Mechanics Research Communications 124 (2022) 103943

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

On the instability for an incremental problem in elastodynamics

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ARTICLE INFO

Keywords:
Incremental problem
Elasticity
Uniqueness
Instability
Logarithmic convexity
Numerical example

ABSTRACT

In this short note, we consider some issues regarding the instability of some elastodynamical problems when the elasticity tensor is not positive definite. By using the so-called logarithmic convexity argument, we prove the instability of solutions when the time derivative of the elasticity tensor is semi-definite negative or it satisfies another restriction on the coefficients. The uniqueness of the solution is also concluded. Finally, a simple one-dimensional example is provided to demonstrate the numerical behaviour of the instability.

1. Introduction

In recent years, few results have been established in the theory of small deformations superimposed on large deformations of elastic materials. It is suspected that this is due to the fact that it leads to a difficult to deal with and, for this reason, it is appropriate to consider the approximate problem [1]. We recall that the first contributions in this line were provided by Green [2,3], Knops and Wilkes [4], and, more recently, by Ieşan [5]. An existence result was obtained by Navarro and Quintanilla [6] (see also the work of Quintanilla and Williams [7] for the viscoelastodynamics). A couple of considerations should be given in the case of the isothermal elasticity. The theory of incremental elasticity has given impetus to theoretical research into equations of elastic bodies for which little or no information is known concerning elasticities [4]. The case when the elasticity tensor is not positive definite but time-independent has been deeply studied by Knops [8,9], but the problem when it also depends on the time has not received too much attention.

The analysis provided by Knops to obtain the instability of solutions uses the so-called logarithmic convexity argument. However, to our knowledge the extension of this argument to the case when the elasticity tensor also depends on the time has not been considered yet. In this short note, we will give a couple of situations where the classical argument of logarithmic convexity can be adapted. Therefore, we will provide sufficient conditions to guarantee the exponential growth of the solutions to the problem of the small deformations, superimposed on a large deformation in the case that the primary state depends on the time. In fact, our arguments are strongly based on the analysis for the case when the elasticity tensor is independent of the time, but we adapt the method to some cases when it also depends on the time.

2. Basic equations

Let B be a bounded domain in $\mathbb{R}^3$ with a boundary smooth enough to apply the divergence theorem. As we want to study the incremental problem of isothermal elasticity in the case that the primary state is not at equilibrium, we consider the system

$$\rho \ddot{u}_i(\boldsymbol{x},t) = \left(C_{ijkl}(\boldsymbol{x},t) u_{k,l}(\boldsymbol{x},t) \right)_{,j} \quad \boldsymbol{x} \in B, \quad t \geq 0, \tag{1}$$

where ρ is the mass density, u_i is the displacement vector and C_{ijkl} is the elasticity tensor which depends on the material point and the time. As usual, we assume that the elasticity tensor satisfies the symmetry property:

$$C_{ijkl} = C_{klij}. \tag{2}$$

We want to study the problem determined by the system (1) with the initial conditions:

$$u_i(\boldsymbol{x},0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x},0) = v_i^0(\boldsymbol{x}) \quad \boldsymbol{x} \in B, \tag{3}$$

and homogeneous Dirichlet boundary conditions:

$$u_i(\boldsymbol{x},t) = 0 \quad t > 0, \quad \boldsymbol{x} \in \partial B. \tag{4}$$

We are going to prove the instability of the solutions under suitable conditions on the constitutive functions.

We assume that

(i) $\rho \geq \rho_0 > 0$.

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https://doi.org/10.1016/j.mechrescom.2022.103943
Received 23 June 2022; Received in revised form 16 July 2022; Accepted 18 July 2022
Available online 21 July 2022
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(ii) The time derivative of the elasticity tensor is semi-definite negative; that is,

$$\dot{C}_{ijkl}\xi_{ij}\xi_{kl} \leq 0$$

for every tensor ξ_{ij}.

(iii) There exists a positive constant $k \geq 0$ such that

$$(C_{ijkl} - k\dot{C}_{ijkl})\xi_{ij}\xi_{kl} \geq 0$$

for every tensor ξ_{ij}.

We are going to prove the instability of the solutions under either assumptions (i)–(ii) or either (i)–(iii).

The meaning of assumption (i) is clear. Condition (ii) states that the elasticity tensor is stronger with the time. Assumption (iii) is a technical condition which can be understood as a restriction on the class of the elasticity tensors. However, we will give some examples.

It is known that, if we assume that the elasticity tensor is not positive definite, we can obtain the instability of solutions in the case that C_{ijkl} does not depend on time. Hence, our aim here is to extend this kind of results when the elasticity tensor also depends on the time.

In order to find possible applications of this analysis, it will useful to propose examples on the elasticity tensor satisfying conditions (ii) or (iii).

We note that the tensors of the form

$$C_{ijkl}(\boldsymbol{x},t) = C_{ijkl}(\boldsymbol{x},0)e^{\alpha t}, \tag{5}$$

where $\alpha \geq 0$ and $C_{ijkl}(\boldsymbol{x},0)$ is a negative semi-definite tensor, satisfy conditions (ii) or (iii). In general, if we assume that $C_{ijkl}(\boldsymbol{x},t) = C_{ijkl}(\boldsymbol{x},0)f(t)$, where $f(t)$ is a non-decreasing function such that $f(0) \geq 0$ and $C_{ijkl}(\boldsymbol{x},0)$ is a negative semi-definite tensor, condition (ii) holds. However, it is worth noting that we can define some other tensors satisfying condition (iii) even if the tensor $C_{ijkl}(\boldsymbol{x},0)$ is not negative semi-definite. In fact, all the tensors of the form (5) always satisfy condition (iii).

3. Instability result

In this section, we prove a result of exponential instability in the cases (i)–(ii) or (i) and (iii) for the solutions to the problem (1)–(4).

We first give the proof in the case (i)–(ii), which is easier, and later we will provide suitable arguments to study the other case.

The analysis needs the energy equation

$$\begin{aligned} E(t) = \frac{1}{2}\int_B \left(\rho\dot{u}_i\dot{u}_i + C_{ijkl}u_{i,j}u_{k,l}\right) dv \\ -\frac{1}{2}\int_0^t \int_B \dot{C}_{ijkl}u_{i,j}u_{k,l}\, dv ds = E(0). \end{aligned} \tag{6}$$

Since we assume that condition (ii) holds, we obtain

$$E_1(t) \leq E_1(0) = E(0), \tag{7}$$

where

$$E_1(t) = \frac{1}{2}\int_B \left(\rho\dot{u}_i\dot{u}_i + C_{ijkl}u_{i,j}u_{k,l}\right) dv.$$

We are going to prove our result by means of the logarithmic convexity argument. In fact, we will follow similar arguments to the ones proposed in [9]. It is known that this method is strongly based on a suitable selection of the function to evaluate. In our case, we define the function

$$F_{\omega,t_0}(t) = \frac{1}{2}\int_B \rho u_i u_i\, dv + \omega(t+t_0)^2, \quad t \geq 0,$$

where ω and t_0 are two positive constants to be chosen later.

A direct differentiation gives

$$\dot{F}_{\omega,t_0}(t) = \int_B \rho u_i \dot{u}_i\, dv + 2\omega(t+t_0), \quad t \geq 0,$$

and

$$\ddot{F}_{\omega,t_0}(t) = \int_B (\rho\dot{u}_i\dot{u}_i + \rho u_i\ddot{u}_i)\, dv + 2\omega, \quad t \geq 0.$$

In view of system (1) we see that

$$\ddot{F}_{\omega,t_0}(t) = \int_B (\rho\dot{u}_i\dot{u}_i - C_{ijkl}u_{i,j}u_{k,l})\, dv + 2\omega, \quad t \geq 0,$$

and, after the use of the energy inequality (7), we obtain

$$\ddot{F}_{\omega,t_0}(t) \geq 2\int_B \rho\dot{u}_i\dot{u}_i\, dv + 2(\omega - E(0)), \quad t \geq 0.$$

It then follows that

$$\ddot{F}_{\omega,t_0}(t)F_{\omega,t_0}(t) - (\dot{F}_{\omega,t_0}(t))^2 \geq 2(\omega + E(0))F_{\omega,t_0}(t), \quad t \geq 0.$$

In the case that $E(0) < 0$, we can choose $\omega = -E(0)$ to obtain that

$$\frac{d^2}{dt^2} ln F_{\omega,t_0}(t) \geq 0, \quad t \geq 0. \tag{8}$$

This inequality implies that

$$F_{\omega,t_0}(t) \geq F_{\omega,t_0}(0)\exp\frac{\dot{F}_{\omega,t_0}(0)}{F_{\omega,t_0}(0)}t,$$

which is an estimate of exponential instability. We note that we can always select t_0 large enough to guarantee that $F_{\omega,t_0}(0) > 0$. Moreover, in the case that $E(0) = 0$ and $\dot{F}_{0,0}(0) > 0$ we can also obtain the exponential growth.

It is also worth noting that, in the case that we will assume null initial conditions, we obtain

$$\frac{d^2}{dt^2} ln F_{0,0}(t) \geq 0, \quad t \geq 0.$$

It is also known that

$$F_{0,0}(t) \leq F_{0,0}(0)^{1-t/T}F_{0,0}(T)^{t/T}, \quad 0 \leq t \leq T.$$

Therefore, $F_{0,0}(t) = 0$ for $0 \leq t \leq T$. It then follows the uniqueness of solutions under conditions (i) and (ii).

In the remain of this section, we prove the exponential instability and the uniqueness of the solutions to problem (1)–(4) under conditions (i) and (iii). It is worth noting that the key point to prove it is to show the inequality (7). In view of the equality (6) it will be enough to show that

$$G(t) = \int_0^t \int_B \dot{C}_{ijkl}u_{i,j}u_{k,l}\, dv ds \leq 0 \tag{9}$$

for every solution such that $E(0) \leq 0$.

Thanks to inequality (6) we see that

$$\int_B C_{ijkl}u_{i,j}u_{k,l}\, dv - \int_0^t \int_B \dot{C}_{ijkl}u_{i,j}u_{k,l}\, dv ds \leq 0,$$

and, using condition (iii), we have

$$k\int_B \dot{C}_{ijkl}u_{i,j}u_{k,l}\, dv - \int_0^t \int_B \dot{C}_{ijkl}u_{i,j}u_{k,l}\, dv ds \leq 0.$$

We can write this inequality in the form

$$k\dot{G}(t) - G(t) \leq 0,$$

which implies that

$$G(t) \leq G(0)e^{-k^{-1}t}.$$

Since $G(0) = 0$, we obtain the inequality (9). Therefore, we can adapt the previous analysis to this new case to obtain the exponential instability and the uniqueness results proposed before.

Remark 1. This uniqueness result can be adapted to include also elasticity tensors of the form $C_{ijkl}(\boldsymbol{x},t) = C_{ijkl}(\boldsymbol{x},0)f(t)$, where $C_{ijkl}(\boldsymbol{x},0)$ is a negative semi-definite tensor, and $f(t)$ is the solution to the ordinary differential equation:

$$\dot{f} = f^{\alpha},$$

for a given $\alpha > 1$. However, in this case we note that this function f tends to infinity at a finite time.

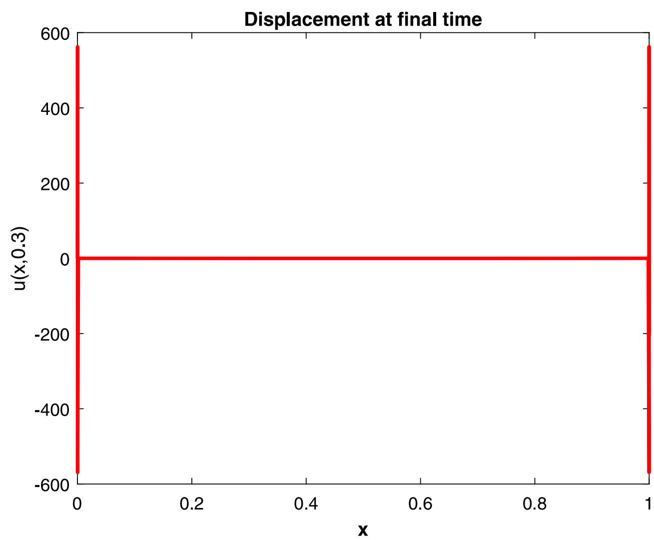

Fig. 1. Displacement at final time.

4. A numerical example

In this final section, our aim is to perform some numerical simulations to show the theoretical behaviour obtained in the previous sections. For the sake of simplicity, we restrict ourselves to the one-dimensional case, and so we have to study the following problem:

$$\rho\ddot{u} + e^t C u_{xx} = 0 \quad \text{in} \quad (0, L) \times (0, T),$$
$$u(0, t) = u(L, t) = 0 \quad \text{for a.e. } t \in (0, T),$$
$$u(x, 0) = u^0(x), \quad \dot{u}(x, 0) = v^0(x) \quad \text{for a.e. } x \in (0, L).$$

Here, L represents the length of the bar, $(0, T)$, $T > 0$, denotes the time interval, and C is a positive constant which represents the spatial part of the elasticity coefficient. As we can see, the elastic coefficient is $C_{1111} = -e^t C$ and so condition (ii) is met.

In the simulations, we have used the following data:

$$\rho = 100, \quad L = 1, \quad T = 0.3, \quad C = 10^{-4}$$

and the initial conditions:

$$v^0(x) = 0, \quad u^0(x) = x(x-1) \quad \text{for a.e. } x \in (0, 1).$$

We note that, with the prescribed initial conditions, it is easy to conclude that $E(0) < 0$.

We will not provide details regarding the numerical approximation of this problem since it is quite standard. We only note that we have used continuous and piecewise affine finite elements for the spatial approximation, and the implicit Euler scheme for the time discretization of the first-order time derivatives (we write the problem in terms of the velocity field).

Therefore, taking the spatial discretization parameter $h = 10^{-5}$ and the time discretization parameter $k = 0.3 \times 10^{-4}$, in Fig. 1 we plot the displacement field at final time. As we can see, the solution has increased drastically at that time, with big oscillations near the corners.

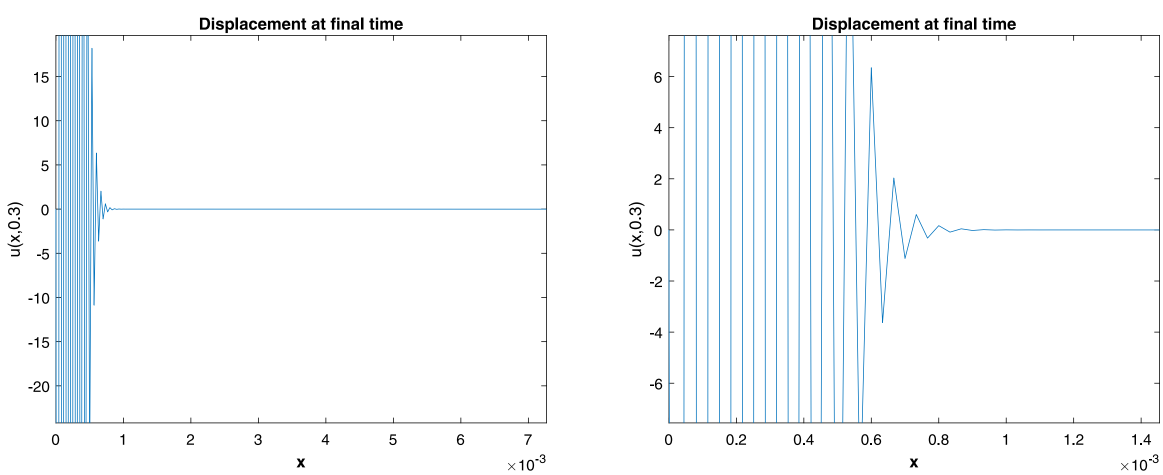

Fig. 2. Displacements at final time (zooms near the left corner $x = 0$).

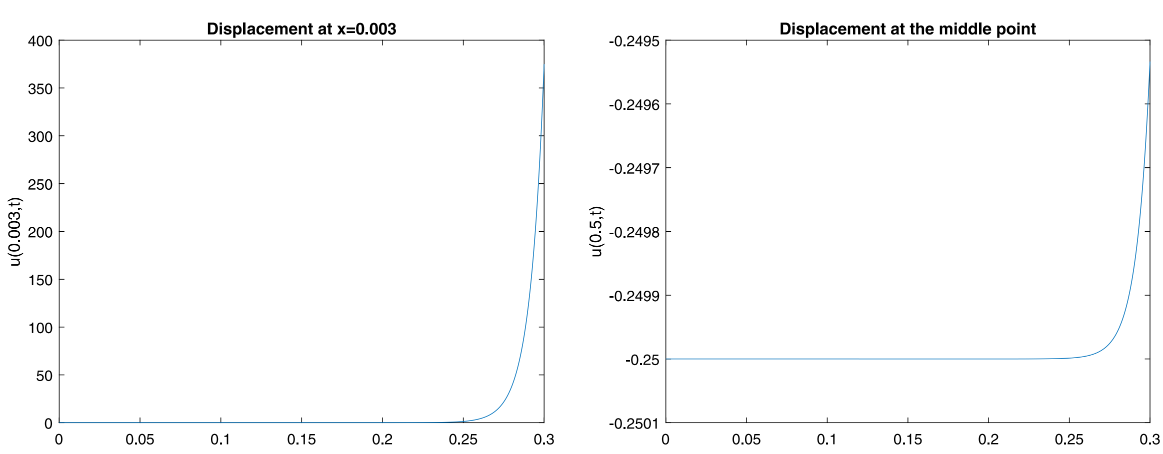

Fig. 3. Evolution in time of the displacements of points $x = 0.003$ (left) and $x = 0.5$ (right).

This is possibly due to the negative value of the elastic coefficient, which leads to the concentration of the deformation far away from the point where the maximum initial displacement is applied (the middle point of the bar).

Moreover, in order to better observe the oscillations of the deformation, we show two zooms near the left corner $x = 0$. In Fig. 2 we can see the results obtained. We can clearly appreciate how the oscillations are really high and we note that, for the sake of clarity, we also performed this example with a reduced mesh size, although a similar oscillating behaviour was also found.

Finally, in Fig. 3 the evolution in time of the displacements at points $x = 0.003$ and $x = 0.5$ (the middle point of the bar) are shown. We can see how the deformation at point $x = 0.003$ changes very slowly until time $t = 0.25$, where it begins to increase quickly (in fact, at time $t = 0.9$ it reaches value 10^{230}). However, the displacement at the middle point is almost constant and equal to the initial value -0.25 even if a small change is also produced again at time $t = 0.25$.

5. Conclusions

In this short note, we considered an incremental problem arising in elastodynamics. The main difficulty of this study was the assumption that the elasticity tensor was not positive definite. Then, by using logarithmic convexity arguments, we proved that this kind of problems were unstable for the cases where the tensor was semi-definite negative or satisfying a restriction on its coefficients. The uniqueness of the solution for both cases was also proved. Finally, we presented a numerical example involving a simple one-dimensional problem to demonstrate numerically the unstable behaviour of the solution.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors would like to express their gratitude to the anonymous referee, whose comments improved the quality of the paper.

This paper is part of the project PID2019-105118GB-I00, funded by the Spanish Ministry of Science, Innovation and Universities and FEDER “A way to make Europe”.

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