}^{hk}_0 = \boldsymbol{v}^{0h}$, $\zeta^{hk}_0 = \zeta^{0h}$, $\psi^{hk}_0 = \psi^{0h}$, and, for $n = 1, \ldots, N$ and for all $\boldsymbol{w}^h \in V^h$ and $r^h, l^h \in E^h$,*

$$\begin{aligned} &\rho(\delta \boldsymbol{v}^{hk}_n, \boldsymbol{w}^h)_H + \mu^*(\nabla \boldsymbol{u}^{hk}_n, \nabla \boldsymbol{w}^h)_Q + (\lambda^* + \mu^*)(\operatorname{div} \boldsymbol{u}^{hk}_n, \operatorname{div} \boldsymbol{w}^h)_Y \\ &\quad + \beta^*(\nabla(\theta^{hk}_n + \tau \zeta^{hk}_n), \boldsymbol{w}^h)_H = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &A^*(\zeta^{hk}_n + \tau \delta \zeta^{hk}_n, l^h)_Y + \epsilon(\nabla \zeta^{hk}_n, \nabla l^h)_H + \kappa(\nabla \theta^{hk}_n, \nabla l^h)_H = -\beta^*(\operatorname{div} \boldsymbol{v}^{hk}_n, l^h)_Y \\ &\quad - \kappa^*(\nabla \alpha^{hk}_n, \nabla l^h)_H + Q^*(\nabla \phi^{hk}_n, \nabla l^h)_H + Q(\nabla \psi^{hk}_n, \nabla l^h)_H, \end{aligned} \tag{18}$$

$$\begin{aligned} &\gamma(\nabla \psi^{hk}_n, \nabla r^h)_H + \gamma^*(\nabla \phi^{hk}_n, \nabla r^h)_H = Q^*(\nabla \alpha^{hk}_n, \nabla r^h)_H \\ &\quad + Q(\nabla \theta^{hk}_n, \nabla r^h)_H, \end{aligned} \tag{19}$$

where the discrete displacement field $\boldsymbol{u}^{hk}_n$, the discrete temperature θ^{hk}_n, the discrete thermal displacement α^{hk}_n and the discrete electric potential ϕ^{hk}_n are then recovered from the relations:

$$\begin{aligned} &\boldsymbol{u}^{hk}_n = k \sum_{j=1}^n \boldsymbol{v}^{hk}_n + \boldsymbol{u}^{0h}, \quad \theta^{hk}_n = k \sum_{j=1}^n \zeta^{hk}_n + \theta^{0h}, \\ &\alpha^{hk}_n = k \sum_{j=1}^n \theta^{hk}_n + \alpha^{0h}, \quad \phi^{hk}_n = k \sum_{j=1}^n \psi^{hk}_n + \phi^{0h}. \end{aligned} \tag{20}$$

In the previous fully discrete problem the artificial "discrete initial condition" for the electric potential speed ψ^{0h} is obtained solved the following discrete problem:

$$\begin{aligned} &\gamma(\nabla \psi^{0h}, \nabla r^h)_H + \gamma^*(\nabla \phi^{0h}, \nabla r^h)_H = Q^*(\nabla \alpha^{0h}, \nabla r^h)_H \\ &\quad + Q(\nabla \theta^{0h}, \nabla r^h)_H \quad \forall r^h \in E^h. \end{aligned} \tag{21}$$

The existence of a unique discrete solution to the above discrete problem can be shown straightforwardly applying the well-known Lax–Milgram lemma, taking into account the assumptions of Theorem 1.

Now, we prove a discrete stability property.

Lemma 2. *Let the assumptions of Theorem 1 hold. It follows that the sequences $\{\boldsymbol{u}^{hk}, \boldsymbol{v}^{hk}, \phi^{hk}, \psi^{hk}, \alpha^{hk}, \theta^{hk}, \zeta^{hk}\}$, generated by discrete problem VP^{hk}, satisfy the stability estimate:*

$$\begin{aligned} &\|\boldsymbol{v}^{hk}_n\|^2_H + \|\nabla \boldsymbol{u}^{hk}_n\|^2_Q + \|div\, \boldsymbol{u}^{hk}_n\|^2_Y + \|\nabla \psi^{hk}_n\|^2_H + \|\nabla \phi^{hk}_n\|^2_H + \|\nabla \alpha^{hk}_n\|^2_H \\ &\quad + \|\nabla \theta^{hk}_n\|^2_H + \|\zeta^{hk}_n\|^2_Y \le C, \end{aligned}$$

where C is a positive constant which is independent of the discretization parameters h and k.

Proof. In order to simplify the calculations, we assume that $\tau = 1$. The extension to the general case can be done easily with some minor modifications.

Taking $\boldsymbol{v}^{hk}_n$ as a test function in variational equation (17) we find that

$$\begin{aligned} &\rho(\delta \boldsymbol{v}^{hk}_n, \boldsymbol{v}^{hk}_n)_H + \mu^*(\nabla \boldsymbol{u}^{hk}_n, \nabla \boldsymbol{v}^{hk}_n)_Q + (\lambda^* + \mu^*)(\operatorname{div} \boldsymbol{u}^{hk}_n, \operatorname{div} \boldsymbol{v}^{hk}_n)_Y \\ &\quad + \beta^*(\nabla(\theta^{hk}_n + \zeta^{hk}_n), \boldsymbol{v}^{hk}_n)_H = 0, \end{aligned}$$

and taking into account that

$$\rho(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk})_H \geq \frac{\rho}{2k}\left\{\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\},$$
$$\mu^*(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{v}_n^{hk})_Q \geq \frac{\mu^*}{2k}\left\{\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{u}_{n-1}^{hk}\|_Q^2\right\},$$
$$(\lambda^* + \mu^*)(\operatorname{div} \boldsymbol{u}_n^{hk}, \operatorname{div} \boldsymbol{v}_n^{hk})_Y \geq \frac{\lambda^*+\mu^*}{2k}\left\{\|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 - \|\operatorname{div} \boldsymbol{u}_{n-1}^{hk}\|_Y^2\right\},$$

using the Cauchy's inequality

$$ab \leq \eta a^2 + \frac{1}{4\eta} b^2 \quad \forall a, b \in \mathbb{R}, \ \ \eta > 0, \tag{22}$$

we obtain

$$\begin{aligned} &\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\} + \frac{\mu^*}{2k}\left\{\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{u}_{n-1}^{hk}\|_Q^2\right\} \\ &\qquad + \frac{\lambda^*+\mu^*}{2k}\left\{\|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 - \|\operatorname{div} \boldsymbol{u}_{n-1}^{hk}\|_Y^2\right\} \\ &\quad \leq C\left(\|\nabla \theta_n^{hk}\|_H^2 + \|\boldsymbol{v}_n^{hk}\|_H^2\right) - \beta^*(\nabla \zeta_n^{hk}, \boldsymbol{v}_n^{hk})_H. \end{aligned} \tag{23}$$

Now, taking ζ_n^{hk} as a test function in discrete variational equation (18) we have

$$\begin{aligned} &A^*(\zeta_n^{hk} + \delta \zeta_n^{hk}, \zeta_n^{hk})_Y + \epsilon(\nabla \zeta_n^{hk}, \nabla \zeta_n^{hk})_H + \kappa(\nabla \theta_n^{hk}, \nabla \zeta_n^{hk})_H = -\beta^*(\operatorname{div} \boldsymbol{v}_n^{hk}, \zeta_n^{hk})_Y \\ &\quad -\kappa^*(\nabla \alpha_n^{hk}, \nabla \zeta_n^{hk})_H + Q^*(\nabla \phi_n^{hk}, \nabla \zeta_n^{hk})_H + Q(\nabla \psi_n^{hk}, \nabla \zeta_n^{hk})_H, \end{aligned}$$

and, keeping in mind that

$$A^*(\delta \zeta_n^{hk}, \zeta_n^{hk})_Y \geq \frac{A^*}{2k}\left\{\|\zeta_n^{hk}\|_Y^2 - \|\zeta_{n-1}^{hk}\|_Y^2\right\},$$
$$\kappa(\theta_n^{hk}, \zeta_n^{hk}) \geq \frac{\kappa}{2k}\left\{\|\nabla \theta_n^{hk}\|_H^2 - \|\nabla \theta_{n-1}^{hk}\|_H^2\right\},$$
$$-\beta^*(\operatorname{div} \boldsymbol{v}_n^{hk}, \zeta_n^{hk})_Y = \beta^*(\boldsymbol{v}_n^{hk}, \nabla \zeta_n^{hk})_H,$$

using Cauchy's inequality (22) several times it leads to the following estimates:

$$\begin{aligned} &\frac{A^*}{2k}\left\{\|\zeta_n^{hk}\|_Y^2 - \|\zeta_{n-1}^{hk}\|_Y^2\right\} + \frac{\kappa}{2k}\left\{\|\nabla \theta_n^{hk}\|_H^2 - \|\nabla \theta_{n-1}^{hk}\|_H^2\right\} \\ &\quad \leq C\left(\|\nabla \alpha_n^{hk}\|_H^2 + \|\nabla \phi_n^{hk}\|_H^2 + \|\nabla \psi_n^{hk}\|_H^2\right) + \beta^*(\boldsymbol{v}_n^{hk}, \nabla \zeta_n^{hk})_H. \end{aligned} \tag{24}$$

Finally, we obtain the estimates for the electric potential speed. So, we take ψ_n^{hk} as a test function in discrete variational equation (19) and we have

$$\gamma(\nabla \psi_n^{hk}, \nabla \psi_n^{hk})_H = -(\nabla \phi_n^{hk}, \nabla \psi_n^{hk})_H + Q^*(\nabla \alpha_n^{hk}, \nabla \psi_n^{hk})_H + Q(\nabla \theta_n^{hk}, \nabla \psi_n^{hk})_H,$$

and, using again inequality (22), we get the estimates

$$\|\nabla \psi_n^{hk}\|_H^2 \leq C\left(\|\nabla \phi_n^{hk}\|_H^2 + \|\nabla \alpha_n^{hk}\|_H^2 + \|\nabla \theta_n^{hk}\|_H^2\right). \tag{25}$$

Proceeding in a similar form, from the discrete variational equation (21) we can also show that

$$\|\nabla \psi^{0h}\|_H^2 \leq C\left(\|\nabla \phi^{0h}\|_H^2 + \|\nabla \alpha^{0h}\|_H^2 + \|\nabla \theta^{0h}\|_H^2\right). \tag{26}$$

Combining estimates (23)–(24) we find that

$$\begin{aligned} &\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\} + \frac{\mu^*}{2k}\left\{\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{u}_{n-1}^{hk}\|_Q^2\right\} \\ &\quad + \frac{\lambda^*+\mu^*}{2k}\left\{\|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 - \|\operatorname{div} \boldsymbol{u}_{n-1}^{hk}\|_Y^2\right\} + \frac{A^*}{2k}\left\{\|\zeta_n^{hk}\|_Y^2 - \|\zeta_{n-1}^{hk}\|_Y^2\right\} \\ &\quad + \frac{\kappa}{2k}\left\{\|\nabla \theta_n^{hk}\|_H^2 - \|\nabla \theta_{n-1}^{hk}\|_H^2\right\} \\ &\leq C\left(\|\nabla \theta_n^{hk}\|_H^2 + \|\boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla \alpha_n^{hk}\|_H^2 + \|\nabla \phi_n^{hk}\|_H^2 + \|\nabla \psi_n^{hk}\|_H^2\right). \end{aligned}$$

Summing up to n and keeping in mind (26) we have

$$\begin{aligned}&\|\boldsymbol{v}_n^{hk}\|_H^2+\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2+\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2+\|\zeta_n^{hk}\|_Y^2+\|\nabla\theta_n^{hk}\|_H^2\\&\quad\leq Ck\sum_{j=1}^{n}\Big(\|\nabla\theta_j^{hk}\|_H^2+\|\boldsymbol{v}_j^{hk}\|_H^2+\|\nabla\alpha_j^{hk}\|_H^2+\|\nabla\phi_j^{hk}\|_H^2+\|\nabla\psi_j^{hk}\|_H^2\Big)\\&\qquad+C\Big(\|\boldsymbol{v}^{0h}\|_H^2+\|\boldsymbol{u}^{0h}\|_V^2+\|\zeta^{0h}\|_Y^2+\|\nabla\theta^{0h}\|_H^2\Big).\end{aligned}$$

Now, taking into account that

$$\|\nabla\phi_n^{hk}\|_H^2+\|\nabla\alpha_n^{hk}\|_H^2\leq Ck\sum_{j=1}^{n}\Big(\|\nabla\theta_j^{hk}\|_H^2+\|\nabla\psi_j^{hk}\|_H^2\Big)+C\|\nabla\alpha^{0h}\|_H^2+C\|\nabla\phi^{0h}\|_H^2,$$

and estimates (25), we find that

$$\begin{aligned}&\|\boldsymbol{v}_n^{hk}\|_H^2+\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2+\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2+\|\zeta_n^{hk}\|_Y^2+\|\nabla\theta_n^{hk}\|_H^2+\|\nabla\psi_n^{hk}\|_H^2+\|\nabla\phi_n^{hk}\|_H^2\\&\quad\leq Ck\sum_{j=1}^{n}\Big(\|\nabla\theta_j^{hk}\|_H^2+\|\boldsymbol{v}_j^{hk}\|_H^2+\|\nabla\psi_j^{hk}\|_H^2\Big)+C\Big(\|\nabla\alpha^{0h}\|_H^2+\|\nabla\phi^{0h}\|_H^2\\&\qquad+\|\boldsymbol{v}^{0h}\|_H^2+\|\boldsymbol{u}^{0h}\|_V^2+\|\zeta^{0h}\|_Y^2+\|\nabla\theta^{0h}\|_H^2\Big).\end{aligned}$$

Using a discrete version of Gronwall's inequality (see, for instance, [22]) and estimates (26) we conclude the discrete stability property.

In what follows, we provide an a priori error analysis of Problem VP. So we will obtain some estimates on the numerical errors $\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}$, $\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}$, $\psi_n-\psi_n^{hk}$, $\phi_n-\phi_n^{hk}$, $\alpha_n-\alpha_n^{hk}$, $\theta_n-\theta_n^{hk}$ and $\zeta_n-\zeta_n^{hk}$. We have the following.

Theorem 3. *Let the assumptions of Theorem 1 still hold. If we denote by $(\boldsymbol{v},\zeta,\psi)$ the solution to Problem VP and by $(\boldsymbol{v}^{hk},\zeta^{hk},\psi^{hk})$ the solution to Problem VP^{hk}, then we have the following a priori error estimates, for all $\boldsymbol{w}^h=\{\boldsymbol{w}_j^h\}_{j=0}^N\subset V^h$ and $r^h=\{r_j^h\}_{j=0}^N$, $l^h=\{l_j^h\}_{j=0}^N\subset E^h$,*

$$\begin{aligned}&\max_{0\leq n\leq N}\Big\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|div\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2\\&\quad+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2+\|\nabla(\psi_n-\psi_n^{hk})\|_H^2+\|\nabla(\phi_n-\phi_n^{hk})\|_H^2+\|\nabla(\alpha_n-\alpha_n^{hk})\|_H^2\Big\}\\&\leq Ck\sum_{j=1}^{N}\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\dot{\zeta}_j-\delta\zeta_j\|_Y^2\\&\quad+\|\dot{\theta}_j-\delta\theta_j\|_E^2+\|\zeta_j-l_j^h\|_V^2+\|\psi_j-r_j^h\|_E^2+I_j+J_j\Big)+C\max_{0\leq n\leq N}\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_Y^2\\&\quad+C\max_{0\leq n\leq N}\|\zeta_n-l_n^h\|_H^2+\frac{C}{k}\sum_{j=1}^{N-1}\Big(\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H^2\\&\quad+\|\zeta_j-l_j^h-(\zeta_{j+1}-l_{j+1}^h)\|_Y^2\Big)+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2+\|\boldsymbol{u}^0-\boldsymbol{u}^{0h}\|_V^2\\&\quad+\|\zeta^0-\zeta^{0h}\|_Y^2+\|\nabla(\theta^0-\theta^{0h})\|_H^2+\|\nabla(\phi^0-\phi^{0h})\|_H^2\\&\quad+\|\nabla(\alpha^0-\alpha^{0h})\|_H^2\Big),\end{aligned}\tag{27}$$

where C is again a positive constant which does not depend on parameters h and k, and I_j and J_j are the integration errors given by

$$I_j=\left\|\int_0^{t_j}\nabla\theta(s)\,ds-k\sum_{l=1}^{j}\nabla\theta_l\right\|_H^2,\tag{28}$$

$$J_j=\left\|\int_0^{t_j}\nabla\psi(s)\,ds-k\sum_{l=1}^{j}\nabla\psi_l\right\|_H^2.\tag{29}$$

Proof. Assume that, again for simplicity in the calculations, $\tau=1$.

First, we obtain the error estimates on the velocity field. Thus, subtracting variational equation (10) at time $t = t_n$ for a test function $\boldsymbol{w} = \boldsymbol{w}^h \in V^h \subset V$ and discrete variational equation (17) we have, for all $\boldsymbol{w}^h \in V^h$,

$$\begin{aligned}&\rho(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + \mu^*(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla\boldsymbol{w}^h)_Q + (\lambda^* + \mu^*)(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,\boldsymbol{w}^h)_Y\\ &\quad+\beta^*(\nabla(\theta_n - \theta_n^{hk} + (\zeta_n - \zeta_n^{hk})), \boldsymbol{w}^h)_H = 0,\end{aligned}$$

and so, we find that, for all $\boldsymbol{w}^h \in V^h$,

$$\begin{aligned}&\rho(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \mu^*(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q\\ &\quad+(\lambda^* + \mu^*)(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y\\ &\quad+\beta^*(\nabla(\theta_n - \theta_n^{hk} + (\zeta_n - \zeta_n^{hk})), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H\\ &= \rho(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H + \mu^*(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{w}^h))_Q\\ &\quad+(\lambda^* + \mu^*)(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{w}^h))_Y\\ &\quad+\beta^*(\nabla(\theta_n - \theta_n^{hk}(\zeta_n - \zeta_n^{hk})), \boldsymbol{v}_n - \boldsymbol{w}^h)_H.\end{aligned}$$

Taking into account that

$$\begin{aligned}&(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H = (\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H,\\ &\left(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\right)_H \geq \frac{1}{2k}\left\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right\},\\ &(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y \geq (\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n))_Y\\ &\quad+\frac{1}{2k}\left\{\|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\},\\ &(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \geq (\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n))_Q\\ &\quad+\frac{1}{2k}\left\{\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\},\\ &(\nabla(\zeta_n - \zeta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{w}^h)_H = -(\zeta_n - \zeta_n^{hk}, \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{w}^h))_Y,\end{aligned}$$

using several times Cauchy's inequality (22) we have, for all $\boldsymbol{w}^h \in V^h$,

$$\begin{aligned}&\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right\}\\ &\quad+\frac{\lambda^* + \mu^*}{2k}\left\{\|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\}\\ &\quad+\frac{\mu^*}{2k}\left\{\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\}\\ &\leq C\Big(\|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H^2 + \|\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2\\ &\quad+\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2\\ &\quad+(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H\Big) - \beta^*(\nabla(\zeta_n - \zeta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H.\end{aligned} \tag{30}$$

Now, we obtain the error estimates for the temperature speed. So, we subtract variational equation (11) at time $t = t_n$ for a test function $l = l^h \in E^h \subset E$, and discrete variational equation (18) to obtain, for all $l^h \in E^h$,

$$\begin{aligned}&A^*(\zeta_n - \zeta_n^{hk} + (\dot{\zeta}_n - \delta\zeta_n^{hk}), l^h)_Y + \epsilon(\nabla(\zeta_n - \zeta_n^{hk}), \nabla l^h)_H + \kappa(\nabla(\theta_n - \theta_n^{hk}), \nabla l^h)_H\\ &\quad+\beta^*(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), l^h)_Y + \kappa^*(\nabla(\alpha_n - \alpha_n^{hk}), \nabla l^h)_H - Q^*(\nabla(\phi_n - \phi_n^{hk}), \nabla l^h)_H\\ &\quad-Q(\nabla(\psi_n - \psi_n^{hk}), \nabla l^h)_H = 0,\end{aligned}$$

and so, for all $l^h \in E^h$ it leads

$$\begin{aligned}&A^*(\zeta_n - \zeta_n^{hk} + (\dot{\zeta}_n - \delta\zeta_n^{hk}), \zeta_n - \zeta_n^{hk})_Y + \epsilon(\nabla(\zeta_n - \zeta_n^{hk}), \nabla(\zeta_n - \zeta_n^{hk}))_H\\ &\quad+\kappa(\nabla(\theta_n - \theta_n^{hk}), \nabla(\zeta_n - \zeta_n^{hk}))_H + \beta^*(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \zeta_n - \zeta_n^{hk})_Y\\ &\quad+\kappa^*(\nabla(\alpha_n - \alpha_n^{hk}), \nabla(\zeta_n - \zeta_n^{hk}))_H - Q^*(\nabla(\phi_n - \phi_n^{hk}), \nabla(\zeta_n - \zeta_n^{hk}))_H\\ &\quad-Q(\nabla(\psi_n - \psi_n^{hk}), \nabla(\zeta_n - \zeta_n^{hk}))_H\\ &= A^*(\zeta_n - \zeta_n^{hk} + (\dot{\zeta}_n - \delta\zeta_n^{hk}), \zeta_n - l^h)_Y + \epsilon(\nabla(\zeta_n - \zeta_n^{hk}), \nabla(\zeta_n - l^h))_H\\ &\quad+\kappa(\nabla(\theta_n - \theta_n^{hk}), \nabla(\zeta_n - l^h))_H + \beta^*(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \zeta_n - l^h)_Y\\ &\quad+\kappa^*(\nabla(\alpha_n - \alpha_n^{hk}), \nabla(\zeta_n - l^h))_H - Q^*(\nabla(\phi_n - \phi_n^{hk}), \nabla(\zeta_n - l^h))_H\\ &\quad-Q(\nabla(\psi_n - \psi_n^{hk}), \nabla(\zeta_n - l^h))_H.\end{aligned}$$

Keeping in mind that

$$(\dot{\zeta}_n - \delta\zeta_n^{hk}, \zeta_n - \zeta_n^{hk})_Y = (\dot{\zeta}_n - \delta\zeta_n, \zeta_n - \zeta_n^{hk})_Y + (\delta\zeta_n - \delta\zeta_n^{hk}, \zeta_n - \zeta_n^{hk})_Y,$$
$$\left(\delta\zeta_n - \delta\zeta_n^{hk}, \zeta_n - \zeta_n^{hk}\right)_Y \geq \frac{1}{2k}\left\{\|\zeta_n - \zeta_n^{hk}\|_Y^2 - \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Y^2\right\},$$
$$(\nabla(\theta_n - \theta_n^{hk}), \nabla(\zeta_n - \zeta_n^{hk}))_H \geq (\nabla(\theta_n - \theta_n^{hk}), \nabla(\dot{\theta}_n - \delta\theta_n))_H$$
$$+\frac{1}{2k}\left\{\|\nabla(\theta_n - \theta_n^{hk})\|_H^2 - \|\nabla(\theta_{n-1} - \theta_{n-1}^{hk})\|_H^2\right\},$$
$$\beta^*(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \zeta_n - \zeta_n^{hk})_Y = -\beta^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \nabla(\zeta_n - \zeta_n^{hk}))_H,$$
$$\beta^*(\operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \zeta_n - l^h)_Y = -\beta^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \nabla(\zeta_n - l^h))_H,$$

we obtain

$$\begin{aligned}
&\frac{A^*}{2k}\left\{\|\zeta_n - \zeta_n^{hk}\|_Y^2 - \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Y^2\right\}\\
&\quad+\frac{\kappa}{2k}\left\{\|\nabla(\theta_n - \theta_n^{hk})\|_H^2 - \|\nabla(\theta_{n-1} - \theta_{n-1}^{hk})\|_H^2\right\}\\
&\leq C\Big(\|\dot{\zeta}_n - \delta\zeta_n\|_Y^2 + \|\dot{\theta}_n - \delta\theta_n\|_E^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\nabla(\alpha_n - \alpha_n^{hk})\|_H^2\\
&\quad+\|\zeta_n - \zeta_n^{hk}\|_Y^2 + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\nabla(\psi_n - \psi_n^{hk})\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2\\
&\quad+\|\zeta_n - l^h\|_V^2 + (\delta\zeta_n - \delta\zeta_n^{hk}, \zeta_n - l^h)_Y\Big)\\
&\quad+\beta^*(\nabla(\zeta_n - \zeta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H.
\end{aligned}\tag{31}$$

Combining estimates (30) and (31) it follows that

$$\begin{aligned}
&\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right\}\\
&\quad+\frac{\lambda^* + \mu^*}{2k}\left\{\|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\operatorname{div}(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\}\\
&\quad+\frac{\mu^*}{2k}\left\{\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\}\\
&\quad+\frac{A^*}{2k}\left\{\|\zeta_n - \zeta_n^{hk}\|_Y^2 - \|\zeta_{n-1} - \zeta_{n-1}^{hk}\|_Y^2\right\}\\
&\quad+\frac{\kappa}{2k}\left\{\|\nabla(\theta_n - \theta_n^{hk})\|_{H^2} - \|\nabla(\theta_{n-1} - \theta_{n-1}^{hk})\|_{H^2}\right\}\\
&\leq C\Big(\|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_H^2 + \|\dot{\boldsymbol{u}}_n - \delta\boldsymbol{u}_n\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2\\
&\quad+\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H\\
&\quad+\|\dot{\zeta}_n - \delta\zeta_n\|_Y^2 + \|\dot{\theta}_n - \delta\theta_n\|_E^2 + \|\zeta_n - l^h\|_V^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2\\
&\quad+\|\zeta_n - \zeta_n^{hk}\|_Y^2 + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\nabla(\psi_n - \psi_n^{hk})\|_H^2 + \|\nabla(\alpha_n - \alpha_n^{hk})\|_H^2\\
&\quad+(\delta\zeta_n - \delta\zeta_n^{hk}, \zeta_n - l^h)_Y\Big).
\end{aligned}$$

Summing up to n the previous estimates it leads, for all $\{\boldsymbol{w}_j^h\}_{j=0}^n \subset V^h$ and $\{l_j^h\}_{j=0}^n \subset E^h$,

$$\begin{aligned}
&\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\zeta_n - \zeta_n^{hk}\|_Y^2\\
&\quad+\|\nabla(\theta_n - \theta_n^{hk})\|_H^2\\
&\leq Ck\sum_{j=1}^n\Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_H^2 + \|\dot{\boldsymbol{u}}_j - \delta\boldsymbol{u}_j\|_V^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\operatorname{div}(\boldsymbol{u}_j - \boldsymbol{u}_j^{hk})\|_Y^2\\
&\quad+\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_H^2 + \|\nabla(\boldsymbol{u}_j - \boldsymbol{u}_j^{hk})\|_Q^2 + (\delta\boldsymbol{v}_j - \delta\boldsymbol{v}_j^{hk}, \boldsymbol{v}_j - \boldsymbol{w}_j^h)_H\\
&\quad+\|\dot{\zeta}_j - \delta\zeta_j\|_Y^2 + \|\dot{\theta}_j - \delta\theta_j\|_E^2 + \|\zeta_j - l_j^h\|_V^2 + \|\nabla(\theta_j - \theta_j^{hk})\|_H^2\\
&\quad+\|\zeta_j - \zeta_j^{hk}\|_Y^2 + \|\nabla(\phi_j - \phi_j^{hk})\|_H^2 + \|\nabla(\psi_j - \psi_j^{hk})\|_H^2 + \|\nabla(\alpha_j - \alpha_j^{hk})\|_H^2
\end{aligned}$$

$$+(\delta\zeta_j-\delta\zeta_j^{hk},\zeta_j-l_j^h)_Y\Big)+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2+\|\boldsymbol{u}^0-\boldsymbol{u}^{0h}\|_V^2+\|\zeta^0-\zeta^{0h}\|_Y^2$$
$$+\|\nabla(\theta^0-\theta^{0h})\|_H^2\Big). \tag{32}$$

Next, we obtain the error estimates for the electric potential speed. Subtracting variational equation (12) at time $t=t_n$ for a test function $r=r^h\in E^h\subset E$ and discrete variational equation (19) we have, for all $r^h\in E^h$,

$$\gamma(\nabla(\psi_n-\psi_n^{hk}),\nabla r^h)_H+\gamma^*(\nabla(\phi_n-\phi_n^{hk}),\nabla r^h)_H-Q^*(\nabla(\alpha_n-\alpha_n^{hk}),\nabla r^h)_H$$
$$-Q(\nabla(\theta_n-\theta_n^{hk}),\nabla r^h)_H=0.$$

Therefore, we obtain, for all $r^h\in E^h$,

$$\gamma(\nabla(\psi_n-\psi_n^{hk}),\nabla(\psi_n-\psi_n^{hk}))_H+\gamma^*(\nabla(\phi_n-\phi_n^{hk}),\nabla(\psi_n-\psi_n^{hk}))_H$$
$$-Q^*(\nabla(\alpha_n-\alpha_n^{hk}),\nabla(\psi_n-\psi_n^{hk}))_H-Q(\nabla(\theta_n-\theta_n^{hk}),\nabla(\psi_n-\psi_n^{hk}))_H$$
$$=\gamma(\nabla(\psi_n-\psi_n^{hk}),\nabla(\psi_n-r^h))_H+\gamma^*(\nabla(\phi_n-\phi_n^{hk}),\nabla(\psi_n-r^h))_H$$
$$-Q^*(\nabla(\alpha_n-\alpha_n^{hk}),\nabla(\psi_n-r^h))_H-Q(\nabla(\theta_n-\theta_n^{hk}),\nabla(\psi_n-r^h))_H,$$

and, using several times Cauchy's inequality (22), we find that

$$\|\nabla(\psi_n-\psi_n^{hk})\|_H^2\le C\Big(\|\psi_n-r^h\|_E^2+\|\nabla(\phi_n-\phi_n^{hk})\|_H^2+\|\nabla(\alpha_n-\alpha_n^{hk})\|_H^2$$
$$+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2\Big). \tag{33}$$

Proceeding in a similar form, using the regularity of the electric potential speed ψ (which implies that $\psi_0=\psi(0)$ is the solution to variational equation (12) at time $t=0$), and subtracting it to the discrete variational equation (21) we find that

$$\|\nabla(\psi_0-\psi^{0h})\|_H^2\le C\Big(\|\psi_0-r^h\|_E^2+\|\nabla(\phi^0-\phi^{0h})\|_H^2+\|\nabla(\alpha^0-\alpha^{0h})\|_H^2$$
$$+\|\nabla(\theta^0-\theta^{0h})\|_Y^2\Big)\quad\forall r^h\in E^h.$$

Combining (32) and (33) and taking into account that, for $n=1,\dots,N$,

$$\|\nabla(\alpha_n-\alpha_n^{hk})\|_H^2\le C\Big(I_j+k\sum_{j=1}^n\|\nabla(\theta_j-\theta_j^{hk})\|_H^2+\|\nabla(\alpha^0-\alpha^{0h})\|_H^2\Big),$$
$$\|\nabla(\phi_n-\phi_n^{hk})\|_H^2\le C\Big(J_j+k\sum_{j=1}^n\|\nabla(\psi_j-\psi_j^{hk})\|_H^2+\|\nabla(\psi_0-\psi^{0h})\|_H^2\Big),$$

where I_j and J_j are the integration errors defined in (28) and (29), respectively, it follows that

$$\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\zeta_n-\zeta_n^{hk}\|_Y^2$$
$$+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2+\|\nabla(\psi_n-\psi_n^{hk})\|_H^2+\|\nabla(\alpha_n-\alpha_n^{hk})\|_H^2+\|\nabla(\phi_n-\phi_n^{hk})\|_H^2$$
$$\le Ck\sum_{j=1}^n\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\mathrm{div}\,(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Y^2$$
$$+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H^2+\|\nabla(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Q^2+(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H$$
$$+\|\dot{\zeta}_j-\delta\zeta_j\|_Y^2+\|\dot{\theta}_j-\delta\theta_j\|_E^2+\|\zeta_j-l_j^h\|_V^2+\|\nabla(\theta_j-\theta_j^{hk})\|_H^2$$
$$+\|\zeta_j-\zeta_j^{hk}\|_Y^2+\|\nabla(\phi_j-\phi_j^{hk})\|_H^2+\|\nabla(\psi_j-\psi_j^{hk})\|_H^2+I_j+J_j$$
$$+\|\nabla(\alpha_j-\alpha_j^{hk})\|_H^2+\|\psi_j-r_j^h\|_E^2+(\delta\zeta_j-\delta\zeta_j^{hk},\zeta_j-l_j^h)_Y\Big)+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2$$
$$+\|\boldsymbol{u}^0-\boldsymbol{u}^{0h}\|_V^2+\|\zeta^0-\zeta^{0h}\|_Y^2+\|\nabla(\theta^0-\theta^{0h})\|_H^2+\|\nabla(\alpha^0-\alpha^{0h})\|_Y^2$$
$$+\|\nabla(\phi^0-\phi^{0h})\|_H^2\Big).$$

Finally, keeping in mind the estimates:

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H &= \sum_{j=1}^{n}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H\\
&=(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+(\boldsymbol{v}_0^h-\boldsymbol{v}^0,\boldsymbol{v}_1-\boldsymbol{w}_1^h)_H\\
&\quad+\sum_{j=1}^{n-1}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h))_H,\\
k\sum_{j=1}^{n}(\delta\zeta_j-\delta\zeta_j^{hk},\zeta_j-l_j^h)_Y &= \sum_{j=1}^{n}(\zeta_j-\zeta_j^{hk}-(\zeta_{j-1}-\zeta_{j-1}^{hk}),\zeta_j-l_j^h)_Y\\
&=(\zeta_n-\zeta_n^{hk},\zeta_n-l_n^h)_Y+(\zeta_0^h-\zeta^0,\zeta_1-l_1^h)_Y\\
&\quad+\sum_{j=1}^{n-1}(\zeta_j-\zeta_j^{hk},\zeta_j-l_j^h-(\zeta_{j+1}-l_{j+1}^h))_Y,
\end{aligned}
$$

we use again a discrete version of Gronwall's inequality [22] and we conclude a priori error estimates (27).

From estimates (27), under suitable regularity conditions on the continuous solution, we can obtain the convergence order of the approximations and so, we have the following result.

Corollary 1. *Under the assumptions of Theorem 3 and the additional regularity conditions:*

$$
\begin{aligned}
&\zeta\in H^2(0,T;Y)\cap L^\infty(0,T;H^2(\Omega))\cap H^1(0,T;H^1(\Omega)),\\
&\boldsymbol{u}\in H^3(0,T;H)\cap W^{1,\infty}(0,T;[H^2(\Omega)]^d)\cap H^2(0,T;[H^1(\Omega)]^d),\\
&\psi\in L^\infty(0,T;H^2(\Omega))\cap H^1(0,T;H^1(\Omega)),
\end{aligned}
$$

we conclude the linear convergence of the approximations given by Problem VP^{hk}; that is, there exists a constant $C>0$, independent of parameters h and k, such that

$$
\begin{aligned}
\max_{0\le n\le N}\Big\{&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\zeta_n-\zeta_n^{hk}\|_Y+\|\theta_n-\theta_n^{hk}\|_E\\
&+\|\alpha_n-\alpha_n^{hk}\|_E+\|\psi_n-\psi_n^{hk}\|_E+\|\phi_n-\phi_n^{hk}\|_E\Big\}\le C(h+k).
\end{aligned}
$$

Remark 1. We note that estimates (27) and the linear convergence obtained in Corollary 1 are just an example by using the finite element spaces V^h and E^h defined in (14) and (15), respectively, and assuming some regularity conditions on the continuous solution. Proceeding in a similar way and using, for instance, the Ritz projection and that the space $[C^1(\Omega)]^d$ is dense in V, we could obtain other error estimates for the L^2-norm. Similarly, if we also assume that the solution belongs to the space $H^r(\Omega)$, for a given exponent r, we also could deduce general error estimates depending on this regularity.

5. Numerical results

We note that the numerical scheme provided in discrete problem (17)–(21) was implemented on a 3.2 GHz PC using MATLAB, and a typical one-dimensional run ($h=k=0.001$) took about 0.53 s of CPU time.

5.1. Numerical convergence in a one-dimensional example

As an academical example, in order to show the accuracy of the approximations we solve the following one-dimensional version of problem (4)–(8):

$$
\begin{aligned}
&\rho\ddot{u}=(\lambda+2\mu^*)u_{xx}-\beta^*(\theta_x+\tau\dot{\theta}_x)+F_1\quad\text{in}\quad(0,1)\times(0,1),\\
&A^*(\dot{\theta}+\tau\ddot{\theta})=-\beta^*\dot{u}_x+\epsilon\dot{\theta}_{xx}+\kappa^*\alpha_{xx}+\kappa\theta_{xx}-Q^*\phi_{xx}-Q\dot{\phi}_{xx}+F_2\\
&\qquad\qquad\qquad\qquad\text{in}\quad(0,1)\times(0,1),\\
&-\gamma\dot{\phi}_{xx}-\gamma^*\phi_{xx}=-Q^*\alpha_{xx}-Q\theta_{xx}\quad\text{in}\quad(0,1)\times(0,1),\\
&u(x,t)=\theta(x,t)=\phi(x,t)=0\quad\text{for}\quad x\in\{0,1\},\ t\in(0,1),\\
&u(x,0)=\dot{u}(x,0)=\alpha(x,0)=\theta(x,0)=\dot{\theta}(x,0)=\phi(x,0)=x(x-1)\\
&\qquad\qquad\qquad\qquad\text{for a.e.}\quad x\in(0,1),
\end{aligned}
$$

with the following data:

$$
\begin{aligned}
&\rho=1,\quad \mu^*=1,\quad \lambda^*=1,\quad \kappa^*=3,\quad \gamma^*=2,\quad \tau=1,\quad \kappa=1,\\
&\gamma=1,\quad \beta^*=1,\quad A^*=1,\quad \epsilon=1,\quad Q^*=2,\quad Q=1.
\end{aligned}
$$

Table 1
Example 1: Numerical errors for some h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.821931	0.814403	0.810371	0.809133	0.808539	0.808191	0.808077
$1/2^4$	0.414581	0.405456	0.400797	0.399412	0.398766	0.398399	0.398281
$1/2^5$	0.217743	0.206055	0.200543	0.198988	0.198273	0.197873	0.197748
$1/2^6$	0.124399	0.108693	0.101742	0.099940	0.099148	0.098713	0.098577
$1/2^7$	0.082725	0.062251	0.053022	0.050805	0.049893	0.049415	0.049269
$1/2^8$	0.065524	0.041478	0.029410	0.026501	0.025388	0.024846	0.024687
$1/2^9$	0.058879	0.032904	0.018565	0.014707	0.013249	0.012595	0.012419
$1/2^{10}$	0.056574	0.029594	0.013980	0.009288	0.007354	0.006505	0.006297
$1/2^{11}$	0.055884	0.028446	0.012170	0.006996	0.004645	0.003518	0.003252
$1/2^{12}$	0.055700	0.028102	0.011516	0.006092	0.003500	0.002115	0.001759
$1/2^{13}$	0.055653	0.028010	0.011311	0.005766	0.003048	0.001506	0.001058

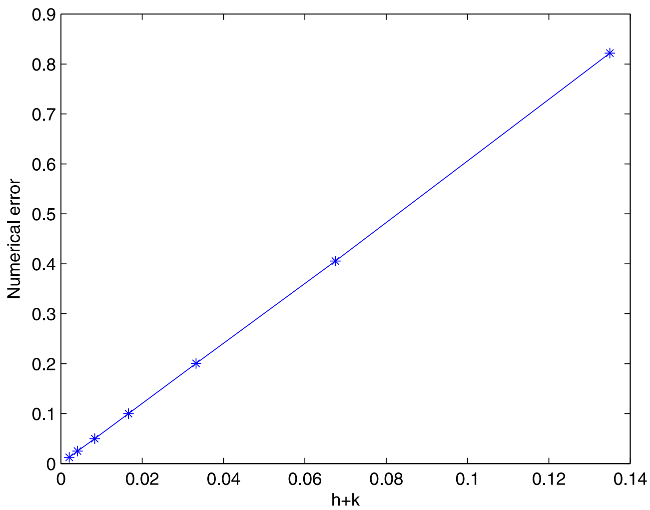

Fig. 1. Example 1: Asymptotic constant error.

In the previous problem, the artificial supply terms F_i, $i = 1, 2$, are given by, for $(x, t) \in [0, 1] \times [0, 1]$:

$$F_1(x, t) = e^t(4x + x(x - 1) - 8),$$
$$F_2(x, t) = e^t(2x + 2x(x - 1) - 5).$$

Thus, the exact solution to Problem (4)–(8) can be easily calculated and it has the form, for $(x, t) \in [0, 1] \times [0, 1]$:

$$u(x, t) = \theta(x, t) = \phi(x, t) = e^t x(x - 1).$$

Therefore, the approximation errors estimated by

$$\max_{0\leq n\leq N} \Big\{ \|v_n - v_n^{hk}\|_Y + \|u_n - u_n^{hk}\|_E + \|\zeta_n - \zeta_n^{hk}\|_Y + \|\theta_n - \theta_n^{hk}\|_E + \|\alpha_n - \alpha_n^{hk}\|_E + \|\psi_n - \psi_n^{hk}\|_E + \|\phi_n - \phi_n^{hk}\|_E \Big\}$$

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 1, is achieved.

If we assume now that there are not supply terms, and we use the final time $T = 30$, the following data:

$$\rho = 1, \quad \mu^* = 1, \quad \lambda^* = 1, \quad \kappa^* = 1, \quad \gamma^* = 2, \quad \tau = 1,$$
$$\kappa = 5, \quad \gamma = 5, \quad \beta^* = 2, \quad A^* = 2, \quad Q^* = 1, \quad Q = 1,$$

and the initial conditions, for all $x \in (0, 1)$,

$$\phi^0(x) = u^0(x) = v^0(x) = \zeta^0(x) = \theta^0(x) = \alpha^0(x) = x(x - 1),$$

taking the discretization parameter $k = 10^{-4}$, the evolution in time of the discrete energy

$$E_n^{hk} = \|v_n^{hk}\|_Y^2 + 3\|(u_n^{hk})_x\|_Y^2 + \|\zeta_n^{hk}\|_Y^2 + 3\|(\theta_n^{hk})_x\|_Y^2 + \|(\psi_n^{hk})_x\|_Y^2 + 2\|(\phi_n^{hk})_x\|_Y^2$$

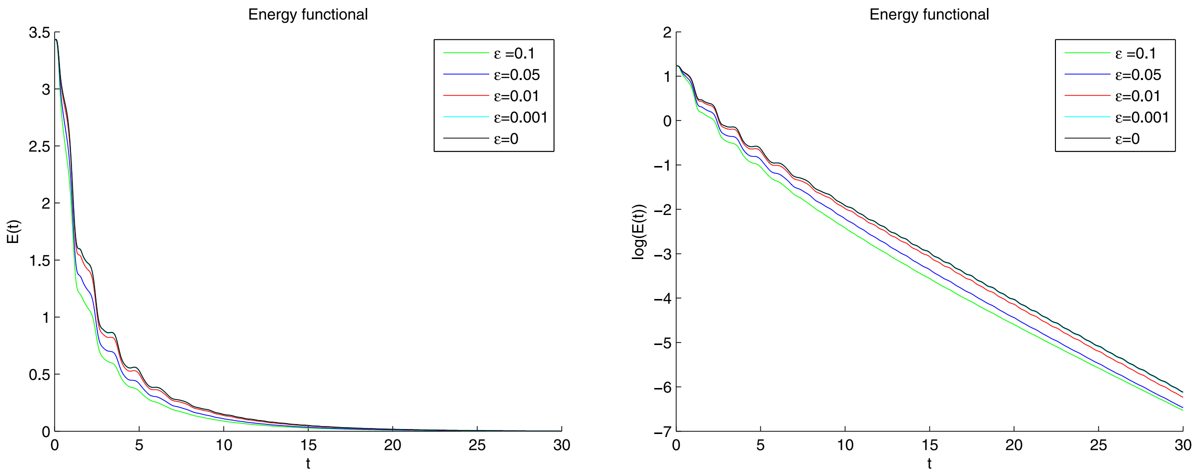

Fig. 2. Example 2: Evolution in time of the discrete energy (natural and semi-log scales) for different values of ϵ.

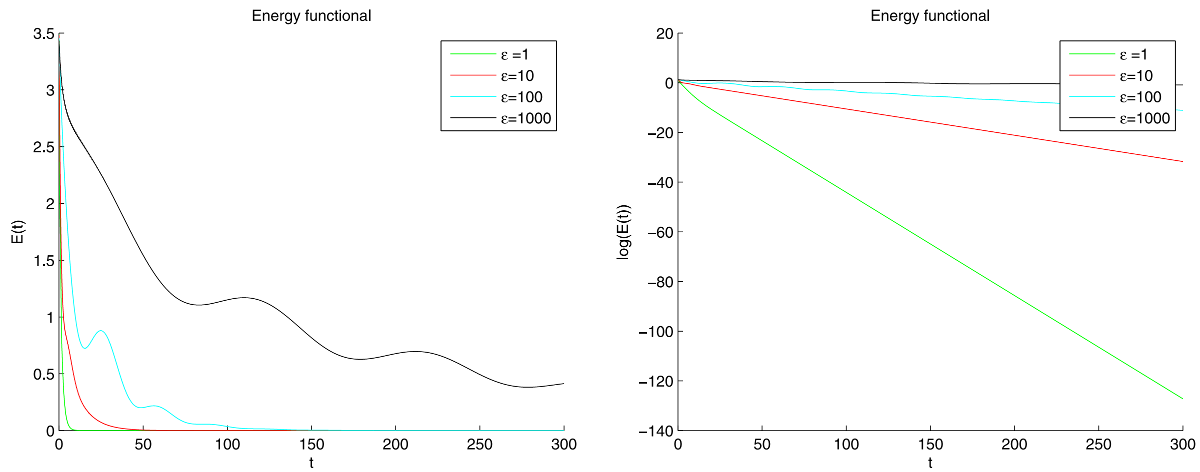

Fig. 3. Example 3: Evolution in time of the discrete energy (natural and semi-log scales) for large values of ϵ.

is plotted in Fig. 2 (in both natural and semi-log scales) for different values of ϵ ($\epsilon = 0.1, 0.05, 0.01, 0.001, 0$). As can be seen, it converges to zero and an exponential decay seems to be achieved. Moreover, we also observe that the energy converges to the energy obtained with parameter $\epsilon = 0$.

Finally, our aim is to investigate the behavior of the solution when parameter ϵ becomes large. Although the most interesting case is for values close to zero, here we study the influence of this regularized term. So, we use the same data than in the previous case, varying parameter $\epsilon = 1, 10, 100, 1000$ with large values. Therefore, taking parameter $k = 10^{-6}$, in Fig. 3 we show the discrete energy in both natural and semi-log scales. Again, we can see that, if ϵ is less than 10, then an exponential energy decay is clearly achieved. However, for values $\epsilon = 100, 1000$ an oscillating behavior is found for the first part of the time interval. Even, for the largest value of the parameter, the energy asymptotic behavior is not observed yet. One possible explanation for this issue, which has been also found in other problems, is due to the fact that the dissipation mechanism becomes too rigid and so, the dissipation is drastically reduced.

5.2. A two-dimensional example

As a two-dimensional example, our aim is to show the evolution in time of the deformation when some initial conditions are prescribed only for the displacements and velocity. Therefore, let us consider the domain $\Omega = (0, 1)\times(0, 1)$. We assume homogeneous null boundary conditions for all the variables and we use the initial conditions:

$$\boldsymbol{u}(x, y) = \boldsymbol{v}(x, y) = (x(x-1)y(y-1), x(x-1)y(y-1)),$$
$$\alpha^0 = \theta^0 = \zeta^0 = \phi^0 = 0.$$

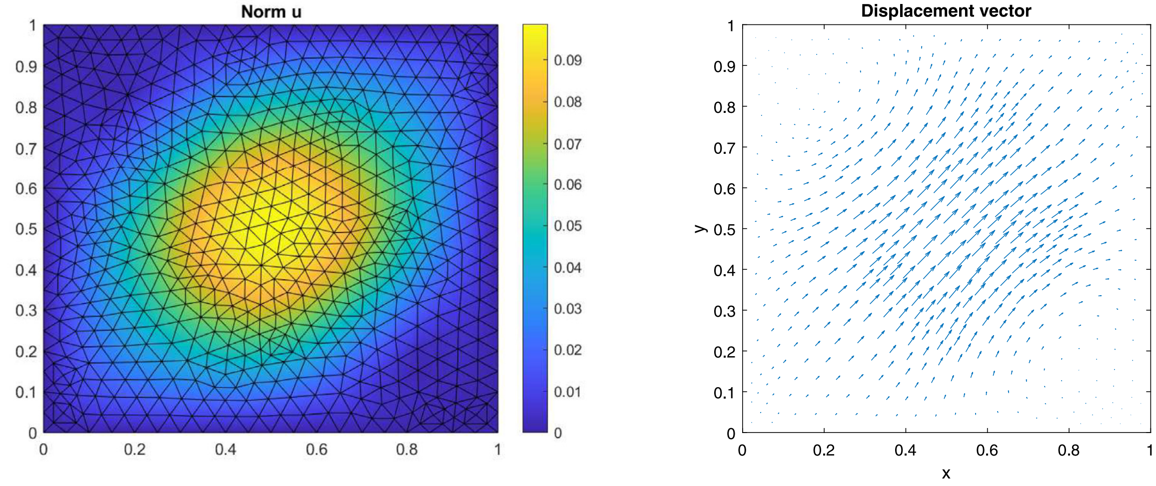

Fig. 4. Example 4: Norm of the displacements and arrows at final time.

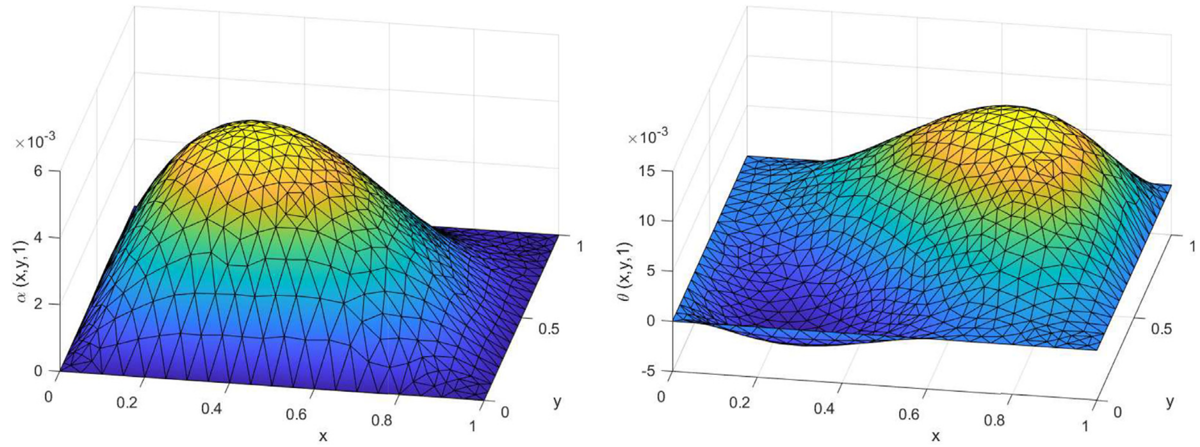

Fig. 5. Example 4: Thermal displacement and temperature at final time.

Moreover, we employ the following data in the simulations:

$$\rho = 1, \quad \mu^* = 1, \quad \lambda^* = 1, \quad \kappa^* = 1, \quad \gamma^* = 2, \quad \tau = 1,$$
$$\kappa = 5, \quad \gamma = 5, \quad \beta^* = 2, \quad A^* = 2, Q^* = 1, \quad Q = 1, \quad \epsilon = 1.$$

Taking the time discretization parameter $k = 10^{-3}$, in Fig. 4 we plot the displacements in norm (left) and arrows (right) at final time. We can observe the influence of the boundary condition and a certain orientation of the displacements.

Now, in Fig. 5 we plot the thermal displacement and the temperature at final time. Those are generated by the deformation of the body. We can observe that the thermal displacement has a quadratic shape due to the clamping conditions. However, the temperature has a clear oscillation due to the orientation of the deformation.

Finally, in Fig. 6 the electric potential is shown at final time. Again, it has been produced by the deformation of the body and, as in the case of the thermal displacements, a quadratic shape is found.

Acknowledgments

This paper is part of the projects PGC2018-096696-B-I00 and PID2019-105118GB-I00, funded by the Spanish Ministry of Science, Innovation and Universities and FEDER "A way to make Europe". Funding for open access charge: Universidade de Vigo/CISUG.

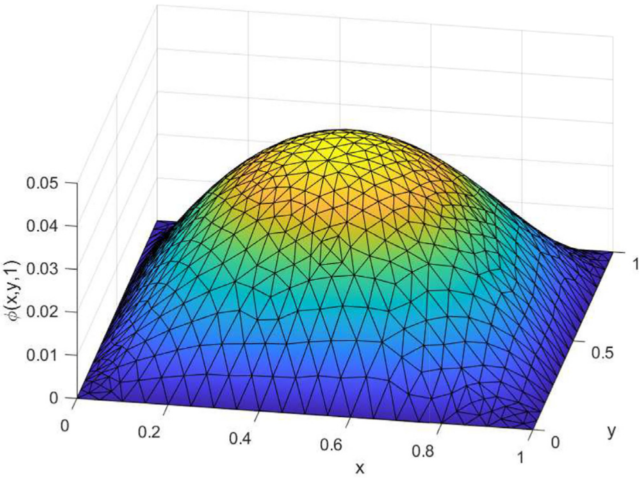

Fig. 6. Example 4: Electric potential at final time.

References

[1] A.E. Abouelregal, I.-E. Ahmed, M.E. Nasr, K.M. Khalil, A. Zakria, F.A. Mohammed, Thermoelastic processes by a continuous heat source line in an infinite solid via Moore–Gibson–Thompson thermoelasticity, Materials 13 (2020) 4463.
[2] A.E. Abouelregal, H.M. Sedighi, The effect of variable properties and rotation in a visco-thermoelastic orthotropic annular cylinder under the Moore–Gibson–Thompson heat conduction model, Proc. Inst. Mech. Eng. 235 (2021) 1004–1020.
[3] N. Bazarra, J.R. Fernández, A. Magaña, R. Quintanilla, A poro-thermoelastic problem with dissipative heat conduction, J. Thermal Stresses 43 (2020) 1415–1436.
[4] N. Bazarra, J.R. Fernández, A. Magaña, R. Quintanilla, Time decay for several porous thermoviscoelastic systems of MGT-type, Asymptotic Anal. (2021) in press.
[5] N. Bazarra, J.R. Fernández, R. Quintanilla, Analysis of a Moore–Gibson–Thompson thermoelastic problem, J. Comput. Appl. Math. 382 (2021) 113058.
[6] M. Conti, V. Pata, R. Quintanilla, Thermoelasticity of Moore–Gibson–Thompson type with history dependence in temperature, Asymptotic Anal. 120 (2020) 1–21.
[7] M. Conti, V. Pata, M. Pellicer, R. Quintanilla, On the analyticity of the MGT-viscoelastic plate with heat conduction, J. Differ. Equ. 269 (2020) 7862–7880.
[8] M. Conti, V. Pata, M. Pellicer, R. Quintanilla, A new approach to MGT-thermeolasticity, Discrete Cont. Dyn. Syst. 41 (2021) 4645–4666.
[9] K. Jangid, S. Mukhopadhyay, A domain of influence theorem under MGT thermoelasticity theory, Math. Mech. Solids 26 (2021) 285–295.
[10] K. Jangid, S. Mukhopadhyay, A domain of influence theorem for a natural stress-heat-flux problem in the MGT thermoelasticity theory, Acta Mech. 232 (2021) 177–187.
[11] H. Kumar, S. Mukhopadhyay, Thermoelastic damping analysis in microbeam resonators based on Moore–Gibson–Thompson generalized thermoelasticity theory, Acta Mech. 231 (2020) 3003–3015.
[12] M. Pellicer, R. Quintanilla, On uniqueness and instability for some thermomechanical problems involving the Moore–Gibson–Thompson equation, ZAMP-Z. Angew. Math. Phys. 71 (2020) 84.
[13] R. Quintanilla, Moore–Gibson–Thompson thermoelasticity, Math. Mech. Solids 24 (2019) 4020–4031.
[14] R. Quintanilla, Moore–Gibson–Thompson thermoelasticity with two temperatures, Appl. Engng. Sci. 1 (2020) 100006.
[15] B. Sing. S. Mukhopadhyay, Galerkin-type solution for the Moore–Gibson–Thompson thermoelasticity theory, Acta Mech. 232 (2021) 1273–1283.
[16] M.E. Gurtin, Time-reversal and symmetry in the thermodynamics of materials with memory, Arch. Ration. Mech. Anal. 44 (1972) 387–399.
[17] R. Borghesani, A. Morro, Relaxation functions and time-reversal invariance in thermal and electric conduction, Ann. Mat. Pare Appl. 114 (1977) 271–288.
[18] R. Borghesani, A. Morro, Time-reversal invariance in thermodynamics of electromagnetic fields in materials with memory, Ann. Mat. Pare Appl. 99 (1974) 65–80.
[19] M. Ciarletta, D. Ieşan, A theory of thermoviscoelastic dielectrics, J. Thermal Stresses 14 (1991) 589–606.
[20] J.R. Fernández, R. Quintanilla, Moore–Gibson–Thompson theory for thermoelastic dielectrics, Appl. Math. Mech. 42 (2021) 309–316.
[21] P.G. Ciarlet, Basic error estimates for elliptic problems, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, vol II, 1993, pp. 17–351.
[22] M. Campo, J.R. Fernández, K.L. Kuttler, M. Shillor, J.M. Viaño, Numerical analysis and simulations of a dynamic frictionless contact problem with damage, Comput. Methods Appl. Mech. Engrg. 196 (1–3) (2006) 476–488.