ICT, ELECTRONICS, PROGRAMMING

UDC 519.854

ALGEBRAIC APPROACH TO THE SOLUTION OF RENUMERALS OF COMBINATORY TASKS

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Abstract – In the article available means showed the way of the solution of combinatory tasks on transfer based on use of internal symmetry of the body given in a condition that, in turn, reflects close connection between objects of research of the theory of final groups and the combinatory analysis. A key step of the solution of similar tasks – heuristic finding of shifts.

The theory of enumerations developed by Polya and other mathematicians is widely used in quantum physics, equipment, cybernetics, organic chemistry, biology. One of basic provisions of this theory is Burnside's lemma. That to formulate it, we will define concept of an orbit of permutation group.

Let $G = \{\alpha_0 = \varepsilon, \alpha_1, ..., \alpha_{k-1}\}$ – permutation group on a set $M = \{1, 2, ..., n\}$. The subset $O \subset M$ is called as *an orbit of group* G, if:

1) action of permutations of G on the elements of O does not bring them out of O;
2) any two elements of O can be transformed into each other by some permutation of G.

Burnside's lemma [1]. *For any permutation group performed the equality*

$$t(G) = \frac{1}{|G|}\sum_{\alpha\in G}\chi(\alpha),$$

where $\chi(\alpha)$ *– number of fixed points of permutations,* $t(G)$ *– number of orbits of permutation* G*, acting on the set* M*.*

It's easy to prove that every permutation group has the orbit. It's clear that any two orbits of the group are either disjoint or coincide. It follows that the set M is the union of disjoint subsets – orbits of the group G. In connection with the division M into the orbit of the permutation group G, two questions arise.

1) How many orbits has the group G on the set M?

2) What is the length of each of these orbits?

The answer to the second question can be found using Lagrange's theorem. The answer to the first question gives Burnside's lemma.

Consider the problem of the number of ways that you can paint the top of the cube in four colors.

Discuss the condition and solution of this problem. One vertex can be painted in four ways. For other vertex coloring are all the same four ways that don't depend on the color of the remaining vertices. According to the rule works have

$$\underbrace{4\cdot 4\cdot\ldots\cdot 4}_{8\ \text{раз}} = 4^8 = 65536$$

ways of coloring the vertices of a cube in four colors. So it is possible to solve the problem, if we have all the vertices are different, that is, the cube is fixed in space and its vertices are numbered.

If the cube can freely rotate, some ways of a coloring become isomorphic each other, that is at turns of a cube coincide (fig. 1). It is clear that when calculating ways for coloring the vertices are isomorphic options need to be exclude.

Turns of a cube can be carried out round its axes of symmetry. The set of turns at which the cube itself is combined, forms group of symmetry under multiplication of turns. Turns are described by permutations. For example, the rotation of the cube in fig. 1 corresponds to a permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 4 & 8 & 6 & 2 & 3 & 7 \end{pmatrix} = (1\,5\,6\,2)(3\,4\,8\,7).$$

Orbit of vertex will call a sequence of vertices obtained repeated performance of the same turning. *A cycle* is called the smallest sequence of vertices orbit that maps initial vertex into itself. For example: $(1\,5\,6\,2)$ – the cycle of vertex 1 at turn of a cube on a corner $\pi/2$ round an axis O_1O_2 (fig. 1).

If as a result of turn color of vertex hasn't changed, it's called *a fixed point* of rotation. Otherwise – *moving point*. To apply Burnside's lemma to the solution of a task, you need to determine the number of fixed points of each permutation.

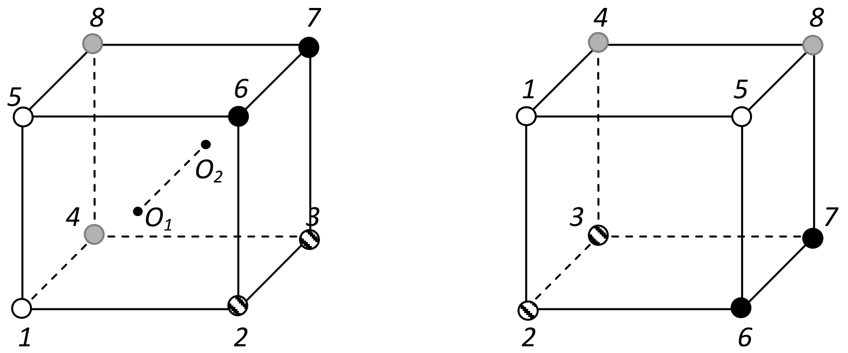

Fig. 1. Rotate the cube on a corner $\frac{\pi}{2}$ around the symmetry axis connecting the O_1 and O_2 centers of the opposite faces

For example, concerning permutation $\alpha = (1\,2\,3\,4)(5\,6\,7\,8)$, a coloring of vertices from one color and a coloring of vertices from two colors, in one of which are colored the vertices of the first cycle in the other - the vertices of the second cycle, will be fixed points. Count all the possible options:

$$4 + A_4^2 = 16 = 4^2 .$$

Arguing similarly, we conclude that the number of fixed points of permutations α equal 4^k, where k – *type permutation* (sequence of lengths of cycles of decomposition of permutation in product of cycles).

Let the cubes of the same size, which are fixed in space, having different coloring of the vertices form a set M. Then $|M| = 4^8$.

We show that the group G of rotations of the cube consists of 24 permutations. We number the vertices of the cube (fig. 2).

In table 1 will write down permutations corresponding to rotations around the cube axes connecting opposite vertices (fig. 2).

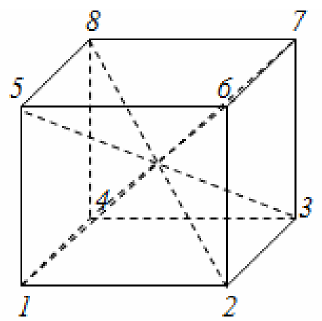

Fig. 2. Cube with axes connecting opposite vertices

In table 2 we will place permutations corresponding to rotations the cube around axes a, b, c, connecting the centers of opposite faces (fig. 3).

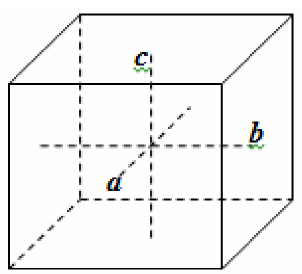

Fig. 3. Cube with axes connecting the centers of opposite faces

Table 1

axes	angle	permutation	decomposition into product of cycles	type permutation
1-7	$\frac{2\pi}{3}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\1&5&6&2&4&8&7&3\end{pmatrix}$	(2 5 4)(3 6 8)(1)(7)	$<3,3,1,1>$
1-7	$\frac{4\pi}{3}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\1&4&8&5&2&3&7&6\end{pmatrix}$	(2 4 5)(3 8 6)(1)(7)	$<3,3,1,1>$
2-8	$\frac{2\pi}{3}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\6&2&1&5&7&3&4&8\end{pmatrix}$	(1 6 3)(4 5 7)(2)(8)	$<3,3,1,1>$
2-8	$\frac{4\pi}{3}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\3&2&6&7&4&1&5&8\end{pmatrix}$	(1 3 6)(4 7 5)(2)(8)	$<3,3,1,1>$
3-5	$\frac{2\pi}{3}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\6&7&3&2&5&8&4&1\end{pmatrix}$	(1 6 8)(2 7 4)(3)(5)	$<3,3,1,1>$
3-5	$\frac{4\pi}{3}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\8&4&3&7&5&6&2&6\end{pmatrix}$	(1 8 6)(2 4 7)(3)(5)	$<3,3,1,1>$
4-6	$\frac{2\pi}{3}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\8&5&1&4&7&6&2&3\end{pmatrix}$	(1 8 3)(2 5 7)(4)(6)	$<3,3,1,1>$
4-6	$\frac{4\pi}{3}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\3&7&8&4&2&6&5&1\end{pmatrix}$	(1 3 8)(2 7 5)(4)(6)	$<3,3,1,1>$

Table 2

axes	angle	permutation	decomposition into product of cycles	type permutation
a	$\frac{\pi}{2}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\5&1&4&8&6&2&3&7\end{pmatrix}$	(1 5 6 2)(3 4 8 7)	$<4,4>$
a	π	$\begin{pmatrix}1&2&3&4&5&6&7&8\\6&5&8&7&2&1&4&3\end{pmatrix}$	(1 6) (2 5) (3 8) (4 7)	$<2,2,2,2>$
a	$\frac{3\pi}{2}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\2&6&7&3&1&5&8&4\end{pmatrix}$	(1 2 6 5)(3 7 8 4)	$<4,4>$
b	$\frac{\pi}{2}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\5&6&2&1&8&7&3&4\end{pmatrix}$	(1 5 8 4)(2 6 7 3)	$<4,4>$
b	π	$\begin{pmatrix}1&2&3&4&5&6&7&8\\8&7&6&5&4&3&2&1\end{pmatrix}$	(1 8) (2 7) (3 6) (4 5)	$<2,2,2,2>$
b	$\frac{3\pi}{2}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\4&3&7&8&1&2&6&5\end{pmatrix}$	(1 4 8 5)(2 3 7 6)	$<4,4>$
c	$\frac{\pi}{2}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\4&1&2&3&8&5&6&7\end{pmatrix}$	(1 4 3 2)(5 8 7 6)	$<4,4>$
c	π	$\begin{pmatrix}1&2&3&4&5&6&7&8\\3&4&1&2&7&8&5&6\end{pmatrix}$	(1 3) (2 4) (5 7) (6 8)	$<2,2,2,2>$
c	$\frac{3\pi}{2}$	$\begin{pmatrix}1&2&3&4&5&6&7&8\\2&3&4&1&6&7&8&5\end{pmatrix}$	(1 2 3 4)(5 6 7 8)	$<4,4>$

Table 3 contains a permutation of the cube corresponds to a rotation around axes l_1, l_2, l_3, l_4, l_5, l_6, connecting the midpoints of opposite ribs (fig. 4).

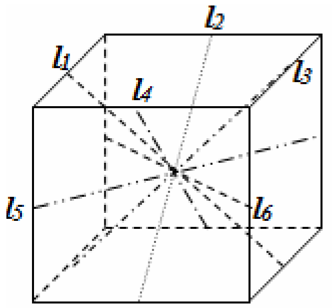

Fig. 4. Cube with axes connecting the midpoints of opposite edges

Table 3

axes	angle	permutation	decomposition into product of cycles	type permutation
l_3	π	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 8 & 5 & 1 & 3 & 7 & 6 & 2 \end{pmatrix}$	(1 4) (2 8) (3 5) (6 7)	< 2, 2, 2, 2 >
l_2	π	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 4 & 3 & 6 & 5 & 1 & 2 \end{pmatrix}$	(1 7) (2 8) (3 4) (5 6)	< 2, 2, 2, 2 >
l_1	π	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 3 & 2 & 6 & 8 & 4 & 1 & 5 \end{pmatrix}$	(1 7) (2 3) (4 6) (5 8)	< 2, 2, 2, 2 >
l_4	π	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 5 & 6 & 3 & 4 & 8 & 7 \end{pmatrix}$	(1 2) (3 5) (4 6) (7 8)	< 2, 2, 2, 2 >
l_5	π	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 7 & 6 & 1 & 4 & 3 & 2 \end{pmatrix}$	(1 5) (2 8) (3 7) (4 6)	< 2, 2, 2, 2 >
l_6	π	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 8 & 3 & 2 & 1 & 4 \end{pmatrix}$	(1 7) (2 6) (3 5) (4 8)	< 2, 2, 2, 2 >

In Table 4we define thenumber of fixed pointsfor each type ofpermutations.

Table 4

typepermutation	the number of permutations of this type	number of fixed points
< 1, 1, 1, 1, 1, 1, 1, 1 >	1	4^8
< 2, 2, 2, 2 >	9	4^4
< 3, 3, 1, 1 >	8	4^4
< 4, 4 >	6	4^2

By Burnside's Lemma we obtain:

$$t(G) = \frac{1}{24}(4^8 + 9 \cdot 4^4 + 8 \cdot 4^4 + 6 \cdot 4^2) = 2916 .$$

Consequently, there is a 2916 different ways of coloring the vertices of a cube in four colors.

In article developed a method based on the use of Burnside's lemma, which allows the use of algebraic tools for solving problems on sets of a more general nature than the set of natural numbers, which are formulated and solved problems of classical combinatorics.

REFERENCES

Калужнин, Л.А. Преобразования и перестановки / Л.А. Калужнин, В.И. Сущанский ; пер. с укр. – М. : Наука, 1979. – 112 с.