Cooperation, allocation and strategy in interactive decision-making

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Proefschrift

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door

Johannes Schouten

geboren op 12 januari 1993 te Hengelo.

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Cooperation, allocation and strategy in interactive decision-making

Illustrations in chapter headings: Maaike Schouten, `www.artbymaaike.nl`.

This is the day that the Lord has made;
we will rejoice and be glad in it.

— *Psalm 118:24*

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After five years, my PhD journey comes to an end. It was a wonderful journey and a huge learning experience, in both research and teaching. It all started with a one-day decision to apply for the Research Master program in Operations Research on 13 April 2016, two weeks after the deadline. I would like to thank everyone who encouraged me and supported me to get on that train. And one thing is for sure: I could not have reached its final destination without the amazing support of many people in many different ways.

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I once read that it is best to talk about your family in your native language and so will I. *Papa, mama, Rianne, Maaike en Roos, dank jullie wel voor alle hulp, adviezen en aanmoedigingen, in het bijzonder in de afgelopen jaren, en voor alle gezellige momenten samen! Papa en mama, dank jullie wel voor de fijne thuishaven die Rijssen altijd is geweest. Maaike, dankjewel voor de mooie* string art *op mijn kantoor en de lijntekeningen in mijn proefschrift.*

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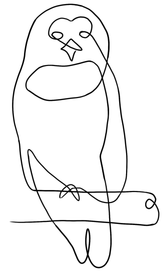
Jop Schouten
Beuningen, July 2022

Contents

1 Introduction



— *A wise old owl*

1.1 Game theory

Game theory is the mathematical theory to analyze the behavior of rational decision-makers in both cooperative and strategic interactive situations. It aims to resolve these situations by developing mathematical models and applying mathematical tools to provide insights in the interactive decision-making process. As a research field, it all started with the paper of Von Neumann (1928), followed by a more comprehensive book of Von Neumann and Morgenstern (1944). In the latter, both cooperative and strategic aspects in interactive decision-making are discussed.

Roughly, game theory can be divided into two subfields: cooperative game theory and non-cooperative game theory. Within *non-cooperative game theory*, the focus is on situations of conflict in which individuals, or players, independently have to make a strategic choice. Typically, the pay-offs to a single player result from the strategy combination of both the player itself and all the other players. Usually it is assumed that the players have complete information in the sense that all strategy combinations and the corresponding pay-offs are known beforehand. Non-cooperative game theory takes a global perspective and tries to answer the question which strategy combination will be a rational outcome.

The most established answer to this question is provided by the Nash equilibrium concept and is based on the general assumption that all players are (selfish) pay-off maximizing individuals. A Nash equilibrium (Nash, 1950, 1951) is a strategy combination of the players that is stable in the sense that no player unilaterally wants to deviate from the equilibrium strategy. More specifically, in a Nash equilibrium, every player maximizes his own pay-off by playing the equilibrium strategy, given the equilibrium strategy combination of the other players.

Cooperative game theory on the other hand is all about possible cooperation between the players. Generally, cooperation leads to joint revenues or joint cost savings. This poses a new type of question: how to fairly allocate these joint gains among the players? Answering this question adequately is essential to establish effective and stable cooperation. In order to settle this *interactive joint allocation problem*, the commonly used model of a transferable utility (TU) game is employed. In this model, the joint utilities of all subgroups are explicitly taken into account in finding an adequate allocation vector. This joint utility of a subgroup is called the worth and for example reflects the economic possibilities of the subgroup.

Regularly, allocation vectors are determined by a global perspective based on general principles for various classes of TU-games. For specific types of cooperative allocation problems however, a more tailor-made approach might be favorable. In particular, in *interactive operations research problems*, the allocation problem arises after first a joint optimization problem is solved and subsequently, a context specific approach of utilizing the structure of an optimal solution settles the joint allocation problem.

In this dissertation, both non-cooperative and cooperative game theory play a role. Furthermore, we deal with several interactive allocation problems, two of which we illustrate in Section 1.2 and Section 1.3 below. These two sections are representative for the two common approaches that are typical for this dissertation: a strategic approach and a cooperative approach. An overview of all topics discussed in this dissertation can be found in Section 1.4.

1.2 Claims problems

One of the simplest and most natural allocation problems is a claims problem. In a *claims problem*, a (monetary) estate has to be divided among several claimants, who each have a non-negative, justifiable (monetary) claim on the estate. One can think of a bankruptcy situation in which a firm goes bankrupt and leaves an estate to be divided among the creditors. A problem occurs if the estate is insufficient to cover all the claims. The natural question is then how to divide the estate among the claimants, hereafter called players. The following example illustrates such a claims problem and describes various solutions.

Example 1.1 Consider a bankruptcy situation where a bankrupt firm leaves an estate of 10 (e.g. millions of dollars) to be divided among two creditors. The first creditor, hereafter player 1, claims an amount of 8, while the second creditor, hereafter player 2, claims an amount of 4. Consequently, the claims sum up to 12, whereas there is only 10 available to be divided.

To solve this problem, one could reason that player 1 claims twice the amount of player 2 and hence, should therefore be awarded twice as much as player 2. In other words, player 1's claim constitutes $\frac{8}{8+4} = \frac{2}{3}$ of the total claims and is, by following this *proportionality* principle, thus awarded two-thirds of the estate: $\frac{2}{3} \cdot 10 = 6\frac{2}{3}$. On the other hand, player 2's claim constitutes one-third of the total claims and he is thus awarded one-third of the estate: $\frac{1}{3} \cdot 10 = 3\frac{1}{3}$.

Instead of the above-used proportionality principle, one could also use a *(constrained) equality* principle as the leading principle: each player receives half of the estate. However, in that case, player 2 would receive an amount of 5, which is more than his claim. To avoid this, we award player 2 his full claim and give the remainder of the estate to player 1. Player 1 thus receives 6, while player 2 receives 4.

Yet another way of dividing the estate among the players is to use the *concede and divide* principle. First, one could argue that player 1 only claims part of the estate, leaving an amount of $10 - 8 = 2$ unclaimed. This part is conceded to player 2. Similarly, player 2 concedes an amount of $10 - 4 = 6$ to player 1. Then there is still an amount of 2 left, claimed by both players. Since the two remaining claims are equal, it makes sense to divide this amount equally among the two players. In total, player 1 is thus awarded $6 + 1 = 7$ and player 2 gets $2 + 1 = 3$.

Table 1.1 provides a schematic overview of the problem as well as the discussed solutions. △

Estate: 10	player 1	player 2
Claim	8	4
Solutions		
Proportionality	$6\frac{2}{3}$	$3\frac{1}{3}$
Constrained equality	6	4
Concede and divide	7	3

Table 1.1 – *The claims problem and several solutions of Example 1.1.*

Example 1.1 already indicates that there is a wide variety of possible solutions for a general claims problem. Solutions to particular cases of a claims problem already occur in the 2000-year-old Babylonian Talmud. Studying these examples, O'Neill (1982) and Aumann and Maschler (1985) initiated a new line of research and formulated several general rules on how to tackle a claims problem. The most basic one is the one that is discussed first in Example 1.1 and is known as the *proportional rule*. For an arbitrary claims problem, it divides the estate proportionally to the claims.

Instead of proportionality, one could also focus on (constrained) equality as leading principle. This is reflected in the second solution of Example 1.1 and is known as the *constrained equal awards rule*. For an arbitrary claims problem, it divides the estate as equal as possible among the players, provided that each player does not receive more than his claim.

This reasoning can also be reversed by focusing on the losses rather than on the awards. In particular, for an arbitrary claims problem, the *constrained equal losses rule* divides the total losses, that is, the total claims minus the estate, as equal as possible among the players, provided that each player receives a non-negative amount. One can readily verify that in Example 1.1 this principle leads to the same outcome as the concede and divide principle.

In general however, the constrained equal losses rule might specify a different outcome than the *concede and divide rule*. The idea of first conceding the non-claimed parts of the estate and then dividing the possible remainder equally among the players, is only applicable for claims problems with two players. Interestingly, this idea is extended to claims problems with an arbitrary number of players by Aumann and Maschler (1985), leading to the famous *Talmud rule*. For a comprehensive overview of claims rules suggested in the literature, we refer to Thomson (2015).

In this dissertation, in Chapter 6 to be precise, we propose a new model that extends a standard claims problem and naturally leads to a strategic element. In this new model, called a *claims problem with estate holders*, we assume that the estate is separated in smaller parts, kept by several estate holders. In other words, there are multiple estate holders, each holding a different estate that has to be divided among the players. This forces the players to divide their claim over the estates, leading to a non-cooperative model. The following example illustrates this new model and explains the possible strategic decisions of the players.

Example 1.2 Reconsider the bankruptcy situation as described in Example 1.1, where the bankrupt firm leaves an amount of 10 to be divived among the two players, claiming 8 and 4, respectively. To introduce the claims problem with estate holders, suppose that the amount of 10 is separated over two different banks: bank A holding an amount of 4 and bank B holding an amount of 6:

$$\text{Total estate of } 10 \begin{cases} 4 \text{ at bank } A; \\ 6 \text{ at bank } B. \end{cases}$$

Furthermore, suppose that bank A uses the constrained equal awards rule to divide the estate, whereas bank B uses the proportional rule.

Players 1 and 2 now have to decide how to divide their claims over the two banks. For example, player 1 can choose to claim an amount of 5 at bank A and an amount of 3 at bank B:

$$\text{Player 1's claim of 8} \begin{cases} 5 \text{ at bank } A; \\ 3 \text{ at bank } B. \end{cases}$$

Player 2 can choose to claim 1 at bank A and 3 at bank B (also dividing his whole claim of 4):

$$\text{Player 2's claim of 4} \begin{cases} 1 \text{ at bank } A; \\ 3 \text{ at bank } B. \end{cases}$$

This strategy combination is visualized in the left part of Table 1.2.

Next, both banks divide their estate using the pre-specified claims rule: bank A divides the estate of 4 over the two players, leading to an award of 3 for player 1 and 1 for player 2. This is visualized in the right part of Table 1.2.

For bank B, both players claim an equal amount of 3 at this bank. The available estate of 6 is thus equally divided over the two players: 3 for player 1 and 3 for player 2, as is seen in Table 1.2.

Summarizing, these choices of players 1 and 2 lead to a pay-off of 6 for player 1 and 4 for player 2.

	Claims		Awards	
	player 1	player 2	player 1	player 2
Bank A	5	1	3	1
Bank B	3	3	3	3
Total	8	4	6	4

Table 1.2 – *Division of the claims in the first strategy combination and the corresponding awards of Example 1.2.*

However, if player 1 believes that player 2 is going to divide his claim of 4 in the way as prescribed above, then player 1 can improve his total pay-off by changing his strategy. Instead of claiming 5 at A and 3 at B, player 1 can also decide to claim only 3 at bank A and 5 at bank B:

$$\text{Player 1's claim of 8} \begin{cases} 3 \text{ at bank } A; \\ 5 \text{ at bank } B. \end{cases}$$

Table 1.3 summarizes this division of claims and the corresponding awards for both players for this second option. Interestingly, the allocation of bank A remains the same as in the first option: 3 to player 1 and 1 to player 2. However, bank B now allocates more to player 1: he receives, by following the proportional rule, an amount of $\frac{5}{5+3} \cdot 6 = 3\frac{3}{4}$. Player 2 still claims 3 at bank B, but he now only receives an amount of $\frac{3}{5+3} \cdot 6 = 2\frac{1}{4}$. In total, this leads to a pay-off of $6\frac{3}{4}$ for player 1, which is indeed more than in the first option, and $3\frac{1}{4}$ for player 2. In game theoretical terms, this means that the first strategy combination is not a Nash equilibrium.

	Claims		Awards	
	player 1	player 2	player 1	player 2
Bank A	3	1	3	1
Bank B	5	3	$3\frac{3}{4}$	$2\frac{1}{4}$
Total	8	4	$6\frac{3}{4}$	$3\frac{1}{4}$

Table 1.3 – *Division of the claims in the second strategy combination and the corresponding awards of Example 1.2.*

To conclude this example, we analyze one more strategy combination, which is summarized in Table 1.4. This third strategy combination gives player 1 a total pay-off of $6\frac{1}{2}$ and player 2 receives a total pay-off of $3\frac{1}{2}$. It can be shown that this strategy combination is a Nash equilibrium, which means that, given the equilibrium strategy of the other player, no player has an incentive to choose a strategy different from his equilibrium strategy.

	Claims		Awards	
	player 1	player 2	player 1	player 2
Bank A	2	2	2	2
Bank B	6	2	$4\frac{1}{2}$	$1\frac{1}{2}$
Total	8	4	$6\frac{1}{2}$	$3\frac{1}{2}$

Table 1.4 – *Division of the claims in the third strategy combination and the corresponding awards of Example 1.2.*

$\triangle$

Example 1.2 illustrates the strategic choices of the players in a claims problem with estate holders. General questions that are answered in Chapter 6 include whether there always exist Nash equilibria and whether the corresponding Nash equilibria pay-offs always allocate the total estate.

1.3 Sequencing situations

A different example of an allocation problem appears in so-called sequencing situations. Here, the allocation problem appears after solving a joint optimization problem. In a *(standard) sequencing situation*, several jobs need to be processed by a single machine. Each job has its own processing time, specifying how long the machine takes to process the job, and linear cost coefficient, specifying the cost per time unit the job spent in the system. It is assumed that all jobs are present in the system at the time the machines starts processing and the time a job spends in the system thus consists of the waiting time before the job is processed by the machine and the processing time of the job itself. The optimization goal is to find an order of the jobs that minimizes the total joint costs of the jobs. Such an order is called an optimal order.

An allocation problem arises if one includes an initial order that provides the initial processing rights on the machine. By rearranging the jobs from the initial order to an optimal order, cost savings can be obtained. The additional question is then how to allocate these cost savings among the jobs. The following example is inspired by an outsourcing context and illustrates an integrated approach for finding an optimal order and solving the corresponding allocation problem.

Example 1.3 Consider a highly specialized machine which is able to produce chips for computers. Three companies (e.g. Apple, Dell and HP), named player 1, 2 and 3, want to outsource the production of chips for their computers and use the specialized machine. The processing times and cost per time unit are specified in Table 1.5 below. In this example, one might think of the processing times representing the number of chips needed. The cost per time unit might represent the costs of delaying the process of building the computers.

	player 1	player 2	player 3
Processing time	4	3	1
Cost per time unit	2	6	1

Table 1.5 – *The sequencing situation of Example 1.3.*

To make use of the specialized machine, the companies have to reserve the machine to produce the chips needed for their computers. These reservations provide the initial processing rights on the machine and are treated on a 'first come, first serve' basis. Here, we assume that player 1 made a reservation first, then player 2 and finally, player 3. This initial order is denoted by $(1, 2, 3)$.

Given this initial order, it is clear that the job of player 1 has no waiting time and is thus completed after his processing time of 4. After these 4 time units, the job of player 2 is processed. The completion time of this job is thus after 7 time units. Finally, it takes one more time unit to finish the job of player 3, resulting in a completion time of 8 for player 3. The initial order and the corresponding completion times are visualized in Figure 1.1. Using the cost per time unit, the individual costs are given by 8 for player 1, 42 for player 2 and 8 for player 3. Consequently, the total costs for the initial order are given by $8 + 42 + 8 = 58$.

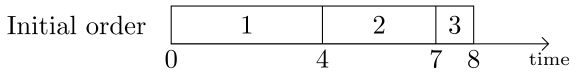


Figure 1.1 – *The initial order* $(1, 2, 3)$ *in the sequencing situation of Example 1.3.*

Interestingly, after a careful deliberation of the processing times and cost per time unit, the two neighboring players 1 and 2 decide to work together and switch positions. The reason for this switch is the fact that after the switch, the individual costs of player 1 and player 2 become 14 and 18, respectively. Hence, the total costs for the order $(2, 1, 3)$ are given by $14 + 18 + 8 = 40$, since the neighbor switch between players 1 and 2 will clearly not influence the completion time of player 3. Thus, as a consequence of the switch of players 1 and 2, a cost savings of 18 is obtained. A natural way of directly allocating these cost savings is equally between players 1 and 2. Both players thus receive 9 for this switch.

After this first switch, players 1 and 3 become neighbors and decide to work together. A switch between these players lead to the order $(2, 3, 1)$ and individual costs of 16 for player 1 and 4 for player 3. Hence, the total costs for this order are given by $16 + 18 + 4 = 38$, which is again lower than before. This switch thus generates a cost savings of 2, which is again equally divided between players 1 and 3. Both players thus receive 1 for this switch.

At this point, there are no profitable neighbor switches available for the players anymore. In the standard sequencing situation of this example, the players now have reached the optimal order, that is, the order that minimizes the total costs. This can readily be verified by computing the total costs of all possible orders. This is shown in Table 1.6. Indeed, the lowest total costs are given by 38 and obtained for the unique optimal order $(2, 3, 1)$, which is schematically visualized in Figure 1.2.

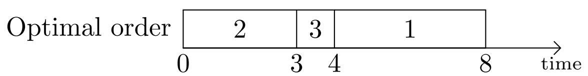


Figure 1.2 – *The optimal order* $(2, 3, 1)$ *in the sequencing situation of Example 1.3.*

We can thus reach the optimal order $(2, 3, 1)$ from the initial order $(1, 2, 3)$ by consec-

Processing order	Total costs
$(1, 2, 3)$	58
$(1, 3, 2)$	61
$(2, 1, 3)$	40
$(2, 3, 1)$	38
$(3, 1, 2)$	59
$(3, 2, 1)$	41

Table 1.6 – *The total costs of all processing orders in the sequencing situation of Example 1.3.*

utively switch neighboring players:

$$(1, 2, 3) \xrightarrow[(1,2)]{18} (2, 1, 3) \xrightarrow[(1,3)]{2} (2, 3, 1).$$

The corresponding cost savings are given by 18 for the first switch and 2 for the second switch, which were equally divided among the involved players. This leads to the following natural allocation of the total cost savings:

$$\text{Total cost savings of 20} \begin{cases} 9 + 1 = 10 \text{ for player 1;} \\ 9 \text{ for player 2;} \\ 1 \text{ for player 3.} \end{cases}$$

Player 1 thus gets allocated 10, player 2 receives 9 and player 3 receives 1. $\triangle$

Example 1.3 illustrates a natural solution to the allocation problem: recursively allocate the cost savings, or gains, of each (profitable) neighbor switch equally among the two players involved. This solution is formally called the *equal gain splitting rule* (Curiel, Pederzoli, and Tijs, 1989). If one allows for an arbitrary division of the neighbor switching gains among the two players involved, then we obtain the so-called *gain splitting rules* (Hamers, Suijs, Tijs, and Borm, 1996). Note that there, in principle, might be several paths from the initial order to an optimal order that repairs all profitable neighbor switches. However, the order in which these switches are repaired is not relevant for the outcome of the gains splitting rules.

All of the above is based on the assumption that the costs of the players are linear, that is, that there is a fixed amount of cost per time unit. In Chapter 4 of this dissertation, we relax this assumption and consider *non-linear cost functions* instead. In particular, we focus on finding a natural allocation of the total cost savings obtained from the neighbor switches to reach an optimal order from the initial order. For this, we use the principle of the equal gain splitting rule and divide the cost savings for each of the neighbor switches equally among the two players involved. The following example illustrates this process for a sequencing situation with exponential cost functions and shows that this process is more challenging than in the case of standard sequencing situations.

Example 1.4 Reconsider the situation with the highly specialized machine and the three companies as described in Example 1.3. That is, the processing times are specified in Table 1.5: player 1 needs 4 millions of chips to be processed by the specialized machine, player 2 needs 3 millions of chips and player 3 only needs 1 million of chips. Moreover, the initial order is given by $(1, 2, 3)$.

Now, however, the individual costs of the companies are assumed to be exponential in the completion time. In particular, we assume that for all three companies it holds that the costs are doubled every time unit. We start with an initial costs of 1. After one time unit, the costs double to 2. After another time unit, the costs are equal to 4 and so on. This can be described by the cost function $c(t) = 2^t$ where t represents the completion time.

Processing order	Total costs
$(1, 2, 3)$	400
$(1, 3, 2)$	304
$(2, 1, 3)$	392
$(2, 3, 1)$	280
$(3, 1, 2)$	290
$(3, 2, 1)$	274

Table 1.7 – *The total costs of all processing orders in the sequencing situation of Example 1.4.*

To find an optimal order, we compute the total costs of all possible processing orders. This is shown in Table 1.7. For example, the total costs for the initial order (see also Figure 1.1) are given by 400: the completion time of player 1 is equal to 4, so his individual costs are $2^4 = 16$, the completion time of player 2 is 7 with corresponding individual costs of $2^7 = 128$ and the completion time of player 3 is 8 with individual costs of $2^8 = 256$. Consequently, the total costs for the initial order are given by
$16 + 128 + 256 = 400$.

Table 1.7 reveals that the order $(3, 2, 1)$ is the unique optimal order with a total costs of 274. If the players decide to cooperate, then a total cost savings of $400 - 274 = 126$ can be obtained. To obtain these maximal total cost savings, we try to follow the recursive process as described in Example 1.3 using profitable neighbor switches. This is possible in two different ways:

$$(1,2,3) \xrightarrow[8]{(1,2)} (2,1,3) \xrightarrow[112]{(1,3)} (2,3,1) \xrightarrow[6]{(2,3)} (3,2,1),$$

and

$$(1,2,3) \xrightarrow[96]{(2,3)} (1,3,2) \xrightarrow[14]{(1,3)} (3,1,2) \xrightarrow[16]{(1,2)} (3,2,1).$$

In both ways, we can compute the corresponding cost savings (below each arrow) for each neighbor switch (above each arrow) and, by following the principle of the equal gain splitting rule, divide these cost savings equally among the two players involved.

For the first path, players 1 and 2 are in the first step responsible for a cost savings of 8 and are thus each allocated an amount of 4. Furthermore, in the second step of the first path, players 1 and 3 are each allocated an amount of 56. Finally, players 2 and 3 each receive an amount of 3. In total, the first path thus leads to the following allocation of the total cost savings among the players:

$$\text{Total cost savings of 126} \begin{cases} 4+56=60 \text{ for player 1;} \\ 4+3=7 \text{ for player 2;} \\ 56+3=59 \text{ for player 3.} \end{cases}$$

For the second path, players 2 and 3 first switch positions to obtain a cost savings of 96 together. Both players are thus allocated an amount of 48. In the second step, players 1 and 3 are each allocated an amount of 7. Finally, players 1 and 2 each receive an amount of 8. In total, we thus obtain the following allocation of the total cost savings among the players:

$$\text{Total cost savings of 126} \begin{cases} 7+8=15 \text{ for player 1;} \\ 48+8=56 \text{ for player 2;} \\ 48+7=55 \text{ for player 3.} \end{cases}$$

Note that the two allocations differ. In fact, the cost savings obtained from a switch between two players, for example players 1 and 2, differ in both paths: in the first path, a switch of players 1 and 2 yields a cost savings of 8, while in the second path, this switch yields a cost savings of 16. In the exponential sequencing situation of this example, the order in which the profitable neighbor switches are repaired is thus relevant.

Next, we analyze the two natural allocations, as summarized in Table 1.8, from the perspective of stability. An allocation vector satisfies *stability* if every subgroup of players is allocated at least as much as the amount this subgroup can obtain by stop working together with the other players and only cooperate with each other.

First note that players 1 and 2 can obtain a cost savings of 8 by working together. Without the help of player 3, they can switch positions in the initial order to reach the order $(2,1,3)$. The corresponding cost savings of this switch are equal to 8.

	player 1	player 2	player 3
First allocation	60	7	59
Second allocation	15	56	55

Table 1.8 – *The two natural allocations in the sequencing situation of Example 1.4.*

In both allocation vectors, players 1 and 2 receive more than 8: in the first allocation, the players receive in total an amount of 67, whereas in the second allocation, the players receive a total amount of 71. This implies that players 1 and 2 prefer a cooperation with player 3 rather than working together with the two of them only.

Secondly, note that players 1 and 3 can not obtain any cost savings without the help of player 2. For, in the initial order, players 1 and 3 are separated by player 2.

Finally, players 2 and 3 are neighbors in the initial order and therefore can switch positions. This leads to a cost savings of 96. In the first allocation, players 2 and 3 jointly receive only an amount of 66. Since this is less than the cost savings the two players can obtain by directly switching positions without the help of player 1, players 2 and 3 will not agree with the first allocation. In other words, the first allocation is not a stable one. On the other hand, in the second allocation, players 2 and 3 receive a total amount of 111, which does exceed 96. This means that the second allocation vector is a stable one. △

Example 1.4 shows that there exists a stable allocation of the cost savings in the *exponential sequencing situation* under consideration. In Chapter 4, we provide more insights in determining the paths from the initial order to an optimal order and the corresponding cost savings allocation. We are in particular interested in the question whether one of these allocations is stable.

1.4 Overview

This dissertation starts with a survey of the relevant preliminaries on cooperative games, claims problems and strategic games in Chapter 2. The remaining chapters are structured in the following way. First, in Chapter 3, we focus on the theoretical model of TU-games, its modifications and the corresponding commonly used allocation vectors. Secondly, we introduce our first sequencing related interactive operations research problem in Chapter 4, for which we take a cooperative approach. In particular, we solve the joint allocation problem by employing the corresponding optimization problem and relate our solutions to the associated TU-game. Next, in Chapter 5, we use cooperative cost sharing techniques to solve a purchasing related interactive joint allocation problem. In contrast, in Chapter 6, we take a strategic approach to a claims related allocation problem. Finally, Chapter 7 concludes this dissertation with a study to the theoretical model of strategic games.

Below, we provide a brief overview per chapter. For the exact results we refer to the introduction section of each chapter.

In Chapter 3, we study the nucleolus of *graph-restricted games*. The model of a TU-game is extended by Myerson (1977) to *communication situations*. In a communication situation, it is assumed that there are communication restrictions on the players, which limit the cooperation possibilities. This leads to a modified TU-game, so that the traditional solution concepts like the Shapley value or the nucleolus provide a solution to the communication situation. For example, applying the Shapley value to this modified TU-game results in the so-called Myerson value (Myerson, 1977). In Chapter 3, we apply the nucleolus to this modified TU-game. In particular, we study both the *invariance* of the nucleolus and the *inheritance* of the related properties of strong compromise admissibility and compromise stability.

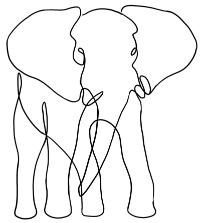
Chapter 4 studies *interactive sequencing situations with non-linear cost functions*, as described in Section 1.3 above. Interactive sequencing situations are a perfect example of an interactive operations research problem. The first goal is to find an optimal order that minimizes the total processing costs. The second goal is to find a suitable, stable allocation for the cost savings that can be obtained by recursively rearranging the initial order to an optimal order. The main focus of this chapter is on interactive sequencing situations with exponential (Saavedra-Nieves, Schouten, and Borm, 2020), discounting (Rothkopf, 1966) or logarithmic cost functions, for which both goals are achieved.

Chapter 5 introduces the concept of capacity restricted cooperative purchasing (CRCP) situations. In a *cooperative purchasing situation*, a group of players, each having individual order quantities, is cooperating in order to benefit from a higher discount. In this chapter, we focus on cooperative purchasing situations with *two suppliers with limited capacities*. We solve both the optimization problem of splitting the total order over the two suppliers to minimize the total costs and the allocation problem of dividing these costs over the players. For the latter, we model a CRCP-situation as a cost sharing problem to exploit cost sharing rules in order to find a suitable cost allocation method. We modify one of the main cost sharing rules, the serial rule, in different piecewise serial rules based on different claims rules.

In Chapter 6, we formally introduce a claims problem with estate holders as described in Section 1.2 above. In contrast to the standard *claims problems* where there is a single estate on which the claimants have a claim, there are several estates in a *claims problem with estate holders* over which the claimants have to divide their claims. Since each estate is separately allocated using a claims rule, this leads to a strategic choice for the players of how to divide their claims over the estates. We study the existence of Nash equilibria of the resulting strategic game and are particularly interested in the corresponding pay-offs.

Chapter 7 concludes this dissertation with the introduction of a new equilibrium concept for strategic games. The standard Nash equilibrium concept is based on selfish behavior among the players. In contrast, Berge (1957) proposed an alternative equilibrium concept based on altruistic behavior. The basic idea of a *Berge equilibrium* is *group support*, following the famous idea of 'one for all, and all for one'. Instead, we propose a new equilibrium concept based on *individual support*. In a so-called *unilateral support equilibrium*, every player is supported by every other player individually. We characterize unilateral support equilibria in several ways and focus on the relation with both the Berge and Nash equilibrium concepts.

2 Preliminaries



— *An elephant has an excellent memory*

2.1 Cooperative games

A (transferable utility) *cooperative game* is a pair (N, v) where N is a non-empty, finite set of *players* and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is a *characteristic function* which assigns to every *coalition* $S \in 2^N$ the *worth* of the coalition. Here, 2^N is the collection of all subsets of N. The set of all cooperative games with player set N is denoted by TU^N and a cooperative game (N, v) is also denoted by $v \in TU^N$.

For a cooperative game $v \in TU^N$, the *imputation set* is given by

$$I(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N \right\},$$

the *core* is given by

$$C(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\},$$

and the *core cover* (cf. Tijs and Lipperts, 1982) is given by

$$CC(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N) \text{ and } m(v) \leq x \leq M(v) \right\},$$

where $M(v), m(v) \in \mathbb{R}^N$ are, for all $i \in N$, defined by

$$M_i(v) = v(N) - v(N \setminus \{i\}),$$

and

$$m_i(v) = \max_{S \in 2^N : i \in S} \left\{ v(S) - \sum_{j \in S, j \neq i} M_j(v) \right\}.$$

A cooperative game $v \in TU^N$ is called

- *imputation admissible* if $I(v) \neq \emptyset$;
- *balanced* if $C(v) \neq \emptyset$ (cf. Bondareva, 1963; Shapley, 1967);
- *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$;
- *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \in 2^N$ (cf. Shapley, 1971);
- *compromise stable* if $CC(v) \neq \emptyset$ and $C(v) = CC(v)$, or equivalently if $CC(v) \neq \emptyset$ and $v(S) \leq \max\left\{\sum_{i \in S} m_i(v), v(N) - \sum_{j \in N \setminus S} M_j(v)\right\}$ for all $S \in 2^N \setminus \{\emptyset\}$ (cf. Quant, Borm, Reijnierse, and Van Velzen, 2005);
- *strongly compromise admissible* if $CC(v) \neq \emptyset$ and $v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v)$ for all $S \in 2^N \setminus \{\emptyset\}$ (introduced by Driessen (1988) as 1-convex, but we adopt the terminology of Quant et al. (2005)).

Figure 2.1 provides an overview of the relation between the above-mentioned properties of a cooperative game. In the figure, each ellipse reflects the class of all cooperative games for which the corresponding property is satisfied. One can see that, e.g., strong compromise admissibility implies compromise stability, convexity implies superadditivity and balancedness implies imputation admissibility.

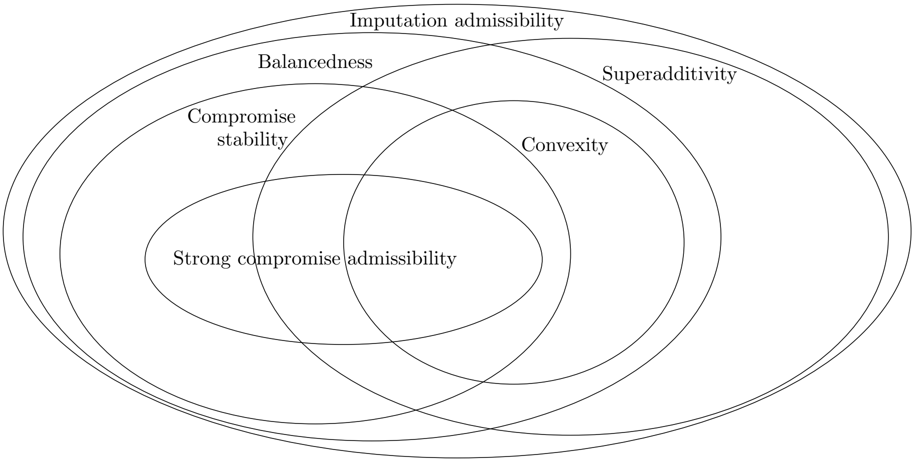


Figure 2.1 – *An overview of the relation between several properties of a cooperative game.*

2.2 Claims problems

A *claims problem* is a tuple (N, E, c) where N is a non-empty, finite set of *players*, $E \in \mathbb{R}_+$ is the *estate* and $c \in \mathbb{R}_+^N$ is the *claims vector* summarizing the claims c_i on the estate for every player $i \in N$.[1] The set of all claims problems with player set N is denoted by $\mathcal{C}^N$ and a claims problem (N, E, c) is also denoted by $(E, c) \in \mathcal{C}^N$.

A *claims rule* $\varphi : \mathcal{C}^N \to \mathbb{R}_+^N$ assigns to each claims problem $(E, c) \in \mathcal{C}^N$ an *awards vector* $\varphi(E, c) \in \mathbb{R}_+^N$ such that, for all $i \in N$,

$$0 \leq \varphi_i(E, c) \leq c_i, \tag{2.1}$$

and, if $\sum_{i \in N} c_i \leq E$,

$$\varphi(E, c) = c, \tag{2.2}$$

and finally, if $\sum_{i \in N} c_i > E$,

$$\sum_{i \in N} \varphi_i(E, c) = E. \tag{2.3}$$

Here, the first inequality of Equation (2.1) is referred to as *non-negativity* and the second inequality as *claims boundedness*. Moreover, Equation (2.2) requires a claims rule to fulfill all claims if the estate is sufficiently large.[2] Finally, the condition as formulated in Equation (2.3) is referred to as *efficiency*.

[1] Note that we do not impose $\sum_{i \in N} c_i > E$ from the onset as standard in *bankruptcy problems*.

[2] This condition requires a claims rule to solve any trivial problem in the obvious way, which is a technically convenient and natural extension of the standard bankruptcy assumptions.

Moreover, we assume from the onset that a claims rule φ satisfies *continuity*, which requires that, for all claims problems $(E, c) \in \mathcal{C}^N$ and for any sequence of claims problems $\{(E^k, c^k)\}_{k=1}^{\infty} \subseteq \mathcal{C}^N$ that converges to (E, c), it holds that $\{\varphi(E^k, c^k)\}_{k=1}^{\infty}$ converges to $\varphi(E, c)$.

Well-known claims rules include the *constrained equal awards rule*, the *constrained equal losses rule*, the *Talmud rule* and the *proportional rule*.[3]

For a claims problem $(E, c) \in \mathcal{C}^N$, the *constrained equal awards rule* is denoted by CEA and, for all $i \in N$, defined by

$$\mathrm{CEA}_i(E, c) = \begin{cases} c_i, & \text{if } E \geq \sum_{j \in N} c_j; \\ \min\{\lambda, c_i\}, & \text{if } E < \sum_{j \in N} c_j, \end{cases}$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \min\{\lambda, c_i\} = E$. The constrained equal awards rule divides the estate as equal as possible under the restriction that no player is awarded more than his claim.

On the other hand, the constrained equal losses rule divides the total loss as equal as possible under the restriction that no player loses more than this claim. Formally, for a claims problem $(E, c) \in \mathcal{C}^N$, the *constrained equal losses rule* is denoted by CEL and, for all $i \in N$, defined by

$$\mathrm{CEL}_i(E, c) = \begin{cases} c_i, & \text{if } E \geq \sum_{j \in N} c_j; \\ \max\{c_i - \lambda, 0\}, & \text{if } E < \sum_{j \in N} c_j, \end{cases}$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \max\{c_i - \lambda, 0\} = E$.

Incorporating both ideas of the constrained equal awards rule and the constrained equal losses rule leads, for a claims problem $(E, c) \in \mathcal{C}^N$, to the *Talmud rule* (cf. Aumann and Maschler, 1985), which is denoted by TAL and, for all $i \in N$, defined by

$$\mathrm{TAL}_i(E, c) = \begin{cases} c_i, & \text{if } E \geq \sum_{j \in N} c_j; \\ \frac{1}{2}c_i + \mathrm{CEL}_i\left(E - \frac{1}{2}\sum_{j \in N} c_j, \frac{1}{2}c\right), & \text{if } \frac{1}{2}\sum_{j \in N} c_j < E < \sum_{j \in N} c_j; \\ \mathrm{CEA}_i\left(E, \frac{1}{2}c\right), & \text{if } E \leq \frac{1}{2}\sum_{j \in N} c_j. \end{cases}$$

Note that, since $\mathrm{CEL}_i(E, c) = c_i - \mathrm{CEA}_i\left(\sum_{j \in N} c_j - E, c\right)$, it holds that

$$\mathrm{TAL}_i(E, c) = \begin{cases} c_i, & \text{if } E \geq \sum_{j \in N} c_j; \\ c_i - \mathrm{CEA}_i\left(\sum_{j \in N} c_j - E, \frac{1}{2}c\right), & \text{if } \frac{1}{2}\sum_{j \in N} c_j < E < \sum_{j \in N} c_j; \\ \mathrm{CEA}_i\left(E, \frac{1}{2}c\right), & \text{if } E \leq \frac{1}{2}\sum_{j \in N} c_j. \end{cases}$$

[3] For an overview on claims rules, see Thomson (2003, 2013, 2015), among others.

For claims problems with two players, the Talmud rule boils down to the *concede and divide rule* in which both players first concede the possible amount that is not claimed and secondly, divide the possible remainder equally. Formally, for a claims problem $(E,c) \in \mathcal{C}^N$ with $N = \{1,2\}$, the *concede and divide rule* (cf. Aumann and Maschler, 1985) is denoted by CD and defined by

$$\mathrm{CD}_1(E,c) = \begin{cases} c_1, & \text{if } E \geq c_1 + c_2; \\ \max\{E - c_2, 0\} + \frac{E - \max\{E-c_1,0\} - \max\{E-c_2,0\}}{2}, & \text{if } E < c_1 + c_2, \end{cases}$$

and

$$\mathrm{CD}_2(E,c) = \begin{cases} c_2, & \text{if } E \geq c_1 + c_2; \\ \max\{E - c_1, 0\} + \frac{E - \max\{E-c_1,0\} - \max\{E-c_2,0\}}{2}, & \text{if } E < c_1 + c_2. \end{cases}$$

Exchanging the roles of the constrained equal awards rule and the constrained equal losses rule in the Talmud rule leads, for a claims problems $(E,c) \in \mathcal{C}^N$, to the *reverse Talmud rule* (cf. Chun, Schummer, and Thomson, 2001), which is denoted by RTAL and, for all $i \in N$, defined by

$$\mathrm{RTAL}_i(E,c) = \begin{cases} c_i, & \text{if } E \geq \sum_{j \in N} c_j; \\ \frac{1}{2}c_i + \mathrm{CEA}_i\left(E - \frac{1}{2}\sum_{j \in N} c_j, \frac{1}{2}c\right), & \text{if } \frac{1}{2}\sum_{j \in N} c_j < E < \sum_{j \in N} c_j; \\ \mathrm{CEL}_i\left(E, \frac{1}{2}c\right), & \text{if } E \leq \frac{1}{2}\sum_{j \in N} c_j. \end{cases}$$

Again, by using that $\mathrm{CEL}_i(E,c) = c_i - \mathrm{CEA}_i\left(\sum_{j \in N} c_j - E, c\right)$, it holds that

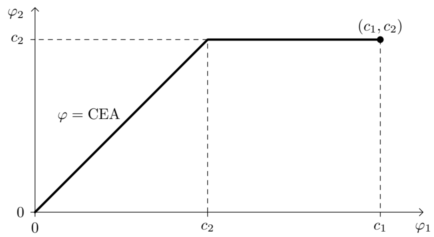
$$\mathrm{RTAL}_i(E,c) = \begin{cases} c_i, & \text{if } E \geq \sum_{j \in N} c_j; \\ c_i - \mathrm{CEL}_i\left(\sum_{j \in N} c_j - E, \frac{1}{2}c\right), & \text{if } \frac{1}{2}\sum_{j \in N} c_j < E < \sum_{j \in N} c_j; \\ \mathrm{CEL}_i\left(E, \frac{1}{2}c\right), & \text{if } E \leq \frac{1}{2}\sum_{j \in N} c_j. \end{cases}$$

Finally, for a claims problem $(E,c) \in \mathcal{C}^N$, the *proportional rule* is denoted by PROP and, for all $i \in N$, defined by

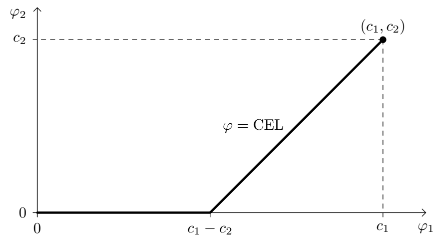
$$\mathrm{PROP}_i(E,c) = \begin{cases} c_i, & \text{if } E \geq \sum_{j \in N} c_j; \\ \lambda c_i, & \text{if } E < \sum_{j \in N} c_j, \end{cases}$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{j \in N} \lambda c_j = E$. Rewriting leads, for all $i \in N$, to

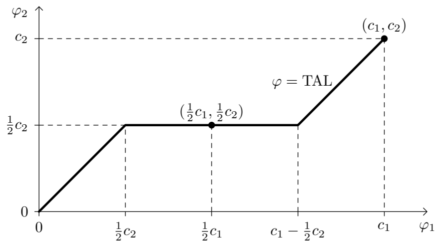
$$\mathrm{PROP}_i(E,c) = \begin{cases} c_i, & \text{if } E \geq \sum_{j \in N} c_j; \\ \frac{c_i}{\sum_{j \in N} c_j} E, & \text{if } E < \sum_{j \in N} c_j. \end{cases}$$



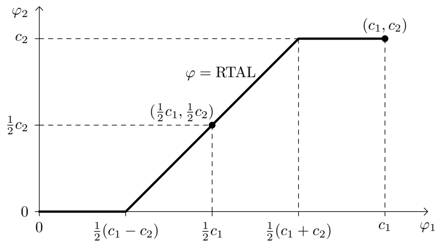
(a) *For the constrained equal awards rule.*



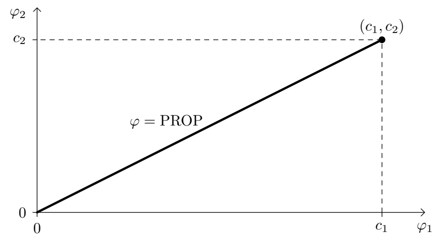
(b) *For the constrained equal losses rule.*



(c) *For the Talmud rule.*



(d) *For the reverse Talmud rule.*



(e) *For the proportional rule.*

Figure 2.2 – *The paths of awards for five claims rules.*

The proportional rule thus divides the estate proportionally to the claims.

For claims problem with only two players, we can visualize the claims rules by the so-called *paths of awards* (Thomson, 2015). These paths indicate the awards of the two players for an increasing level of the estate. The paths of awards of the constrained equal awards rule, the constrained equal losses rule, the Talmud rule, the reverse Talmud rule and the proportional rule are depicted in Figure 2.2.

A claims rule φ satisfies

- *claims monotonicity* if, for all $(E, c) \in \mathcal{C}^N$, all $i \in N$ and all $c'_i \in \mathbb{R}_+$ with $c'_i \geq c_i$,
$$\varphi_i(E, (c_{-i}, c'_i)) \geq \varphi_i(E, c);$$
- *estate monotonicity* if, for all $(E, c) \in \mathcal{C}^N$, all $i \in N$ and all $E' \in \mathbb{R}_+$ with $E' \geq E$,
$$\varphi_i(E', c) \geq \varphi_i(E, c);$$
- *order preservation* if, for all $(E, c) \in \mathcal{C}^N$ and all $i, j \in N$ with $c_i \leq c_j$,
$$\begin{cases} \varphi_i(E, c) \leq \varphi_j(E, c); \\ c_i - \varphi_i(E, c) \leq c_j - \varphi_j(E, c); \end{cases}$$
- *exemption* if, for all $(E, c) \in \mathcal{C}^N$ and all $i \in N$ with $c_i \leq \frac{1}{|N|}E$ (cf. Herrero and Villar, 2001),
$$\varphi_i(E, c) = c_i;$$
- *consistency* if, for all $(E, c) \in \mathcal{C}^N$, all $N' \subseteq N$ and all $i \in N'$,
$$\varphi_i(E, c) = \varphi_i(\sum_{j \in N'} \varphi_j(E, c), (c_j)_{j \in N'}).^4$$

All claims rules mentioned above satisfy claims monotonicity, estate monotonicity, order preservation and consistency. Moreover, the constrained equal awards rule satisfies exemption, while the constrained equal losses rule, the proportional rule, the Talmud rule and the reverse Talmud rule do not satisfy exemption (Herrero and Villar, 2001).

2.3 Strategic games

A *strategic (non-cooperative) game* is a triple $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$, where N is a non-empty, finite set of players, with $|N| \geq 2$, X_i is the set of *strategies* for player $i \in N$ and $\pi_i : X \to \mathbb{R}$ is the *pay-off function* of player $i \in N$. Here, X is the product of all sets of strategies, $X = \prod_{j \in N} X_j$, and is called the *set of strategy combinations*. A strategy combination $x \in X$ is sometimes written as $x = (x_{-i}, x_i) = (x_i, x_{-i})$ for a certain $i \in N$, where $x_{-i} = (x_j)_{j \in N \setminus \{i\}} \in X_{-i}$, with $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$ denoting the set of strategy combinations of the players in $N \setminus \{i\}$.

[4] For consistency, we allow for a variable player set and extend φ accordingly.

For a strategic game $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$, a strategy combination $\hat{x} \in X$ is called a *Nash equilibrium* (cf. Nash, 1950, 1951) if, for all $i \in N$, it holds that

$$\pi_i(\hat{x}_{-i}, \hat{x}_i) \geq \pi_i(\hat{x}_{-i}, x_i),$$

for all $x_i \in X_i$. The set of Nash equilibria for G is denoted by $NE(G)$. A Nash equilibrium is thus a strategy combination in which every player maximizes his own pay-off by playing the equilibrium strategy, given the equilibrium strategy combination of the other players. In other words, in a Nash equilibrium, no player has an incentive to unilaterally deviate.

Clearly, with

$$BR_i(x_{-i}) = \{x_i \in X_i \mid \pi_i(x_{-i}, x_i) \geq \pi_i(x_{-i}, x_i') \quad \text{for all } x_i' \in X_i\},$$

for all $x_{-i} \in X_{-i}$ and all $i \in N$, denoting the *set of best reply strategies against* x_{-i}, we have that $\hat{x} \in NE(G)$ if and only if $\hat{x}_i \in BR_i(\hat{x}_{-i})$ for all $i \in N$.

Nash equilibria do not always exist. A sufficient condition on a strategic game to guarantee the existence of Nash equilibria is provided in the following theorem.

Theorem 2.1 [cf. Rosen, 1965] *Let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be a strategic game. If, for all $i \in N$, the following four conditions hold:*

i) $X_i \subseteq \mathbb{R}^{m_i}$ with $m_i \in \mathbb{N}$;

ii) X_i is non-empty, convex, closed and bounded;

iii) $\pi_i : X \to \mathbb{R}$ is continuous;

iv) $g_i : X_i \to \mathbb{R}$ defined by $g_i(x_i) = \pi_i(x_i, x_{-i})$ is concave for all $x_{-i} \in X_{-i}$,

then $NE(G) \neq \emptyset$.

Strategic games for which the conditions of Theorem 2.1 are satisfied include all bimatrix and trimatrix games. An $m_1 \times m_2$ *bimatrix game* (A, B) for $A, B \in \mathbb{R}^{m_1 \times m_2}$ is formally prescribed by $(N, \{\Delta_1, \Delta_2\}, \{\pi_1, \pi_2\})$ with $N = \{1, 2\}$ and

$$\begin{cases} \Delta_1 = \{p = p_1 e_1 + p_2 e_2 + \ldots p_{m_1} e_{m_1} \in \mathbb{R}^{m_1} \mid p \geq 0,\ \sum_{i=1}^{m_1} p_i = 1\}; \\ \Delta_2 = \{q = q_1 f_1 + q_2 f_2 + \ldots + q_{m_2} f_{m_2} \in \mathbb{R}^{m_2} \mid q \geq 0,\ \sum_{j=1}^{m_2} q_j = 1\}, \end{cases}$$

where, for $i \in \{1, 2, \ldots, m_1\}$, $e_i \in \mathbb{R}^{m_1}$ is the unit vector of length m_1, i.e.,

$$(e_i)_k = \begin{cases} 1, & \text{if } k = i; \\ 0, & \text{otherwise,} \end{cases}$$

for all $k \in \{1, 2, \ldots, m_1\}$, and represents a *pure strategy* of player 1. A *mixed strategy* $p \in \Delta_1$ for player 1 is then a probability distribution on the m_1 pure strategies.

Similarly, a mixed strategy $q \in \Delta_2$ for player 2 is a probability distribution on the m_2 different unit vectors of length m_2, i.e., $f_1, f_2, \ldots, f_{m_2}$, that represent the pure strategies of player 2. Finally, the pay-off functions are given by the expected pay-offs:

$$\begin{cases} \pi_1(p,q) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} p_i q_j A_{ij}; \\ \pi_2(p,q) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} p_i q_j B_{ij}, \end{cases}$$

for all $(p,q) \in \Delta_1 \times \Delta_2$ where A_{ij} and B_{ij} denote the entries in the ith row and jth column of matrix A and B, respectively.

It can be readily verified that, for an $m_1 \times m_2$ bimatrix game (A,B) and a strategy combination $(\hat{p},\hat{q}) \in \Delta_1 \times \Delta_2$, it holds that $(\hat{p},\hat{q}) \in NE(A,B)$ if and only if

$$\begin{cases} \pi_1(\hat{p},\hat{q}) \geq \pi_1(e_i,\hat{q}) & \text{for all } i \in \{1,2,\ldots,m_1\}; \\ \pi_2(\hat{p},\hat{q}) \geq \pi_2(\hat{p},f_j) & \text{for all } j \in \{1,2,\ldots,m_2\}. \end{cases} \tag{2.4}$$

An $m_1 \times m_2 \times m_3$ *trimatrix game* (A,B,C) for $A,B,C \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ is formally prescribed by $(N, \{\Delta_1, \Delta_2, \Delta_3\}, \{\pi_1, \pi_2, \pi_3\})$ with $N = \{1,2,3\}$ and

$$\begin{cases} \Delta_1 = \{p = p_1 e_1 + p_2 e_2 + \ldots p_{m_1} e_{m_1} \in \mathbb{R}^{m_1} \mid p \geq 0,\ \sum_{i=1}^{m_1} p_i = 1\}; \\ \Delta_2 = \{q = q_1 f_1 + q_2 f_2 + \ldots + q_{m_2} f_{m_2} \in \mathbb{R}^{m_2} \mid q \geq 0,\ \sum_{j=1}^{m_2} q_j = 1\}; \\ \Delta_3 = \{r = r_1 g_1 + r_2 g_2 + \ldots r_{m_3} g_{m_3} \in \mathbb{R}^{m_3} \mid r \geq 0,\ \sum_{k=1}^{m_3} r_k = 1\}, \end{cases}$$

where again $e_1, e_2, \ldots, e_{m_1} \in \mathbb{R}^{m_1}$ are the unit vectors of length m_1, representing the pure strategies of player 1, $f_1, f_2, \ldots, f_{m_2} \in \mathbb{R}^{m_2}$ are the unit vectors of length m_2, representing the pure strategies of player 2, and similarly, $g_1, g_2, \ldots, g_{m_3} \in \mathbb{R}^{m_3}$ are the unit vectors of length m_3, representing the pure strategies of player 3. The pay-off functions are again given by the expected pay-offs:

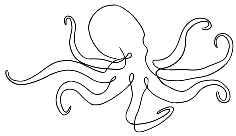
$$\begin{cases} \pi_1(p,q,r) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{k=1}^{m_3} p_i q_j r_k A_{ijk}; \\ \pi_2(p,q,r) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{k=1}^{m_3} p_i q_j r_k B_{ijk}; \\ \pi_3(p,q,r) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{k=1}^{m_3} p_i q_j r_k C_{ijk}, \end{cases}$$

for all strategy combinations $(p,q,r) \in \Delta_1 \times \Delta_2 \times \Delta_3$.

Similar to bimatrix games, it can be readily verified that, for an $m_1 \times m_2 \times m_3$ trimatrix game (A,B,C) and a strategy combination $(\hat{p},\hat{q},\hat{r}) \in \Delta_1 \times \Delta_2 \times \Delta_3$, it holds that $(\hat{p},\hat{q},\hat{r}) \in NE(A,B,C)$ if and only if

$$\begin{cases} \pi_1(\hat{p},\hat{q},\hat{r}) \geq \pi_1(e_i,\hat{q},\hat{r}) & \text{for all } i \in \{1,2,\ldots,m_1\}; \\ \pi_2(\hat{p},\hat{q},\hat{r}) \geq \pi_2(\hat{p},f_j,\hat{r}) & \text{for all } j \in \{1,2,\ldots,m_2\}; \\ \pi_3(\hat{p},\hat{q},\hat{r}) \geq \pi_3(\hat{p},\hat{q},g_k) & \text{for all } k \in \{1,2,\ldots,m_3\}. \end{cases} \tag{2.5}$$

3 Properties of graph-restricted games



— An octopus changes color to communicate

3.1 Introduction

One of the main issues in cooperative game theory is the allocation of the joint pay-off among all players, taking into account the worths of all coalitions. Two distinguished solutions that solve this issue are the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969).

Myerson (1977) extended cooperative games by introducing communication situations in which the communication restrictions of the players are modeled by a communication graph. The corresponding graph-restricted game is a modified cooperative game in which the communication restrictions are taken into account. Throughout this chapter we will assume that the communication graph is connected.

The Myerson value (Myerson, 1977) of a communication situation is defined as the Shapley value of the corresponding graph-restricted game. This value is axiomatically characterized by Myerson (1980) and studied in several other contexts as well: hypergraphs (Van den Nouweland, Borm, and Tijs, 1992), union stable structures (Algaba, Bilbao, Borm, and López, 2001), antimatroids (Algaba, Bilbao, Van den Brink, and Jiménez-Losada, 2004), bipartite graphs (Van den Brink and Pintér, 2015), two-level communication structures (Van den Brink, Khmelnitskaya, and Van der Laan, 2016) and communication situations in which the players have different bargaining abilities (Manuel and Martín, 2021). Moreover, several studies are devoted to the inheritance of properties of cooperative games that are related to the Shapley value. In particular, Owen (1986) studied the inheritance of superadditivity, Van den Nouweland

and Borm (1991) studied convexity and Slikker (2000) studied, among others, average convexity. The inheritance of convexity is also studied in a unified approach by Algaba, Bilbao, and López (2001).

Also the nucleolus of the graph-restricted game is studied in the context of communication situations. Potters and Reijnierse (1995) showed that the nucleolus is the unique element of the kernel if the communication graph is a tree. Reijnierse and Potters (1998) and Katsev and Yanovskaya (2013) studied the collection of coalitions that determine the nucleolus and prenucleolus, respectively. Khmelnitskaya and Sudhölter (2013) provided an axiomatic characterization of the prenucleolus for games with communication structures.

In this chapter, based on Schouten, Dietzenbacher, and Borm (2022), we first focus on the *inheritance* of two properties of cooperative games that are related to the nucleolus: strong compromise admissibility and compromise stability. In general, computing the nucleolus of a cooperative game is not straightforward. However, interestingly, for cooperative games satisfying strong compromise admissibility or compromise stability, there exists a direct, closed formula for the nucleolus, based on the Talmud rule for claims problems (Driessen, 1988; Quant, Borm, Reijnierse, and Van Velzen, 2005). In particular, when these properties are inherited, computation of the nucleolus can still be facilitated. For strongly compromise admissible games it holds that the nucleolus coincides with the compromise value (Tijs, 1981), as shown by Driessen (1988). Furthermore, the class of strongly compromise admissible games contains, among others, the class of simple games with one veto-player, while the larger class of compromise stable games contains several interesting classes of economic games, like big boss games (Muto, Nakayama, Potters, and Tijs, 1988), clan games (Potters, Poos, Tijs, and Muto, 1989) and bankruptcy games (O'Neill, 1982; Curiel, Maschler, and Tijs, 1987).

With regard to the inheritance of *strong compromise admissibility*, we show that the graph-restricted game is strongly compromise admissible for every communication situation with an underlying strongly compromise admissible game, if the graph is *biconnected*. In fact, for every connected graph that is not biconnected, we explicitly construct a communication situation with an underlying strongly compromise admissible game such that the graph-restricted game is not strongly compromise admissible.

Starting with a communication situation with an underlying strongly compromise admissible game, we automatically ensure compromise stability for the graph-restricted game in case of a *biconnected* graph. In addition, we show that compromise stability is also ensured if the graph is a *star*. In fact, it is exactly the family of graphs that is biconnected or a star that guarantees compromise stability for the graph-restricted game if the underlying game of a communication situation is strongly compromise admissible. This is again shown by constructing, for every connected graph that is not biconnected and not a star, a communication situation with an underlying strongly compromise admissible game such that the graph-restricted game is not compromise stable.

However, with regard to the inheritance of *compromise stability*, we actually derive an impossibility result: only for communication situations with three players or with a *complete* communication graph, the graph-restricted game is compromise stable if the underlying game is compromise stable. For every connected graph that is not complete (and has at least four players), we explicitly construct a communication situation with an underlying compromise stable game such that the graph-restricted game is not compromise stable.

Next, we study the *invariance* of the nucleolus, that is, the feature that the nucleolus of the graph-restricted game equals the nucleolus of the underlying game of a communication situation. In that way, we investigate the robustness of the nucleolus to communication restrictions. We identify exactly the families of graphs for which the invariance of the nucleolus is guaranteed for communication situations with an underlying strongly compromise admissible or compromise stable game.

For communication situations with an underlying *strongly compromise admissible* game and a biconnected graph, we use the inheritance result for strong compromise admissibility to employ the direct formula for the nucleolus for both the graph-restricted game and the underlying game. We show that in these situations, the nucleolus is invariant. In fact, benefiting from the construction for the inheritance of strong compromise admissibility, we provide, for every connected graph that is not biconnected, a communication situation with an underlying strongly compromise admissible game for which the nucleolus of the graph-restricted game is not equal to the nucleolus of the underlying game.

If the underlying game is *compromise stable*, it is again impossible to guarantee the invariance of the nucleolus in a non-trivial way. Only for *complete* graphs, invariance is trivially guaranteed. For every connected graph that is not complete, we provide a communication situation with an underlying compromise stable game for which the nucleoli are different. Interestingly, invariance can be guaranteed for communication situations with an underlying game that is both compromise stable and *simple*. In that case, the family of *biconnected* graphs is characterized as the largest family of connected graphs that guarantees the invariance of the nucleolus for communication situations with an underlying simple game that is compromise stable.

This chapter is structured in the following way. Section 3.2 formally introduces the notions of a graph-restricted game and a communication situation. Section 3.3 studies the inheritance of strong compromise admissibility and compromise stability and Section 3.4 studies the invariance of the nucleolus.

3.2 Graph-restricted games and communication situations

To incorporate communication restrictions in the model of TU-games, Myerson (1977) introduced the notions of graph-restricted games and communication situations. Roughly, these notions bring together two different concepts: a cooperative game and a graph. Therefore, we first formally introduce the concept of a graph.

A *graph* is a pair (N, E), where N is a non-empty, finite set of *players*, with $|N| \geq 3$ and $E \subseteq \{\{i, j\} \mid i, j \in N, i \neq j\}$ is a finite set of *edges*. For a graph (N, E) and a subset of players $S \in 2^N \setminus \{\emptyset\}$, the *induced subgraph* on S is defined as the graph (S, E_S), where $E_S = \{\{i, j\} \in E \mid i, j \in S\}$. A *path* in a graph (N, E) is defined as a sequence of players $(i_0, \ldots, i_m)$ such that $i_k \neq i_\ell$ for all $k, \ell \in \{0, 1, \ldots, m\}, k \neq \ell$ and $\{i_{k-1}, i_k\} \in E$ for all $k \in \{1, \ldots, m\}$.

A graph (N, E) is called

- *connected* if, for all $i, j \in N$ with $i \neq j$, there is a path $(i, \ldots, j)$ in (N, E);
- *complete* if $\{i, j\} \in E$ for all $i, j \in N$ with $i \neq j$;
- *biconnected* if, for all $i \in N$, the induced subgraph $(N \setminus \{i\}, E_{N \setminus \{i\}})$ is connected;
- a *star* if there exists a player $k \in N$ such that $E = \{\{i, k\} \mid i \in N \setminus \{k\}\}$.

For a graph (N, E), a *component* $C \in 2^N \setminus \{\emptyset\}$ is defined as a maximal (inclusion-wise) subset of players such that the induced subgraph (C, E_C) is connected. For a graph (N, E) and a subset of players $S \in 2^N \setminus \{\emptyset\}$, let S/E denote the set of all components in the induced subgraph (S, E_S).

Next, we define a graph-restricted game of a graph and a cooperative game as the modified cooperative game where the worth of each coalition is determined by the sum of the worths of its components in the graph. Formally, for a graph (N, E) and a cooperative game $v \in TU^N$, the *graph-restricted game* $v^E \in TU^N$ is (cf. Myerson, 1977), for all $S \in 2^N \setminus \{\emptyset\}$, defined by

$$v^E(S) = \sum_{C \in S/E} v(C). \tag{3.1}$$

Note that for a connected graph (N, E) and a cooperative game $v \in TU^N$, it holds that $v^E(N) = v(N)$, since $N/E = \{N\}$.

Finally, we formally define a communication situation, using a slightly modified version of the definition as stated by Myerson (1977): an *essential communication situation* is a triple (N, v, E) where $|N| \geq 3$, $v \in TU^N$ and (N, E) is a connected graph such that, for all $S \in 2^N$,

$$v^E(S) \leq v(S). \tag{3.2}$$

The set of all essential communication situations is denoted by $\mathcal{ECS}^N$.[1]

In the definition of an essential communication situation, Equation (3.2) reflects the idea that, by restricting communication, it is natural to assume that the joint pay-off of the players is reduced. Note that, for any graph (N, E) and any cooperative game $v \in TU^N$ that is superadditive, it holds that $v^E(S) \leq v(S)$ for all $S \in 2^N$. In other words, Equation (3.2) is satisfied for sure if the underlying game is superadditive.

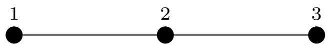


Figure 3.1 – *The communication graph* (N, E) *of Example 3.1.*

Example 3.1 Consider the graph (N, E) as depicted in Figure 3.1 and the cooperative game $v \in TU^N$ as shown in Table 3.1.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	1	1	0	1

Table 3.1 – *The cooperative game* $v \in TU^N$ *of Example 3.1.*

Using Equation (3.1), we can compute the graph-restricted game $v^E \in TU^N$. Trivially, $v^E(\{i\}) = 0$ for all $i \in N$. Moreover, since ${\{1,2\}}/_{E} = \{\{1,2\}\}$ and ${\{2,3\}}/_{E} = \{\{2,3\}\}$, it immediately follows that

$$v^E(\{1,2\}) = v(\{1,2\}) = 1,$$

and

$$v^E(\{2,3\}) = v(\{2,3\}) = 0.$$

Finally, ${\{1,3\}}/_{E} = \{\{1\},\{3\}\}$ and hence,

$$v^E(\{1,3\}) = v(\{1\}) + v(\{3\}) = 0.$$

Thus, we obtain the graph-restricted game $v^E \in TU^N$ as shown in Table 3.2.

Clearly, $v^E(S) \leq v(S)$ for all $S \in 2^N$ and hence, Equation (3.2) is satisfied. Consequently, the triple (N, v, E) is an essential communication situation, i.e., $(N, v, E) \in \mathcal{ECS}^N$. $\triangle$

[1]Here, the extra letter compared to the more common notation in the literature, $\mathcal{CS}^N$, is to emphasize that Equation (3.2) is part of an essential communication situation.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v^E(S)$	0	0	0	1	0	0	1

Table 3.2 – *The graph-restricted game $v^E \in TU^N$ of Example 3.1.*

3.3 Inheritance of strong compromise admissibility and compromise stability

This section studies the inheritance of two properties: strong compromise admissibility and compromise stability. For each of these properties, we identify the family of graphs for which the inheritance of this property from the underlying game to the graph-restricted game is guaranteed. This is useful for the computation of the nucleolus for both the graph-restricted game and the underlying game of an essential communication situation, as is seen in Section 3.4.

Note that strong compromise admissibility implies compromise stability, compromise stability implies balancedness, and balancedness implies imputation admissibility, as illustrated in Figure 2.1. First, we want to remark that both balancedness and imputation admissibility are always inherited. That is, for every essential communication situation with an underlying balanced (imputation admissible) game it holds that the graph-restricted game is balanced (imputation admissible) as well. This was first observed by Van den Nouweland and Borm (1991).

In Theorem 3.1, we show that for every essential communication situation with an underlying strongly compromise admissible game it holds that the graph-restricted game is strongly compromise admissible as well, if the graph is biconnected. Moreover, for every connected graph that is not biconnected, we construct an essential communication situation with an underlying strongly compromise admissible game such that the graph-restricted game is not strongly compromise admissible. Thus, we can conclude that it is exactly the family of biconnected graphs that guarantees the inheritance of strong compromise admissibility.

In the proof of Theorem 3.1, we use the following lemma.

Lemma 3.1 *Let $(N, v, E) \in \mathcal{ECS}^N$. Then $M(v^E) \geq M(v)$ and $m(v^E) \leq m(v)$. Consequently, $CC(v) \subseteq CC(v^E)$.*

Proof: Since $v^E(N) = v(N)$ and $v^E(S) \leq v(S)$ for all $S \in 2^N$, we have that

$$M_i(v^E) = v^E(N) - v^E(N \setminus \{i\}) \geq v(N) - v(N \setminus \{i\}) = M_i(v),$$

for all $i \in N$. Using this, we have that

$$\begin{aligned} m_i(v^E) &= \max_{S \in 2^N : i \in S} \left\{ v^E(S) - \sum_{j \in S, j \neq i} M_j(v^E) \right\} \\ &\leq \max_{S \in 2^N : i \in S} \left\{ v(S) - \sum_{j \in S, j \neq i} M_j(v) \right\} \\ &= m_i(v), \end{aligned}$$

for all $i \in N$. Hence, $M(v^E) \geq M(v)$ and $m(v^E) \leq m(v)$.

Furthermore, for $x \in CC(v)$, we have

$$\sum_{i \in N} x_i = v(N) = v^E(N),$$

and

$$m(v^E) \leq m(v) \leq x \leq M(v) \leq M(v^E).$$

Hence, $x \in CC(v^E)$ and thus $CC(v) \subseteq CC(v^E)$. $\square$

According to Lemma 3.1, we have that $M(v^E) \geq M(v)$ for any essential communication situation $(N, v, E) \in \mathcal{ECS}^N$. Interestingly, if the communication graph (N, E) is biconnected, it even holds that $M(v^E) = M(v)$, since the induced subgraph on all players except one is connected. This is used in Theorem 3.1 below, which characterizes the family of graphs that guarantees the inheritance of strong compromise admissibility from the underlying game to the graph-restricted game.

Theorem 3.1 *The following two statements hold:*

i) Let $(N, v, E) \in \mathcal{ECS}^N$ be an essential communication situation with (N, E) biconnected and v strongly compromise admissible. Then v^E is strongly compromise admissible;

ii) Let (N, E) be a connected graph that is not biconnected. Then there exists an essential communication situation $(N, v, E) \in \mathcal{ECS}^N$ where v is strongly compromise admissible such that v^E is not strongly compromise admissible.

Proof: *i)* Since v is strongly compromise admissible, we have that $CC(v) \neq \emptyset$. This implies that $CC(v^E) \neq \emptyset$, by using Lemma 3.1. Moreover, since (N, E) is biconnected, $v^E(N \setminus \{i\}) = v(N \setminus \{i\})$ for all $i \in N$ and hence, $M_i(v^E) = M_i(v)$ for all $i \in N$. Consequently, for all $S \in 2^N \setminus \{\emptyset\}$,

$$v^E(S) \leq v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v) = v^E(N) - \sum_{j \in N \setminus S} M_j(v^E).$$

Here, the second inequality is due to the fact that v is strongly compromise admissible. Consequently, v^E is strongly compromise admissible.

ii) Since (N, E) is not biconnected, we can set $N = \{1, 2, \ldots, n\}$ and assume w.l.o.g. that $\{1,2\}, \{2,3\} \in E$, while the induced subgraph on $N \setminus \{2\}$ is not connected and that players 1 and 3 are in two different components in the induced subgraph on $N \setminus \{2\}$. Consider the essential communication situation[2] $(N, v_1, E) \in \mathcal{ECS}^N$ with, for all $S \in 2^N$,

$$v_1(S) = \begin{cases} 1, & \text{if } \{1,2\} \subseteq S \text{ or } \{1,3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

First, we check that v_1 is indeed strongly compromise admissible. For this, note that $CC(v_1) \neq \emptyset$, since it can be readily verified that $M(v_1) = m(v_1) = (1, 0, 0, \ldots, 0)$.

Moreover, for $S \in 2^N$ for which $1 \in S$, $v_1(S) \leq 1 = v_1(N) - \sum_{j \in N \setminus S} M_j(v_1)$. For $S \in 2^N$ for which $1 \notin S$, $v_1(S) = 0 = v_1(N) - \sum_{j \in N \setminus S} M_j(v_1)$. Hence, v_1 is strongly compromise admissible.

Next, we show that v_1^E is not strongly compromise admissible, by showing that

$$v_1^E(\{3\}) > v_1^E(N) - \sum_{j \in N \setminus \{3\}} M_j(v_1^E).$$

First, note that $v_1^E(\{3\}) = v_1(\{3\}) = 0$.

Secondly, we have that $v_1^E(N \setminus \{2\}) = 0$, due to the fact that players 1 and 3 are in two different components of the induced subgraph on $N \setminus \{2\}$. Consequently, $M_2(v_1^E) = 1$.

Moreover, by using Lemma 3.1, $M(v_1^E) \geq M(v_1) \geq 0$ and in particular, $M_1(v_1^E) \geq M_1(v_1) = 1$. Hence,

$$v_1^E(N) - \sum_{j \in N \setminus \{3\}} M_j(v_1^E) \leq v_1^E(N) - M_1(v_1^E) - M_2(v_1^E) \leq -1.$$

Consequently, v_1^E is not strongly compromise admissible.

This finishes the construction of the essential communication situation $(N, v_1, E) \in \mathcal{ECS}^N$ where v_1 is strongly compromise admissible, while v_1^E is not strongly compromise admissible. □

Before turning to the inheritance of compromise stability, we first characterize the family of graphs that guarantees compromise stability for the graph-restricted game of any essential communication situation with an underlying strongly compromise admissible game. This family of course includes all biconnected graphs (cf. Theorem 3.1) and, in addition, it is seen that it also contains all stars.

[2]Note that it can be readily verified that $v_1 \in TU^N$ is a superadditive game, implying that Equation (3.2) is satisfied.

Theorem 3.2 *The following two statements hold:*

i) Let $(N, v, E) \in \mathcal{ECS}^N$ be an essential communication situation with (N, E) biconnected or a star and v strongly compromise admissible. Then v^E is compromise stable;

ii) Let (N, E) be a connected graph that is not biconnected and not a star. Then there exists an essential communication situation $(N, v, E) \in \mathcal{ECS}^N$ where v is strongly compromise admissible such that v^E is not compromise stable.

Proof: *i)* If (N, E) is biconnected, then v^E is strongly compromise admissible, according to part i) of Theorem 3.1. Hence, v^E is compromise stable.

If (N, E) is a star, then let $k \in N$ such that $E = \{\{i, k\} \mid i \in N \setminus \{k\}\}$. First, note that $CC(v^E) \neq \emptyset$, by using Lemma 3.1 and the fact that v is strongly compromise admissible. Secondly, it remains to prove that for all $S \in 2^N \setminus \{\emptyset\}$,

$$v^E(S) \leq \max \left\{ \sum_{i \in S} m_i(v^E), v^E(N) - \sum_{j \in N \setminus S} M_j(v^E) \right\}. \tag{3.3}$$

Let $S \in 2^N \setminus \{\emptyset\}$. If $k \notin S$, then

$$v^E(S) = \sum_{i \in S} v(\{i\}) = \sum_{i \in S} v^E(\{i\}) \leq \sum_{i \in S} m_i(v^E),$$

where the inequality follows from the fact that, for all $i \in S$,

$$m_i(v^E) = \max_{T \in 2^N : i \in T} \left\{ v^E(T) - \sum_{j \in T, j \neq i} M_j(v^E) \right\} \geq v^E(\{i\}).$$

If $k \in S$, then $M_j(v^E) = M_j(v)$ for all $j \in N \setminus S$ and hence,

$$v^E(S) \leq v(S) \leq v(N) - \sum_{j \in N \setminus S} M_j(v) = v^E(N) - \sum_{j \in N \setminus S} M_j(v^E),$$

where the second inequality follows from the fact that v is strongly compromise admissible.

Together, this proves Equation (3.3) and consequently, that v^E is compromise stable.

ii) Since (N, E) is connected, but neither biconnected, nor a star, it readily follows that $|N| \geq 4$. Set $N = \{1, 2, 3, 4, \ldots, n\}$ and since (N, E) is not biconnected, we can assume w.l.o.g. that the induced subgraph on $N \setminus \{3\}$ is not connected and that $\{2, 3\}, \{3, 4\} \in E$. Moreover, since (N, E) is not a star, we can assume that $\{1, 2\} \in E$ and in particular, that players 1 and 2 are in one component of the induced subgraph on $N \setminus \{3\}$ and player 4 is in another. Figure 3.2 provides a schematic representation in case $|N| = 4$. If $|N| > 4$, then the graph contains at least the part as indicated in Figure 3.2. Consider the essential communication situation[3] $(N, v_2, E) \in \mathcal{ECS}^N$ with, for all $S \in 2^N$,

$$v_2(S) = \begin{cases} 8, & \text{if } S = N; \\ 8, & \text{if } S = N \setminus \{j\} \text{ for } j \in N \setminus \{1,2,3,4\}; \\ 6, & \text{if } S \in \{N \setminus \{1\}, N \setminus \{2\}, N \setminus \{3\}, N \setminus \{4\}\}; \\ 3, & \text{if } |S| \leq n-2 \text{ and } \{1,2\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

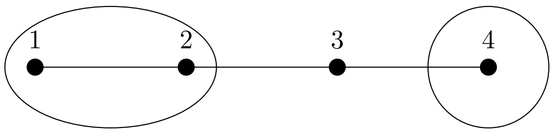


Figure 3.2 – *A schematic representation of the graph (N, E) for $N = \{1, 2, 3, 4\}$ in the proof of Theorem 3.2.*

First, we check that v_2 is indeed strongly compromise admissible. For this, note that $CC(v_2) \neq \emptyset$, since it can be readily verified that $M(v_2) = m(v_2) = (2, 2, 2, 2, 0, \ldots, 0)$. Moreover, for $S \in 2^N$ for which $|S| > n - 2$, we obviously have that

$$v_2(N) - \sum_{j \in N \setminus S} M_j(v_2) \geq v_2(S).$$

For $S \in 2^N$ for which $|S| \leq n - 2$ and $\{1, 2\} \subseteq S$, we have that

$$v_2(N) - \sum_{j \in N \setminus S} M_j(v_2) \geq 8 - 4 > 3 = v_2(S).$$

Finally, for $S \in 2^N$ for which $|S| \leq n - 2$ and $\{1, 2\} \not\subseteq S$, we have that

$$v_2(N) - \sum_{j \in N \setminus S} M_j(v_2) \geq 8 - 8 = 0 = v_2(S).$$

Hence, v_2 is strongly compromise admissible.

[3]Note that it can be readily verified that $v_2 \in TU^N$ is a superadditive game, implying that Equation (3.2) is satisfied.

Next, we show that v_2^E is not compromise stable, by showing that

$$v_2^E(\{1,2\}) > \max\left\{m_1(v_2^E) + m_2(v_2^E), v_2^E(N) - \sum_{j\in N\setminus\{1,2\}} M_j(v_2^E)\right\}. \tag{3.4}$$

First, note that $v_2^E(\{1,2\}) = v_2(\{1,2\}) = 3$.

Secondly, we show that

$$v_2^E(N) - \sum_{j\in N\setminus\{1,2\}} M_j(v_2^E) \leq 1. \tag{3.5}$$

For this, note that $v_2^E(N\setminus\{3\}) = v_2(\{1,2\}) = 3$, due to the fact that the induced subgraph on $N\setminus\{3\}$ is not connected, but consists of at least one component with $\{1,2\}\in E$. Consequently, $M_3(v_2^E) = 5$.

Moreover, by using Lemma 3.1, we have that $M(v_2^E) \geq M(v_2) \geq 0$ and in particular, $M_4(v_2^E) \geq M_4(v_2) = 2$. Hence,

$$v_2^E(N) - \sum_{j\in N\setminus\{1,2\}} M_j(v_2^E) \leq v_2^E(N) - M_3(v_2^E) - M_4(v_2^E) \leq 8-5-2 = 1.$$

Next, we show that $m_1(v_2^E) \leq 1$ by proving that, for all $S\in 2^N$ with $1\in S$,

$$v_2^E(S) - \sum_{j\in S, j\neq 1} M_j(v_2^E) \leq 1.$$

For $S = N$ and $S = N\setminus\{j\}$ for $j\in N\setminus\{1,2,3\}$, we see that $\{2,3\}\subseteq S$ and $v_2^E(S)\leq 8$, and consequently,

$$v_2^E(S) - \sum_{j\in S, j\neq 1} M_j(v_2^E) \leq v_2^E(S) - M_2(v_2^E) - M_3(v_2^E) \leq 8-2-5 = 1.$$

For $S = N\setminus\{3\}$, we have that $v_2^E(S) = 3$ and $M_2(v_2^E)\geq 2$, and consequently,

$$v_2^E(S) - \sum_{j\in S, j\neq 1} M_j(v_2^E) \leq v_2^E(S) - M_2(v_2^E) \leq 3-2 = 1.$$

For $S = N\setminus\{2\}$, we have that $\{3,4\}\subseteq S$ and $v_2^E(S)\leq 6$, and consequently,

$$v_2^E(S) - \sum_{j\in S, j\neq 1} M_j(v_2^E) \leq v_2^E(S) - M_3(v_2^E) - M_4(v_2^E) \leq 6-5-2 = -1.$$

For all $S\in 2^N$ with $|S|\leq n-2$ and $\{1,2\}\subseteq S$, we have that $v_2^E(S)\leq v_2(S) = 3$ and $2\in S$, and consequently,

$$v_2^E(S) - \sum_{j\in S, j\neq 1} M_j(v_2^E) \leq v_2(S) - M_2(v_2^E) \leq 3-2 = 1.$$

Finally, for $S \in 2^N$ with $|S| \leq n-2$ and $\{1,2\} \not\subseteq S$, we have that $v_2^E(S) = 0$ and $M(v^E) \geq 0$, and consequently,

$$v_2^E(S) - \sum_{j \in S, j \neq 1} M_j(v_2^E) \leq 0.$$

We may conclude that $m_1(v_2^E) = \max_{S \in 2^N : 1 \in S} \left\{ v_2^E(S) - \sum_{j \in S, j \neq 1} M_j(v_2^E) \right\} \leq 1.$

Similarly, one can show that $m_2(v_2^E) \leq 1$ and thus

$$m_1(v_2^E) + m_2(v_2^E) \leq 2. \tag{3.6}$$

Consequently, by combining Equation (3.5) and Equation (3.6), we can conclude that Equation (3.4) is satisfied. Thus, v_2^E is not compromise stable.

This finishes the construction of the essential communication situation $(N, v_2, E) \in \mathcal{ECS}^N$ where v_2 is strongly compromise admissible, while v_2^E is not compromise stable. □

Finally, we focus on the inheritance of compromise stability. For a three player essential communication situation, compromise stability is equivalent with balancedness and thus always inherited. For essential communication situations with more than three players, inheritance of compromise stability is only (trivially) guaranteed for the family of complete graphs. Theorem 3.3 below formalizes this impossibility result and in the proof, we construct an essential communication situation with an underlying compromise stable game such that the graph-restricted game is not compromise stable.

Theorem 3.3 *Let (N, E) be a connected graph that is not complete and $|N| > 3$. Then there exists an essential communication situation $(N, v, E) \in \mathcal{ECS}^N$ where v is compromise stable such that v^E is not compromise stable.*

Proof: Since (N, E) is not complete and $|N| > 3$, set $N = \{1, 2, 3, 4, \ldots, n\}$ and assume w.l.o.g. that $\{1, 2\} \notin E$, while $\{1, 3\} \in E$. Consider the essential communication situation[4] $(N, v_3, E) \in \mathcal{ECS}^N$ with, for all $S \in 2^N$,

$$v_3(S) = \begin{cases} 7, & \text{if } S = N; \\ 6, & \text{if } S = N \setminus \{1\}; \\ 5, & \text{if } S = N \setminus \{2\}; \\ 4, & \text{if } S = N \setminus \{4\}; \\ 3, & \text{if } S \notin \{N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{4\}\}, \text{ and } \{1,2\} \subseteq S \text{ or } \{1,3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

[4]Note that it can be readily verified that $v_3 \in TU^N$ is a superadditive game, implying that Equation (3.2) is satisfied.

First, we check that v_3 is indeed compromise stable. For this, note that $CC(v_3) \neq \emptyset$, since it can be readily verified that $M(v_3) = (1, 2, 4, 3, 4, \ldots, 4)$ and $m(v_3) = (1, 2, 2, 0, 0, \ldots, 0)$.

Moreover, for $S \in \{N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{4\}\}$ and for $S \in 2^N$ with $v_3(S) = 0$, we obviously have that

$$v_3(S) \leq \max \left\{ \sum_{i \in S} m_i(v_3), v_3(N) - \sum_{j \in N \setminus S} M_j(v_3) \right\}.$$

For $S \in 2^N, S \notin \{N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{4\}\}$ with $\{1, 2\} \subseteq S$, it holds that

$$v_3(S) \leq m_1(v_3) + m_2(v_3).$$

Similarly, for $S \in 2^N, S \notin \{N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{4\}\}$ with $\{1, 3\} \subseteq S$, it holds that

$$v_3(S) \leq m_1(v_3) + m_3(v_3).$$

Hence, v_3 is compromise stable.

Next, we show that v_3^E is not compromise stable, by showing that

$$v_3^E(\{1, 3\}) > \max \left\{ m_1(v_3^E) + m_3(v_3^E), v_3^E(N) - \sum_{j \in N \setminus \{1,3\}} M_j(v_3^E) \right\}. \tag{3.7}$$

First, note that $v_3^E(\{1, 3\}) = v_3(\{1, 3\}) = 3$.

Secondly, we show that $m_1(v_3^E) = 0$. For $S \in 2^N$ with $1 \in S$ and $S \neq \{1, 2\}$, we have that

$$v_3^E(S) - \sum_{j \in S, j \neq 1} M_j(v_3^E) \leq v_3(S) - \sum_{j \in S, j \neq 1} M_j(v_3) \leq 0,$$

where the first inequality follows from our basic assumption that $v_3^E \leq v_3$ and Lemma 3.1. Moreover, for $S = \{1, 2\}$, it holds that $v_3^E(\{1, 2\}) = 0$, since $\{1, 2\} \notin E$ and consequently,

$$v_3^E(\{1, 2\}) - M_2(v_3^E) \leq 0 - M_2(v_3) \leq 0,$$

where we also used Lemma 3.1 for the first inequality. Hence, since $v_3^E(\{1\}) = 0$, we have that

$$m_1(v_3^E) = \max_{S \in 2^N : 1 \in S} \left\{ v_3^E(S) - \sum_{j \in S, j \neq 1} M_j(v_3^E) \right\} = 0.$$

Furthermore, by using Lemma 3.1, it follows that $m_3(v_3^E) \leq m_3(v_3) = 2$. Consequently,

$$m_1(v_3^E) + m_3(v_3^E) \leq 2. \tag{3.8}$$

Next, we prove that

$$v_3^E(N) - \sum_{j \in N \setminus \{1,3\}} M_j(v_3^E) \leq 2. \tag{3.9}$$

For this, note that, by using Lemma 3.1, $v_3^E(N) = v_3(N) = 7$, $M_2(v_3^E) \geq M_2(v_3) = 2$, $M_4(v_3^E) \geq M_4(v_3) = 3$ and $M_j(v_3^E) \geq M_j(v_3) \geq 0$ for all $j \in N$. Consequently,

$$v_3^E(N) - \sum_{j \in N \setminus \{1,3\}} M_j(v_3^E) \leq v_3^E(N) - M_2(v_3^E) - M_4(v_3^E) \leq 7 - 2 - 3 = 2.$$

By combining Equation (3.8) and Equation (3.9), we can conclude that Equation (3.7) is satisfied. Thus, v_3^E is not compromise stable.

This finishes the construction of the essential communication situations $(N, v_3, E) \in \mathcal{ECS}^N$ with v_3 is compromise stable, while v_3^E is not compromise stable.

□

3.4 Invariance of the nucleolus

In this section, we focus on the nucleolus of graph-restricted games as an alternative for the Shapley value to evaluate essential communication situations. In contrast to the Shapley value, the nucleolus is more likely to be robust to changes in the communication restrictions. Therefore, we study the invariance of the nucleolus. That is, we focus on necessary and sufficient conditions on an essential communication situation such that the nucleolus of the graph-restricted game equals the nucleolus of the game underlying the essential communication situation. First, we formally recall the definition and various results on the nucleolus.

Let $v \in TU^N$ be an imputation admissible game. The *excess* of a coalition $S \in 2^N$ with respect to an imputation $x \in I(v)$ is defined as $\text{Exc}(S, x, v) = v(S) - \sum_{i \in S} x_i$, while the *excess vector* $\theta(x) \in \mathbb{R}^{2^{|N|}}$ is defined as the vector consisting of all excesses in non-increasing order, i.e., $\theta_k(x) \geq \theta_{k+1}(x)$ for all $k \in \{1, \ldots, 2^{|N|} - 1\}$. The *nucleolus* (cf. Schmeidler, 1969) $\text{nuc}(v) \in \mathbb{R}^N$ is the unique imputation for which $\theta(\text{nuc}(v)) \preceq \theta(x)$ for all $x \in I(v)$, where $\preceq$ denotes the lexicographical order. It is known that $\text{nuc}(v) \in C(v)$ for all balanced games $v \in TU^N$.

For compromise stable games and strongly compromise admissible games, the nucleolus can be described by a direct, closed formula. This formula is based on the Talmud rule for an associated claims problem.

Proposition 3.1 [cf. Quant et al., 2005; Driessen, 1988] *Let $v \in TU^N$.*

i) If v is compromise stable, then, for all $i \in N$,

$$nuc_i(v) = m_i(v) + TAL_i\left(v(N) - \sum_{j\in N} m_j(v), M(v) - m(v)\right);$$

ii) If v is strongly compromise admissible, then, for all $i \in N$,

$$nuc_i(v) = M_i(v) - \frac{1}{|N|}\left(\sum_{j\in N} M_j(v) - v(N)\right).$$

For the study of the invariance of the nucleolus, it is important to recall first that imputation admissibility is always inherited. That is, for every essential communication situation with an underlying imputation admissible game it holds that the graph-restricted game is also imputation admissible.

Recall from Theorem 3.1 that the family of biconnected graphs guarantees the inheritance of strong compromise admissibility from the underlying game to the graph-restricted game. For essential communication situations with an underlying strongly compromise admissible game, we show that this family of biconnected graphs also guarantees the invariance of the nucleolus. Moreover, for every connected graph that is not biconnected, we explicitly construct an essential communication situation with an underlying strongly compromise admissible game for which the nucleolus of the graph-restricted game is not equal to the nucleolus of the underlying game. For this, we benefit from the construction in the proof of Theorem 3.1.

Theorem 3.4 *The following two statements hold:*

i) Let $(N, v, E) \in \mathcal{ECS}^N$ be an essential communication situation with (N, E) biconnected and v strongly compromise admissible. Then $nuc(v^E) = nuc(v)$;

ii) Let (N, E) be a connected graph that is not biconnected. Then there exists an essential communication situation $(N, v, E) \in \mathcal{ECS}^N$ with v strongly compromise admissible such that $nuc(v^E) \neq nuc(v)$.

Proof: *i)* First, note that v^E is strongly compromise admissible, according to part i) of Theorem 3.1. Hence, by using Proposition 3.1, for all $i \in N$,

$$\mathrm{nuc}_i(v^E) = M_i(v^E) - \frac{1}{|N|}\left(\sum_{j\in N} M_j(v^E) - v^E(N)\right)$$

$$= M_i(v) - \frac{1}{|N|}\left(\sum_{j\in N} M_j(v) - v(N)\right)$$
$$= \text{nuc}_i(v),$$

since $M(v^E) = M(v)$ for any biconnected graph (N,E). Consequently, $\text{nuc}(v^E) = \text{nuc}(v)$.

ii) Since (N,E) is not biconnected, set $N = \{1,2,\dots,n\}$ and assume w.l.o.g. that $\{1,2\},\{2,3\} \in E$, that the induced subgraph on $N \setminus \{2\}$ is not connected and that players 1 and 3 are in two different components in the induced subgraph on $N \setminus \{2\}$. Reconsider the essential communication situation $(N, v_1, E) \in \mathcal{ECS}^N$ with, for all $S \in 2^N$,

$$v_1(S) = \begin{cases} 1, & \text{if } \{1,2\} \subseteq S \text{ or } \{1,3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that v_1 is strongly compromise admissible and since $M(v_1) = (1,0,0,\dots,0)$, we have that, using Proposition 3.1,

$$\text{nuc}(v_1) = (1,0,0,\dots,0).$$

Moreover, for all $S \in 2^N$,

$$v_1^E(S) = \begin{cases} 1, & \text{if } \{1,2\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $M(v_1^E) = (1,1,0,\dots,0)$, $m(v_1^E) = (0,0,0,\dots,0)$ and v_1^E is compromise stable. Consequently,

$$\text{nuc}(v_1^E) = (\tfrac{1}{2}, \tfrac{1}{2}, 0, \dots, 0).$$

Hence, $\text{nuc}(v_1^E) \neq \text{nuc}(v_1)$. This concludes the construction of the essential communication situation $(N, v_1, E) \in \mathcal{ECS}^N$ with v_1 strongly compromise admissible such that $\text{nuc}(v_1^E) \neq \text{nuc}(v_1)$. □

With regard to the class of essential communication situations with an underlying compromise stable game, we derive an impossibility result. For every connected graph that is not complete, we construct an essential communication situation with an underlying compromise stable game for which the nucleolus is not invariant. This construction is mainly based on Theorem 3.3. It involves, among other things, the use of the so-called Kohlberg criterion (Kohlberg, 1971).

A collection $B \subseteq 2^N \setminus \{\emptyset\}$ is called *balanced* if there exists a function $\lambda : B \to \mathbb{R}_{++}$ such that $\sum_{S \in B: i \in S} \lambda(S) = 1$ for all $i \in N$. Let $v \in TU^N$ be a balanced game and let $x \in I(v)$ be an imputation. Define

$$B_1(x,v) = \left\{ S \in 2^N \setminus \{\emptyset, N\} \;\middle|\; \text{Exc}(S,x,v) \geq \text{Exc}(T,x,v) \text{ for all } T \in 2^N \setminus \{\emptyset, N\} \right\},$$

as the set of coalitions (non-empty and not the grand coalition) with the highest excess with respect to the imputation x. Proceeding recursively, define, for all $k \in \{2, 3, \ldots\}$,

$$B_k(x,v) = \Bigg\{ S \in 2^N \setminus \{\emptyset, N\} \;\Bigg|\; S \notin \bigcup_{r=1}^{k-1} B_r(x,v) \text{ and } \mathrm{Exc}(S,x,v) \geq \mathrm{Exc}(T,x,v) \text{ for all } T \in 2^N \setminus \{\emptyset, N\} \text{ with } T \notin \bigcup_{r=1}^{k-1} B_r(x,v) \Bigg\}.$$

Clearly, there exists a unique number $t(x) \in \mathbb{N}$ such that $B_k(x,v) \neq \emptyset$ for all $k \in \{1, \ldots, t(x)\}$ and $B_{t(x)+1}(x,v) = \emptyset$.

Proposition 3.2 [cf. Kohlberg, 1971] *Let $v \in TU^N$ be a balanced game and let $x \in I(v)$ be an imputation. Then $x = nuc(v)$ if and only if the collection $\bigcup_{k=1}^{s} B_k(x,v)$ is balanced for all $s \in \{1, \ldots, t(x)\}$.*

Theorem 3.5 formalizes the impossibility result with regard to the invariance of the nucleolus for essential communication situations with an underlying compromise stable game.

Theorem 3.5 *Let (N, E) be a connected graph that is not complete. Then there exists an essential communication situation $(N, v, E) \in \mathcal{ECS}^N$ with v compromise stable such that $nuc(v^E) \neq nuc(v)$.*

Proof: For $|N| = 3$, we can just (re)consider the essential communication situation $(N, v_1, E) \in \mathcal{ECS}^N$ as in the proof of Theorem 3.4. Compromise stability of v_1 is implied by the strong compromise admissibility of v_1. Recall that $\mathrm{nuc}(v_1) = (1, 0, 0)$.

Moreover, the fact that (N, E) is not complete but connected implies w.l.o.g. that $\{1, 2\}, \{2, 3\} \in E$, while $\{1, 3\} \notin E$. Hence, $\mathrm{nuc}(v_1^E) = (\frac{1}{2}, \frac{1}{2}, 0)$ and consequently, $\mathrm{nuc}(v_1^E) \neq \mathrm{nuc}(v_1)$.

In the remainder, suppose that $|N| \geq 4$. For this, we revert to the construction as in the proof of Theorem 3.3.

Set $N = \{1, 2, 3, 4, \ldots, n\}$ and assume w.l.o.g. that $\{1, 2\} \notin E$ and $\{1, 3\} \in E$. Reconsider the essential communication situation $(N, v_3, E) \in \mathcal{ECS}^N$ with, for all $S \in 2^N$,

$$v_3(S) = \begin{cases} 7, & \text{if } S = N; \\ 6, & \text{if } S = N \setminus \{1\}; \\ 5, & \text{if } S = N \setminus \{2\}; \\ 4, & \text{if } S = N \setminus \{4\}; \\ 3, & \text{if } S \notin \{N, N \setminus \{1\}, N \setminus \{2\}, N \setminus \{4\}\}, \text{ and } \{1,2\} \subseteq S \text{ or } \{1,3\} \subseteq S; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that v_3 is compromise stable. Since $M(v_3) = (1, 2, 4, 3, 4, \ldots, 4)$ and $m(v_3) = (1, 2, 2, 0, 0, \ldots, 0)$, we have that, using Proposition 3.1,

$$\mathrm{nuc}_i(v_3) = \begin{cases} 1, & \text{if } i = 1; \\ 2, & \text{if } i = 2; \\ 2 + \frac{2}{n-2}, & \text{if } i = 3; \\ \frac{2}{n-2}, & \text{otherwise.} \end{cases} \tag{3.10}$$

To show that $\mathrm{nuc}(v_3^E) \neq \mathrm{nuc}(v_3)$, we use the Kohlberg criterion as formulated in Proposition 3.2 and show that $B_1(\mathrm{nuc}(v_3), v_3^E)$ is not balanced. For this, we need to identify the (non-trivial) coalitions with the highest excess. To structure this identification process, for $S \in 2^N \setminus \{\emptyset, N\}$ we distinguish between seven cases, in which players $1, 2$ and 3 play an important role:

I) $|S| = 1$ or $S \in \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$;

II) $|S| = 3$ with $\{1, 2\} \subseteq S$ and $3 \notin S$;

III) $3 < |S| < n - 1$ with $\{1, 2\} \subseteq S$ and $3 \notin S$;

IV) $2 < |S| < n - 1$ with $\{1, 3\} \subseteq S$;

V) $1 < |S| < n - 1$ with $\{1, 2\} \not\subseteq S$, $\{1, 3\} \not\subseteq S$ and $S \neq \{2, 3\}$;

VI) $|S| = n - 1$ with $S = N \setminus \{j\}$ for $j \in N \setminus \{1, 2, 3, 4\}$;

VII) $S \in \{N \setminus \{1\}, N \setminus \{2\}, N \setminus \{3\}, N \setminus \{4\}\}$.

Note that these seven cases indeed cover all coalitions: cases I, VI and VII deal with all coalitions with exactly 1 or $n - 1$ players. Case I also includes three specific 2-player coalitions. For the other coalitions, we distinguish whether $\{1, 2\} \subseteq S$ (and $3 \notin S$) or $\{1, 3\} \subseteq S$ or neither of the two inclusions. In particular, case II deals with the 3-player coalitions that contain both players 1 and 2, but not 3 and case III deals with similar coalitions that contain at least 4 players. In case IV, $\{1, 3\} \subseteq S$ and finally, case V deals with all coalitions for which both $\{1, 2\} \not\subseteq S$ and $\{1, 3\} \not\subseteq S$.

Next, we deal with each of the seven cases separately.

Case I) For this first case, we know that

$$v_3^E(S) = \begin{cases} 0, & \text{if } S = \{j\} \text{ for } j \in N; \\ 0, & \text{if } S = \{1, 2\}; \\ 3, & \text{if } S = \{1, 3\}; \\ 0, & \text{if } S = \{2, 3\}, \end{cases}$$

and hence, by using Equation (3.10), one readily checks that

$$\mathrm{Exc}(S, \mathrm{nuc}(v_3), v_3^E) = \begin{cases} -1, & \text{if } S = \{1\}; \\ -2, & \text{if } S = \{2\}; \\ \frac{-2}{n-2} - 2, & \text{if } S = \{3\}; \\ \frac{-2}{n-2}, & \text{if } S = \{j\} \text{ for } j \in N \setminus \{1,2,3\}; \\ -3, & \text{if } S = \{1,2\}; \\ \frac{-2}{n-2}, & \text{if } S = \{1,3\}; \\ \frac{-2}{n-2} - 4, & \text{if } S = \{2,3\}. \end{cases}$$

Note that, due to the fact that $n \geq 4$, the highest excess listed above equals $\frac{-2}{n-2}$.

Case II) For all $S \in 2^N$ with $|S| = 3$, $\{1,2\} \subseteq S$ and $3 \notin S$, it holds that

$$v_3^E(S) = \begin{cases} 3, & \text{if the induced subgraph on } S \text{ is connected;} \\ 0, & \text{otherwise,} \end{cases}$$

and hence, by using Equation (3.10),

$$\mathrm{Exc}(S, \mathrm{nuc}(v_3), v_3^E) = \begin{cases} \frac{-2}{n-2}, & \text{if the induced subgraph on } S \text{ is connected;} \\ \frac{-2}{n-2} - 3, & \text{otherwise.} \end{cases}$$

Note that the induced subgraph on S is connected if and only if $S = \{1,2,j\}$ and $\{1,j\}, \{2,j\} \in E$ for a certain $j \in N \setminus \{1,2,3\}$.

Case III) For all $S \in 2^N$ with $3 < |S| < n-1$, $\{1,2\} \subseteq S$ and $3 \notin S$, it holds that

$$v_3^E(S) \leq 3.$$

Moreover, since $\mathrm{nuc}_i(v_3) > 0$ for all $i \in N$, we have that

$$\sum_{i \in S} \mathrm{nuc}_i(v_3) > \mathrm{nuc}_1(v_3) + \mathrm{nuc}_2(v_3) + \mathrm{nuc}_j(v_3),$$

for a certain player $j \in S, j \neq 1, 2$. Combining this and using Equation (3.10), we see that

$$\begin{aligned} \mathrm{Exc}(S, \mathrm{nuc}(v_3), v_3^E) &= v_3^E(S) - \sum_{i \in S} \mathrm{nuc}_i(v_3) \\ &< v_3^E(S) - \mathrm{nuc}_1(v_3) - \mathrm{nuc}_2(v_3) - \mathrm{nuc}_j(v_3) \\ &\leq 3 - 1 - 2 - \frac{2}{n-2} \\ &= \frac{-2}{n-2}. \end{aligned}$$

Case IV) For all $S \in 2^N$ with $2 < |S| < n-1$ and $\{1,3\} \subseteq S$, it holds that $v_3^E(S) = 3$ and hence, following a similar reasoning as before,

$$\begin{aligned}\mathrm{Exc}(S, \mathrm{nuc}(v_3), v_3^E) &< v_3^E(S) - \mathrm{nuc}_1(v_3) - \mathrm{nuc}_3(v_3)\\ &= 3 - 1 - 2 - \frac{2}{n-2}\\ &= \frac{-2}{n-2}.\end{aligned}$$

Case V) For all $S \in 2^N$ with $1 < |S| < n-1$, $\{1,2\} \nsubseteq S$, $\{1,3\} \nsubseteq S$ and $S \neq \{2,3\}$, it holds that $v_3^E(S) = 0$ and hence, following a similar reasoning as before, there is a player $j \in S \setminus \{1,2,3\}$ such that

$$\mathrm{Exc}(S, \mathrm{nuc}(v_3), v_3^E) < v_3^E(S) - \mathrm{nuc}_j(v_3) = 0 - \frac{2}{n-2} = \frac{-2}{n-2}.$$

Case VI) For all $S \in 2^N$ with $|S| = n-1$ and $S = N \setminus \{j\}$ for $j \in N \setminus \{1,2,3,4\}$, it holds that $v_3^E(S) = 3$, since $\{1,3\} \subseteq S$ and hence, as before,

$$\mathrm{Exc}(S, \mathrm{nuc}(v_3), v_3^E) < v_3^E(S) - \mathrm{nuc}_1(v_3) - \mathrm{nuc}_3(v_3) = 3 - 1 - 2 - \frac{2}{n-2} = \frac{-2}{n-2}.$$

Case VII) Finally, for all $S \in 2^N$ with $S \in \{N \setminus \{1\}, N \setminus \{2\}, N \setminus \{3\}, N \setminus \{4\}\}$, the worth of the coalition S in the graph-restricted game depends on whether the induced subgraph is connected or not:

$$v_3^E(N \setminus \{4\}) = \begin{cases} 4, & \text{if the induced subgraph on } N \setminus \{4\} \text{ is connected;}\\ 3, & \text{otherwise,}\end{cases}$$

$$v_3^E(N \setminus \{3\}) = \begin{cases} 3, & \text{if the induced subgraph on } N \setminus \{3\} \text{ is connected;}\\ 0, & \text{otherwise,}\end{cases}$$

$$v_3^E(N \setminus \{2\}) = \begin{cases} 5, & \text{if the induced subgraph on } N \setminus \{2\} \text{ is connected;}\\ 3, & \text{otherwise,}\end{cases}$$

$$v_3^E(N \setminus \{1\}) = \begin{cases} 6, & \text{if the induced subgraph on } N \setminus \{1\} \text{ is connected;}\\ 0, & \text{otherwise.}\end{cases}$$

Consequently, by using Equation (3.10),

$$\mathrm{Exc}(N \setminus \{4\}, \mathrm{nuc}(v_3), v_3^E) = \begin{cases} \frac{2}{n-2} - 3, & \text{if the induced subgraph on } N \setminus \{4\}\\ & \hfill \text{is connected;}\\ \frac{2}{n-2} - 4, & \text{otherwise,}\end{cases}$$

$$\mathrm{Exc}(N\setminus\{3\},\mathrm{nuc}(v_3),v_3^E)=\begin{cases}\frac{2}{n-2}-2, & \text{if the induced subgraph on } N\setminus\{3\}\\ & \text{is connected;}\\ \frac{2}{n-2}-5, & \text{otherwise,}\end{cases}$$

$$\mathrm{Exc}(N\setminus\{2\},\mathrm{nuc}(v_3),v_3^E)=\begin{cases}0, & \text{if the induced subgraph on } N\setminus\{2\}\\ & \text{is connected;}\\ -2, & \text{otherwise,}\end{cases}$$

$$\mathrm{Exc}(N\setminus\{1\},\mathrm{nuc}(v_3),v_3^E)=\begin{cases}0, & \text{if the induced subgraph on } N\setminus\{1\}\\ & \text{is connected;}\\ -6, & \text{otherwise.}\end{cases}$$

In order to determine which of the above excesses is the highest, note that, since $n \geq 4$,

$$\mathrm{Exc}(N\setminus\{4\},\mathrm{nuc}(v_3),v_3^E) \leq \frac{2}{n-2}-3 \leq -2 < \frac{-2}{n-2},$$

and, if $n > 4$,

$$\mathrm{Exc}(N\setminus\{3\},\mathrm{nuc}(v_3),v_3^E) \leq \frac{2}{n-2}-2 < -1 < \frac{-2}{n-2}.$$

This concludes the analysis of the seven cases. We may conclude that all coalitions under consideration in cases III, IV, V and VI can not be coalitions with the highest excess.

Furthermore, if the induced subgraph on $N\setminus\{1\}$ or the induced subgraph on $N\setminus\{2\}$ is connected, the highest excess equals 0 and

$$\begin{aligned}B_1(\mathrm{nuc}(v_3),v_3^E) &= \{N\setminus\{1\}, N\setminus\{2\}\},\\ B_1(\mathrm{nuc}(v_3),v_3^E) &= \{N\setminus\{1\}\}, \text{ or}\\ B_1(\mathrm{nuc}(v_3),v_3^E) &= \{N\setminus\{2\}\}.\end{aligned}$$

Clearly, for each of these cases, $B_1(\mathrm{nuc}(v_3),v_3^E)$ is not a balanced collection. Note that, if $n = 4$, it indeed holds that the induced subgraph on $N\setminus\{1\}$ or the induced subgraph on $N\setminus\{2\}$ is connected, due to the connectedness of the graph and the fact that $\{1,2\} \notin E$.

Finally, assume that $n > 4$ and that both induced subgraphs on $N\setminus\{1\}$ and $N\setminus\{2\}$ are not connected. Then the highest excess equals $\frac{-2}{n-2}$ (> -1) and according to cases I and II,

$$\begin{aligned}B_1(\mathrm{nuc}(v_3),v_3^E) = &\{\{j\} \mid j \in N\setminus\{1,2,3\}\}\\ &\cup \{\{1,3\}\}\\ &\cup \{\{1,2,j\} \mid j \in N\setminus\{1,2,3\} \text{ and both } \{1,j\}\in E \text{ and } \{2,j\}\in E\}.\end{aligned}$$

Note that if there is no $j \in N \setminus \{1,2,3\}$ such that both $\{1,j\} \in E$ and $\{2,j\} \in E$, then $B_1(\mathrm{nuc}(v_3), v_3^E)$ is not balanced, since $2 \notin S$ for all $S \in B_1(\mathrm{nuc}(v_3), v_3^E)$.

So let $j \in N \setminus \{1,2,3\}$ be such that both $\{1,j\} \in E$ and $\{2,j\} \in E$. Suppose $\lambda : B_1(\mathrm{nuc}(v_3), v_3^E) \to \mathbb{R}_{++}$ is such that $\sum_{S \in B_1(\mathrm{nuc}(v_3), v_3^E): i \in S} \lambda(S) = 1$ for all $i \in N$. For $i = 3$, this condition boils down to $\lambda(\{1,3\}) = 1$. Then, however,

$$\sum_{S \in B_1(\mathrm{nuc}(v_3), v_3^E): 1 \in S} \lambda(S) \geq \lambda(\{1,3\}) + \lambda(\{1,2,j\}) > 1,$$

and it follows that $B_1(\mathrm{nuc}(v_3), v_3^E)$ is not balanced.

Together, we can conclude that $\mathrm{nuc}(v_3^E) \neq \mathrm{nuc}(v_3)$ according to Proposition 3.2. $\square$

Interestingly, a possibility result can be obtained if we restrict attention to essential communication situations with an underlying simple game.

A cooperative game $v \in TU^N$ is called *simple* if $v(S) \in \{0,1\}$ for all $S \in 2^N$, $v(N) = 1$ and v is monotonic: $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subseteq T$. Moreover, for a simple game $v \in TU^N$, the set of *veto-players* is given by

$$\mathrm{veto}(v) = \{i \in N \mid v(N \setminus \{i\}) = 0\}.$$

Equivalently, for all $i \in N$, it holds that $i \in \mathrm{veto}(v)$ if and only if $v(S) = 0$ for all $S \in 2^N$ with $i \notin S$. For simple games, having veto-players is equivalent to balancedness, which in turn is equivalent to compromise stability. Moreover, if a simple game has veto-players, then the nucleolus is, for all $i \in N$, given by

$$\mathrm{nuc}_i(v) = \begin{cases} \frac{1}{|\mathrm{veto}(v)|}, & \text{if } i \in \mathrm{veto}(v); \\ 0, & \text{otherwise.} \end{cases}$$

Finally, note that if the game underlying an essential communication situation is simple, then the graph-restricted game is simple too.

If, in addition to compromise stability, we also require that the underlying game is simple, Theorem 3.6 shows that the nucleolus is invariant for all such essential communication situations if the graph is biconnected. Furthermore, for every connected graph that is not biconnected, we construct an essential communication situation with an underlying game that is both compromise stable and simple for which the nucleolus of the graph-restricted game is not equal to the nucleolus of the underlying game.

Theorem 3.6 *The following two statements hold:*

i) Let $(N, v, E) \in \mathcal{ECS}^N$ be an essential communication situation with (N, E) biconnected and v both compromise stable and simple. Then $nuc(v^E) = nuc(v)$;

ii) Let (N, E) be a connected graph that is not biconnected. Then there exists an essential communication situation $(N, v, E) \in \mathcal{ECS}^N$ with v both compromise stable and simple such that $nuc(v^E) \neq nuc(v)$.

Proof: *i)* Clearly, it suffices to show that $\text{veto}(v) = \text{veto}(v^E)$. Evidently, for all $i \in N$, the induced subgraph on $N \setminus \{i\}$ is connected, since (N, E) is biconnected. Hence, $v^E(N \setminus \{i\}) = v(N \setminus \{i\})$ and consequently, $\text{veto}(v) = \text{veto}(v^E)$.

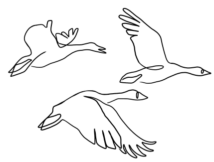
ii) Reconsider the essential communication situation $(N, v_1, E) \in \mathcal{ECS}^N$ as in the proof of Theorem 3.4. As seen in the proof of Theorem 3.5, v_1 is compromise stable. Furthermore, v_1 is clearly simple.

Consequently,

$$\text{nuc}(v_1^E) = (\tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0) \neq (1, 0, 0, \ldots, 0) = \text{nuc}(v_1),$$

which concludes the proof. $\square$

4 Interactive sequencing with non-linear cost functions



— *Geese frequently fly in optimal V-formation*

4.1 Introduction

In interactive sequencing situations, two issues need to be addressed: an optimization problem of finding an optimal order and an allocation problem of finding a suitable allocation. In the optimization problem, the goal is to find an optimal processing order that minimizes the total processing costs. Given an initial processing order, the players can obtain cost savings by rearranging to an optimal order. In the allocation problem, the goal is to find a suitable allocation for these cost savings.

Traditionally, in (interactive) sequencing situations, individual cost functions are assumed to be linear. For these *standard sequencing situations*, Smith (1956) showed that in an optimal order, the players are arranged according to a (weakly) decreasing urgency index. For each player, the urgency index is the ratio of the linear cost coefficient and the processing time. Moreover, a Smith order can be reached from the initial order by consecutively repairing neighbor misplacements. By dividing the corresponding neighbor switching gains equally among the two players involved, the allocation prescribed by the equal gain splitting (EGS) rule for a standard sequencing situation is obtained, defined by Curiel, Pederzoli, and Tijs (1989).

Hamers, Suijs, Tijs, and Borm (1996) generalized the notion of the EGS-rule and defined the gain splitting (GS) rules for standard sequencing situations. Instead of equal division, the GS-rules divide the neighbor switching gains in an arbitrary way among the two neighbors. The allocations specified by the GS-rules, and in particular the EGS-rule, for a standard sequencing situation (with linear cost functions) turn out to be core-elements of the associated standard sequencing game. A sequencing game (Curiel et al., 1989) is a cooperative game in which the worth of a coalition is defined as the maximal cost savings the coalition can obtain by admissible rearrangements with respect to the initial order. Curiel et al. (1989) showed that a standard sequencing game, that is, the game associated to a standard sequencing situation, is a convex game.

Interactive sequencing problems are studied from different perspectives. Considerable research is done in the direction of adding elements to the model. For example, Hamers, Borm, and Tijs (1995) added ready times, Borm, Fiestras-Janeiro, Hamers, Sánchez, and Voorneveld (2002) studied due dates, Hamers, Klijn, and Van Velzen (2005) added precedence relations, Estévez-Fernández, Borm, Calleja, and Hamers (2008) incorporated repeated players, and Liu, Lu, and Qi (2018) added unavailable periods for the machine. A different direction is the stream of research in which assumptions of the standard sequencing model are relaxed or modified. For example, Slikker (2006) relaxed the assumption of cooperation between players, Lohmann, Borm, and Slikker (2014) modified the definition of the time a job spends in the system, Musegaas, Borm, and Quant (2015) relaxed the set of admissible rearrangements, and Yang, Sun, Hou, and Xu (2019) analyzed external influence on the worth of the coalition. A review of the literature on scheduling with learning effects is done by Biskup (2008). Finally, Van den Brink, Van der Laan, and Vasil'ev (2007) studied interactive sequencing problems from the perspective of line-graph games.

In this chapter, based on Schouten, Saavedra-Nieves, and Fiestras-Janeiro (2021) and building on Saavedra-Nieves, Schouten, and Borm (2020), we deal with specific classes of interactive sequencing situations with non-linear cost functions. We follow the above-mentioned lines (cf. Curiel et al., 1989) in the sense that we focus on conditions that guarantee convexity of the associated sequencing game and that provide core-allocations for the associated sequencing game. In particular, we deal with three specific examples of non-linear cost functions, namely exponential, discounting and logarithmic cost functions. For an exponential cost function, it holds that the marginal costs are increasing over time, for example due to deterioration effects on the machine. On the other hand, learning effects can lead to decreasing marginal costs over time, which is the case for a logarithmic cost function. Furthermore, discounting cost functions model situations in which future costs are discounted with a certain discount rate.

Exponential sequencing situations were introduced by Saavedra-Nieves et al. (2020) and *discounting sequencing situations* were introduced by Rothkopf (1966). In the same spirit, we consider sequencing situations with logarithmic cost functions. For such *logarithmic sequencing situations*, we explicitly derive the expression for the cost savings obtained by two players if they interchange their positions. Moreover, we show that in an optimal order, players are arranged according to (weakly) increasing processing times. A common feature among the three types of sequencing situations mentioned above is the fact that in all of them it is possible to directly obtain partial optimal orders for any subgroup of players once an optimal order for the set of all players is determined. Notice that this feature does not need to be satisfied by sequencing situations in general.

For the study of these classes of interactive sequencing situations and games, we focus on two fundamental tasks. First, we provide a specific result for convexity of the associated sequencing games. Saavedra-Nieves et al. (2020) showed that, by imposing a set of conditions on the neighbor switching gains, the associated sequencing game of a sequencing situation is convex. These conditions require the neighbor switching gains to be non-negative and non-decreasing for misplacements and non-positive for non-misplacements. In this chapter, we provide an analogous and complementing result that establishes a new collection of conditions on the gains to ensure convexity for the associated sequencing games. This guarantees convexity for a much wider class of sequencing games and, in particular, for discounting and logarithmic sequencing games.

The second fundamental issue in this context is the definition of allocation rules for sequencing situations with non-linear cost functions that satisfy the common feature mentioned above. For standard sequencing situations with linear cost functions, the cost savings obtained by interchanging a neighbor misplacement are independent of the position of the two players involved. In other words, these neighbor switching gains are not dependent on the moment in time the players interchange their positions. In contrast, for sequencing situations with non-linear cost functions, the neighbor switching gains may be time-dependent. As a result, due to the non-linearity of the cost functions, the neighbor switching gains depend on the path from the initial order to an optimal order.

The preceding observation implies that it is not possible to directly apply the GS-rules to sequencing situations with non-linear cost functions in the sense that the properties satisfied by the GS-rules in the standard case are not maintained in general. In a standard sequencing situation, every path from the initial order to an optimal order leads to the same neighbor switching gains. This is not generally true for any non-linear cost function, so the specifications of the path that has been selected from the initial order to an optimal one becomes fundamental. Given an optimal order, we provide two different procedures that specify a path from the initial order to the given optimal order:

- The *growing head* procedure. This procedure starts with the player that occupies the first position in the optimal order and consecutively moves this player to that position. Secondly, the player that is in the second position of the optimal order moves to that position and so on, successively until all players are in their positions in the optimal order.

- The *growing tail* procedure. This procedure reverses the idea of the growing head procedure and starts with the player that is in the last position of the optimal order.

To obtain an allocation from these two procedures, we adopt the idea of the GS-rules for standard sequencing situations. That is, we divide the neighbor switching gains in every step of a path from the initial order to an optimal order among the two players involved using a distribution of weights not necessarily equal. This leads to two different type of allocations, depending on the procedure that is used: the *gain splitting head rules (GSH-rules)* and the *gain splitting tail rules (GST-rules)*. We show that the two sets of conditions on the neighbor switching gains required for convexity also ensure that the respective type of allocation rules prescribe core-elements of the associated sequencing game. In particular, we show that for discounting and logarithmic sequencing situations, the GST-rules lead to core-elements, while for the three subclasses of exponential sequencing situations as defined in Saavedra-Nieves et al. (2020), the GSH-rules result in a core-elements.

This chapter is structured as follows. Section 4.2 contains preliminaries on general interactive sequencing situations. Section 4.3 provides an analysis of exponential, discounting and logarithmic sequencing situations. Section 4.4 provides a result on convexity. Finally, Section 4.5 introduces two types of allocation rules for sequencing situations with arbitrary non-linear cost functions.

4.2 General interactive sequencing situations

In a *(general) interactive sequencing situation*, there is a non-empty, finite set of players N that each have a job that needs to be processed on a single machine.[1] A *(processing) order* of the players is described by a bijective function $\sigma : N \to \{1, 2, \ldots, |N|\}$ in which $\sigma(i) = k$ means that the job of player i is in position k of the order σ.[2] The set of all orders of N is denoted by $\Pi(N)$. Moreover, let $\sigma_0 \in \Pi(N)$ denote the *initial (processing) order* of the players, providing the initial processing rights on the machine. For every player $i \in N$, let $p_i \in \mathbb{R}_{++}$ denote the *processing time* of the job of player i and let the *cost function* of player i be given by $c_i : [0, \infty) \to \mathbb{R}$, where the argument $t \in [0, \infty)$ is the number of time units player i has spent in the system. Here, it is assumed that the machine starts processing at time $t = 0$ and that all jobs are present at $t = 0$.

[1] Here, the word 'interactive' is added to emphasize the fact that we address two issues: the joint optimization problem and the joint cost savings allocation problem. Therefore, we include an initial order from the onset.

[2] With, e.g., $N = \{1, 2, 3\}$, $\sigma = (3, 2, 1)$ denotes the order in which player 3 is processed first, then player 2 and finally, player 1.

Following Saavedra-Nieves (2019) and Saavedra-Nieves et al. (2020), a *(general) interactive sequencing situation* is represented by a tuple (N, σ_0, p, c), where $p = (p_i)_{i \in N}$ and $c = (c_i)_{i \in N}$ summarize the processing times and cost functions, respectively. The set of all interactive sequencing situations with player set N is denoted by SEQ^N and an interactive sequencing situation (N, σ_0, p, c) is also denoted by $(\sigma_0, p, c) \in SEQ^N$.

Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation and let $\sigma \in \Pi(N)$ be an order. The *set of predecessors* of player $i \in N$ with respect to σ is denoted by $P(\sigma, i)$ and given by $P(\sigma, i) = \{h \in N \mid \sigma(h) < \sigma(i)\}$, while the *set of followers* is denoted by $F(\sigma, i)$ and given by $F(\sigma, i) = \{h \in N \mid \sigma(h) > \sigma(i)\}$. The *starting time* of player $i \in N$ with respect to σ is denoted by $t_i(\sigma)$ and given by $t_i(\sigma) = \sum_{h \in P(\sigma,i)} p_h$. Similarly, the starting time of a group of players $I \subseteq N$ with respect to σ is denoted by $t_I(\sigma)$ and given by

$$t_I(\sigma) = \min_{i \in I} \{t_i(\sigma)\}.$$

Furthermore, the time player $i \in N$ spends in the system when the players follow the order σ is called the *completion time*, denoted by $C_i(\sigma)$ and given by $C_i(\sigma) = t_i(\sigma) + p_i$. The *total costs* of the order σ are denoted by $TC(\sigma)$ and given by

$$TC(\sigma) = \sum_{i \in N} c_i(C_i(\sigma)).$$

An order for which the total costs are minimized is called an *optimal order* and denoted by $\hat{\sigma}$, that is, $TC(\hat{\sigma}) \leq TC(\sigma)$ for all $\sigma \in \Pi(N)$. Given an optimal order $\hat{\sigma} \in \Pi(N)$, the *set of misplacements* contains all pairs of players that need to be interchanged in order to reach optimal order $\hat{\sigma}$ from the initial order σ_0:

$$MP(\sigma_0, \hat{\sigma}) = \{(i, j) \in N \times N \mid \sigma_0(i) < \sigma_0(j) \text{ and } \hat{\sigma}(i) > \hat{\sigma}(j)\}.$$

As a result of basic permutation theory, it is possible to reach an optimal order from the initial order by recursively interchanging pairs of consecutive players, i.e., by recursively switching neighbors only. Formally, a *neighbor switch* associated to two orders $\sigma, \sigma' \in \Pi(N)$ is defined as a pair of players $(i, j) \in N \times N$ for which it holds that $\sigma(j) = \sigma(i) + 1$ and $\sigma'(j) = \sigma(i), \sigma'(i) = \sigma(j)$ and $\sigma'(h) = \sigma(h)$ for all $h \in N \setminus \{i, j\}$. Now, we adopt the notion of a *path* (cf. Saavedra-Nieves et al., 2020) from the initial order σ_0 to an optimal order $\hat{\sigma}$ as a sequence of orders $(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_m)$ with $\sigma_m = \hat{\sigma}$ corresponding to neighbor switches (i_k, j_k), for all $k \in \{1, 2, \ldots, m\}$, associated to orders σ_{k-1} and σ_k such that there does not exist $k, \ell \in \{1, 2, \ldots, m\}$, $k \neq \ell$, such that $i_k = j_\ell$ and $j_k = i_\ell$. In other words, a path from the initial order to an optimal order consecutively interchanges neighbor misplacements and interchanges a particular misplacement exactly once. Hence, $m = |MP(\sigma_0, \hat{\sigma})|$.

By following such a path from the initial order to an optimal order, the players can jointly obtain cost savings. More specifically, the *neighbor switching gain* of a neighbor

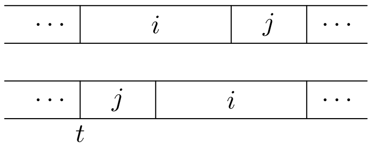


Figure 4.1 – *Interchanging players i and j, leading to the neighbor switching gain $g_{ij}(t)$.*

switch $(i, j) \in N \times N$ at time $t \in [0, \infty)$ (i.e. the starting time of player i, who is directly in front of player j, see Figure 4.1) is given by

$$g_{ij}(t) = c_i(t + p_i) + c_j(t + p_i + p_j) - c_i(t + p_i + p_j) - c_j(t + p_j). \tag{4.1}$$

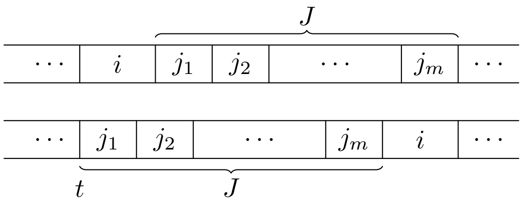


Figure 4.2 – *Interchanging player i with a group of players J, leading to the gain $g_{iJ}(t)$.*

For notational convenience, we also consider the consecutive neighbor switching gains of one player with a group of players. Let $i \in N$ and $J \subseteq N$ be such that player i is directly in front of the group $J = \{j_1, \ldots, j_m\}$ at time $t \in [0, \infty)$ (see Figure 4.2 for the exact ordering of the players in J). Then the neighbor switching gains of player i and group J is denoted by $g_{iJ}(t)$ and given by

$$g_{iJ}(t) = g_{ij_1}(t) + g_{ij_2}(t + p_{j_1}) + \ldots + g_{ij_m}(t + p_{j_1} + \ldots + p_{j_{m-1}}). \tag{4.2}$$

Moreover, to also keep track of each individual gain (that is, each individual term of the above-mentioned sum), we summarize these gains in a vector of length $|J|$, denoted by $\overline{g_{iJ}}(t)$ and given by

$$\overline{g_{iJ}}(t) = \big(g_{ij_1}(t),\ g_{ij_2}(t + p_{j_1}),\ \ldots,\ g_{ij_m}(t + p_{j_1} + \ldots + p_{j_{m-1}})\big). \tag{4.3}$$

Similarly, for a player $j \in N$ and a group $I \subseteq N$ at time $t \in [0, \infty)$ with player j directly behind the group $I = \{i_1, \ldots, i_m\}$ (see Figure 4.3), the corresponding neighbor switching gains are denoted by $g_{Ij}(t)$ and given by

$$g_{Ij}(t) = g_{i_m j}(t + p_{i_1} + \ldots + p_{i_{m-1}}) + g_{i_{m-1}j}(t + p_{i_1} + \ldots + p_{i_{m-2}}) \\ + \ldots + g_{i_2 j}(t + p_{i_1}) + g_{i_1 j}(t). \quad (4.4)$$

Correspondingly, the vector of length $|I|$ summarizing the individual gains is denoted by $\overline{g_{Ij}}(t)$ and given by

$$\overline{g_{Ij}}(t) = \big(g_{i_m j}(t + p_{i_1} + \ldots + p_{i_{m-1}}),\ g_{i_{m-1}j}(t + p_{i_1} + \ldots + p_{i_{m-2}}), \\ \ldots,\ g_{i_2 j}(t + p_{i_1}),\ g_{i_1 j}(t)\big). \quad (4.5)$$

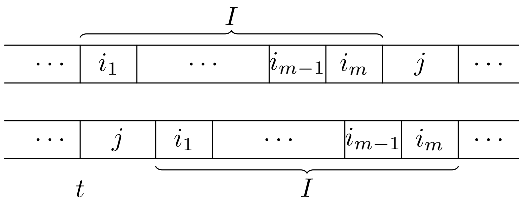


Figure 4.3 – *Interchanging player j with a group of players I, leading to the gain $g_{Ij}(t)$.*

4.3 Interactive sequencing situations with non-linear cost functions

This section is devoted to interactive sequencing situations with non-linear cost functions. Traditionally, the focus is on *standard sequencing situations*, in which the cost functions are linear: $c_i(t) = \alpha_i t$ for all $t \in [0, \infty)$, where $\alpha_i \in \mathbb{R}_{++}$ is the *linear cost coefficient* of player $i \in N$. The set of all standard sequencing situations is denoted by $SSEQ^N$. For a standard sequencing situation $(\sigma_0, p, c) \in SSEQ^N$, Smith (1956) showed that an optimal order can be reached by arranging the players according to weakly decreasing urgency indices, where the *urgency index* u_i is defined by $u_i = \frac{\alpha_i}{p_i}$ for all $i \in N$. The neighbor switching gains as considered in Equation (4.1) can be expressed in terms of the linear cost coefficients and processing times only:

$$g_{ij}(t) = \alpha_j p_i - \alpha_i p_j, \quad (4.6)$$

where $i, j \in N$ are two neighbors at time $t \in [0, \infty)$, as depicted in Figure 4.1. Note that the neighbor switching gains are not time-dependent.

The following example illustrates a standard sequencing situation.

Example 4.1 Consider the standard sequencing situation $(\sigma_0, p, c) \in SSEQ^N$ with $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, and the linear cost coefficients and processing times as shown in Table 4.1.

	player 1	player 2	player 3
α_i	0.25	1	3
p_i	1	2	3

Table 4.1 – *The standard sequencing situation of Example 4.1.*

It readily follows that the urgency index of player 1 is $u_1 = 0.25$, the urgency index of player 2 is $u_2 = 0.5$ and the urgency index of player 3 is $u_3 = 1$. Hence, by arranging the players according to weakly decreasing urgency indices, the unique optimal order is given by $\hat{\sigma} = (3, 2, 1)$. This can also be seen from Table 4.2, which provides the total costs for all possible orders.

σ	$TC(\sigma)$
$(1, 2, 3)$	21.25
$(1, 3, 2)$	18.25
$(2, 1, 3)$	20.75
$(2, 3, 1)$	18.5
$(3, 1, 2)$	16
$(3, 2, 1)$	15.5

Table 4.2 – *The total costs of all processing orders in the sequencing situation of Example 4.1.*

The set of misplacements is given by $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (2, 3)\}$. Hence, there are two paths from the initial order to the optimal order that repairs all neighbor misplacements:

$$\sigma_0 = (1, 2, 3) \xrightarrow[3]{g_{23}(p_1)} (1, 3, 2) \xrightarrow[2.25]{g_{13}(0)} (3, 1, 2) \xrightarrow[0.5]{g_{12}(p_3)} (3, 2, 1) = \hat{\sigma},$$

corresponding to neighbor switches $(2, 3), (1, 3)$ and $(1, 2)$ respectively, and

$$\sigma_0 = (1, 2, 3) \xrightarrow[0.5]{g_{12}(0)} (2, 1, 3) \xrightarrow[2.25]{g_{13}(p_2)} (2, 3, 1) \xrightarrow[3]{g_{23}(0)} (3, 2, 1) = \hat{\sigma},$$

corresponding to neighbor switches $(1, 2), (1, 3)$ and $(2, 3)$ respectively. Below each arrow, the value of the corresponding neighbor switching gain according to Equation

(4.6) is indicated. Note that these values are similar for both paths in the sense that only the order in which they occur is different. $\triangle$

Recently, Saavedra-Nieves et al. (2020) studied *exponential sequencing situations*, in which the cost functions are exponential: $c_i(t) = e^{\alpha_i t}$ for all $t \in [0, \infty)$, where $\alpha_i \in \mathbb{R}_{++}$ is called the *exponential cost coefficient* of player $i \in N$. The set of all exponential sequencing situations is denoted by $ESEQ^N$. For an exponential sequencing situation $(\sigma_0, p, c) \in ESEQ^N$, the neighbor switching gain of two consecutive players $i, j \in N$ at time $t \in [0, \infty)$ as provided in Equation (4.1) can be reformulated as

$$g_{ij}(t) = e^{\alpha_i(t+p_i)} + e^{\alpha_j(t+p_i+p_j)} - e^{\alpha_i(t+p_j+p_i)} - e^{\alpha_j(t+p_j)}. \tag{4.7}$$

For interactive sequencing situations with exponential cost functions, three specific subclasses allow for a comparison index for determining an optimal order, like the urgency index, which is only based on the processing times and exponential cost coefficients. The following proposition summarizes the main result with regard to the neighbor switching gains for each of these three subclasses.

Proposition 4.1 [cf. Saavedra-Nieves et al., 2020] *Let $(\sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that one of the following three cases holds:*

i) there is an $\alpha \in \mathbb{R}_{++}$ such that, for all $i \in N$ and all $t \in [0, \infty)$, $c_i(t) = e^{\alpha t}$;

ii) there is a $p \in \mathbb{R}_{++}$ such that, for all $i \in N$, $p_i = p$;

iii) there are $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ with $\alpha_L < \alpha_H$ and $p_L < p_H$ such that, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}, p_i \in \{p_L, p_H\}$ and

$$e^{\alpha_H p_H} - e^{\alpha_L p_L} \leq e^{\alpha_H(p_L+p_H)} - e^{\alpha_L(p_L+p_H)}.$$

Let $i, j \in N$ be two players such that $\sigma_0(i) < \sigma_0(j)$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then it holds that

1) $g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \in MP(\sigma_0, \hat{\sigma})$;

2) $g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \notin MP(\sigma_0, \hat{\sigma})$.

Thus, for the three subclasses of exponential sequencing situations, all neighbor switching gains corresponding to misplacements are non-negative and non-decreasing in time, while all neighbor switching gains corresponding to non-misplacements are non-positive and non-increasing in time.

The following example illustrates an exponential sequencing situation that belongs to the third subclass as indicated in Proposition 4.1 above.

Example 4.2 Consider the exponential sequencing situation $(\sigma_0, p, c) \in ESEQ^N$ with $N = \{1,2,3\}$, $\sigma_0 = (1,2,3)$, and, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}$ and $p_i \in \{p_L, p_H\}$ with $\alpha_L = p_L = 0.25$ and $\alpha_H = p_H = 2$. The processing times and the exponential cost coefficients of the players are specified in Table 4.3.

	player 1	player 2	player 3
α_i	α_L	α_H	α_H
p_i	p_L	p_H	p_L

Table 4.3 – *The exponential sequencing situation of Example 4.2.*

Note that this exponential sequencing situation is in the third subclass as formulated in Proposition 4.1, since

$$\begin{aligned} e^{\alpha_H p_H} - e^{\alpha_L p_L} = e^4 - e^{0.0625} &\approx 53.5337 \\ &\leq 88.2621 \approx e^{4.5} - e^{0.5625} = e^{\alpha_H(p_L+p_H)} - e^{\alpha_L(p_L+p_H)}. \end{aligned}$$

σ	$TC(\sigma)$
$(1,2,3)$	239.4948
$(1,3,2)$	152.1959
$(2,1,3)$	204.7664
$(2,3,1)$	146.4835
$(3,1,2)$	151.1950
$(3,2,1)$	93.5341

Table 4.4 – *The total costs of all processing orders in the sequencing situation of Example 4.2.*

The approximate total costs for all possible orders are given in Table 4.4. Obviously, $\hat{\sigma} = (3,2,1)$ is the unique optimal order. Furthermore, the set of misplacements is given by $MP(\sigma_0, \hat{\sigma}) = \{(1,2),(1,3),(2,3)\}$ and there are again two paths from the initial order to the optimal order that repairs all neighbor misplacements:

$$\sigma_0 = (1,2,3) \xrightarrow{g_{23}(p_1)} (1,3,2) \xrightarrow{g_{13}(0)} (3,1,2) \xrightarrow{g_{12}(p_3)} (3,2,1) = \hat{\sigma},$$

corresponding to neighbor switches $(2,3),(1,3)$ and $(1,2)$ respectively, and

$$\sigma_0 = (1,2,3) \xrightarrow{g_{12}(0)} (2,1,3) \xrightarrow{g_{13}(p_2)} (2,3,1) \xrightarrow{g_{23}(0)} (3,2,1) = \hat{\sigma},$$

$g_{12}(0)$	34.7284
$g_{12}(p_3)$	57.6609
$g_{13}(0)$	1.0009
$g_{13}(p_2)$	58.2828
$g_{23}(0)$	52.9494
$g_{23}(p_1)$	87.2988

Table 4.5 – *The neighbor switching gains in the sequencing situation of Example 4.2.*

corresponding to neighbor switches $(1,2),(1,3)$ and $(2,3)$ respectively.

Table 4.5 provides the approximate values of the neighbor switching gains, which can be computed using Equation (4.7). Rather than having three similar values in a different order for each path as was the case for standard sequencing situations, we now see that for an exponential sequencing situation, all six values differ. However, the total cost savings for both paths from the initial order to the optimal order are identical.

Note that, for example, $0 \leq g_{12}(0) \leq g_{12}(p_3)$. Indeed, all neighbor switching gains according to misplacements are non-negative and non-decreasing, according to Proposition 4.1. △

Next, we study *discounting sequencing situations*, as introduced by Rothkopf (1966), in which the cost function of player $i \in N$ is given by $c_i(t) = \alpha_i(1 - e^{-rt})$ for all $t \in [0, \infty)$, where $r \in \mathbb{R}_{++}$ denotes the *discount rate* and $\alpha_i \in \mathbb{R}_{++}$ the *discounting cost coefficient* of player $i \in N$. The set of all discounting sequencing situations with player set N is denoted by $DSEQ^N$.

For a discounting sequencing situation $(\sigma_0, p, c) \in DSEQ^N$, the neighbor switching gain of two consecutive players $i, j \in N$ at time $t \in [0, \infty)$ as provided in Equation (4.1) can be reformulated as

$$g_{ij}(t) = \alpha_i e^{-r(t+p_i+p_j)} + \alpha_j e^{-r(t+p_j)} - \alpha_i e^{-r(t+p_i)} - \alpha_j e^{-r(t+p_i+p_j)}. \tag{4.8}$$

Rothkopf (1966) showed that, for a discounting sequencing situation $(\sigma_0, p, c) \in DSEQ^N$ and an order $\hat{\sigma} \in \Pi(N)$, it holds that $\hat{\sigma}$ is optimal if and only if, for all $i, j \in N$,

$$\frac{\alpha_j e^{-rp_j}}{1 - e^{-rp_j}} < \frac{\alpha_i e^{-rp_i}}{1 - e^{-rp_i}} \Rightarrow \hat{\sigma}(i) < \hat{\sigma}(j). \tag{4.9}$$

Equation (4.9) can be used to define the pairs of players that should be interchanged in order to reach an optimal order from the initial order and hence, are in the set of misplacements. In the following proposition, we show that in a discounting sequencing situation, all neighbor switching gains corresponding to misplacements are non-negative and non-increasing in time. On the other hand, the neighbor switching gains corresponding to non-misplacements are non-positive and non-decreasing in time. Note that this partly contrasts Proposition 4.1: in the three subclasses of exponential sequencing situations, the gains of misplacements are non-decreasing (rather than non-increasing) in time and the gains of non-misplacements non-increasing (rather than non-decreasing) in time.

Proposition 4.2 *Let $(\sigma_0, p, c) \in DSEQ^N$ be a discounting sequencing situation. Let $i, j \in N$ be such that $\sigma_0(i) < \sigma_0(j)$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then it holds that*

1) *$g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \in MP(\sigma_0, \hat{\sigma})$;*

2) *$g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \notin MP(\sigma_0, \hat{\sigma})$.*

Proof: First, from Equation (4.8), it readily follows that $g'_{ij}(t) = -r g_{ij}(t)$ for all $t \in [0, \infty)$.

1) Let $(i, j) \in MP(\sigma_0, \hat{\sigma})$. Then player j is processed before player i in $\hat{\sigma}$ and hence, using Equation (4.9),

$$\frac{\alpha_i e^{-rp_i}}{1 - e^{-rp_i}} \leq \frac{\alpha_j e^{-rp_j}}{1 - e^{-rp_j}}.$$

Hence,

$$\alpha_i e^{-rp_i} - \alpha_i e^{-r(p_i+p_j)} \leq \alpha_j e^{-rp_j} - \alpha_j e^{-r(p_i+p_j)},$$

and since $e^{-rt} \geq 0$ for all $t \in [0, \infty)$,

$$e^{-rt}\left(\alpha_i e^{-r(p_i+p_j)} + \alpha_j e^{-rp_j} - \alpha_i e^{-rp_i} - \alpha_j e^{-r(p_i+p_j)}\right) \geq 0.$$

Consequently, $g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$. Moreover, since $r > 0$, $g'_{ij}(t) \leq 0$ for all $t \in [0, \infty)$, which implies that $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$.

2) Let $(i, j) \notin MP(\sigma_0, \hat{\sigma})$. Then player i is processed before player j in $\hat{\sigma}$ and hence, using Equation (4.9),

$$\frac{\alpha_i e^{-rp_i}}{1 - e^{-rp_i}} \geq \frac{\alpha_j e^{-rp_j}}{1 - e^{-rp_j}}.$$

Hence,

$$\alpha_i e^{-rp_i} - \alpha_i e^{-r(p_i+p_j} \geq \alpha_j e^{-rp_j} - \alpha_j e^{-r(p_i+p_j)},$$

and consequently, $g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$. Moreover, since $r > 0$, $g'_{ij}(t) \geq 0$ for all $t \in [0, \infty)$, which implies that $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$. $\square$

The following example illustrates a discounting sequencing situation and in particular, illustrates the behavior of the neighbor switching gains according to Proposition 4.2.

Example 4.3 Consider the discounting sequencing situation $(\sigma_0, p, c) \in DSEQ^N$ with $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $r = 0.1$, $\alpha_i = 1$ for all $i \in N$, and $p_1 = 3, p_2 = 2$ and $p_3 = 1$. The approximate total costs for all possible orders are given in Table 4.6.

σ	$TC(\sigma)$
$(1, 2, 3)$	1.4265
$(1, 3, 2)$	1.3743
$(2, 1, 3)$	1.3560
$(2, 3, 1)$	1.2461
$(3, 1, 2)$	1.2259
$(3, 2, 1)$	1.1682

Table 4.6 – *The total costs of all processing orders in the sequencing situation of Example 4.3.*

Obviously, $\hat{\sigma} = (3, 2, 1)$ is the unique optimal order, which indeed satisfies Equation (4.9). The latter can be seen from computing $\frac{\alpha_i e^{-rp_i}}{1-e^{-rp_i}}$ for all $i \in N$, resulting in 2.8583 for player 1, 4.5167 for player 2 and 9.55083 for player 3.

The set of misplacements is given by $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (2, 3)\}$ and there are thus again two paths from the initial order to the optimal order:

$$\sigma_0 = (1, 2, 3) \xrightarrow{g_{23}(p_1)} (1, 3, 2) \xrightarrow{g_{13}(0)} (3, 1, 2) \xrightarrow{g_{12}(p_3)} (3, 2, 1) = \hat{\sigma},$$

and

$$\sigma_0 = (1, 2, 3) \xrightarrow{g_{12}(0)} (2, 1, 3) \xrightarrow{g_{13}(p_2)} (2, 3, 1) \xrightarrow{g_{23}(0)} (3, 2, 1) = \hat{\sigma}.$$

Table 4.7 provides the approximate values of the neighbor switching gains, which can be computed using Equation (4.8).

Note that, for example, $g_{12}(0) \geq g_{12}(p_3) \geq 0$. Indeed, all neighbor switching gains according to misplacements are non-negative and non-increasing, according to Proposition 4.2. $\triangle$

Finally, we introduce the notion of a *logarithmic sequencing situation* which deals with logarithmic cost functions: $c_i(t) = \ln(\alpha_i t)$ for all $t \in (0, \infty)$, where α_i is called the *logarithmic cost coefficient* of player $i \in N$ and is such that $\ln(\alpha_i t) > 0$ for all $t \geq p_i$. The set of all logarithmic sequencing situations with player set N is denoted by $LSEQ^N$.

$g_{12}(0)$	0.0705
$g_{12}(p_3)$	0.0577
$g_{13}(0)$	0.1484
$g_{13}(p_2)$	0.1100
$g_{23}(0)$	0.0779
$g_{23}(p_1)$	0.0522

Table 4.7 – *The neighbor switching gains in the sequencing situation of Example 4.3.*

For a logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$, the neighbor switching gain of two consecutive players $i, j \in N$ at time $t \in [0, \infty)$ as provided in Equation (4.1) can be reformulated as

$$g_{ij}(t) = \ln(t + p_i) - \ln(t + p_j). \tag{4.10}$$

Equation (4.10) shows that the neighbor switching gains for a logarithmic sequencing situation are time-dependent. Moreover, the logarithmic cost coefficients turn out to be irrelevant from an optimization perspective. Even more, the total costs consist of a fixed part of the sum of the logarithmic cost coefficients and another part that is dependent on the order. Formally, for a logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$ and an order $\sigma \in \Pi(N)$, we have that

$$TC(\sigma) = \sum_{i \in N} \ln(\alpha_i) + \sum_{i \in N} \ln(C_i(\sigma)). \tag{4.11}$$

Consequently, an optimal order can be determined by considering the processing times only. The following lemma shows that in an optimal order, the players are arranged according to weakly increasing processing times. This is also known as the so-called shortest processing time first (SPT) rule.

Lemma 4.1 *Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an order. Then $\hat{\sigma}$ is optimal if and only if, for all $i, j \in N$,*

$$p_i < p_j \Rightarrow \hat{\sigma}(i) < \hat{\sigma}(j). \tag{4.12}$$

Proof: The proof uses a standard exchange argument. First, assume that $\hat{\sigma}$ is an optimal order and suppose for the sake of contradiction that Equation (4.12) is not satisfied for $\hat{\sigma}$. Then there are two neighbors i and j for which it holds that $p_i < p_j$, while $\hat{\sigma}(i) > \hat{\sigma}(j)$. Then it is beneficial for them to interchange positions, since interchanging players i and j leads to an order $\tau \in \Pi(N)$ (see Figure 4.4) for which the total costs are less than the total costs of $\hat{\sigma}$:

$$TC(\hat{\sigma}) - TC(\tau) = g_{ij}(t_j(\hat{\sigma})) = \ln(t_j(\hat{\sigma}) + p_j) - \ln(t_j(\hat{\sigma}) + p_i) > 0,$$

contradicting the statement that $\hat{\sigma}$ is an optimal order. Consequently, Equation (4.12) is satisfied for $\hat{\sigma}$. As a result, all optimal orders satisfy Equation (4.12).

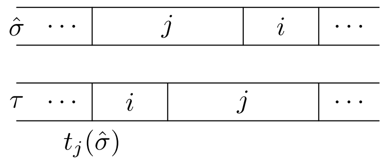


Figure 4.4 – *Interchanging players i and j from $\hat{\sigma}$ to τ.*

Secondly, let $\sigma, \sigma' \in \Pi(N)$ be two orders for which Equation (4.12) is satisfied. Then the only differences between σ and σ' can be within a block of players with identical processing times. Hence, using Equation (4.11), it follows that $TC(\sigma) = TC(\sigma')$. Subsequently, all orders for which Equation (4.12) is satisfied have identical total costs and, by using the first part of the proof, these costs are minimal. As a result, all orders for which Equation (4.12) is satisfied are optimal. $\square$

Using Lemma 4.1, it is easily seen that, to reach an optimal order from the initial order, at least the pairs of players $i, j \in N$ for which it holds that $\sigma_0(i) < \sigma_0(j)$ and $p_i > p_j$ should be interchanged. Obviously, other optimal orders can be obtained by interchanging even more pairs of players with identical processing times. Similar to Proposition 4.2 for discounting sequencing situations, we show that all neighbor switching gains corresponding to misplacements are non-negative and non-increasing, while the neighbor switching gains are non-positive and non-decreasing for non-misplacements.

Proposition 4.3 *Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation. Let $i, j \in N$ be such that $\sigma_0(i) < \sigma_0(j)$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then it holds that*

1) $g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \in MP(\sigma_0, \hat{\sigma})$;

2) $g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \notin MP(\sigma_0, \hat{\sigma})$.

Proof: First, we have that $g_{ij}(t) = \ln(t + p_i) - \ln(t + p_j)$ and $g'_{ij}(t) = \frac{1}{t+p_i} - \frac{1}{t+p_j}$ for all $t \in [0, \infty)$.

1) Assume that $(i, j) \in MP(\sigma_0, \hat{\sigma})$. Then $p_i \geq p_j$ and hence, $\ln(t + p_i) \geq \ln(t + p_j)$ and $\frac{1}{t+p_i} \leq \frac{1}{t+p_j}$. Consequently, $g_{ij}(t) \geq 0$ and $g'_{ij}(t) \leq 0$ for all $t \in [0, \infty)$. The latter implies that $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$.

2) Assume that $(i,j) \notin MP(\sigma_0, \hat{\sigma})$. Then $p_i \leq p_j$ and hence, $\ln(t+p_i) \leq \ln(t+p_j)$ and $\frac{1}{t+p_i} \geq \frac{1}{t+p_j}$. Consequently, $g_{ij}(t) \leq 0$ and $g'_{ij}(t) \geq 0$ for all $t \in [0, \infty)$. The latter implies that $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$. □

The following example illustrates a logarithmic sequencing situation, providing more insights in Proposition 4.3.

Example 4.4 Consider the logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$ with $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $\alpha_i = 1$ for all $i \in N$, and $p_1 = 4, p_2 = 3$ and $p_3 = 2$. The approximate total costs for all possible orders are given in Table 4.8.

σ	$TC(\sigma)$
$(1, 2, 3)$	5.5294
$(1, 3, 2)$	5.3753
$(2, 1, 3)$	5.2417
$(2, 3, 1)$	4.9053
$(3, 1, 2)$	4.6821
$(3, 2, 1)$	4.4998

Table 4.8 – *The total costs of all processing orders in the sequencing situation of Example 4.4.*

Obviously, $\hat{\sigma} = (3, 2, 1)$ is the unique optimal order, which indeed satisfies Equation (4.12) from Lemma 4.1 as $p_3 < p_2 < p_1$. Furthermore, the set of misplacements is given by $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (2, 3)\}$. Hence, there are two paths from the initial order to the optimal order that repairs all neighbor misplacements:

$$\sigma_0 = (1, 2, 3) \xrightarrow{g_{23}(p_1)} (1, 3, 2) \xrightarrow{g_{13}(0)} (3, 1, 2) \xrightarrow{g_{12}(p_3)} (3, 2, 1) = \hat{\sigma},$$

corresponding to neighbor switches $(2, 3), (1, 3)$ and $(1, 2)$ respectively, and

$$\sigma_0 = (1, 2, 3) \xrightarrow{g_{12}(0)} (2, 1, 3) \xrightarrow{g_{13}(p_2)} (2, 3, 1) \xrightarrow{g_{23}(0)} (3, 2, 1) = \hat{\sigma},$$

corresponding to neighbor switches $(1, 2), (1, 3)$ and $(2, 3)$ respectively. Table 4.9 provides the approximate values of the neighbor switching gains.

Note that, for example, $g_{12}(0) \geq g_{12}(p_3) \geq 0$. Indeed, all neighbor switching gains according to misplacements are non-negative and non-increasing, according to Proposition 4.3. △

Interestingly, Propositions 4.1, 4.2 and 4.3 have two statements in common: for an exponential sequencing situation that belongs to one of the three subclasses as specified in Proposition 4.1, a discounting sequencing situation or a logarithmic sequencing

$g_{12}(0)$	0.2877
$g_{12}(p_3)$	0.1823
$g_{13}(0)$	0.6932
$g_{13}(p_2)$	0.3365
$g_{23}(0)$	0.4055
$g_{23}(p_1)$	0.1542

Table 4.9 – *The neighbor switching gains in the sequencing situation of Example 4.4.*

situation, it holds that the neighbor switching gains corresponding to misplacements are non-negative and the ones corresponding to non-misplacements are non-positive. Formally, for such a sequencing situation (σ_0, p, c), two players $i, j \in N$ such that $\sigma_0(i) < \sigma_0(j)$ and an optimal order $\hat{\sigma} \in \Pi(N)$ it holds that

$$\begin{cases} g_{ij}(t) \geq 0 \text{ for all } t \in [0, \infty), \text{ if } (i,j) \in MP(\sigma_0, \hat{\sigma}); \\ g_{ij}(t) \leq 0 \text{ for all } t \in [0, \infty), \text{ if } (i,j) \notin MP(\sigma_0, \hat{\sigma}). \end{cases} \tag{4.13}$$

The two conditions of Equation (4.13) guarantee that it is beneficial to interchange misplacements at any moment in time, while it is not beneficial for non-misplacements to be interchanged at any moment in time. As a consequence, an optimal order for all players implies optimal orders for coalitions, as we see in the next section.

4.4 Sequencing games

In this section, we study sequencing games that are associated to general interactive sequencing situations. Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation, $S \in 2^N \setminus \{\emptyset\}$ be a coalition and $\sigma \in \Pi(N)$ be an order. Then, following Curiel et al. (1989), σ is called *admissible* for S with respect to σ_0 if $P(\sigma, i) = P(\sigma_0, i)$ for all $i \in N \setminus S$. The set of all admissible orders for S with respect to σ_0 is denoted by $\mathcal{A}(\sigma_0, S)$. We define the associated *sequencing game* $v \in TU^N$ by

$$v(S) = \max_{\sigma \in \mathcal{A}(\sigma_0, S)} \left\{ \sum_{i \in S} c_i(C_i(\sigma_0)) - \sum_{i \in S} c_i(C_i(\sigma)) \right\},$$

for all $S \in 2^N \setminus \{\emptyset\}$, that is, the worth of a coalition is equal to the maximal cost savings the coalition can achieve by admissible rearrangements with respect to σ_0.

Admissibility implies that all players outside the coalition remain in the same position compared to the initial order, whereas players in the coalition can only interchange with other players in the coalition if all players in between, according to the initial order, also belong to the coalition. The latter condition is called connectedness and formalized as follows: S is called *connected* with respect to σ if, for all $i, j \in S$ and $k \in N$ for which $\sigma(i) < \sigma(k) < \sigma(j)$, it holds that $k \in S$. Moreover, a connected coalition $T \subseteq S$ is called a *(maximally connected) component* of S with respect to σ if, for all connected coalitions $T' \subseteq S$ with respect to σ, it holds that $T \subseteq T'$ implies that $T = T'$. The set of all components of S is denoted by S/σ. Summarizing, in an admissible rearrangement with respect the initial order, players in the coalition can only interchange within the maximally connected components of the coalition with respect to the initial order.

Next, σ *induces* the order $\sigma_S \in \Pi(N)$ if $\sigma_S \in \mathcal{A}(\sigma_0, S)$ and, for all $T \in S/\sigma_0$ and all $i \in T$, it holds that $P(\sigma, i) \cap T = P(\sigma_S, i) \cap T$. In other words, all players outside S are in the same position in σ_S compared to σ_0 and the order of all players within the components of S with respect to σ_0 are in the same order in σ_S compared to σ. An *optimal order for* S, denoted by $\hat{\sigma}_S \in \Pi(N)$, is an admissible order for S with respect to σ_0 that minimizes the total costs for S, i.e., $\hat{\sigma}_S \in \mathcal{A}(\sigma_0, S)$ for which $TC(\hat{\sigma}_S) \leq TC(\sigma)$ for all $\sigma \in \mathcal{A}(\sigma_0, S)$.[3] Combining these two concepts leads to the concept of *optimal order consistency*, which holds when, for an optimal order $\hat{\sigma} \in \Pi(N)$ and a coalition $S \in 2^N \setminus \{\emptyset\}$, the induced order $\hat{\sigma}_S$ is optimal for S. Thus, under optimal order consistency, an optimal order for N implies optimal orders for all coalitions S. Saavedra-Nieves et al. (2020) provided a link between the neighbor switching gains and optimal order consistency, as shown in the following lemma.

Lemma 4.2 [cf. Saavedra-Nieves et al., 2020] *Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. If the following two conditions hold:*

i) for all $t \in [0, \infty)$, $g_{ij}(t) \geq 0$ for all $(i, j) \in MP(\sigma_0, \hat{\sigma})$;

ii) for all $t \in [0, \infty)$, $g_{ij}(t) \leq 0$ for all $(i, j) \notin MP(\sigma_0, \hat{\sigma})$,

then optimal order consistency is satisfied, that is, for all $S \in 2^N \setminus \{\emptyset\}$, $\hat{\sigma}_S$ is optimal for S.

Importantly, the two conditions in Lemma 4.2 coincide with the conditions as stated in Equation (4.13), the overlap between Propositions 4.1, 4.2 and 4.3. This means that all interactive sequencing situations under consideration satisfy optimal order consistency: standard sequencing situations, the three subclasses of exponential sequencing situations, discounting sequencing situations and logarithmic sequencing situations.

Example 4.5 Consider the interactive sequencing situation $(\sigma_0, p, c) \in SEQ^N$ with $N = \{1, 2, 3, 4\}$ and $\sigma_0 = (1, 2, 3, 4)$. Consider the coalition $S = \{1, 2, 4\} \in 2^N$. Note

[3] Note that both $TC(\hat{\sigma}_S)$ and $TC(\sigma)$ include the costs of players in $N \setminus S$. However, due to the admissibility for S, these costs are the same for both $\hat{\sigma}_S$ and σ.

that S is not connected with respect to σ_0, since $3 \notin S$ while $\sigma(2) < \sigma(3) < \sigma(4)$. The set of components of S is given by $S/\sigma_0 = \{\{1,2\},\{4\}\}$. Then it can be readily seen that $\mathcal{A}(\sigma_0, S) = \{(1,2,3,4),(2,1,3,4)\}$.

Next, assume that $\hat{\sigma} = (4,3,2,1)$ is optimal. Then $\hat{\sigma}$ induces the order $\hat{\sigma}_S = (2,1,3,4)$: for the component $T = \{1,2\}$ it follows that players 2 is ordered before player 1 according to $\hat{\sigma}$. Hence, $\hat{\sigma}_S(2) < \hat{\sigma}_S(1)$. The other component $T = \{4\}$ consists of only player 4 and thus player 4 is still ordered in fourth position. Finally, player 3 is in third position, as already the case in σ_0. Under optimal order consistency, $\hat{\sigma}_S$ is optimal for S. $\triangle$

In this section, we particularly study sequencing games that are associated to interactive sequencing situations with non-linear cost functions. Therefore, the associated sequencing game of a standard sequencing situation (with linear cost functions) is called a *standard sequencing game*, the associated game of an exponential sequencing situation (with exponential cost functions) is called an *exponential sequencing game*, the one associated with a discounting sequencing situation is called a *discounting sequencing game* and finally, the associated game of a logarithmic sequencing situation is called a *logarithmic sequencing game*.

For standard sequencing situations, Curiel et al. (1989) showed that the standard sequencing games are convex. For exponential sequencing situations, Saavedra-Nieves et al. (2020) showed that the three subclasses of exponential sequencing situations yield convex exponential sequencing games. The latter result is based on a more general result for interactive sequencing situations with arbitrary non-linear cost functions:

Theorem 4.1 [cf. Saavedra-Nieves et al., 2020] *Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation and let $v \in TU^N$ be the associated sequencing game. Let $\hat{\sigma} \in \Pi(N)$ be an optimal order. If the following three conditions hold:*

i) for all $t \in [0,\infty)$, $g_{ij}(t) \geq 0$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$;

ii) for all $t \in [0,\infty)$, $g_{ij}(t) \leq 0$ for all $(i,j) \notin MP(\sigma_0, \hat{\sigma})$;

iii) for all $s,t \in [0,\infty)$ with $s \leq t$, $g_{ij}(s) \leq g_{ij}(t)$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$,

then v is convex.

In Theorem 4.1, the first two conditions ensure optimal order consistency and the third condition refers to the non-decreasing character of the neighbor switching gains for misplacements. Together, these conditions imply convexity of the associated sequencing game. For both standard and exponential sequencing situations (at least, restricted to the three subclasses), this suffices to prove convexity. However, for discounting and logarithmic sequencing situations, only the first two conditions are satisfied, that is, optimal order consistency is satisfied. In contrast, Propositions 4.2

and 4.3 imply a non-increasing character of the neighbor switching gains for misplacements. Theorem 4.2 below shows that for convexity it also suffices to require that the neighbor switching gains are non-increasing for misplacements, together with optimal order consistency, i.e., non-negativity for misplacements and non-positivity for non-misplacements. The proof of this result follows the same structure as the proof of Theorem 4.1 and makes use of the following notation.

Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation and let $\sigma \in \Pi(N)$ be an order. Following Borm et al. (2002), we use the following notation for some special connected coalitions:

$$\begin{cases} (i,j)_\sigma = \{k \in N \mid \sigma(i) < \sigma(k) < \sigma(j)\}\,; \\ (i,j]_\sigma = \{k \in N \mid \sigma(i) < \sigma(k) \leq \sigma(j)\}\,; \\ [i,j)_\sigma = \{k \in N \mid \sigma(i) \leq \sigma(k) < \sigma(j)\}\,; \\ [i,j]_\sigma = \{k \in N \mid \sigma(i) \leq \sigma(k) \leq \sigma(j)\}\,, \end{cases}$$

where $i, j \in N$ are two players such that $\sigma(i) < \sigma(j)$. We also benefit from their convexity result:

Proposition 4.4 [cf. Borm et al., 2002] *Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation and let $v \in TU^N$ be the associated sequencing game. Then v is convex if and only if, for all $i, j \in N$ such that $\sigma_0(i) < \sigma_0(j)$,*

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) - v((i,j]_{\sigma_0}) + v((i,j)_{\sigma_0}) \geq 0.$$

Theorem 4.2 *Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation and let $v \in TU^N$ be the associated sequencing game. Let $\hat{\sigma} \in \Pi(N)$ an optimal order. If the following three conditions hold:*

i) for all $t \in [0,\infty)$, $g_{ij}(t) \geq 0$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$;

ii) for all $t \in [0,\infty)$, $g_{ij}(t) \leq 0$ for all $(i,j) \notin MP(\sigma_0, \hat{\sigma})$;

iii) for all $s, t \in [0,\infty)$ with $s \leq t$, $g_{ij}(s) \geq g_{ij}(t)$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$,

then v is convex.

Proof: Assume that conditions i), ii) and iii) are satisfied. Then, by using Proposition 4.4, it suffices to prove that

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) \geq v((i,j]_{\sigma_0}) - v((i,j)_{\sigma_0}), \tag{4.14}$$

for all $i, j \in N$ with $\sigma_0(i) < \sigma_0(j)$. Let $i, j \in N$ with $\sigma_0(i) < \sigma_0(j)$ (see Figure 4.5).

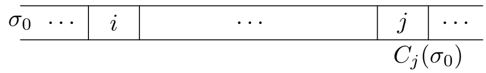


Figure 4.5 – *Players $i, j \in N$ in the initial order σ_0.*

Similar to Theorem 4.1, conditions i) and ii) imply optimal order consistency and hence, the induced orders $\hat{\sigma}_{[i,j]_{\sigma_0}}$, $\hat{\sigma}_{[i,j)_{\sigma_0}}$, $\hat{\sigma}_{(i,j]_{\sigma_0}}$ and $\hat{\sigma}_{(i,j)_{\sigma_0}}$ are optimal for the coalitions $[i,j]_{\sigma_0}, [i,j)_{\sigma_0}, (i,j]_{\sigma_0}$ and $(i,j)_{\sigma_0}$, respectively.

To prove Equation (4.14), we distinguish between two cases: either I) $\hat{\sigma}(i) < \hat{\sigma}(j)$ or II) $\hat{\sigma}(i) > \hat{\sigma}(j)$. For both cases, we show that Equation (4.14) is satisfied.

Case I) First, assume that $\hat{\sigma}(i) < \hat{\sigma}(j)$.

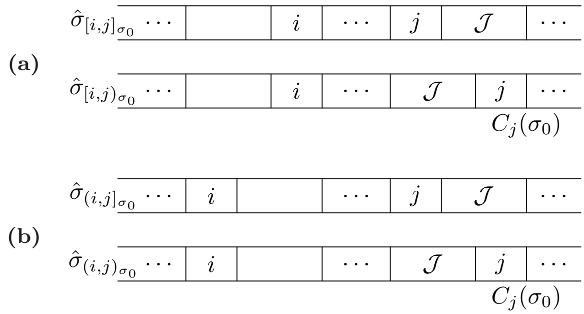


Figure 4.6 – *Schematic overview of the first case in the proof of Theorem 4.2.*

In Figure 4.6, the order of the relevant players is shown for the different induced orders. First note that in Figure 4.6b, $\hat{\sigma}_{(i,j)_{\sigma_0}}$ is the optimal order in which all players that are in between i and j according to σ_0 are now ordered according to $\hat{\sigma}$. In $\hat{\sigma}_{(i,j]_{\sigma_0}}$, we also order player j according to $\hat{\sigma}$. Since $\hat{\sigma}(i) < \hat{\sigma}(j)$ in this first case, this means that player j has to switch with the players in $\mathcal{J} \subseteq N$, given by

$$\mathcal{J} = \{h \in (i,j)_{\sigma_0} \mid (h,j) \in MP(\sigma_0, \hat{\sigma})\}.$$

In Figure 4.6a, the same switch between j and $\mathcal{J}$ is visible, since in $\hat{\sigma}_{[i,j]_{\sigma_0}}$, player j is ordered according to $\hat{\sigma}$, while in $\hat{\sigma}_{[i,j)_{\sigma_0}}$, player j is ordered according to σ_0. Furthermore, in both orders, player i is ordered according to $\hat{\sigma}$.

Ultimately, this means that in both Figure 4.6a and Figure 4.6b, the only difference is the switch between j and the players in $\mathcal{J}$. Hence,

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) = g_{Jj}(C_j(\sigma_0) - p_j - \sum_{h\in\mathcal{J}} p_h)$$
$$= v((i,j]_{\sigma_0}) - v((i,j)_{\sigma_0}),$$

where $g_{Jj}(C_j(\sigma_0)-p_j-\sum_{h\in\mathcal{J}} p_h)$ is the consecutive neighbor switching gain of player j and group $\mathcal{J}$ at time $C_j(\sigma_0) - p_j - \sum_{h\in\mathcal{J}} p_h$ according to Equation (4.2). So, Equation (4.14) is satisfied with equality.

Case II) Secondly, assume that $\hat{\sigma}(i) > \hat{\sigma}(j)$.

Figure 4.7 provides the order of the relevant players for the different induced orders. First, define $\mathcal{I}, \mathcal{J} \subseteq N$ as follows:

$$\begin{cases} \mathcal{I} = \{\ell \in (i,j)_{\sigma_0} \mid \hat{\sigma}(j) < \hat{\sigma}(\ell) < \hat{\sigma}(i)\}\,; \\ \mathcal{J} = \{k \in (i,j)_{\sigma_0} \mid \hat{\sigma}(i) < \hat{\sigma}(k)\}\,. \end{cases}$$

Then, since in this case $\hat{\sigma}(i) > \hat{\sigma}(j)$, while $\sigma_0(i) < \sigma_0(j)$, player j has to switch with player i at some point. In Figure 4.7a, this switch becomes visible: player j first has to switch with the players in $\mathcal{J}$, then with player i and finally, with the players in $\mathcal{I}$. In Figure 4.7b, player i is ordered according to σ_0, while player j has to switch with the players in $\mathcal{J}$ and $\mathcal{I}$ respectively, to be ordered according to $\hat{\sigma}$.

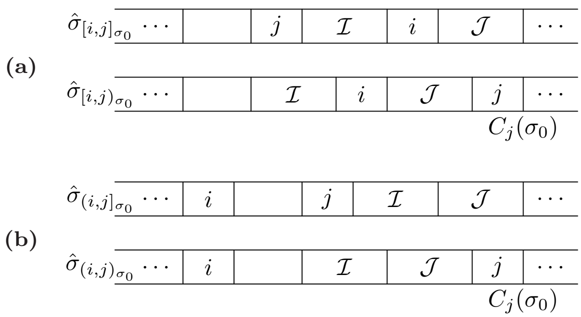


Figure 4.7 – *Schematic overview of the second case in the proof of Theorem 4.2.*

Hence, by following Figure 4.7a,

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) = g_{Jj}(C_j(\sigma_0) - p_j - \sum_{k\in\mathcal{J}} p_k)$$
$$+ g_{ij}(C_j(\sigma_0) - p_i - p_j - \sum_{k\in\mathcal{J}} p_k)$$

$$+ g_{Ij}(C_j(\sigma_0) - p_i - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k),$$

and, by following Figure 4.7b,

$$\begin{aligned} v((i,j]_{\sigma_0}) - v((i,j)_{\sigma_0}) = {} & g_{Jj}(C_j(\sigma_0) - p_j - \sum_{k \in \mathcal{J}} p_k) \\ & + g_{Ij}(C_j(\sigma_0) - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k). \end{aligned}$$

First, note that $g_{ij}(C_j(\sigma_0) - p_i - p_j - \sum_{k \in \mathcal{J}} p_k) \geq 0$ according to condition i), since $(i,j) \in MP(\sigma_0, \hat{\sigma})$. Secondly,

$$C_j(\sigma_0) - p_i - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k < C_j(\sigma_0) - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k,$$

since $p_i > 0$. Using condition iii), we then have that

$$g_{Ij}(C_j(\sigma_0) - p_i - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k) \geq g_{Ij}(C_j(\sigma_0) - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k),$$

and hence,

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) \geq v((i,j]_{\sigma_0}) - v((i,j)_{\sigma_0}),$$

which concludes the second case. □

As a direct consequence of Theorem 4.2, we have that, using Propositions 4.2 and 4.3, any discounting sequencing game and any logarithmic sequencing game is convex.

Corollary 4.1 *The following two statements hold:*

i) Let $(\sigma_0, p, c) \in DSEQ^N$ be a discounting sequencing situation and let $v \in TU^N$ be the associated discounting sequencing game. Then v is convex.

ii) Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation and let $v \in TU^N$ be the associated logarithmic sequencing game. Then v is convex.

4.5 Cost savings allocation rules

In this section, we introduce two different types of allocation rules that can be directly computed from the sequencing situation itself and thus not using the associated sequencing game. Both types of rules use the ideas of the gain splitting rules for standard sequencing situations, now applied in the more general setting. We are also interested in the game-theoretical properties that they satisfy. In particular, we study if the allocations we obtain are stable, in the sense that they belong to the core of the associated cooperative game.

For standard sequencing situations, the *gain splitting (GS) rules* (Hamers et al., 1996) provide such stable allocations. Formally, for a standard sequencing situation $(\sigma_0, p, c) \in SSEQ^N$, the allocations prescribed by the GS-rules are given by

$$GS^\lambda(\sigma_0, p, c) = \sum_{(i,j)\in MP(\sigma_0,\hat{\sigma})} \lambda_{ij}(\alpha_j p_i - \alpha_j p_i)e^{\{i\}} + (1-\lambda_{ij})(\alpha_j p_i - \alpha_j p_i)e^{\{j\}},$$

where $\lambda_{ij} \in [0,1]$ for all $i, j \in N, i \neq j$. Recall that, for all $i \in N$, $e^{\{i\}} \in \mathbb{R}^N$ is such that, for all $h \in N$,

$$(e^{\{i\}})_h = \begin{cases} 1, & \text{if } h = i; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, by choosing $\lambda_{ij} = \frac{1}{2}$ for all $i, j \in N, i \neq j$, we obtain the allocation as prescribed by the *equal gain splitting (EGS) rule* (Curiel et al., 1989). Note that all GS-rules are independent of the choice of an optimal order $\hat{\sigma} \in \Pi(N)$. Furthermore, all GS-rules yield stable allocations, which means that, for $(\sigma_0, p, c) \in SSEQ^N$, we have that $GS^\lambda(\sigma_0, p, c) \in C(v)$, where v denotes the associated standard sequencing game, i.e.,

$$v(S) \leq \sum_{i\in S} GS_i^\lambda(\sigma_0, p, c),$$

for all $S \in 2^N \setminus \{\emptyset\}$ and

$$v(N) = \sum_{i\in N} GS_i^\lambda(\sigma_0, p, c).$$

Since for a standard sequencing situation it holds that $g_{ij}(t) = \alpha_j p_i - \alpha_i p_j$ for every misplacement $(i,j) \in MP(\sigma_0, \hat{\sigma})$ at time $t \in [0, \infty)$, the GS-rules thus divide the neighbor switching gains for every misplaced pair of players between the two players involved. Note that every choice of λ_{ij} can possibly lead to another allocation. However, given such a choice, every path from the initial order to an optimal order leads to the same neighbor switching gains and hence, to the same allocation, since the gains are not dependent on the moment in time both players interchange their positions, as was also seen in Example 4.1.

In sequencing situations with non-linear cost functions, the neighbor switching gains may be time-dependent. This is indeed the case in sequencing situations with exponential, discounting or logarithmic cost functions (see Examples 4.2, 4.3 and 4.4). Hence, the gains depend on the path from the initial order to an optimal order. Therefore, we need to specify which path to choose to reach an optimal order from the initial order. Given such a path, we adopt the idea behind the GS-rules of splitting these gains between the two neighbors involved. Together, this yields allocation rules for sequencing situations with non-linear cost functions.

4.5.1 Specifying a path

We start out by focusing on the choice of the path from the initial order to an optimal order that repairs all neighbor misplacements. Below, we prescribe two possible procedures that specify such a path. For the first procedure, named the *growing head* procedure, we start with the player that occupies the first position in an optimal order. In the initial order, this player may not be in the first position, but in a different position. The growing head procedure starts by consecutively moving this player to the first position, that is, this player consecutively switches with the players in front (according to the initial order) of him, until he reaches the first position. Secondly, we consider the player that is in second position in the given optimal order. Again, we consecutively move this player to the second position. We continue this process until all players are positioned in the position according to the given optimal order. Formally, the growing head procedure is defined as follows:

Procedure 4.1 [growing head procedure] *Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an optimal order.*

σ_0	I	j	$F(\sigma_0, j)$
σ_1	j	I	$F(\sigma_0, j)$

Figure 4.8 – *Step 1 of the growing head procedure.*

Step 1: *For the first step, set $j = \hat{\sigma}^{-1}(1)$, i.e., j is the player that is in the first position according to $\hat{\sigma}$. Consider the path $(\sigma_0, \ldots, \sigma_1)$ corresponding to neighbor switches (i, j) for every $i \in I$, where $I = P(\sigma_0, j)$. Here, $\sigma_1 \in \Pi(N)$ is the order in which $\sigma_1(j) = 1$, $\sigma_1(h) = \sigma_0(h) + 1$ for all $h \in P(\sigma_0, j)$ and $\sigma_1(h) = \sigma_0(h)$ for all $h \in F(\sigma_0, j)$ (see also Figure 4.8).*

σ_{k-1}			$\cdots$	h	I	j	$F(\sigma_{k-1}, j)$
σ_k			$\cdots$	h	j	I	$F(\sigma_{k-1}, j)$

Figure 4.9 – *Step k of the growing head procedure.*

For $k > 1$ until $k = |N| - 1$, perform the following step:

Step k: *Set $j = \hat{\sigma}^{-1}(k)$ and $h = \hat{\sigma}^{-1}(k-1)$, i.e., j is the player that is in position k according to $\hat{\sigma}$ and h the player that is in position $k-1$. Consider the path $(\sigma_{k-1}, \ldots, \sigma_k)$ corresponding to neighbor switches (i, j) for every $i \in I$, where $I = (h, j)_{\sigma_{k-1}}$. Here, $\sigma_k \in \Pi(N)$ is the order in which $\sigma_k(j) = k$, $\sigma_k(i) = \sigma_{k-1}(i) + 1$*

for all $i \in I$, $\sigma_k(g) = \sigma_{k-1}(g)$ *for all* $g \in F(\sigma_{k-1}, j)$ *and* $\sigma_k(g) = \sigma_{k-1}(g)$ *for all* $g \in P(\sigma_{k-1}, h) \cup \{h\}$ *(see also Figure 4.9).* $\triangleleft$

The second procedure, named the *growing tail* procedure, reverses the idea of the first procedure: instead of starting with the player that is in the first position in the optimal order, we now start with the player that is in the last position. We consecutively move this player to the back, in a similar way as in the growing head procedure. Formally, the growing tail procedure is defined as follows:

Procedure 4.2 [growing tail procedure] *Let* $(\sigma_0, p, c) \in SEQ^N$ *be an interactive sequencing situation and let* $\hat{\sigma} \in \Pi(N)$ *be an optimal order.*

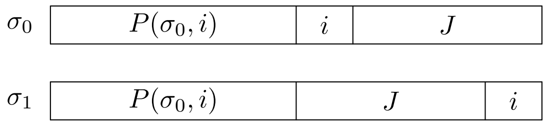


Figure 4.10 – *Step 1 of the growing tail procedure.*

Step 1: *For the first step, set* $i = \hat{\sigma}^{-1}(|N|)$, *i.e.,* i *is the player that is in the last position according to* $\hat{\sigma}$. *Consider the path* $(\sigma_0, \ldots, \sigma_1)$ *corresponding to neighbor switches* (i, j) *for every* $j \in J$, *where* $J = F(\sigma_0, i)$. *Here,* $\sigma_1 \in \Pi(N)$ *is the order in which* $\sigma_1(i) = |N|$, $\sigma_1(h) = \sigma_0(h)$ *for all* $h \in P(\sigma_0, i)$ *and* $\sigma_1(h) = \sigma_0(h) - 1$ *for all* $h \in F(\sigma_0, i)$ *(see also Figure 4.10).*

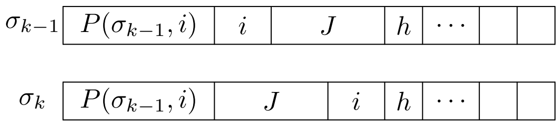


Figure 4.11 – *Step* k *of the growing tail procedure.*

For $k > 1$ *until* $k = |N| - 1$, *perform the following step:*

Step k***:*** *Set* $i = \hat{\sigma}^{-1}(|N| - k + 1)$ *and* $h = \hat{\sigma}^{-1}(|N| - k + 2)$, *i.e.,* i *is the player that is in position* $|N| - k + 1$ *according to* $\hat{\sigma}$ *and* h *the player that is in position* $|N| - k + 2$. *Consider the path* $(\sigma_{k-1}, \ldots, \sigma_k)$ *corresponding to neighbor switches* (i, j) *for every* $j \in J$, *where* $J = (i, h)_{\sigma_{k-1}}$. *Here,* $\sigma_k \in \Pi(N)$ *is the order in which* $\sigma_k(i) = |N| - k + 1$, $\sigma_k(j) = \sigma_{k-1}(j) - 1$ *for all* $j \in J$, $\sigma_k(g) = \sigma_{k-1}(g)$ *for all* $g \in P(\sigma_{k-1}, i)$ *and* $\sigma_k(g) = \sigma_{k-1}(g)$ *for all* $g \in F(\sigma_{k-1}, h) \cup \{h\}$ *(see also Figure 4.11).* $\triangleleft$

Starting with an initial order, one can follow either one of the two procedures in order to reach a given optimal order. For example, by consecutively moving players to the front, forming an optimal order by letting the head grow larger and larger, the growing head procedure specifies a path from the initial order to an optimal order. Similarly, the growing tail procedure forms an optimal order by letting the tail grow larger and larger. This is illustrated for a logarithmic sequencing situation in the following example, which is a continuation of Example 4.4.

Example 4.6 Reconsider the logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$, as described in Example 4.4, with $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $\alpha_i = 1$ for all $i \in N$, and $p_1 = 4, p_2 = 3$ and $p_3 = 2$. Recall that $\hat{\sigma} = (3, 2, 1)$ is the unique optimal order and $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (2, 3)\}$. We first perform the growing head procedure, according to Procedure 4.1. Note that $\hat{\sigma}(3) = 1$, so we start by consecutively moving player 3 to the front:

$$\sigma_0 = (1, 2, 3) \to (1, 3, 2) \to (3, 1, 2) = \sigma_1.$$

In the second step, we move player 2 to the second position, since $\hat{\sigma}(2) = 2$:

$$\sigma_1 = (3, 1, 2) \to (3, 2, 1) = \sigma_2 = \hat{\sigma}.$$

Now, player 1, as the last player in $\hat{\sigma}$, is automatically in the position according to $\hat{\sigma}$. Note that this path corresponds to the first path as described in Example 4.4.

Next, we perform the growing tail procedure, following Procedure 4.2. Note that $\hat{\sigma}(1) = 3$, so we start by consecutively moving player 1 to the back:

$$\sigma_0 = (1, 2, 3) \to (2, 1, 3) \to (2, 3, 1) = \sigma_1.$$

In the second step, we move player 2 to the second position, since $\hat{\sigma}(2) = 2$:

$$\sigma_1 = (2, 3, 1) \to (3, 2, 1) = \sigma_2 = \hat{\sigma}.$$

Note that this path corresponds to the second path as described in Example 4.4. △

4.5.2 Extending the gain splitting rules

After choosing a path from the initial order to an optimal order, we can adopt the idea behind the GS-rules to divide the corresponding neighbor switching gains in every step in such a path. Hence, we obtain two kinds of cost savings allocation rules based on the two procedures. In order to properly define these rules, we need a sophisticated way of summarizing several divisions for consecutive gains.

Fix a choice of $\lambda_{ij} \in [0, 1]$ for all $i, j \in N, i \neq j$. Given a particular player $i \in N$ and a group $J \subseteq N$ with player i directly in front of this group $J = \{j_1, \ldots, j_m\}$, which are ordered consecutively (see also Figure 4.2), define the vector $\overline{\lambda_{iJ}}$ of length $|J|$ as follows:

$$\overline{\lambda_{iJ}} = \left(\lambda_{ij_1}, \lambda_{ij_2}, \ldots, \lambda_{ij_m}\right).$$

By using the inner product $\langle \cdot, \cdot \rangle$, this vector can now be linked to the neighbor switching gains of player i and group J by using the vector $\overline{g_{iJ}}(t)$. In particular, $\left\langle \overline{\lambda_{iJ}}, \overline{g_{iJ}}(t) \right\rangle$ links the individual gains of the players $j \in J$ to the matching division λ_{ij}.

Similarly, given a player $j \in N$ and a group $I \subseteq N$ with player j directly behind the group $I = \{i_1, \dots, i_m\}$, which are ordered consecutively (see also Figure 4.3), define the vector $\overline{\lambda_{Ij}}$ of length $|I|$ as follows:

$$\overline{\lambda_{Ij}} = \big((1 - \lambda_{i_m j}), (1 - \lambda_{i_{m-1} j}), \dots, (1 - \lambda_{i_2 j}), (1 - \lambda_{i_1 j})\big).$$

Again, $\left\langle \overline{\lambda_{Ij}}, \overline{g_{Ij}}(t) \right\rangle$ links the individual gains of the players $i \in I$ to the matching division λ_{ij}.

Definition 4.1 Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then, for every choice of $\lambda_{ij} \in [0, 1]$ for all $i, j \in N, i \neq j$, the *gain splitting head (GSH) rule* is defined as follows:

$$GSH^{\lambda,\hat{\sigma}}(\sigma_0, p, c) = \sum_{k=1}^{|N|-1} \Big(\left\langle \overline{\lambda_{I_k j_k}}, \overline{g_{I_k j_k}}(t_{I_k}(\sigma_{k-1})) \right\rangle e^{\{j_k\}} + \sum_{i \in I_k} \lambda_{i j_k} g_{i j_k}(t_i(\sigma_{k-1})) e^{\{i\}} \Big),$$

where $j_k = \hat{\sigma}^{-1}(k)$ for every $k \in \{1, 2, \dots, |N|-1\}$, σ_k for every $k \in \{1, 2, \dots, |N|-1\}$ according to the growing head procedure, and

$$I_k = \begin{cases} P(\sigma_0, j_k), & \text{if } k = 1; \\ (h_k, j_k)_{\sigma_{k-1}}, & \text{if } k \in \{2, 3, \dots, |N| - 1\}, \end{cases}$$

with $h_k = \hat{\sigma}^{-1}(k - 1)$.

Similarly, for every choice of $\lambda_{ij} \in [0, 1]$ for all $i, j \in N, i \neq j$, the *gain splitting tail (GST) rule* is defined as follows:

$$\begin{aligned} GST^{\lambda,\hat{\sigma}}(\sigma_0, p, c) = \sum_{k=1}^{|N|-1} \Big(& \left\langle \overline{\lambda_{i_k J_k}}, \overline{g_{i_k J_k}}(t_{i_k}(\sigma_{k-1})) \right\rangle e^{\{i_k\}} \\ & + \sum_{j \in J_k} (1 - \lambda_{i_k j}) g_{i_k j}(t_j(\sigma_{k-1}) - p_{i_k}) e^{\{j\}} \Big), \end{aligned}$$

where $i_k = \hat{\sigma}^{-1}(|N| - k + 1)$ for every $k \in \{1, 2, \dots, |N| - 1\}$, σ_k for every $k \in \{1, 2, \dots, |N| - 1\}$ according to the growing tail procedure, and

$$J_k = \begin{cases} F(\sigma_0, i), & \text{if } k = 1; \\ (i, h_k)_{\sigma_{k-1}}, & \text{if } k \in \{2, 3, \dots, |N| - 1\}, \end{cases}$$

with $h_k = \hat{\sigma}^{-1}(|N| - k + 2)$. $\triangleleft$

In particular, by choosing $\lambda_{ij} = \frac{1}{2}$ for all $i, j \in N, i \neq j$, we obtain two extensions of the EGS-rule: the *equal gain splitting head (EGSH) rule* and the *equal gain splitting tail (EGST) rule*.

Note that for standard sequencing situations every path from the initial order to an optimal order leads to the same allocation and hence, the cost savings allocations specified by both the EGSH-rule and the EGST-rule boil down to the cost savings allocation prescribed by the EGS-rule. Example 4.7 shows that for logarithmic sequencing situations, both cost allocations of Definition 4.1 can differ. The example deals with a logarithmic sequencing situation as described earlier in Examples 4.4 and 4.6.

Example 4.7 Reconsider the logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$, as described in Examples 4.4 and 4.6, with $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $\alpha_i = 1$ for all $i \in N$, and $p_1 = 4, p_2 = 3$ and $p_3 = 2$. Recall that $\hat{\sigma} = (3, 2, 1)$ and that $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (2, 3)\}$ and the approximate neighbor switching gains as given in Table 4.10.

$g_{12}(0)$	0.2877
$g_{12}(p_3)$	0.1823
$g_{13}(0)$	0.6932
$g_{13}(p_2)$	0.3365
$g_{23}(0)$	0.4055
$g_{23}(p_1)$	0.1542

Table 4.10 – *The neighbor switching gains in the sequencing situation of Example 4.7.*

Recall from Example 4.6 that the growing head procedure specifies the following path from the initial order to the optimal order:

$$\sigma_0 = (1, 2, 3) \xrightarrow{g_{23}(p_1)} (1, 3, 2) \xrightarrow{g_{13}(0)} (3, 1, 2) = \sigma_1 \xrightarrow{g_{12}(p_3)} (3, 2, 1) = \sigma_2 = \hat{\sigma}.$$

Hence, by using Definition 4.1, the cost allocation prescribed by the EGSH-rule is

$$\begin{aligned}
EGSH^{\hat{\sigma}}(\sigma_0, p, c) &= \left\langle \overline{\lambda_{\{1,2\}3}}, \overline{g_{\{1,2\}3}}(0) \right\rangle e^{\{3\}} + \lambda_{13} g_{13}(0) e^{\{1\}} + \lambda_{23} g_{23}(p_1) e^{\{2\}} \\
&\quad + \left\langle \overline{\lambda_{12}}, \overline{g_{12}}(p_3) \right\rangle e^{\{2\}} + \lambda_{12} g_{12}(p_3) e^{\{1\}} \\
&= \Big((1 - \lambda_{23}) g_{23}(p_1) + (1 - \lambda_{13}) g_{13}(0) \Big) e^{\{3\}} + \lambda_{13} g_{13}(0) e^{\{1\}} \\
&\quad + \lambda_{23} g_{23}(p_1) e^{\{2\}} + \lambda_{12} g_{12}(p_3) e^{\{2\}} + \lambda_{12} g_{12}(p_3) e^{\{1\}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\tfrac{1}{2}g_{23}(p_1) + \tfrac{1}{2}g_{13}(0)\right)e^{\{3\}} + \tfrac{1}{2}g_{13}(0)e^{\{1\}} + \tfrac{1}{2}g_{23}(p_1)e^{\{2\}} \\
&\qquad + \tfrac{1}{2}g_{12}(p_3)e^{\{2\}} + \tfrac{1}{2}g_{12}(p_3)e^{\{1\}} \\
&= \left(\tfrac{1}{2}g_{13}(0) + \tfrac{1}{2}g_{12}(p_3)\right)e^{\{1\}} + \left(\tfrac{1}{2}g_{23}(p_1) + \tfrac{1}{2}g_{12}(p_3)\right)e^{\{2\}} \\
&\qquad + \left(\tfrac{1}{2}g_{23}(p_1) + \tfrac{1}{2}g_{13}(0)\right)e^{\{3\}} \\
&= 0.4378e^{\{1\}} + 0.1683e^{\{2\}} + 0.4237e^{\{3\}} \\
&= (0.4378, 0.1683, 0.4237)\,.
\end{aligned}
$$

Note that the neighbor switching gains that are involved, i.e., $g_{23}(p_1), g_{13}(0)$ and $g_{12}(p_3)$, are indeed divided equally among the two neighbors.

Example 4.6 also shows that the growing tail procedure specifies the following path from the initial order to the optimal order:

$$
\sigma_0 = (1,2,3) \xrightarrow{g_{12}(0)} (2,1,3) \xrightarrow{g_{13}(p_2)} (2,3,1) \xrightarrow{g_{23}(0)} (3,2,1) = \hat{\sigma}.
$$

Hence, by using Definition 4.1, the cost allocation prescribed by the EGST-rule is

$$
\begin{aligned}
EGST^{\hat{\sigma}}(\sigma_0, p, c) &= \left\langle \overline{\lambda_{1\{2,3\}}}, \overline{g_{1\{2,3\}}}(0) \right\rangle e^{\{1\}} + (1-\lambda_{12})g_{12}(p_1 - p_1)e^{\{2\}} \\
&\qquad + (1-\lambda_{13})g_{13}(p_1 + p_2 - p_1)e^{\{3\}} \\
&\qquad + \left\langle \overline{\lambda_{23}}, \overline{g_{23}}(0) \right\rangle e^{\{2\}} + (1-\lambda_{23})g_{23}(p_2 - p_2)e^{\{3\}} \\
&= \Big(\lambda_{12}g_{12}(0) + \lambda_{13}g_{13}(p_2)\Big)e^{\{1\}} + (1-\lambda_{12})g_{12}(0)e^{\{2\}} \\
&\qquad + (1-\lambda_{13})g_{13}(p_2)e^{\{3\}} + \lambda_{23}g_{23}(0)e^{\{2\}} + (1-\lambda_{23})g_{23}(0)e^{\{3\}} \\
&= \Big(\tfrac{1}{2}g_{12}(0) + \tfrac{1}{2}g_{13}(p_2)\Big)e^{\{1\}} + \tfrac{1}{2}g_{12}(0)e^{\{2\}} + \tfrac{1}{2}g_{13}(p_2)e^{\{3\}} \\
&\qquad + \tfrac{1}{2}g_{23}(0)e^{\{2\}} + \tfrac{1}{2}g_{23}(0)e^{\{3\}} \\
&= \left(\tfrac{1}{2}g_{12}(0) + \tfrac{1}{2}g_{13}(p_2)\right)e^{\{1\}} + \left(\tfrac{1}{2}g_{12}(0) + \tfrac{1}{2}g_{23}(0)\right)e^{\{2\}} \\
&\qquad + \left(\tfrac{1}{2}g_{13}(p_2) + \tfrac{1}{2}g_{23}(0)\right)e^{\{3\}} \\
&= 0.3121e^{\{1\}} + 0.3466e^{\{2\}} + 0.3710e^{\{3\}} \\
&= (0.3121, 0.3466, 0.3710)\,.
\end{aligned}
$$

Clearly, the allocations specified by the EGSH-rule and the EGST-rule differ. △

By construction, all GSH-rules and GST-rules lead to efficient allocations, that is, for every interactive sequencing situation $(\sigma_0, p, c) \in SEQ^N$, every optimal order $\hat{\sigma} \in \Pi(N)$ and every choice of λ,

$$
\sum_{i \in N} GSH_i^{\lambda,\hat{\sigma}}(\sigma_0, p, c) = v(N) = \sum_{i \in N} GST_i^{\lambda,\hat{\sigma}}(\sigma_0, p, c),
$$

where $v(N)$ is the worth of the grand coalition of the associated sequencing game. Additionally, under optimal order consistency, we can guarantee stability for the GSH-rules, that is,

$$\sum_{i \in S} GSH_i^{\lambda,\hat{\sigma}}(\sigma_0, p, c) \geq v(S),$$

for all $S \in 2^N \setminus \{\emptyset\}$, if the neighbor switching gains corresponding to misplacements are non-decreasing. Similarly, again under optimal order consistency,

$$\sum_{i \in S} GST_i^{\lambda,\hat{\sigma}}(\sigma_0, p, c) \geq v(S),$$

for all $S \in 2^N \setminus \{\emptyset\}$, if the neighbor switching gains corresponding to misplacements are non-increasing. This is formulated by the following theorem.

Theorem 4.3 *Let $(\sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation and let $v \in TU^N$ be the associated sequencing game. Let $\hat{\sigma} \in \Pi(N)$ be an optimal order. If, for all $t \in [0, \infty)$, the following two conditions hold:*

i) $g_{ij}(t) \geq 0$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$;

ii) $g_{ij}(t) \leq 0$ for all $(i,j) \notin MP(\sigma_0, \hat{\sigma})$,

then the following two statements hold for every choice of λ:

1) if, for all $s, t \in [0, \infty)$ with $s \leq t$, $g_{ij}(s) \leq g_{ij}(t)$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$, then $GSH^{\lambda,\hat{\sigma}}(\sigma_0, p, c) \in C(v)$;

2) if, for all $s, t \in [0, \infty)$ with $s \leq t$, $g_{ij}(s) \geq g_{ij}(t)$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$, then $GST^{\lambda,\hat{\sigma}}(\sigma_0, p, c) \in C(v)$.

Proof: By definition, all allocations prescribed by the GSH-rules and the GST-rules are efficient. For stability, it suffices to restrict to connected coalitions. Hence, let $S \in 2^N \setminus \{\emptyset\}$ be a connected coalition with respect to σ_0. Then we see that

$$v(S) = \sum_{(i,j) \in MP(\sigma_0, \hat{\sigma}); i,j \in S} g_{ij}(t_{ij}^S), \tag{4.15}$$

where t_{ij}^S is the starting time of player i for which players i and j switch positions if the group of players S is rearranging to its optimal position. Note that conditions i) and ii) imply optimal order consistency, according to Lemma 4.2.

1) Assume that, for all $s, t \in [0, \infty)$ with $s \leq t$, $g_{ij}(s) \leq g_{ij}(t)$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$. First, we show that

$$\sum_{(i,j) \in MP(\sigma_0, \hat{\sigma}); i,j \in S} g_{ij}(t_{ij}^S) \leq \sum_{(i,j) \in MP(\sigma_0, \hat{\sigma}); i,j \in S} g_{ij}(t_{ij}), \tag{4.16}$$

where t_{ij} is the starting time of player i for which the players i and j switch positions if all players (of the grand coalition N) are rearranging to its optimal position. We do so by showing that $g_{ij}(t^S_{ij}) \leq g_{ij}(t_{ij})$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$ with $i, j \in S$.

For this, let $(i,j) \in MP(\sigma_0, \hat{\sigma})$ with $i, j \in S$. To find a direct expression for t^S_{ij}, note that in the growing head procedure (according to Procedure 4.1), switching players i and j means moving player j to the front. At the moment of the switch of i and j, there are two types of players in front of i: first, all predecessors of i according to σ_0 are still predecessors of i at the moment of the switch. Secondly, all predecessors of j according to $\hat{\sigma}$ are already in their optimal positions at the head of $\hat{\sigma}$ and hence, are ordered before i at the moment of the switch of i and j. Together, we see that

$$t^S_{ij} = \sum_{h \in P(\sigma_0, i)} p_h + \sum_{h \in F(\sigma_0, i) \cap P(\hat{\sigma}, j) \cap S} p_h. \tag{4.17}$$

On the other hand, by using a similar argument as before, we can derive a direct expression for t_{ij} using the growing head procedure:

$$t_{ij} = \sum_{h \in P(\sigma_0, i)} p_h + \sum_{h \in F(\sigma_0, i) \cap P(\hat{\sigma}, j)} p_h. \tag{4.18}$$

Hence, we see that $t^S_{ij} \leq t_{ij}$, by combining Equation (4.17) and Equation (4.18) and by using the fact that $F(\sigma_0, i) \cap P(\hat{\sigma}, j) \cap S \subseteq F(\sigma_0, i) \cap P(\hat{\sigma}, j)$. Then it follows that $g_{ij}(t^S_{ij}) \leq g_{ij}(t_{ij})$ and consequently, Equation (4.16) is satisfied.

Next, note that, for every $(i,j) \in MP(\sigma_0, \hat{\sigma})$ with $i, j \in S$, we have that the corresponding neighbor switching gain $g_{ij}(t_{ij})$ is divided between only players i and j. This means that, for every choice of λ, when adding all allocations of the players for the GSH-rules, we have that

$$\sum_{h \in S} GSH^{\lambda, \hat{\sigma}}_h(\sigma_0, p, c) \geq \sum_{(i,j) \in MP(\sigma_0, \hat{\sigma}); i,j \in S} g_{ij}(t_{ij}).$$

Consequently, by combining this with Equations (4.15) and (4.16), we have that for every choice of λ

$$\sum_{h \in S} GSH^{\lambda, \hat{\sigma}}_h(\sigma_0, p, c) \geq v(S).$$

This finishes the proof of the first statement.

2) Assume that, for all $s, t \in [0, \infty)$ with $s \leq t$, $g_{ij}(s) \geq g_{ij}(t)$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$. The structure of the proof of the second statement is identical to the proof of the first statement. That is, we first show that Equation (4.16) is satisfied by showing that $g_{ij}(t^S_{ij}) \leq g_{ij}(t_{ij})$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$ with $i, j \in S$.

Let $(i,j) \in MP(\sigma_0, \hat{\sigma})$ with $i,j \in S$. In the growing tail procedure (according to Procedure 4.2), switching players i and j means moving player i to the back. Again, thee are two types of players in front of i: first, all players that are ordered before the coalition S according to σ_0 are still ordered before S and hence, before i at the moment of the switch. Secondly, at that point, all followers of i according to $\hat{\sigma}$ are already in their optimal positions at the tail of $\hat{\sigma}$ and hence, are ordered after i at the moment of the switch of i and j. Hence, the only players that are ordered before player i at that moment are players that are not yet moved to the back and thus, are predecessors of i according to $\hat{\sigma}$ as well as predecessors of j according to σ_0. Together, we see that

$$t_{ij}^S = \sum_{h \in P(\sigma_0, S)} p_h + \sum_{h \in P(\hat{\sigma}, i) \cap P(\sigma_0, j) \cap S} p_h, \tag{4.19}$$

where $P(\sigma_0, S) = \cap_{h \in S} P(\sigma_0, h)$ is the natural extension of the set of predecessors to a group of players.

For t_{ij}, it follows that

$$t_{ij} = \sum_{h \in P(\hat{\sigma}, i) \cap P(\sigma_0, j)} p_h, \tag{4.20}$$

by using the growing tail procedure. Then, by combining Equation (4.19) and Equation (4.20) and by using the fact that for every $h \in P(\hat{\sigma}, i) \cap P(\sigma_0, j)$, we have that either $h \in P(\sigma_0, S)$, if $h \notin S$, or $h \in P(\hat{\sigma}, i) \cap P(\sigma_0, j) \cap S$, if $h \in S$, it follows that $t_{ij} \leq t_{ij}^S$. Consequently, we have that $g_{ij}(t_{ij}^S) \leq g_{ij}(t_{ij})$ and hence, Equation (4.15) is also satisfied in this case.

Then it readily follows, by using a similar argument as before, that

$$\sum_{h \in S} GST_h^{\lambda, \hat{\sigma}}(\sigma_0, p, c) \geq \sum_{(i,j) \in MP(\sigma_0, \hat{\sigma}); i,j \in S} g_{ij}(t_{ij}),$$

and consequently, by combining this with Equations (4.15) and (4.16), we have that for every choice of λ,

$$\sum_{h \in S} GST_h^{\lambda, \hat{\sigma}}(\sigma_0, p, c) \geq v(S).$$

This finishes the proof of the second statement. □

Theorem 4.3 can be applied to sequencing situations with specific types of non-linear cost functions under optimal order consistency. For the three subclasses of exponential sequencing situations as defined in Saavedra-Nieves et al. (2020), we see that, by using Proposition 4.1, the GSH-rules lead to core-elements of the associated exponential sequencing games. For discounting sequencing situations, by using Proposition 4.2, we see that the GST-rules lead to allocations that are core-elements. Finally, for logarithmic sequencing situations, Proposition 4.3 can be used to see that the GST-rules lead to core-elements. Together, this yields the following corollary.

Corollary 4.2 *The following three statements hold:*

1) *Let* $(\sigma_0, p, c) \in ESEQ^N$ *be an exponential sequencing situation such that one of the following three cases holds:*

   i) *there is an* $\alpha \in \mathbb{R}_{++}$ *such that, for all* $i \in N$ *and all* $t \in [0, \infty)$, $c_i(t) = e^{\alpha t}$;

   ii) *there is a* $p \in \mathbb{R}_{++}$ *such that, for all* $i \in N$, $p_i = p$;

   iii) *there are* $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ *with* $\alpha_L < \alpha_H$, $p_L < p_H$ *such that, for all* $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}, p_i \in \{p_L, p_H\}$ *and*

   $$e^{\alpha_H p_H} - e^{\alpha_L p_L} \leq e^{\alpha_H (p_L + p_H)} - e^{\alpha_L (p_L + p_H)}.$$

   Let $\hat{\sigma} \in \Pi(N)$ *be an optimal order and let* $v \in TU^N$ *be the associated exponential sequencing game. Then for every choice of* λ,

   $$GSH_i^{\lambda, \hat{\sigma}}(\sigma_0, p, c) \in C(v);$$

2) *Let* $(\sigma_0, p, c) \in DSEQ^N$ *be a discounting sequencing situation and let* $v \in TU^N$ *be the associated discounting sequencing game. Let* $\hat{\sigma} \in \Pi(N)$ *be an optimal order. Then for every choice of* λ,

   $$GST_i^{\lambda, \hat{\sigma}}(\sigma_0, p, c) \in C(v);$$

3) *Let* $(\sigma_0, p, c) \in LSEQ^N$ *be a logarithmic sequencing situation and let* $v \in TU^N$ *be the associated discounting sequencing game. Let* $\hat{\sigma} \in \Pi(N)$ *be an optimal order. Then for every choice of* λ,

   $$GST_i^{\lambda, \hat{\sigma}}(\sigma_0, p, c) \in C(v).$$

Importantly, the GSH-rules and GST-rules can prescribe different allocations. This is, for example, the case in the three specified subclasses of exponential sequencing situations, in discounting sequencing situations and in logarithmic sequencing situations. Therefore, we end this section with an overview whether one of the two cost savings allocation rules lead to core-elements, as visualized in Table 4.11. To complete Table 4.11, we need two more examples. Example 4.8 shows that the GSH-rules in general do not lead to core-elements for discounting sequencing situations, whereas Example 4.9 shows a similar result for logarithmic sequencing situations. For the sake of completeness, note that the GST-rules do not lead to core-elements for the three specified subclasses of exponential sequencing situations, as shown by Saavedra-Nieves et al. (2020).

Example 4.8 Consider the discounting sequencing situation $(\sigma_0, p, c) \in DSEQ^N$ with $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $r = 0.8838$ and the discounting cost coefficients and processing times as shown in Table 4.12.

	Leading to core-elements	
	gain splitting head rules	gain splitting tail rules
Exponential sequencing situations (for the three specified subclasses)	Yes (Corollary 4.2)	No (Counterexample by Saavedra-Nieves et al., 2020)
Discounting sequencing situations	No (Example 4.8)	Yes (Corollary 4.2)
Logarithmic sequencing situations	No (Example 4.9)	Yes (Corollary 4.2)
Standard sequencing situations	Yes (gain splitting rules)	

Table 4.11 – *Overview of the stability results for the GSH-rules and GST-rules*

	player 1	player 2	player 3
α_i	0.1768	0.9070	0.5041
p_i	0.8371	0.9450	0.6142

Table 4.12 – *The discounting sequencing situation of Example 4.8.*

σ	$TC(\sigma)$
$(1,2,3)$	1.2551
$(1,3,2)$	1.2546
$(2,1,3)$	1.0972
$(2,3,1)$	1.0461
$(3,1,2)$	1.1368
$(3,2,1)$	1.0451

Table 4.13 – *The total costs of all processing orders in the sequencing situation of Example 4.8.*

There are six processing orders, for which the approximate total costs are given in Table 4.13.

Clearly, $\hat{\sigma} = (3, 2, 1)$ is the unique optimal order and $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (2, 3)\}$. The growing head procedure specifies the following path from σ_0 to $\hat{\sigma}$:

$$\begin{aligned}\sigma_0 = (1, 2, 3) \rightarrow (1, 3, 2) \rightarrow (3, 1, 2) &= \sigma_1 \\ \rightarrow (3, 2, 1) &= \hat{\sigma}.\end{aligned}$$

By using Table 4.13, the corresponding approximate neighbor switching gains are $g_{23}(p_1) = 0.0005$, $g_{13}(0) = 0.1178$ and $g_{12}(p_3) = 0.0918$, respectively. Let $\lambda_{ij} = \frac{1}{2}$ for all $i, j \in N, i \neq j$. Then we obtain the following allocation specified by the EGSH-rule:

$$GSH^{\lambda,\hat{\sigma}}(\sigma_0, p, c) = EGSH^{\hat{\sigma}}(\sigma_0, p, c) = (0.1048, 0.0461, 0.0592).$$

Consequently, if we take coalition $\{1, 2\} \in 2^N$, we have that

$$\begin{aligned} v(\{1, 2\}) &= TC(\sigma_0) - TC((2, 1, 3)) = 0.1579 \\ &> 0.1509 = EGSH_1^{\hat{\sigma}}(\sigma_0, p, c) + EGSH_2^{\hat{\sigma}}(\sigma_0, p, c), \end{aligned}$$

where $v \in TU^N$ denotes the associated discounting sequencing game. Thus, we have that $EGSH^{\hat{\sigma}}(\sigma_0, p, c) \notin C(v)$. △

Example 4.9 Consider the logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$ with $N = \{1, 2, 3, 4\}$, $\sigma_0 = (1, 2, 3, 4)$, $\alpha_i = 1$ for all $i \in N$, and $p_1 = 2.96, p_2 = 1.8$, $p_3 = 1.78$ and $p_4 = 1.75$.

The approximate total costs for all 24 orders are given in Table 4.14.

σ	$TC(\sigma)$	σ	$TC(\sigma)$	σ	$TC(\sigma)$
$(1, 2, 3, 4)$	6.6384	$(2, 3, 1, 4)$	5.8561	$(3, 4, 1, 2)$	5.8232
$(1, 2, 4, 3)$	6.6338	$(2, 3, 4, 1)$	5.6516	$(3, 4, 2, 1)$	25.6263
$(1, 3, 2, 4)$	6.6342	$(2, 4, 1, 3)$	5.8431	$(4, 1, 2, 3)$	6.0977
$(1, 3, 4, 2)$	6.6265	$(2, 4, 3, 1)$	5.6431	$(4, 1, 3, 2)$	6.0946
$(1, 4, 2, 3)$	6.6233	$(3, 1, 2, 4)$	6.1256	$(4, 2, 1, 3)$	5.8150
$(1, 4, 3, 2)$	6.6202	$(3, 1, 4, 2)$	6.1180	$(4, 2, 3, 1)$	5.6150
$(2, 1, 3, 4)$	6.1410	$(3, 2, 1, 4)$	5.8450	$(4, 3, 1, 2)$	5.8062
$(2, 1, 4, 3)$	6.1364	$(3, 2, 4, 1)$	5.6404	$(4, 3, 2, 1)$	5.6093

Table 4.14 – *The total costs of all processing orders in the sequencing situation of Example 4.9.*

Obviously, $\hat{\sigma} = (4, 3, 2, 1)$ is the unique optimal order and

$$MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}.$$

$g_{34}(p_1+p_2)$	0.0046
$g_{24}(p_1)$	0.0106
$g_{14}(0)$	0.5256
$g_{23}(p_1+p_4)$	0.0031
$g_{13}(p_4)$	0.2884
$g_{12}(p_3+p_4)$	0.1969

Table 4.15 – *The neighbor switching gains corresponding to the growing head procedure in Example 4.9.*

The growing head procedure specifies the following path from the initial order to the optimal order:

$$\begin{aligned}\sigma_0 = (1,2,3,4) \to (1,2,4,3) \to (1,4,2,3) \to (4,1,2,3) = \sigma_1 \\ \to (4,1,3,2) \to (4,3,1,2) = \sigma_2 \\ \to (4,3,2,1) = \hat{\sigma}.\end{aligned}$$

The corresponding approximate neighbor switching gains are shown in Table 4.15. Let $\lambda_{ij} = \frac{1}{2}$ for all $i,j \in N, i \neq j$. Then,

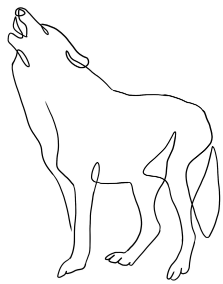
$$GSH_i^{\lambda,\hat{\sigma}}(\sigma_0,p,c) = EGSH^{\hat{\sigma}}(\sigma_0,p,c) = (0.5054, 0.1053, 0.1480, 0.2704).$$

Then we see that for coalition $\{1,2,3\} \in 2^N$, we have that

$$\begin{aligned}v(\{1,2,3\}) &= TC(\sigma_0) - TC((3,2,1,4)) = 6.6384 - 5.845 \\ &= 0.7935 > 0.7587 \\ &= EGSH_1^{\hat{\sigma}}(\sigma_0,p,c) + EGSH_2^{\hat{\sigma}}(\sigma_0,p,c) + EGSH_3^{\hat{\sigma}}(\sigma_0,p,c),\end{aligned}$$

where $v \in TU^N$ denotes the associated logarithmic sequencing game. Consequently, $EGSH^{\hat{\sigma}}(\sigma_0,p,c) \notin C(v)$. △

5 Cost sharing methods for capacity restricted cooperative purchasing situations



— *Wolfs hunt down a prey together and divide the spoils afterwards*

5.1 Introduction

Recently, several studies of different types of cooperative purchasing situations, in which organizations collaborate in their purchasing process, have focused on the bundling of purchasing volumes in order to obtain cost savings, see e.g. Schotanus (2007), Nagarajan, Sošić, and Zhang (2010) and Hezarkhani and Sošić (2019). Reasons for organizations to purchase cooperatively are numerous, but according to Tella and Virolainen (2005) the main motive is the cost savings due to the offered quantity discounts by the supplier. One of the main underlying assumptions in the cooperative purchasing literature in general is that the capacity of the supplier is sufficient to fulfill the total order of the group of purchasers. Although commonly assumed, one should realize that in practice the capacity of a supplier is limited. In particular, while the group of purchasers gets larger, the supplier's capacity might be exceeded and the group has to use a second supplier. In this chapter, based on Schouten, Groote Schaarsberg, and Borm (2020), the focus is on a group of cooperative purchasers that has to deal with two suppliers with limited capacities.

We consider a group of cooperative purchasers in which each organization has an individual order quantity with respect to the same specific commodity. Think of a group of departments or ministries, a group of municipalities with a joined purchasing program or online group-buying markets (the latter is studied by Anand and Aron, 2003). Typically, the sum of the order quantities determines the unit price negotiated with a supplier. Instead of facing one supplier with sufficient supplies, as in the classical cooperative purchasing situations, the group of purchasers faces two suppliers with (possibly) insufficient individual supplies, although the combined capacity of the two suppliers is assumed to be sufficient to cover the demands of the group. The unit price of a supplier weakly decreases with the size of the total order, that is however up to his capacity bound. These unit prices or quantity discount schemes are not necessarily the same for both suppliers. Within these *capacity restricted cooperative purchasing (CRCP) situations*, we are interested in finding the answers to two questions. First, how to split the total order over the two suppliers such that the total purchasing costs are minimized. Secondly, how to adequately allocate the total joint purchasing costs over the group of purchasers.

To solve the optimization problem in the first question, we show that there is a straightforward solution. We show that it is optimal to order as much as possible at one supplier and the possible remainder at the other. The second problem is more involved. For example, Schotanus (2007) argued that finding a fair cost allocation method is one of the critical success factors for cooperative purchasing. For such an allocation method, there are two desirable properties. Firstly, the quantity discounts should be incorporated in the cost allocation, that is, organizations with large order quantities pay a (weakly) lower unit price than organizations with (weakly) smaller order quantities. A second desirable property of an allocation method is that organizations with large order quantities do profit, in terms of cost allocations, from the presence of players with smaller order quantities. Loosely formulated: the smaller players should not be free riders.

To incorporate both properties in a suitable cost allocation method, we model the allocation problem within a CRCP-situation as a *cost sharing problem*. Generally, a cost sharing problem involves a set of users of a certain 'technology' and each of the users has an individual level of demanded output. To produce the total demanded output a certain level of input or costs is needed. The relationship between input and output, is represented by a cost function, where the function describes for each level of output the needed input in terms of associated costs. How to fairly distribute these joint costs is the central theme in the cost sharing literature. Moulin (2002) provided an overview of different types of cost sharing problems and allocation mechanisms.

In the CRCP-setting the output in the corresponding cost sharing problem is the sum of the individual order quantities. The cost function of the cost sharing problem then provides for each level of order quantities, the minimal purchasing costs. These minimal purchasing costs follow from dividing the order quantities optimally over the two suppliers and naturally follow from the straightforward solution to the first question. We show that the cost function of a cost sharing problem corresponding to a CRCP-situation is non-decreasing and piecewise concave, and that the finite number of (maximal) intervals of concavity are determined by the restricted capacity of the suppliers. According to Swoveland (1975), piecewise concave cost functions are a realistic representation of returns to scale in a production environment. Cost sharing problems with (almost) concave functions are studied in Karsten, Slikker, and Borm (2017), from the perspective of coalitional rationality and benefit ordering.

Two of the main cost sharing rules are the *average cost sharing rule* (see e.g. Moulin, 2002) and the *serial cost sharing rule* (Moulin and Shenker, 1992). The average (cost sharing) rule divides the total purchasing costs proportionally to the individual order quantities. By doing so, it neglects the quantity discounts present in the problem and allocates an equal unit price for everyone. Therefore, the main fous in this chapter is on the serial (cost sharing) rule. Interestingly, in the setting of concave cost functions, we show that the serial rule satisfies the two desirable properties from the perspective of CRCP-situations. Specifically, the serial rule satisfies *unit cost monotonicity (UCM)* and *monotonic vulnerability for the absence of the smallest player (MOVASP)*. UCM requires that when a specific organization has a (weakly) higher order quantity than an other organization, the former pays a lower cost per unit than the latter. MOVASP requires that, when the organization with the smallest order quantity is not present in the cooperation, there is a monotonicity relation throughout: the larger the order quantity, the higher the increase (or the smaller the decrease) in cost allocation. In other words, the organization with the largest order quantity has either the least decrease or the highest increase in cost allocation when the smallest player is absent. Hence, MOVASP creates a group cohesiveness in which the organization with the smallest order quantity can contribute to lower cost allocations of organizations with larger order quantities.

Next, we explicitly show that for cost sharing problems with piecewise concave cost functions, the serial rule in general does not satisfy UCM and MOVASP. The serial rule is thus not an appropriate allocation method to solve the allocation problem within a CRCP-situation. For this reason, we introduce a new tailor-made class of cost sharing rules for cost sharing problems with piecewise concave cost functions. Using a claims approach and, in particular, by selecting a specific claims rule (see Section 2.2), we first divide the vector of order quantities into separate vectors for the different maximal intervals of concavity. Subsequently, for each interval and its corresponding vector we use the serial rule to allocate the costs of that specific interval over the organizations. Finally, by summing these allocated costs over all intervals we obtain the allocation prescribed by a so-called piecewise serial rule.

In particular, we consider the *piecewise serial rules* where we divide the vector of order quantities into separate vectors, on the basis of the *proportional rule* and the *constrained equal losses rule*. It is shown that the proportional rule is the only claims rule for which the corresponding piecewise serial rule satisfies UCM on the class of cost sharing problems with piecewise concave cost functions. With regard to the constrained equal losses rule, we show that the corresponding piecewise serial rule satisfies MOVASP. Together, this shows incompatibility of UCM and MOVASP on the class of cost sharing problems with piecewise concave cost functions.

Finally, we address the possible compatibility of UCM and MOVASP on the subclass of cost sharing problems corresponding to CRCP-situations. Note that all cost sharing problems corresponding to CRCP-situations have piecewise concave cost functions, but not vice versa. Consequently, the incompatibility of UCM and MOVASP is not necessarily transferred to this subclass from the above-mentioned results. In particular, we illustrate that the piecewise serial rule based on the proportional rule does not satisfy MOVASP on the subclass of cost sharing problems corresponding to CRCP-situations. To investigate whether the piecewise serial rule based on the constrained equal losses rule satisfies UCM on this specific subclass, we perform a simulation which suggests that this is indeed the case.

Related studies on joint purchasing are Chen and Roma (2011) and Hu, Duenyas, and Beil (2013) on the (dis)advantages of cooperative purchasing, Ghodsypour and O'Brien (1998), Jayaraman and Srivastava (1999), and Ghodsypour and O'Brien (2001) on ordering processes and quantity allocations, and Marvel and Yang (2008) on strategic pricing by suppliers. Coalitional considerations in different purchasing inspired contexts are studied in, e.g., Meca and Sošić (2014) and Nagarajan and Sošić (2007). Studies on the capacity allocation problem, rather than the cost allocation problem, are, among others, Cho and Tang (2014) and Cui and Zhang (2018).

The structure of this chapter is as follows. Section 5.2 formally describes a CRCP-situation and solves the associated joint optimization problem. To address the joint cost allocation problems in Section 5.3, we model CRCP-situations as cost sharing problems in which the cost functions are piecewise concave. As an alternative to the serial cost sharing rule, we introduce piecewise serial rules in Section 5.4 and focus on the properties of UCM and MOVASP for the proportional and constrained equal losses variants of piecewise serial rules on the class of cost sharing problems with piecewise concave cost functions. Section 5.5 reflects on the consequences of the results on the subclass of cost sharing problems corresponding to CRCP-situations and provides a simulation.

5.2 Capacity restrictions in cooperative purchasing: CRCP-situations

In a *capacity restricted cooperative purchasing (CRCP) situation*, there is a finite set of organizations (from now onwards referred to as players) $N = \{1, \dots, n\}$, with $n \geq 2$ and a vector of individual order quantities $q \in \mathbb{R}^N_{++}$. There are two suppliers providing a particular commodity: A and B. The suppliers have capacities $Q_A, Q_B \in \mathbb{R}_{++}$ such that $\sum_{i \in N} q_i \leq Q_A + Q_B$. The suppliers have unit price functions $p_A : [0, Q_A] \to \mathbb{R}_+$ and $p_B : [0, Q_B] \to \mathbb{R}_+$ which are weakly decreasing, i.e., $p'_A(t) \leq 0$ for all quantities $t \in [0, Q_A]$ and $p'_B(t) \leq 0$ for all quantities $t \in [0, Q_B]$, and twice differentiable with a continuous second derivative on their respective domains. Moreover, both suppliers have revenue functions $c_A : [0, Q_A] \to \mathbb{R}_+$ and $c_B : [0, Q_B] \to \mathbb{R}_+$ which are given by

$$c_A(t) = p_A(t) \cdot t,$$

for all $t \in [0, Q_A]$ and

$$c_B(t) = p_B(t) \cdot t,$$

for all $t \in [0, Q_B]$. It is natural to assume that the revenue of a supplier does not decrease if t increases and that the quantity discounts are such that the higher the total order quantity the lower the increase in revenue. This is the case if we assume that the revenue functions are non-decreasing and concave, i.e.,

$$c'_A(t) \geq 0 \quad \text{and} \quad c''_A(t) \leq 0,$$

for all $t \in [0, Q_A]$ and

$$c'_B(t) \geq 0 \quad \text{and} \quad c''_B(t) \leq 0,$$

for all $t \in [0, Q_B]$. For the remainder of this chapter, we assume, without loss of generality, that

$$Q_A \leq Q_B,$$

and that the players are numbered in such a way that

$$q_1 \leq q_2 \leq \ldots \leq q_n.$$

When we refer to a smaller (larger) player, we refer to a player with smaller (larger) order quantity and thus smaller (larger) index.

A CRCP-situation on N as described above, is summarized by $Z = (S, q)$, in which $S = (Q_A, p_A, Q_B, p_B)$ summarizes the suppliers' information. We denote the set of all such CRCP-situations on N by $\mathcal{Z}^N$.

Within these CRCP-situations, we first want to know how to split the total order over the two suppliers such that the total purchasing costs are minimized. To answer this question, let $Z = (S, q) \in \mathcal{Z}^N$ be a CRCP-situation. Using the suppliers' information S, we determine the *minimal purchasing costs* $\gamma(Z)$ as follows:

$$\gamma(Z) = \min\left\{ c_A(t_A) + c_B(t_B) \;\middle|\; t_A + t_B = \sum_{i \in N} q_i,\ \ 0 \le t_A \le Q_A,\ \ 0 \le t_B \le Q_B \right\}. \tag{5.1}$$

The following theorem specifies these minimal purchasing costs by considering three separate cases based on the size of the total order quantity, i.e., $\sum_{i\in N} q_i$, compared to the capacities Q_A and Q_B of supplier A and B respectively.

Theorem 5.1 *Let $Z = (S, q) \in \mathcal{Z}^N$ with $S = (Q_A, p_A, Q_B, p_B)$ be a capacity restricted cooperative purchasing situation. Then it holds that*

$$\gamma(Z) = \begin{cases} \min\left\{c_A\big(\sum_{i\in N} q_i\big), c_B\big(\sum_{i\in N} q_i\big)\right\}, & \text{if } \sum_{i\in N} q_i \le Q_A; \\ \min\left\{c_A(Q_A) + c_B\big(\sum_{i\in N} q_i - Q_A\big),\right. & \\ \qquad \left. c_B\big(\sum_{i\in N} q_i\big)\right\}, & \text{if } Q_A < \sum_{i\in N} q_i \le Q_B; \\ \min\left\{c_A(Q_A) + c_B\big(\sum_{i\in N} q_i - Q_A\big),\right. & \\ \qquad \left. c_A\big(\sum_{i\in N} q_i - Q_B\big) + c_B(Q_B)\right\}, & \text{if } \sum_{i\in N} q_i > Q_B. \end{cases}$$

Consequently,

$$\begin{aligned} \gamma(Z) = \min\Bigg\{ & c_A\left(\min\left\{Q_A, \sum_{i\in N} q_i\right\}\right) + c_B\left(\max\left\{\sum_{i\in N} q_i - Q_A, 0\right\}\right), \\ & c_A\left(\max\left\{\sum_{i\in N} q_i - Q_B, 0\right\}\right) + c_B\left(\min\left\{Q_B, \sum_{i\in N} q_i\right\}\right)\Bigg\}. \end{aligned} \tag{5.2}$$

Proof: First, we reformulate Equation (5.1) of the minimal purchasing costs:

$$\begin{aligned} \gamma(Z) &= \min\left\{ c_A(t_A) + c_B(t_B) \;\middle|\; t_A + t_B = \sum_{i\in N} q_i, 0 \le t_A \le Q_A, 0 \le t_B \le Q_B \right\} \\ &= \min\left\{ c_A(t_A) + c_B\left(\sum_{i\in N} q_i - t_A\right) \;\middle|\; 0 \le t_A \le Q_A, 0 \le \sum_{i\in N} q_i - t_A \le Q_B \right\} \\ &= \min\left\{ c_A(t_A) + c_B\left(\sum_{i\in N} q_i - t_A\right) \;\middle|\; 0 \le t_A \le Q_A, \sum_{i\in N} q_i - Q_B \le t_A \le \sum_{i\in N} q_i \right\} \\ &= \min\left\{ c_A(t_A) + c_B\left(\sum_{i\in N} q_i - t_A\right) \;\middle|\; t_A \in \left[\max\left\{\sum_{i\in N} q_i - Q_B, 0\right\}, \min\left\{Q_A, \sum_{i\in N} q_i\right\}\right] \right\}. \end{aligned}$$

Note that the interval $\big[\max\{\sum_{i\in N} q_i - Q_B, 0\}, \min\{Q_A, \sum_{i\in N} q_i\}\big]$ is non-empty, since $\sum_{i\in N} q_i - Q_B \le Q_A$, and Q_A, Q_B and $\sum_{i\in N} q_i$ are all three strictly positive.

Next, we show that the minimum is obtained at one of the boundaries of this interval. Therefore, let $g : \left[\max\left\{\sum_{i\in N} q_i - Q_B, 0\right\}, \min\left\{Q_A, \sum_{i\in N} q_i\right\}\right] \to \mathbb{R}_+$ be defined by

$$g(t_A) = c_A(t_A) + c_B\left(\sum_{i\in N} q_i - t_A\right),$$

for all $t_A \in \left[\max\left\{\sum_{i\in N} q_i - Q_B, 0\right\}, \min\left\{Q_A, \sum_{i\in N} q_i\right\}\right]$. Note that

$$g'(t_A) = c'_A(t_A) - c'_B\left(\sum_{i\in N} q_i - t_A\right),$$

and that

$$g''(t_A) = c''_A(t_A) + c''_B\left(\sum_{i\in N} q_i - t_A\right).$$

Consequently, due to the fact that $c''_A(t) \leq 0$ for all $t \in [0, Q_A]$ and $c''_B(t) \leq 0$ for all $t \in [0, Q_B]$, we see that $g''(t_A) \leq 0$ for all possible t_A. Hence, g is concave and thus the minimum of g can be found at one of the boundaries of the domain of g: $t_A = \max\left\{\sum_{i\in N} q_i - Q_B, 0\right\}$ or $t_A = \min\left\{Q_A, \sum_{i\in N} q_i\right\}$, in the minimum.

Finally, we distinguish between the three cases with respect to $\sum_{i\in N} q_i$:

I) $\sum_{i\in N} q_i \leq Q_A$;

II) $Q_A < \sum_{i\in N} q_i \leq Q_B$;

III) $\sum_{i\in N} q_i > Q_B$.

Case I) In the first case, we assume that $\sum_{i\in N} q_i \leq Q_A$. Then also $\sum_{i\in N} q_i \leq Q_B$ and thus $\max\left\{\sum_{i\in N} q_i - Q_B, 0\right\} = 0$ and $\min\left\{Q_A, \sum_{i\in N} q_i\right\} = \sum_{i\in N} q_i$. In the minimum, we then have that $t_A = 0$ (and $t_B = \sum_{i\in N} q_i$) or $t_A = \sum_{i\in N} q_i$ (and $t_B = 0$) and consequently,

$$\begin{aligned}\gamma(Z) &= \min\left\{c_A(0) + c_B\left(\sum_{i\in N} q_i\right), c_A\left(\sum_{i\in N} q_i\right) + c_B(0)\right\}\\ &= \min\left\{c_A\left(\sum_{i\in N} q_i\right), c_B\left(\sum_{i\in N} q_i\right)\right\}.\end{aligned}$$

Case II) In the second case, we assume that $Q_A < \sum_{i\in N} q_i \leq Q_B$. Then it follows that $\max\left\{\sum_{i\in N} q_i - Q_B, 0\right\} = 0$ and $\min\left\{Q_A, \sum_{i\in N} q_i\right\} = Q_A$. In the minimum, we then have that $t_A = 0$ (and $t_B = \sum_{i\in N} q_i$) or $t_A = Q_A$ (and $t_B = \sum_{i\in N} q_i - Q_A$). Consequently,

$$\begin{aligned}\gamma(Z) &= \min\left\{c_A(0) + c_B\left(\sum_{i\in N} q_i\right), c_A(Q_A) + c_B\left(\sum_{i\in N} q_i - Q_A\right)\right\}\\ &= \min\left\{c_A(Q_A) + c_B\left(\sum_{i\in N} q_i - Q_A\right), c_B\left(\sum_{i\in N} q_i\right)\right\}.\end{aligned}$$

Case III) In the third and last case, we assume that $\sum_{i\in N} q_i > Q_B$. Then also $\sum_{i\in N} q_i > Q_A$ and thus $\max\left\{\sum_{i\in N} q_i - Q_B, 0\right\} = \sum_{i\in N} q_i - Q_B$ and $\min\left\{Q_A, \sum_{i\in N} q_i\right\} = Q_A$. In the minimum, we then have that $t_A = \sum_{i\in N} q_i - Q_B$ (and $t_B = Q_B$) or $t_A = Q_A$ (and $t_B = \sum_{i\in N} q_i - Q_A$). Consequently,

$$\begin{aligned}\gamma(Z) &= \min\left\{c_A\left(\sum_{i\in N} q_i - Q_B\right) + c_B(Q_B), c_A(Q_A) + c_B\left(\sum_{i\in N} q_i - Q_A\right)\right\}\\ &= \min\left\{c_A(Q_A) + c_B\left(\sum_{i\in N} q_i - Q_A\right), c_A\left(\sum_{i\in N} q_i - Q_B\right) + c_B(Q_B)\right\}.\end{aligned}$$

This concludes the proof of the first part. Equation (5.2) is readily obtained by combining the three cases. □

So, Theorem 5.1 implies that to minimize purchasing costs, one has to compare two extreme policies: order as much as possible at one of the two suppliers first and the remaining part at the other one. Depending on the unit price functions and the total order quantity one might prefer A 'first' or B 'first'. This is illustrated in Example 5.1 below.

Example 5.1 Consider the capacity restricted cooperative purchasing situation $Z = (S, q) \in \mathcal{Z}^N$ with $N = \{1, 2, 3\}$, $q = (3, 5, 7)$ and $S = (Q_A, p_A, Q_B, p_B)$ given by $Q_A = 8$, $Q_B = 12$ and

$$\begin{cases} p_A(t) = 14 - t, & \text{for all } t \in [0, 8];\\ p_B(t) = 10 - \frac{1}{2}t, & \text{for all } t \in [0, 12].\end{cases}$$

One readily verifies that all assumptions on a CRCP-situation are satisfied.

Since $\sum_{i\in N} q_i = 15$, the players can order any amount $t_A \in [3, 8]$ at A and the remaining part $\sum_{i\in N} q_i - t_A \in [7, 12]$ at B. The corresponding purchasing costs are given by

$$\begin{aligned}
c_A(t_A) + c_B(15 - t_A) &= p_A(t_A) \cdot t_A + p_B(15 - t_A) \cdot (15 - t_A) \\
&= (14 - t_A) \cdot t_A + (10 - \tfrac{1}{2}(15 - t_A)) \cdot (15 - t_A) \\
&= (14 - t_A) \cdot t_A + (2\tfrac{1}{2} + \tfrac{1}{2}t_A) \cdot (15 - t_A) \\
&= -\tfrac{3}{2}(t_A)^2 + 19t_A + 37\tfrac{1}{2},
\end{aligned}$$

for all $t_A \in [3, 8]$. These purchasing costs are visualized in Figure 5.1.

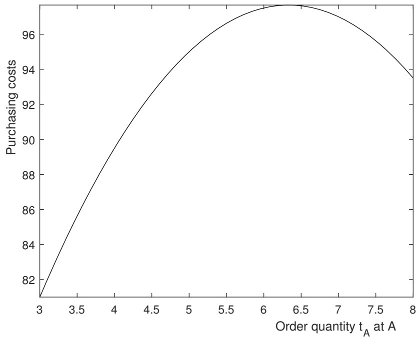


Figure 5.1 – *The purchasing costs in the CRCP-situation of Example 5.1.*

Clearly, the minimum is obtained in the left-hand boundary $t_A = 3$. This means that the players order as much as possible at supplier B, thus ordering $Q_B = 12$ at B, and the remaining part $\sum_{i \in N} q_i - Q_B = 3$ at A. This gives a total purchasing costs of

$$\begin{aligned}
c_A(3) + c_B(12) &= p_A(3) \cdot 3 + p_B(12) \cdot 12 \\
&= (14 - 3) \cdot 3 + (10 - 6) \cdot 12 \\
&= 33 + 48 \\
&= 81.
\end{aligned}$$

Indeed, the other extreme policy to order as much as possible at supplier A, thus ordering $Q_A = 8$ at A and $\sum_{i \in N} q_i - Q_A = 7$ at B, results in higher purchasing costs:

$$\begin{aligned}
c_A(8) + c_B(7) &= p_A(8) \cdot 8 + p_B(7) \cdot 7 \\
&= (14 - 8) \cdot 8 + (10 - 3\tfrac{1}{2}) \cdot 7 \\
&= 48 + 45\tfrac{1}{2} \\
&= 93\tfrac{1}{2}.
\end{aligned}$$

Figure 5.1 also clearly visualizes Equation (5.2) of Theorem 5.1 that the minimum is obtained in either one of the boundary cases due to the concave nature. $\triangle$

5.3 Cost sharing problems corresponding to CRCP-situations

In Section 5.2, our first question concerning the joint optimization problem within a CRCP-situation is solved. In the rest of the chapter, the focus is on the second question: how to allocate the total joint purchasing costs over the group of purchasers. For this, we use a cost sharing approach and model a CRCP-situation as a cost sharing problem. The main goal in a cost sharing problem is to fairly distribute the costs of jointly using a certain 'technology' to produce the sum of the individual levels of demanded output, taking into account the costs associated to every level of demanded output. In this section, we formally model a CRCP-situation as a cost sharing problem.

A *cost sharing problem on* $N = \{1, \ldots, n\}$ is represented by a pair (C, q), with demand vector $q \in \mathbb{R}^N_{++}$ such that $q_1 \leq \ldots \leq q_n$, and cost function $C : [0, \sum_{i \in N} q_i] \to \mathbb{R}_+$ for which it holds that $C(t)$ is non-decreasing and continuous in t and $C(0) = 0$. Here, the argument t in $C(t)$ represents the total demanded output. Let $\mathcal{CS}^N$ denote the corresponding class of cost sharing problems on N. When allowing for a variable finite player set, we use the notation $\mathcal{CS}$ for the class of all such cost sharing problems.

A *cost sharing rule* f *on* $\mathcal{CS}^N$ is a mapping $f : \mathcal{CS}^N \to \mathbb{R}^N$ such that, for all $(C, q) \in \mathcal{CS}^N$, $f(C, q) \geq 0$ and

$$\sum_{i \in N} f_i(C, q) = C\Big(\sum_{i \in N} q_i\Big).$$

A *cost sharing rule* f *on* $\mathcal{CS}$ is a mapping that assigns to each cost sharing problem $(C, q) \in \mathcal{CS}^N$, with an arbitrary finite player set N, such a vector $f(C, q)$. Also for specific subclasses of $\mathcal{CS}^N$, dropping the index N from the notation will mean that we allow for a variable player set.

A CRCP-situation corresponds to a special type of cost sharing problem. Let $Z = (S, q) \in \mathcal{Z}^N$ be a capacity restricted cooperative purchasing situation. We naturally extend the *minimal purchasing costs* and define $\gamma^S(t)$ for all $t \in [0, Q_A + Q_B]$ as follows:

$$\gamma^S(t) = \min \{c_A(t_A) + c_B(t_B) \mid t_A + t_B = t,\ \ 0 \leq t_A \leq Q_A,\ \ 0 \leq t_B \leq Q_B\}. \tag{5.3}$$

Clearly, $\gamma(Z) = \gamma^S\left(\sum_{i \in N} q_i\right)$.

Then we can determine the corresponding cost sharing problem (C^Z, q), in which C^Z is the restriction of γ^S to the domain $[0, \sum_{i \in N} q_i]$. It can be readily verified that $(C^Z, q) \in \mathcal{CS}^N$. Note that the value $C^Z(t)$ does not depend on N and q. Only the domain of the function C^Z is determined by $\sum_{i \in N} q_i$. Similar to Theorem 5.1, we have the following, more general, result. The proof can be easily obtained by following the proof of Theorem 5.1, basically by replacing $\sum_{i \in N} q_i$ with $t \in [0, Q_A + Q_B]$.

Theorem 5.2 *Let $Z = (S, q) \in \mathcal{Z}^N$ with $S = (Q_A, p_A, Q_B, p_B)$ be a capacity restricted cooperative purchasing situation and let $(C^Z, q) \in \mathcal{CS}^N$ be the corresponding cost sharing problem. Then, for all $t \in [0, Q_A + Q_B]$,*

$$\gamma^S(t) = \begin{cases} \min\{c_B(t), c_A(t)\}, & \text{if } t \in [0, Q_A]; \\ \min\{c_B(t), c_A(Q_A) + c_B(t - Q_A)\}, & \text{if } t \in (Q_A, Q_B]; \\ \min\{c_B(Q_B) + c_A(t - Q_B), & \\ \qquad c_A(Q_A) + c_B(t - Q_A)\}, & \text{if } t \in (Q_B, Q_A + Q_B]. \end{cases}$$

Consequently, for all $t \in [0, \sum_{i \in N} q_i]$,

$$C^Z(t) = \gamma^S(t) = \min\Big\{c_A(\min\{Q_A, t\}) + c_B(\max\{t - Q_A, 0\}), \\ c_A(\max\{t - Q_B, 0\}) + c_B(\min\{Q_B, t\})\Big\}. \tag{5.4}$$

Thus, for each level of order quantities, the minimal purchasing costs follow from dividing these order quantities optimally over the two suppliers. To reach this optimal division, one has to compare the two extreme policies: order as much as possible at one of the two suppliers first and the remaining part at the other one, or the other way around. The actual unit price functions and total order quantity determines whether one prefers 'A first' or 'B first'.

For an increasing level of order quantities, switches from one policy to the other and back can occur. We call these switches *policy switches*. They arise from the minimization process according to Equation (5.4) of Theorem 5.2. At these points, switching from one extreme policy to the other, that is going to the other supplier first, result in lower purchasing costs.

The following example illustrates such a policy switch. Besides, it also demonstrates the use of Theorem 5.2 in finding the function of the cost sharing problem corresponding to a CRCP-situation.

Example 5.2 Consider the capacity restricted cooperative purchasing situation $Z = (S, q) \in \mathcal{Z}^N$ with $S = (Q_A, p_A, Q_B, p_B)$ given by $Q_A = 16$, $Q_B = 20$ and

$$\begin{cases} p_A(t) = 18 - \frac{1}{3}t, & \text{for all } t \in [0, 16]; \\ p_B(t) = 20 - \frac{1}{2}t, & \text{for all } t \in [0, 20]. \end{cases}$$

Note that we now do not need exact specifications of N and q in order to compute the value $\gamma^S(t)$ for all $t \in [0, 36]$. Rather, we can use Theorem 5.2 and compare the two extreme policies for three separate cases:

I) $t \in [0, Q_A]$;

II) $t \in (Q_A, Q_B]$;

III) $t \in (Q_B, Q_A + Q_B]$.

Case I) For the first case, let $t \in [0, Q_A] = [0, 16]$ and note that $c_A(t) = 18t - \frac{1}{3}t^2$ and $c_B(t) = 20t - \frac{1}{2}t^2$. Then it is readily verified that $c_A(t) \leq c_B(t)$ for all $t \in [0, 12]$ and $c_B(t) < c_A(t)$ for all $t \in (12, 16]$. In the first part, it is thus optimal to order at A first. Then there is a policy switch at $t = 12$. In the second part, it is optimal to order at B first.

Therefore, we have that, for $t \in [0, 16]$,

$$\gamma^S(t) = \min\{c_B(t), c_A(t)\} = \begin{cases} c_A(t), & \text{if } t \in [0, 12]; \quad (A \text{ first}) \\ c_B(t), & \text{if } t \in (12, 16]. \quad (B \text{ first}) \end{cases}$$

Case II) For the second case, let $t \in (Q_A, Q_B] = (16, 20]$ and note that

$$c_A(16) + c_B(t - 16) = 202\tfrac{2}{3} + 20(t - 16) - \tfrac{1}{2}(t - 16)^2.$$

Then it is readily verified that $c_B(t) < c_A(16) + c_B(t - 16)$ for all $t \in (16, 20]$. Consequently, the optimal policy in this case is to order at B first. Therefore, for all $t \in (16, 20]$,

$$\gamma^S(t) = \min\{c_B(t), c_A(16) + c_B(t - 16)\} = c_B(t). \quad (B \text{ first})$$

Case III) Finally, for the third case, let $t \in (Q_B, Q_A + Q_B] = (20, 36]$ and note that

$$c_B(20) + c_A(t - 20) = 200 + 18(t - 20) - \tfrac{1}{3}(t - 20)^2.$$

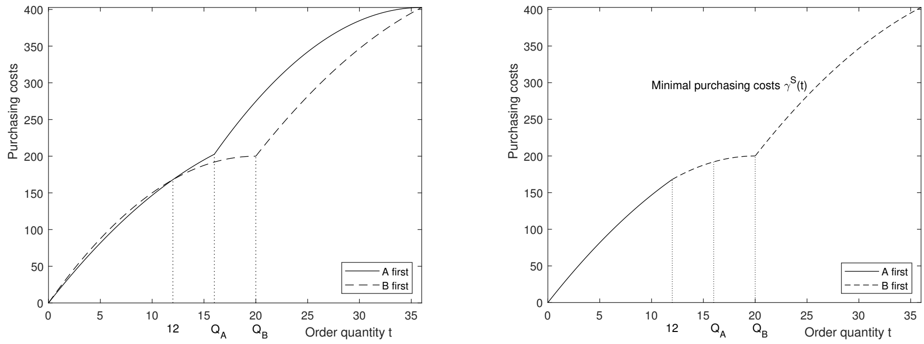
Then it is readily verified that $c_B(20) + c_A(t - 20) \leq c_A(16) + c_B(t - 16)$ for all $t \in (20, 36]$. So also in this case, it is optimal to order B first. Consequently, for all $t \in (20, 36]$,

$$\gamma^S(t) = \min\{c_B(20) + c_A(t-20), c_A(16) + c_B(t-16)\} = c_B(20) + c_A(t-20). \quad (B \text{ first})$$

Summarizing, we find that for all $t \in [0, 36]$,

$$\gamma^S(t) = \begin{cases} c_A(t) = 18t - \frac{1}{3}t^2, & \text{if } t \in [0, 12]; \quad (A \text{ first}) \\ c_B(t) = 20t - \frac{1}{2}t^2, & \text{if } t \in (12, 20]; \quad (B \text{ first}) \\ c_B(20) + c_A(t - 20) & \\ \qquad = 200 + 18(t - 20) - \frac{1}{3}(t - 20)^2, & \text{if } t \in (20, 36], \quad (B \text{ first}) \end{cases} \tag{5.5}$$

as drawn in Figure 5.2b.



(a) *The two extreme policies.* **(b)** *The cost function γ^S.*

Figure 5.2 – *The two extreme policies and the cost function γ^S of the CRCP-situation of Example 5.2.*

As stated in Theorem 5.2, the cost function C^Z, which is the restriction of γ^S to the domain $[0, \sum_{i \in N} q_i]$, is the minimum of the following two policies: order as much as possible at A first and then go to B or order as much as possible at B first and then go to A. The cost functions of these two policies are shown in Figure 5.2a. In this case, there is only one policy switch between the two extreme policies, at $t = 12$. $\triangle$

Note that in Example 5.2, the cost function γ^S is piecewise concave with two maximal intervals of concavity: $[0, 20]$ and $[20, 36]$. These intervals of concavity arise not from switching policies, but from the necessity of changing supplier within one policy due to the capacity restriction. At a certain point, one supplier hits its capacity and players have to order the remaining part at the other supplier. We call such a point a *concavity break*.

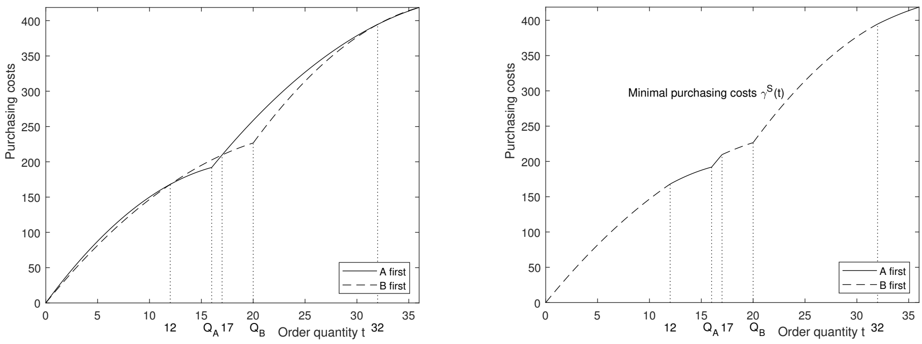
The following example shows that there can be several policy switches as well as several concavity breaks. However, due to the fact that there are only two suppliers, the number of concavity breaks is at most two, leading to at most three maximal intervals of concavity. In fact, this is the case in Example 5.3 below.

Example 5.3 Consider the capacity restricted cooperative purchasing situation $Z = (S, q) \in \mathcal{Z}^N$ with $S = (Q_A, p_A, Q_B, p_B)$ given by $Q_A = 16$, $Q_B = 20$ and[1]

$$\begin{cases} p_A(t) = 20 - \frac{1}{2}t, & \text{for all } t \in [0, 16]; \\ p_B(t) = 18 - \frac{1}{3}t, & \text{for all } t \in [0, 20]. \end{cases}$$

Figure 5.3a depicts the cost functions corresponding to the two extreme policies (A

[1] The careful reader might notice that this CRCP-situation is almost identical to the one discussed in Example 5.2, except that the unit price functions of the suppliers are interchanged.



(a) *The two extreme policies.* **(b)** *The cost function γ^S.*

Figure 5.3 – *The two extreme policies and the cost function γ^S of the CRCP-situation of Example 5.3.*

first or B first). In this situation we see three policy switches: at $t = 12, t = 17$ and $t = 32$. This can also be seen from the following explicit expression of $\gamma^S(t)$, as depicted in Figure 5.3b, for all $t \in [0, 36]$:

$$\gamma^S(t) = \begin{cases} c_B(t) = 18t - \frac{1}{3}t^2, & \text{if } t \in [0,12]; \quad (B \text{ first}) \\ c_A(t) = 20t - \frac{1}{2}t^2, & \text{if } t \in (12,16]; \quad (A \text{ first}) \\ c_A(16) + c_B(t-16) & \\ \qquad = 192 + 18(t-16) - \frac{1}{3}(t-16)^2, & \text{if } t \in (16,17]; \quad (A \text{ first}) \\ c_B(t) = 18t - \frac{1}{3}t^2, & \text{if } t \in (17,20]; \quad (B \text{ first}) \\ c_B(20) + c_A(t-20) & \\ \qquad = 226\frac{2}{3} + 20(t-20) - \frac{1}{2}(t-20)^2, & \text{if } t \in (20,32]; \quad (B \text{ first}) \\ c_A(16) + c_B(t-16) & \\ \qquad = 192 + 18(t-16) - \frac{1}{3}(t-16)^2, & \text{if } t \in (32,36]. \quad (A \text{ first}) \end{cases} \tag{5.6}$$

Note that there are two concavity breaks: at $t = Q_A = 16$ and $t = Q_B = 20$. Furthermore, γ^S is piecewise concave and there are three maximal intervals of concavity: $[0, 16]$, $[16, 20]$ and $[20, 36]$. △

One can readily generalize the observations made in Example 5.2 and 5.3: the cost function γ^S will be non-decreasing and piecewise concave with at most three maximal intervals of concavity and hence, so will the cost function C^Z corresponding to a CRCP-situation $Z = (S, q)$.

5.4 Cost sharing rules for piecewise concave cost functions

In this section, we develop the new class of piecewise serial rules for the class of cost sharing problems with piecewise concave cost functions. These cost sharing rules are all based on the serial cost sharing rule (Moulin and Shenker, 1992) and are tailor-made in the sense that they fit with two context specific properties that are desirable from a CRCP-perspective.

5.4.1 The serial cost sharing rule

The serial cost sharing rule is based on the requirement that a player's costs should not depend on the size of the order quantity of larger players. For a concave cost function, this requirement implies that smaller players profit less from the economies of scale than the larger players. If we think of CRCP-situations in which large players generally account for higher quantity discounts, this seems a suitable solution method for dividing costs that follow from purchasing cooperatively.

Formally, the *serial (cost sharing) rule* (cf. Moulin and Shenker, 1992), Ser, on the class $\mathcal{CS}$ of cost sharing problems[2] is such that for all $(C, q) \in \mathcal{CS}^N$,

$$\begin{cases} \mathrm{Ser}_1(C,q) = \dfrac{C(s_1)}{n}; \\ \mathrm{Ser}_i(C,q) = \mathrm{Ser}_{i-1}(C,q) + \dfrac{C(s_i) - C(s_{i-1})}{n-(i-1)}, \qquad \text{for all } i \in N \setminus \{1\}, \end{cases}$$

where the *intermediate points* $s_0, s_1, \ldots, s_n$ are recursively given by

$$\begin{cases} s_0 = 0; \\ s_i = \sum_{j=1}^{i-1} q_j + \big(n-(i-1)\big)q_i, \qquad \text{for all } i \in N. \end{cases}$$

The serial rule is illustrated in Example 5.4 below.

Example 5.4 [cf. Moulin and Shenker, 1994] Consider the cost sharing problem $(C, q) \in \mathcal{CS}^N$ with $N = \{1, 2, 3\}$, $q = (3, 5, 7)$ and (concave) cost function $C : [0, 15] \to \mathbb{R}_+$ given by

$$C(t) = \min\left\{t, 9 + \tfrac{1}{10}t\right\},$$

for all $t \in [0, 15]$.

First, the intermediate points that are needed to compute $\mathrm{Ser}(C, q)$ are given by $s_1 = 3 \cdot 3 = 9$, $s_2 = 3 + 2 \cdot 5 = 13$ and $s_3 = 3 + 5 + 7 = 15$.

[2] Recall that for all cost sharing problems $(C, q) \in \mathcal{CS}^N$, it holds that $q_1 \leq \ldots \leq q_n$.

Next, it readily follows that $C(s_1) = C(9) = \min\left\{9, 9 + \frac{9}{10}\right\} = 9$ are the costs corresponding to the intermediate point of player 1. Similarly, $C(s_2) = C(13) = 10.3$ and $C(s_3) = C(15) = 10.5$. Consequently,

$$\begin{aligned}
\mathrm{Ser}_1(C,q) =& \frac{C(9)}{3} = \frac{9}{3} = 3,\\
\mathrm{Ser}_2(C,q) =& \mathrm{Ser}_1(C,q) + \frac{C(13) - C(9)}{2} = 3 + \frac{1.3}{2} = 3.65,\\
\mathrm{Ser}_3(C,q) =& \mathrm{Ser}_2(C,q) + \frac{C(15) - C(13)}{1} = 3.65 + 0.2 = 3.85.
\end{aligned}$$

The serial rule thus assigns 3 to player 1, 3.65 to player 2 and 3.85 to player 3. △

There are many equivalent ways of presenting the serial rule. The above-mentioned recursive definition underlines the serial behavior. Alternatively, the serial rule can be reformulated as in the following lemma.

Lemma 5.1 [cf. Moulin and Shenker, 1992, 1994] *Let $(C,q) \in \mathcal{CS}^N$. Then, for all $i \in N$,*

i) $Ser_i(C,q) = \displaystyle\sum_{j=1}^{i} \frac{C(s_j) - C(s_{j-1})}{n - (j-1)}$;

ii) $Ser_i(C,q) = \displaystyle\frac{C(s_i)}{n-(i-1)} - \sum_{j=1}^{i-1} \frac{C(s_j)}{\big(n-(j-1)\big)(n-j)}$.

The serial rule is also studied from an axiomatic point of view by Moulin and Shenker (1992, 1994, 1999) and Friedman and Moulin (1999), among others. In fact, Moulin and Shenker (1992) originally defined the serial rule as the unique rule that satisfies two properties: anonymity and independence of the size of larger demands.

A cost sharing rule satisfies *anonymity* if the outcome only depends on the size of the demand rather than on the identity, or index, of the players.

A cost sharing rule satisfies *independence of the size of larger demands (ISLAD)* if the outcome of a player does not depend on the demands that are larger than his own demand. Formally, a cost sharing rule f satisfies ISLAD if, for all $(C,q) \in \mathcal{CS}^N$, all $i, j \in N$ with $q_i \leq q_j$, and all $(C,\bar{q}) \in \mathcal{CS}^N$ with $\bar{q} = ((q_k)_{k \in N\setminus\{j\}}, r)$, with $r \geq q_j$, it holds that $f_i(C,q) = f_i(C,\bar{q})$.

Theorem 5.3 [cf. Moulin and Shenker, 1992] *The serial rule is the unique cost sharing rule satisfying both anonymity and independence of the size of larger demands.*

Let $\mathcal{CCS}^N$ denote the subclass of all $(C,q) \in \mathcal{CS}^N$ in which the cost function C is concave. On $\mathcal{CCS}$, the serial rule satisfies two attractive properties: unit cost monotonicity and monotonic vulnerability for the absence of the smallest player. Both properties are inspired by the CRCP-context and formulated on the class of all cost sharing problems.

First, we focus on unit cost monotonicity, which requires that for a player with a (weakly) higher demand than an other player, the former is assigned a (weakly) lower cost per unit than the latter. This is formalized in Definition 5.1.

Definition 5.1 A cost sharing rule f satisfies *unit cost monotonicity (UCM)* on $\mathcal{C} \subseteq \mathcal{CS}^N$ if, for all $(C,q) \in \mathcal{C}$ and for all $i \in N \setminus \{n\}$,

$$\frac{f_i(C,q)}{q_i} \geq \frac{f_{i+1}(C,q)}{q_{i+1}}. \qquad \triangleleft$$

The following theorem shows that the serial rule satisfies unit cost monotonicity on the class of cost sharing problems with concave cost functions.[3] The proof of this theorem is based on some basic features of concave functions. These features are first summarized in the following lemma.

Lemma 5.2 *Let X be a convex subset of $\mathbb{R}$ and let $f : X \to \mathbb{R}$ with $f(0) = 0$ be a non-decreasing and concave function. Then the following three statements hold:*

i) $\frac{f(x)}{x} \geq \frac{f(y)}{y}$, *for all $x, y \in X$ with $0 < x \leq y$;*

ii) $f(x+z) - f(x) \geq f(y+z) - f(y)$, *for all $x, y, z \in X$ with $0 < x \leq y$ and $z > 0$;*

iii) $\frac{f(y)-f(x)}{y-x} \geq \frac{f(z)-f(x)}{z-x} \geq \frac{f(z)-f(y)}{z-y}$, *for all $x, y, z \in X$ with $0 < x < y < z$.*

Theorem 5.4 *The serial rule satisfies UCM on $\mathcal{CCS}^N$.*

Proof: Let $(C,q) \in \mathcal{CCS}^N$. For all $i \in N \setminus \{n\}$, we have that

$$\begin{aligned}
\frac{\mathrm{Ser}_i(C,q)}{q_i} - \frac{\mathrm{Ser}_{i+1}(C,q)}{q_{i+1}} &= \frac{\mathrm{Ser}_i(C,q)}{q_i} - \frac{\mathrm{Ser}_i(C,q)}{q_{i+1}} - \frac{C(s_{i+1}) - C(s_i)}{(n-i)q_{i+1}} \\
&= \frac{(q_{i+1}-q_i)\mathrm{Ser}_i(C,q)}{q_i q_{i+1}} - \frac{C(s_{i+1}) - C(s_i)}{(n-i)q_{i+1}} \\
&= \frac{(q_{i+1}-q_i)\mathrm{Ser}_i(C,q)}{q_i q_{i+1}} - \frac{(q_{i+1}-q_i)(C(s_{i+1}) - C(s_i))}{(n-i)(q_{i+1}-q_i)q_{i+1}} \\
&= \frac{(q_{i+1}-q_i)\mathrm{Ser}_i(C,q)}{q_i q_{i+1}} - \frac{(q_{i+1}-q_i)(C(s_{i+1}) - C(s_i))}{(s_{i+1}-s_i)q_{i+1}},
\end{aligned}$$

[3] It is readily seen that the average cost sharing rule satisfies unit cost monotonicity on the class of all cost sharing problems.

by noting that

$$\begin{aligned} s_{i+1} - s_i &= \sum_{j=1}^{i} q_j + (n-i)q_{i+1} - \sum_{j=1}^{i-1} q_j - \big(n-(i-1)\big)q_i \\ &= q_i + (n-i)q_{i+1} - \big(n-(i-1)\big)q_i \\ &= (n-i)(q_{i+1}-q_i), \end{aligned}$$

for the last equality.

Hence, to show that

$$\frac{\mathrm{Ser}_i(C,q)}{q_i} \geq \frac{\mathrm{Ser}_{i+1}(C,q)}{q_{i+1}}, \tag{5.7}$$

for all $i \in N \setminus \{n\}$, it is sufficient to show, by factoring out the common factor $\frac{q_{i+1}-q_i}{q_{i+1}}$, that

$$\frac{\mathrm{Ser}_i(C,q)}{q_i} \geq \frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i}, \tag{5.8}$$

for all $i \in N \setminus \{n\}$.

Next, we proceed by induction and start out with the base case. So, let $i = 1$. Obviously, if $q_1 = q_2$, then $\mathrm{Ser}_1(C,q) = \mathrm{Ser}_2(C,q)$ and hence,

$$\frac{\mathrm{Ser}_1(C,q)}{q_1} = \frac{\mathrm{Ser}_2(C,q)}{q_2}.$$

Consequently, Equation (5.7) is satisfied.

If $q_1 \neq q_2$, then

$$\begin{aligned} \frac{\mathrm{Ser}_1(C,q)}{q_1} - \frac{C(s_2)-C(s_1)}{s_2-s_1} &= \frac{C(s_1)}{nq_1} - \frac{C(s_2)-C(s_1)}{s_2-s_1} \\ &= \frac{C(s_1)}{s_1} - \frac{s_1C(s_2)-s_1C(s_1)}{s_1(s_2-s_1)} \\ &= \frac{(s_2-s_1)C(s_1) - s_1C(s_2) + s_1C(s_1)}{s_1(s_2-s_1)} \\ &= \frac{s_2C(s_1) - s_1C(s_2)}{s_1(s_2-s_1)} \geq 0, \end{aligned}$$

where the inequality follows from Lemma 5.2 part i), by using that $0 < s_1 \leq s_2$ and the fact that C is a concave function. Consequently, Equation (5.8) is satisfied for $i = 1$.

Advancing to the induction step, take $i \in N \setminus \{n\}$, $i > 1$ and assume that for all $j \in N \setminus \{n\}$, $j < i$,

$$\frac{\mathrm{Ser}_j(C,q)}{q_j} \geq \frac{\mathrm{Ser}_{j+1}(C,q)}{q_{j+1}},$$

which is equivalent to

$$\mathrm{Ser}_j(C,q) \geq \frac{q_j}{q_{j+1}}\mathrm{Ser}_{j+1}(C,q).$$

If $q_i = q_{i+1}$, then $\mathrm{Ser}_i(C,q) = \mathrm{Ser}_{i+1}(C,q)$ and hence,

$$\frac{\mathrm{Ser}_i(C,q)}{q_i} = \frac{\mathrm{Ser}_{i+1}(C,q)}{q_{i+1}}.$$

Consequently, Equation (5.7) is satisfied.

If $q_i < q_{i+1}$, we distinguish between two cases:

I) $q_1 = q_2 = \ldots = q_{i-1} = q_i$;

II) there exists a $k \in N$ such that $q_k < q_i$ and $q_\ell = q_i$ for all $\ell \in \{k+1, \ldots, i-1\}$.

Case I) Assume that $q_1 = q_2 = \ldots = q_{i-1} = q_i$. Then $s_1 = s_2 = \ldots = s_i = nq_i$ and hence,

$$\begin{aligned}
\frac{\mathrm{Ser}_i(C,q)}{q_i} - \frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i} &= \frac{C(s_1)}{nq_i} + \frac{C(s_2) - C(s_1)}{(n-1)q_i} + \ldots \\
&\quad + \frac{C(s_i) - C(s_{i-1})}{(n-i)q_i} - \frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i} \\
&= \frac{C(s_1)}{nq_i} - \frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i} \\
&= \frac{C(s_1)}{s_1} - \frac{C(s_{i+1}) - C(s_1)}{s_{i+1} - s_1} \\
&= \frac{s_{i+1}C(s_1) - s_1C(s_{i+1})}{s_1(s_{i+1} - s_1)} \geq 0,
\end{aligned}$$

where the inequality follows from Lemma 5.2 part i), by using that $0 < s_1 \leq s_{i+1}$ and the fact that C is a concave function. Consequently, Equation (5.8) is satisfied.

Case II) Let $k \in N$ be such that $q_k < q_i$ and $q_\ell = q_i$ for all $\ell \in \{k+1, \ldots, i-1\}$. Then $s_k < s_i < s_{i+1}$ and hence,

$$\begin{aligned}
\frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i} &\leq \frac{C(s_i) - C(s_k)}{s_i - s_k} \\
&= \frac{C(s_i) - C(s_k)}{(n-k)(q_i - q_k)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{Ser}_i(C,q) - \mathrm{Ser}_k(C,q)}{q_i - q_k} \\
&\leq \frac{\mathrm{Ser}_i(C,q) - \frac{q_k}{q_i}\mathrm{Ser}_i(C,q)}{q_i - q_k} \\
&= \frac{\mathrm{Ser}_i(C,q)}{q_i},
\end{aligned}
$$

where the second inequality follows from the induction hypothesis and the first inequality from Lemma 5.2 part iii), by using that $0 < s_k < s_i < s_{i+1}$ and the fact that C is a concave function. Consequently, Equation (5.8) is satisfied.

This concludes the induction step and hence, the proof. □

Secondly, we focus on monotonic vulnerability for the absence of the smallest player, which requires that for a player with a (weakly) higher demand than an other player, the former has a (weakly) higher increase in cost allocation when the player with the lowest demand is not present than the latter. This is formalized in Definition 5.2.

Definition 5.2 A cost sharing rule f satisfies *monotonic vulnerability for the absence of the smallest player (MOVASP)* on $\mathcal{C} \subseteq \mathcal{CS}$ if, for all $(C,q) \in \mathcal{C}$ with $(C,q) \in \mathcal{CS}^N$ and $(C, q_{|N\setminus\{1\}}) \in \mathcal{C}$ and for all $i \in N \setminus \{1,2\}$,[4]

$$
f_i(C,q) - f_i(C, q_{|N\setminus\{1\}}) \leq f_{i-1}(C,q) - f_{i-1}(C, q_{|N\setminus\{1\}}). \quad \triangleleft
$$

MOVASP thus ensures that also the player with the lowest demanded output contributes to lower cost allocations of players with larger demanded output. The following theorem shows that the serial rule satisfies MOVASP on the class of cost sharing problems with concave cost functions.[5]

Theorem 5.5 *The serial rule satisfies MOVASP on $\mathcal{CCS}$.*

Proof: Let $(C,q) \in \mathcal{CCS}^N$. Define $\Delta\mathrm{Ser}_j = \mathrm{Ser}_j(C,q) - \mathrm{Ser}_j(C, q_{|N\setminus\{1\}})$ for all $j \in N \setminus \{1\}$. We have to show that

$$
\Delta\mathrm{Ser}_2 \geq \Delta\mathrm{Ser}_3 \geq \ldots \geq \Delta\mathrm{Ser}_n. \tag{5.9}
$$

According to Lemma 5.1 we have, for all $i \in \{2,3,\ldots,n\}$,

$$
\mathrm{Ser}_i(C,q) = \frac{C(s_i)}{n-i+1} - \sum_{j=1}^{i-1} \frac{C(s_j)}{(n-j+1)(n-j)}
$$

[4] Here, $q_{|N\setminus\{1\}} = (q_j)_{j\in N\setminus\{1\}}$. Moreover, with minor abuse of notation and spelling, in $(C, q_{|N\setminus\{1\}})$, the players are labeled $\{2,3,\ldots,n\}$ with $q_2 \leq q_3 \leq \ldots \leq q_n$.

[5] It is readily seen that the average cost sharing rule satisfies MOVASP on the class of cost sharing problems with concave cost functions.

and, also (with some technical variation[6])

$$\mathrm{Ser}_i(C, q_{|N\setminus\{1\}}) = \frac{C(s_i - q_1)}{n-i+1} - \sum_{j=2}^{i-1} \frac{C(s_j - q_1)}{(n-j+1)(n-j)}.$$

Hence, for all $i \in \{2, 3, \ldots, n\}$,

$$\Delta\mathrm{Ser}_i = \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_1)}{n(n-1)} - \sum_{j=2}^{i-1} \frac{C(s_j) - C(s_j - q_1)}{(n-j+1)(n-j)}.$$

Consequently, for $i \in \{3, 4, \ldots, n\}$,

$$\begin{aligned}
\Delta\mathrm{Ser}_i - \Delta\mathrm{Ser}_{i-1} &= \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_1)}{n(n-1)} - \sum_{j=2}^{i-1} \frac{C(s_j) - C(s_j - q_1)}{(n-j+1)(n-j)} \\
&\quad - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+2} + \frac{C(s_1)}{n(n-1)} + \sum_{j=2}^{i-2} \frac{C(s_j) - C(s_j - q_1)}{(n-j+1)(n-j)} \\
&= \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{(n-i+2)(n-i+1)} \\
&\qquad\qquad - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+2} \\
&= \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+1} \\
&\leq \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+1} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+1} = 0,
\end{aligned}$$

where the inequality follows from Lemma 5.2 part ii), by using that $0 < s_{i-1} - q_1 \leq s_i - q_1$ and $q_1 > 0$ and the fact that C is a concave function. Consequently, Equation (5.9) is satisfied. $\square$

Unfortunately, on the larger class of cost sharing problems with piecewise concave cost functions, the serial rule loses both UCM and MOVASP as is seen in the following two examples.[7]

---

[6] Let $s_0, s_1, s_2, \ldots, s_n$ with $s_0 = 0$ be the intermediate points that are used to compute $\mathrm{Ser}(C, q)$. Let $s'_1, s'_2, s'_3, \ldots, s'_n$ with $s'_1 = 0$ be the intermediate points used to compute $\mathrm{Ser}(C, q_{|N\setminus\{1\}})$. Clearly, $s'_j = s_j - q_1$ for all $j \in N \setminus \{1\}$.

[7] Example 5.6 can be adapted to show that the average cost sharing rule does not satisfy MOVASP on the class of cost sharing problems with piecewise concave cost functions. More precisely, by changing the demand of player 4 to 10, i.e. $q_4 = 10$, and restrict the cost function C in Equation (5.11) to $[0, 25]$, the resulting cost allocation violates MOVASP.

Example 5.5 Consider the cost sharing problem $(C, q) \in \mathcal{CS}^N$ with $N = \{1, 2, 3\}$, $q = (8, 9, 15)$ and cost function C given by

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2, & \text{if } t \in [0, 12]; \\ 20t - \frac{1}{2}t^2, & \text{if } t \in (12, 20]; \\ 200 + 18(t - 20) - \frac{1}{3}(t - 20)^2, & \text{if } t \in (20, 32], \end{cases} \tag{5.10}$$

for all $t \in [0, 32]$.

Note that this cost function is the restriction of the function as specified by Equation (5.5) and hence, visualized in Figure 5.2b. As can be seen in Figure 5.2, C is piecewise concave with two maximal intervals of concavity: $[0, 20]$ and $[20, 32]$.

One readily checks that

$$\text{Ser}(C, q) = \left(88\tfrac{8}{9}, 103\tfrac{5}{9}, 175\tfrac{5}{9}\right).$$

Hence, e.g.,

$$\frac{\text{Ser}_1(C, q)}{q_1} = 11\tfrac{9}{81} < \frac{\text{Ser}_2(C, q)}{q_2} = 11\tfrac{41}{81},$$

contradicting UCM. △

Example 5.6 Consider the cost sharing problem $(C, q) \in \mathcal{CS}^N$ with $N = \{1, 2, 3, 4\}$, $q = (2, 4, 9, 15)$ and cost function C given by

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2, & \text{if } t \in [0, 12]; \\ 20t - \frac{1}{2}t^2, & \text{if } t \in (12, 16]; \\ 192 + 18(t - 16) - \frac{1}{3}(t - 16)^2, & \text{if } t \in (16, 17]; \\ 18t - \frac{1}{3}t^2, & \text{if } t \in (17, 20]; \\ 226\frac{2}{3} + 20(t - 20) - \frac{1}{2}(t - 20)^2, & \text{if } t \in (20, 30], \end{cases} \tag{5.11}$$

for all $t \in [0, 30]$.

Note that this cost function is the restriction of the function as specified by Equation (5.6) and hence, visualized in Figure 5.3b. As can be seen in Figure 5.3, C is piecewise concave with three maximal intervals of concavity: $[0, 16]$, $[16, 20]$ and $[20, 30]$.

One readily checks that

$$\text{Ser}(C, q) = \left(30\tfrac{2}{3}, 50\tfrac{4}{9}, 108\tfrac{7}{9}, 186\tfrac{7}{9}\right),$$

while

$$\text{Ser}(C, q_{|N\setminus\{1\}}) = \left(56, 104\tfrac{1}{3}, 194\tfrac{1}{3}\right).$$

Hence, e.g.,

$$\text{Ser}_3(C, q) - \text{Ser}_3(C, q_{|N\setminus\{1\}}) = 4\tfrac{4}{9} > -5\tfrac{5}{9} = \text{Ser}_2(C, q) - \text{Ser}_2(C, q_{|N\setminus\{1\}}),$$

contradicting MOVASP. △

5.4.2 Piecewise serial rules

In this section, we use claims rules (see Section 2.2) to modify the serial rule into piecewise serial rules that are tailor-made for cost sharing problems with piecewise concave cost functions and thus applicable as solution method for the allocation problem within a CRCP-situation. We pinpoint a specific piecewise serial rule that satisfies UCM and a specific piecewise serial rule that satisfies MOVASP. Moreover, it is seen that these two properties are incompatible for piecewise serial rules on the class of all cost sharing problems with piecewise concave cost functions.

Let $\mathcal{CCS}^{N,m} \subseteq \mathcal{CS}^N$ with $m \in \{1, 2, \ldots\}$ denote the class of cost sharing problems where the cost function is piecewise concave with exactly m maximal intervals of concavity.

First, we explain the idea for a piecewise serial rule by means of an example and then present the formal definition.

Example 5.7 Reconsider the cost sharing problem $(C, q) \in \mathcal{CCS}^{N,2}$, as described in Example 5.5, with $N = \{1, 2, 3\}$, $q = (8, 9, 15)$ and cost function C given by

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2, & \text{if } t \in [0, 12]; \\ 20t - \frac{1}{2}t^2, & \text{if } t \in (12, 20]; \\ 200 + 18(t-20) - \frac{1}{3}(t-20)^2, & \text{if } t \in (20, 32], \end{cases}$$

for all $t \in [0, 32]$.

Recall that C is the restriction of the function as visualized in Figure 5.2b. For convenience, C is also visualized in Figure 5.4. Moreover, recall that the two maximal intervals of concavity are given by $[0, 20]$ and $[20, 32]$.

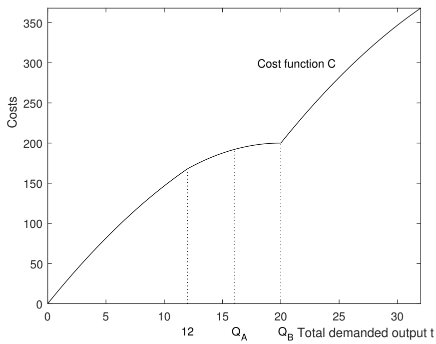


Figure 5.4 – *The cost function C of the cost sharing problem of Example 5.7.*

If we could divide the vector q over these two intervals, i.e., find a suitable vector $x^1 \in \mathbb{R}^N$ with $\sum_{j \in N} x_j^1 = 20$ for the first interval and a suitable vector $x^2 = q - x^1$

for the second interval, we could apply the serial rule on each of these two cost sharing problems separately and then add the resulting vectors. For this, the demands, given by q, are considered as claims on their preferred interval $[0, 20]$ (since in this interval, the returns to scale are larger than in the other interval) and we use a claims rule φ to determine $x^1 = \varphi(20, q)$. Arguing that large players should obtain a lower cost per unit than small players and therefore should be allocated a relatively higher part of the preferred interval $[0, 20]$, we can opt for the constrained equal losses rule. In that case, it gives

$$\begin{cases} x^1 = \mathrm{CEL}\,(20, (8, 9, 15)) = (4, 5, 11); \\ x^2 = (8, 9, 15) - (4, 5, 11) = (4, 4, 4). \end{cases}$$

Subsequently, on the interval $[0, 20]$ we face the cost sharing problem (C^1, x^1) with $C^1(t) = C(t)$ for $t \in [0, 20]$ and on the second interval we face the cost sharing problem (C^2, x^2) with $C^2(t) = C(t + 20) - C(20) = C(t + 20) - 200$ for all $t \in [0, 12]$. Using the serial rule as the leading principle for concave cost functions, it is readily verified that

$$\begin{cases} \mathrm{Ser}(C^1, x^1) = (56, 63, 81); \\ \mathrm{Ser}(C^2, x^2) = (56, 56, 56). \end{cases}$$

Consequently, for the cost allocation specified by this so-called CEL-piecewise serial rule, denoted by $\Psi^{\mathrm{CEL}}(C, q)$, we have

$$\Psi^{\mathrm{CEL}}(C, q) = (56, 63, 81) + (56, 56, 56) = (112, 119, 137).$$

On the other hand, one could also argue that the players should have 'relatively equal' rights to all of the intervals, which can be realized by dividing q proportionally over the intervals. In that case, it gives

$$\begin{cases} x^1 = \mathrm{PROP}\,(20, (8, 9, 15)) = (5, 5\frac{5}{8}, 9\frac{3}{8}); \\ x^2 = (8, 9, 15) - (5, 5\frac{5}{8}, 9\frac{3}{8}) = (3, 3\frac{3}{8}, 5\frac{5}{8}). \end{cases}$$

Subsequently, since

$$\begin{cases} \mathrm{Ser}(C^1, x^1) = (62\frac{1}{2}, 65\frac{15}{64}, 72\frac{17}{64}); \\ \mathrm{Ser}(C^2, x^2) = (45, 49\frac{13}{32}, 73\frac{19}{32}), \end{cases}$$

we obtain the so-called PROP-piecewise serial rule, denoted by $\Psi^{\mathrm{PROP}}(C, q)$ and given by

$$\Psi^{\mathrm{PROP}}(C, q) = (107\frac{1}{2}, 114\frac{41}{64}, 145\frac{55}{64}). \qquad \triangle$$

The idea for a piecewise serial rules based on claims rules is now formalized in Definition 5.3 below.

Definition 5.3 Let $m \in \{1, 2, \ldots\}$ and let φ be a claims rule. Furthermore, let $(C, q) \in \mathcal{CCS}^{N,m}$ be a cost sharing problem with a piecewise concave cost function with exactly m maximal intervals of concavity. Let $[t_0, t_1], [t_1, t_2], \ldots, [t_{m-1}, t_m]$ with $0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = \sum_{i \in N} q_i$ be these maximal intervals of concavity of C.

Then first the separate cost sharing problems on each of these intervals are defined as follows. With $q^1(\varphi) = q$, recursively define the vectors $q^r(\varphi) \in \mathbb{R}^N$ and $x^r(\varphi) \in \mathbb{R}^N$ for $r \in \{1, \ldots, m\}$ in the following way:

$$\begin{cases} x^r(\varphi) & = \varphi(t_r - t_{r-1}, q^r(\varphi)); \\ q^{r+1}(\varphi) & = q^r(\varphi) - x^r(\varphi). \end{cases}$$

Moreover, the modified cost function on the interval $[t_{r-1}, t_r]$ is denoted by $C^r : [0, t_r - t_{r-1}] \to \mathbb{R}_+$ and defined by

$$C^r(t) = C(t + t_{r-1}) - C(t_{r-1}),$$

for all $t \in [0, t_r - t_{r-1}]$. Consequently, $(C^r, x^r(\varphi))$ is a cost sharing problem with a concave cost function.[8]

Next, the *φ-piecewise serial rule* $\Psi^\varphi : \mathcal{CCS}^{N,m} \to \mathbb{R}^N$ is defined by

$$\Psi^\varphi(C, q) = \sum_{r=1}^{m} \mathrm{Ser}(C^r, x^r(\varphi)). \qquad \triangleleft$$

From Definition 5.3, it immediately follows that, for all cost sharing problems with exactly one maximal interval of concavity, all piecewise serial rules boil down to the serial rule.

We specifically focus on Ψ^{PROP} and on Ψ^{CEL} as allocation methods for cost sharing problems with piecewise concave cost functions. Importantly, both the proportional rule and the constrained equal losses rule satisfy *order preservation*. This implies that both claims and allocations underlying the recursive procedure for the piecewise serial rule based on either the proportional rule or the constrained equal losses rule are non-decreasing over the players in each maximal interval of concavity,[9] that is,

$$\begin{cases} x_1^r(\mathrm{PROP}) \leq x_2^r(\mathrm{PROP}) \leq \ldots \leq x_n^r(\mathrm{PROP}); \\ q_1^r(\mathrm{PROP}) \leq q_2^r(\mathrm{PROP}) \leq \ldots \leq q_n^r(\mathrm{PROP}), \end{cases} \tag{5.12}$$

[8]To draw this conclusion, we realize that we need that $x^r(\varphi)$ is non-decreasing over the players. For arbitrary φ, this may not be the case and therefore, the players should be reordered. However, to avoid a notational overburden, this reordering is left implicit in the definition. If needed in the proofs, the reordering of the players will be made explicit.

[9]Thus, if a claims rule φ satisfies order preservation, then there is no need to reorder the players to ensure that $x^r(\varphi)$ is non-decreasing over the players.

and

$$\begin{cases} x_1^r(\text{CEL}) \leq x_2^r(\text{CEL}) \leq \ldots \leq x_n^r(\text{CEL}); \\ q_1^r(\text{CEL}) \leq q_2^r(\text{CEL}) \leq \ldots \leq q_n^r(\text{CEL}), \end{cases} \tag{5.13}$$

for all $r \in \{1, \ldots, m\}$.

The following theorem shows that the piecewise serial rule based on the proportional rule satisfies unit cost monotonicity.

Theorem 5.6 *Ψ^{PROP} satisfies UCM on $\mathcal{CCS}^{N,m}$ for all $m \in \{1, 2, \ldots\}$.*

Proof: The theorem clearly holds for $m = 1$, by using the facts that the serial rule satisfies UCM according to Theorem 5.4 and that $\mathcal{CCS}^{N,1} = \mathcal{CCS}^N$.

Let $m \in \{2, 3, \ldots\}$ and let $(C, q) \in \mathcal{CCS}^{N,m}$ and denote by $[t_0, t_1], [t_1, t_2], \ldots, [t_{m-1}, t_m]$ with $0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = \sum_{i \in N} q_i$ the maximal intervals of concavity of C. For the remainder of the proof, we abbreviate $x^r(\text{PROP})$ and $q^r(\text{PROP})$ to x^r and q^r for all $r \in \{1, \ldots, m\}$.

We first show that, for all $r \in \{1, \ldots, m\}$, $(C^r, x^r) \in \mathcal{CCS}^N$. For this, note that C^r is the restriction of C to the concave interval $[t_{r-1}, t_r]$ and hence, satisfies all requirements of a cost sharing problem. With regard to x^r, note that

$$x_1^r \leq x_2^r \leq \ldots \leq x_n^r,$$

for all $r \in \{1, \ldots, m\}$, in line with Equation (5.12), since the proportional rule satisfies order preservation. Finally, to show that $x_1^r > 0$, for all $r \in \{1, \ldots, m\}$, we first show that $q_1^r > 0$, for all $r \in \{1, \ldots, m\}$. Starting from the fact that $q_1^1 = q_1 > 0$, this can be recursively followed by $q_1^{r+1} = q_1^r - x_1^r$, for all $r \in \{1, \ldots, m-1\}$. Subsequently, for all $r \in \{1, \ldots, m-1\}$, it holds that

$$x_1^r = \text{PROP}_1(t_r - t_{r-1}, q^r) < q_1^r,$$

since $t_r - t_{r-1} < \sum_{i \in N} q_i^r$ for all $r \in \{1, \ldots, m-1\}$:

$$\begin{aligned} t_r - t_{r-1} < t_m - t_{r-1} &= t_m - t_0 - (t_1 - t_0) - (t_2 - t_1) - \ldots (t_{r-1} - t_{r-2}) \\ &= \sum_{i \in N} q_i^1 - \sum_{i \in N} x_i^1 - \sum_{i \in N} x_i^2 - \sum_{i \in N} x_i^3 - \ldots - \sum_{i \in N} x_i^{r-1} \\ &= \sum_{i \in N} q_i^2 - \sum_{i \in N} x_i^2 - \sum_{i \in N} x_i^3 - \ldots - \sum_{i \in N} x_i^{r-1} \\ &= \sum_{i \in N} q_i^3 - \sum_{i \in N} x_i^3 - \ldots - \sum_{i \in N} x_i^{r-1} \\ &= \ldots \\ &= \sum_{i \in N} q_i^r. \end{aligned}$$

Due to the nature of the proportional rule $q_1^r > 0$ implies that, for all $r \in \{1, \ldots, m\}$,

$$x_1^r = \mathrm{PROP}_1(t_r - t_{r-1}, q^r) > 0.$$

Consequently, we now can conclude that, for all $r \in \{1, \ldots, m\}$, $(C^r, x^r) \in \mathcal{CCS}^N$.
Secondly, we ue Theorem 5.4 to conclude that, for all $i \in N \setminus \{n\}$,

$$\frac{\mathrm{Ser}_i(C^r, x^r)}{x_i^r} \geq \frac{\mathrm{Ser}_{i+1}(C^r, x^r)}{x_{i+1}^r}. \tag{5.14}$$

For the rest of the proof, fix $i \in N \setminus \{n\}$. For UCM of Ψ^{PROP}, it suffices to prove that

$$\frac{\Psi_i^{\mathrm{PROP}}(C, q)}{q_i} \geq \frac{\Psi_{i+1}^{\mathrm{PROP}}(C, q)}{q_{i+1}}.$$

We first prove, by using induction, that $\frac{x_i^r}{q_i} = \frac{x_{i+1}^r}{q_{i+1}}$ for all $r \in \{1, 2, \ldots, m\}$. Clearly,

$$\frac{\mathrm{PROP}_i(t_1, q)}{q_i} = \frac{\mathrm{PROP}_{i+1}(t_1, q)}{q_{i+1}},$$

and thus,

$$\frac{x_i^1}{q_i} = \frac{x_{i+1}^1}{q_{i+1}}.$$

This finishes the proof of the base case. Proceeding to the induction step, take $r \in \{2, \ldots, m\}$ and assume that, for all $s \in \{1, \ldots, r-1\}$,

$$\frac{x_i^s}{q_i} = \frac{x_{i+1}^s}{q_{i+1}}.$$

Then it follows that

$$\begin{aligned}
\frac{q_i^r}{q_i} &= \frac{q_i^{r-1} - x_i^{r-1}}{q_i} = \ldots = \frac{q_i - \sum_{s=1}^{r-1} x_i^s}{q_i} = 1 - \sum_{s=1}^{r-1} \frac{x_i^s}{q_i} \\
&\overset{\text{(IH)}}{=} 1 - \sum_{s=1}^{r-1} \frac{x_{i+1}^s}{q_{i+1}} = \frac{q_{i+1} - \sum_{s=1}^{r-1} x_{i+1}^s}{q_{i+1}} = \ldots = \frac{q_{i+1}^{r-1} - x_{i+1}^{r-1}}{q_{i+1}} = \frac{q_{i+1}^r}{q_{i+1}},
\end{aligned}$$

where IH stands for the induction hypothesis. Moreover,

$$\frac{x_i^r}{q_i^r} = \frac{\mathrm{PROP}_i(t_r - t_{r-1}, q^r)}{q_i^r} = \frac{\mathrm{PROP}_{i+1}(t_r - t_{r-1}, q^r)}{q_{i+1}^r} = \frac{x_{i+1}^r}{q_{i+1}^r}.$$

Hence,

$$\frac{x_i^r}{q_i} = \frac{x_i^r}{q_i^r}\frac{q_i^r}{q_i} = \frac{x_{i+1}^r}{q_{i+1}^r}\frac{q_{i+1}^r}{q_{i+1}} = \frac{x_{i+1}^r}{q_{i+1}}.$$

We may conclude that, for all $r \in \{1, \ldots, m\}$,

$$\frac{x_i^r}{q_i} = \frac{x_{i+1}^r}{q_{i+1}}. \tag{5.15}$$

Using Equations (5.14) and (5.15), we find that

$$\begin{aligned}
\frac{\Psi_i^{\mathrm{PROP}}(C,q)}{q_i} &= \sum_{r=1}^{m} \frac{\mathrm{Ser}_i(C^r, x^r)}{q_i} = \sum_{r=1}^{m} \frac{\mathrm{Ser}_i(C^r, x^r)}{x_i^r} \frac{x_i^r}{q_i} \\
&\overset{(5.15)}{=} \sum_{r=1}^{m} \frac{\mathrm{Ser}_i(C^r, x^r)}{x_i^r} \frac{x_{i+1}^r}{q_{i+1}} \\
&\overset{(5.14)}{\geq} \sum_{r=1}^{m} \frac{\mathrm{Ser}_{i+1}(C^r, x^r)}{x_{i+1}^r} \frac{x_{i+1}^r}{q_{i+1}} \\
&= \sum_{r=1}^{m} \frac{\mathrm{Ser}_{i+1}(C^r, x^r)}{q_{i+1}} = \frac{\Psi_{i+1}^{\mathrm{PROP}}(C,q)}{q_{i+1}}.
\end{aligned}$$

Hence, Ψ^{PROP} satisfies UCM on $\mathcal{CCS}^{N,m}$ for all $m \in \{1, 2, \ldots\}$. □

Next, we show that Ψ^{PROP} is the unique piecewise serial rule based on a claims rule that satisfies unit cost monotonicity.

Theorem 5.7 *Let φ be a claims rule such that Ψ^{φ} satisfies UCM on $\mathcal{CCS}^{N,m}$ for all $m \in \{1, 2 \ldots\}$. Then $\varphi = PROP$.*

Proof: Suppose for the sake of contradiction that $\varphi \neq$ PROP. Then there exists $(E, c) \in \mathcal{C}^N$ with $0 < c_1 \leq \ldots \leq c_n$, $\sum_{j \in N} c_j > E$ and $i \in N$ with $c_i \neq c_{i+1}$ such that

$$\frac{\varphi_i(E,c)}{c_i} > \frac{\varphi_{i+1}(E,c)}{c_{i+1}} \tag{5.16}$$

or

$$\frac{\varphi_i(E,c)}{c_i} < \frac{\varphi_{i+1}(E,c)}{c_{i+1}}. \tag{5.17}$$

For both cases, we show that there exists a cost sharing problem $(C, q) \in \mathcal{CCS}^{N,2}$ for which

$$\frac{\Psi_i^{\varphi}(C,q)}{q_i} < \frac{\Psi_{i+1}^{\varphi}(C,q)}{q_{i+1}},$$

contradicting UCM.

Suppose Equation (5.16) holds. Consider $(C, q) \in \mathcal{CCS}^{N,2}$ with $q = c$ and the cost function C given by

$$C(t) = \begin{cases} 2t, & \text{if } t \in [0, E]; \\ 5t - 3E, & \text{if } t \in (E, \sum_{j \in N} c_j], \end{cases}$$

for all $t \in [0, \sum_{j \in N} c_j]$. One readily verifies that all assumptions on a cost sharing problem are satisfied. Note that C has two maximal intervals of concavity: $[0, E]$ and $[E, \sum_{j \in N} c_j]$.

For the first interval, we have that $C^1(t) = 2t$ for all $t \in [0, E]$ and $q^1(\varphi) = q = c$. For $x^1(\varphi)$, the players should be reordered in such a way that $x^1(\varphi)$ is non-decreasing over the players. Let $\sigma : N \to \{1, \dots, n\}$ be a bijection such that for all $k \in \{1, \dots, n\}$, $\sigma^{-1}(k)$ is the player with the kth lowest value allocated by φ, that is,

$$\varphi_{\sigma^{-1}(1)}(E, c) \leq \varphi_{\sigma^{-1}(2)}(E, c) \leq \dots \leq \varphi_{\sigma^{-1}(n)}(E, c).$$

Next, let $x^1_k(\varphi) = \varphi_{\sigma^{-1}(k)}(E, c)$ for all $k \in \{1, \dots, n\}$, or equivalently, for all $j \in N$, $x^1_{\sigma(j)}(\varphi) = \varphi_j(E, c)$, which means that player $j \in N$ is on position $\sigma(j)$ of $x^1(\varphi)$. Then $(C^1, x^1(\varphi))$ is a cost sharing problem on (positions) $\{1, 2, \dots, n\}$ and[10]

$$\mathrm{Ser}(C^1, x^1(\varphi)) = 2x^1(\varphi).$$

For the second interval, we have that $C^2(t) = 5t$ for all $t \in [0, \sum_{j \in N} c_j - E]$. For $q^2(\varphi)$, let $\bar{\sigma} : N \to \{1, \dots, n\}$ be a bijection such that

$$c_{\bar{\sigma}^{-1}(1)} - \varphi_{\bar{\sigma}^{-1}(1)}(E, c) \leq \dots \leq c_{\bar{\sigma}^{-1}(n)} - \varphi_{\bar{\sigma}^{-1}(n)}(E, c).$$

Next, let $q^2_k(\varphi) = c_{\bar{\sigma}^{-1}(k)} - \varphi_{\bar{\sigma}^{-1}(k)}(E, c)$ for all $k \in \{1, \dots, n\}$, or equivalently, for all $j \in N$, $q^2_{\bar{\sigma}(j)}(\varphi) = c_j - \varphi_j(E, c)$. Moreover,

$$x^2(\varphi) = \varphi(\sum_{j \in N} c_j - E, q^2(\varphi)) = q^2(\varphi),$$

since $\sum_{j \in N} q^2_j(\varphi) = \sum_{j \in N} c_j - \sum_{j \in N} \varphi_j(E, c) = \sum_{j \in N} c_j - E$. Then $(C^2, x^2(\varphi))$ is a cost sharing problem on (positions) $\{1, 2, \dots, n\}$ and

$$\mathrm{Ser}(C^2, x^2(\varphi)) = 5x^2(\varphi).$$

Consequently,

$$\begin{aligned}
\Psi^\varphi_i(C, q) &= \mathrm{Ser}_{\sigma(i)}(C^1, x^1(\varphi)) + \mathrm{Ser}_{\bar{\sigma}(i)}(C^2, x^2(\varphi)) \\
&= 2x^1_{\sigma(i)}(\varphi) + 5x^2_{\bar{\sigma}(i)}(\varphi) \\
&= 2\varphi_i(E, c) + 5(c_i - \varphi_i(E, c)) \\
&= 5c_i - 3\varphi_i(E, c),
\end{aligned}$$

and similarly,

$$\Psi^\varphi_{i+1}(C, q) = 5c_{i+1} - 3\varphi_{i+1}(E, c).$$

[10] In general, for a cost sharing problem $(C, q) \in \mathcal{CS}^N$ with an affine cost function $C(t) = \alpha t + \beta$, with $\alpha, \beta \in \mathbb{R}$, it can be readily verified that $\mathrm{Ser}(C, q) = \alpha q$.

Hence, using that $q = c$,

$$\begin{aligned}\frac{\Psi_i^{\varphi}(C,q)}{q_i} - \frac{\Psi_{i+1}^{\varphi}(C,q)}{q_{i+1}} &= 5 - 3\frac{\varphi_i(E,c)}{c_i} - 5 + 3\frac{\varphi_{i+1}(E,c)}{c_{i+1}} \\ &= 3\left(\frac{\varphi_{i+1}(E,c)}{c_{i+1}} - \frac{\varphi_i(E,c)}{c_i}\right) < 0,\end{aligned}$$

where the inequality follows from Equation (5.16). This shows that

$$\frac{\Psi_i^{\varphi}(C,q)}{q_i} < \frac{\Psi_{i+1}^{\varphi}(C,q)}{q_{i+1}},$$

contradicting UCM.

Next, suppose Equation (5.17) holds. In order to define a cost sharing problem for this case, we first need two things: First, reconsider $\bar{\sigma} : N \to \{1,\dots,n\}$, the bijection for which

$$c_{\bar{\sigma}^{-1}(1)} - \varphi_{\bar{\sigma}^{-1}(1)}(E,c) \leq \ldots \leq c_{\bar{\sigma}^{-1}(n)} - \varphi_{\bar{\sigma}^{-1}(n)}(E,c).$$

Secondly, let $\varepsilon > 0$ be such that

$$\varepsilon < \min\left\{r_k \mid r_k > 0, k \in \{1,\dots,n\}\right\},$$

where, for every $k \in \{1,\dots,n\}$,

$$r_k = \sum_{\ell=1}^{k-1}(c_{\bar{\sigma}^{-1}(\ell)} - \varphi_{\bar{\sigma}^{-1}(\ell)}(E,c)) + (n-k+1)(c_{\bar{\sigma}^{-1}(k)} - \varphi_{\bar{\sigma}^{-1}(k)}(E,c)).$$

In particular,

$$\varepsilon < r_n = \sum_{\ell=1}^{n} c_{\bar{\sigma}^{-1}(\ell)} - \sum_{\ell=1}^{n}\varphi_{\bar{\sigma}^{-1}(\ell)}(E,c) = \sum_{j\in N} c_j - E.$$

Consider $(C^{\varepsilon}, q) \in \mathcal{CCS}^{N,2}$ with $q = c$ and the cost function C^{ε} given by

$$C^{\varepsilon}(t) = \begin{cases} 2t, & \text{if } t \in [0,E]; \\ 5t - 3E, & \text{if } t \in (E, E+\varepsilon]; \\ t + E + 4\varepsilon, & \text{if } t \in (E+\varepsilon, \sum_{j\in N} c_j], \end{cases}$$

for all $t \in [0, \sum_{j\in N} c_j]$. One readily verifies that all assumptions on a cost sharing problem are satisfied. Note that C has two maximal intervals of concavity: $[0,E]$ and $[E, \sum_{j\in N} c_j]$.

For the first interval, we have that $C^1(t) = 2t$ for all $t \in [0, E]$ and $q^1(\varphi) = q = c$. For $x^1(\varphi)$, let $\sigma : N \to \{1, \ldots, n\}$ be a bijection such that

$$\varphi_{\sigma^{-1}(1)}(E, c) \leq \varphi_{\sigma^{-1}(2)}(E, c) \leq \ldots \leq \varphi_{\sigma^{-1}(n)}(E, c).$$

Let $x^1_k(\varphi) = \varphi_{\sigma^{-1}(k)}(E, c)$ for all $k \in \{1, \ldots, n\}$. Then $(C^1, x^1(\varphi))$ is a cost sharing problem on (positions) $\{1, 2, \ldots, n\}$ and

$$\mathrm{Ser}(C^1, x^1(\varphi)) = 2x^1(\varphi).$$

For the second interval, we have that

$$C^2(t) = \begin{cases} 5t, & \text{if } t \leq \varepsilon; \\ t + 4\varepsilon, & \text{otherwise,} \end{cases}$$

for all $t \in [0, \sum_{j \in N} c_j - E]$. For $q^2(\varphi)$, we can use the bijection $\bar{\sigma}$ as defined above to reorder the players such that $q^2(\varphi)$ is non-decreasing over the players. That is, let $q^2_k(\varphi) = c_{\bar{\sigma}^{-1}(k)} - \varphi_{\bar{\sigma}^{-1}(k)}(E, c)$ for all $k \in \{1, \ldots, n\}$ and note that again,

$$x^2(\varphi) = q^2(\varphi).$$

To determine which part of the function C^2 applies, we consider the intermediate points used to compute $\mathrm{Ser}(C^2, x^2(\varphi))$: $s^2_0 = 0$,

$$s^2_1 = nx^2_1(\varphi) = n(c_{\bar{\sigma}^{-1}(1)} - \varphi_{\bar{\sigma}^{-1}(1)}(E, c)),$$

and, for all $k \in \{2, \ldots, n\}$,

$$\begin{aligned} s^2_k &= \sum_{\ell=1}^{k-1} x^2_\ell(\varphi) + (n - k + 1)x^2_k(\varphi) \\ &= \sum_{\ell=1}^{k-1} (c_{\bar{\sigma}^{-1}(\ell)} - \varphi_{\bar{\sigma}^{-1}(\ell)}(E, c)) + (n - k + 1)(c_{\bar{\sigma}^{-1}(k)} - \varphi_{\bar{\sigma}^{-1}(k)}(E, c)). \end{aligned}$$

Note that $s^2_k = r_k$ for all $k \in \{1, \ldots, n\}$. Therefore, for all $k \in \{1, \ldots, n\}$, if $s^2_k = 0$, then $C^2(s^2_k) = 0$, and if $s^2_k > 0$, then $s^2_k = r_k > \varepsilon$ and hence, $C^2(s^2_k) = s^2_k + 4\varepsilon$. This implies that,

$$\mathrm{Ser}(C^2, x^2(\varphi)) = x^2(\varphi).$$

Consequently,

$$\begin{aligned} \Psi^\varphi_i(C, q) &= \mathrm{Ser}_{\sigma(i)}(C^1, x^1(\varphi)) + \mathrm{Ser}_{\bar{\sigma}(i)}(C^2, x^2(\varphi)) \\ &= 2x^1_{\sigma(i)}(\varphi) + x^2_{\bar{\sigma}(i)}(\varphi) \\ &= 2\varphi_i(E, c) + c_i - \varphi_i(E, c) \end{aligned}$$

$$= c_i + \varphi_i(E, c),$$

and similarly,

$$\Psi^{\varphi}_{i+1}(C, q) = c_{i+1} + \varphi_{i+1}(E, c).$$

Hence, using that $q = c$,

$$\begin{aligned}\frac{\Psi^{\varphi}_i(C,q)}{q_i} - \frac{\Psi^{\varphi}_{i+1}(C,q)}{q_{i+1}} &= 1 + \frac{\varphi_i(E,c)}{c_i} - 1 - \frac{\varphi_{i+1}(E,c)}{c_{i+1}} \\ &= \frac{\varphi_i(E,c)}{c_i} - \frac{\varphi_{i+1}(E,c)}{c_{i+1}} < 0,\end{aligned}$$

where the inequality follows from Equation (5.17). This shows that

$$\frac{\Psi^{\varphi}_i(C,q)}{q_i} < \frac{\Psi^{\varphi}_{i+1}(C,q)}{q_{i+1}},$$

contradicting UCM.

Together, this leads to the conclusion that $\varphi = \text{PROP}$. □

The next example shows that Ψ^{PROP} does not satisfy MOVASP on $\mathcal{CCS}$.

Example 5.8 Consider the cost sharing problem $(C, q) \in \mathcal{CS}^N$ with $N = \{1, 2, 3, 4\}$, $q = (2, 5, 6, 9)$ and cost function C given by

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2, & \text{if } t \in [0, 12]; \\ 20t - \frac{1}{2}t^2, & \text{if } t \in (12, 16]; \\ 192 + 18(t-16) - \frac{1}{3}(t-16)^2, & \text{if } t \in (16, 17]; \\ 18t - \frac{1}{3}t^2, & \text{if } t \in (17, 18]; \\ 216 + 20(t-18) - \frac{1}{2}(t-18)^2, & \text{if } t \in (18, 22], \end{cases} \tag{5.18}$$

for all $t \in [0, 22]$.

This cost function is visualized in Figure 5.5. As can be seen in Figure 5.5, C is piecewise concave with three maximal intervals of concavity: $[0, 16]$, $[16, 18]$ and $[18, 22]$. It can be calculated that

$$\Psi^{\text{PROP}}(C, q) \approx (33.6, 71.8, 80.5, 102.1),$$

while

$$\Psi^{\text{PROP}}(C, q_{|N\setminus\{1\}}) \approx (72.6, 80.8, 100.6).$$

Hence, e.g.,

$$\begin{aligned}\Psi^{\text{PROP}}_3(C, q) - \Psi^{\text{PROP}}_3(C, q_{|N\setminus\{1\}}) &\approx -0.3 \\ &> -0.8 \approx \Psi^{\text{PROP}}_2(C, q) - \Psi^{\text{PROP}}_2(C, q_{|N\setminus\{1\}}),\end{aligned}$$

contradicting MOVASP. △

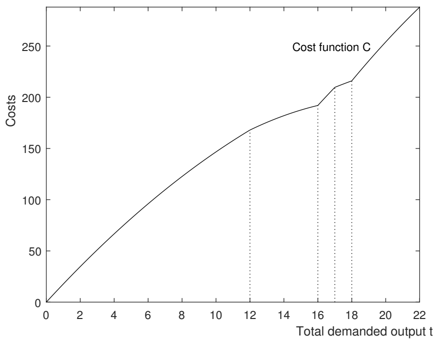


Figure 5.5 – *The cost function C of the cost sharing problem of Example 5.8.*

As a result of both Example 5.8 and Theorem 5.7, there is no piecewise serial rule based on a claims rule that satisfies both UCM and MOVASP. On the class of cost sharing problems with piecewise concave cost functions, these two properties are thus incompatible.

Interestingly, the CEL-piecewise serial rule does satisfy MOVASP on $\mathcal{CCS}^m$ for all $m \in \{1, 2, \ldots\}$, as shown in Theorem 5.8. The proof of Theorem 5.8 uses, among other things, a feature of the constrained equal losses rule as provided below in Lemma 5.3.

Lemma 5.3 *Let $(E, c) \in \mathcal{C}^N$ be a claims problem with $N = \{1, 2, \ldots, n\}$ such that $c_1 \leq c_2 \leq \ldots \leq c_n$ and $c_2 + c_3 + \ldots + c_n \geq E$. Then the following two statements hold:*

i) if $CEL_1(E, c) = 0$, then $CEL_j(E, c_{|N\setminus\{1\}}) = CEL_j(E, c)$ for all $j \in N \setminus \{1\}$;

ii) if $CEL_1(E, c) > 0$, then

$$CEL_j(E, c) = c_j - \frac{\sum_{k\in N} c_k - E}{n},$$

for all $j \in N$, and

$$CEL_j(E, c_{|N\setminus\{1\}}) = CEL_j(E, c) + \frac{CEL_1(E, c)}{n-1},$$

for all $j \in N \setminus \{1\}$.

The following theorem shows that the piecewise serial based on the constrained equal losses rule satisfies monotonic vulnerability for the absence of the smallest player.

Theorem 5.8 *Ψ^{CEL} satisfies MOVASP on $\mathcal{CCS}^m$ for all $m \in \{1, 2, \dots\}$.*

Proof: The theorem clearly holds for $m = 1$, using Theorem 5.5 and the fact that $\mathcal{CCS}^1 = \mathcal{CCS}$.

Let $m \in \{2, 3, \dots\}$ and let $(C, q) \in \mathcal{CCS}^{N,m}$ and denote, as before, the maximal intervals of concavity of C by $[t_0, t_1], [t_1, t_2], \dots, [t_{m-1}, t_m]$ with $0 = t_0 < t_1 < t_2 < \dots < t_{m-1} < t_m = \sum_{i \in N} q_i$, with modified cost functions $C^r : [0, t_r - t_{r-1}] \to \mathbb{R}_+$, for $r \in \{1, \dots, m\}$, defined by $C^r(t) = C(t + t_{r-1}) - C(t_{r-1})$, for all $t \in [0, t_r - t_{r-1}]$. Also, abbreviate for all $r \in \{1, \dots, m\}$, $q^r(\text{CEL})$ to q^r and $x^r(\text{CEL})$ to x^r. Similarly, denote by $\bar{q}^r$ and $\bar{x}^r$, $r \in \{1, \dots, m\}$, the respective vectors in $\mathbb{R}^{N \setminus \{1\}}$ as provided in the definition of $\Psi^{\text{CEL}}(C, q_{|N \setminus \{1\}})$.

Note that, by definition, $\Psi^{\text{CEL}}(C, q) = \sum_{r=1}^m \text{Ser}(C^r, x^r)$ and $\Psi^{\text{CEL}}(C, q_{|N \setminus \{1\}}) = \sum_{r=1}^m \text{Ser}(C^r, \bar{x}^r)$. So, in order to show MOVASP it suffices to show that, for all $i \in \{3, \dots, n\}$ and for all $r \in \{1, \dots, m\}$,

$$\text{Ser}_i(C^r, x^r) - \text{Ser}_i(C^r, \bar{x}^r) \leq \text{Ser}_{i-1}(C^r, x^r) - \text{Ser}_{i-1}(C^r, \bar{x}^r). \tag{5.19}$$

Let $i \in \{3, \dots, n\}$ and $r \in \{1, \dots, m\}$.

We need to fix some notation regarding the intermediate points needed to calculate $\text{Ser}(C^r, x^r)$ and $\text{Ser}(C^r, \bar{x}^r)$. Regarding $\text{Ser}(C^r, x^r)$, these points are denoted by

$$0 = s_0^r, s_1^r, \dots, s_{n-1}^r, s_n^r,$$

with, for $\ell \in \{1, \dots, n\}$,

$$s_\ell^r = \sum_{k=1}^{\ell-1} x_k^r + (n - \ell + 1) x_\ell^r.$$

Similarly, the intermediate points regarding $\text{Ser}(C^r, \bar{x}^r)$ are denoted by

$$0 = \bar{s}_1^r, \bar{s}_2^r, \dots, \bar{s}_{n-1}^r, \bar{s}_n^r,$$

with, for all $\ell \in \{1, \dots, n\}$,

$$\bar{s}_\ell^r = \sum_{k=2}^{\ell-1} \bar{x}_k^r + (n - \ell + 1) \bar{x}_\ell^r.$$

First, assume that $q^r_{|N \setminus \{1\}} \neq \bar{q}^r$. Then, obviously, $r > 1$ and $q_k^r \neq \bar{q}_k^r$ for some $k \in \{2, \dots, n\}$. Then we have that $x_1^s > 0$ for some $s \in \{1, \dots, r-1\}$, since otherwise we would have that $q^r_{|N \setminus \{1\}} = \bar{q}^r$, as a consequence of Lemma 5.3 part i). Assume w.l.o.g. that s is the first index for which this happen, i.e., $x_1^1 = \dots = x_1^{s-1} = 0$. This implies that $q^s_{|N \setminus \{1\}} = \bar{q}^s$ and, by using Lemma 5.3 part ii),

$$\begin{cases} x_j^s = q_j^s - \frac{\sum_{h \in N} q_h^s - (t_s - t_{s-1})}{n}; \\ \bar{x}_j^s = x_j^s + \frac{x_1^s}{n-1}, \end{cases}$$

for all $j \in N \setminus \{1\}$ and consequently, by using that $\bar{q}_j^s = q_j^s$,

$$\begin{cases} q_j^{s+1} = q_j^s - x_j^s = \frac{\sum_{h \in N} q_h^s - (t_s - t_{s-1})}{n}; \\ \bar{q}_j^{s+1} = \bar{q}_j^s - \bar{x}_j^s = q_j^s - x_j^s - \frac{x_1^s}{n-1} = \frac{\sum_{h \in N} q_h^s - (t_s - t_{s-1})}{n} - \frac{x_1^s}{n-1}, \end{cases}$$

are independent of $j \in N \setminus \{1\}$. Thus, in particular,

$$\begin{cases} q_i^{s+1} = q_{i-1}^{s+1}; \\ \bar{q}_i^{s+1} = \bar{q}_{i-1}^{s+1}, \end{cases}$$

and, by using the symmetry of the constrained equal losses rule,

$$\begin{cases} x_i^{s+1} = x_{i-1}^{s+1}; \\ \bar{x}_i^{s+1} = \bar{x}_{i-1}^{s+1}. \end{cases}$$

Subsequently, we have that

$$\begin{cases} q_i^{\ell} = q_{i-1}^{\ell}; \\ \bar{q}_i^{\ell} = \bar{q}_{i-1}^{\ell}, \end{cases}$$

and

$$\begin{cases} x_i^{\ell} = x_{i-1}^{\ell}; \\ \bar{x}_i^{\ell} = \bar{x}_{i-1}^{\ell}, \end{cases}$$

for all $\ell \in \{s+1, \ldots, m\}$ and in particular, $x_i^r = x_{i-1}^r$ and $\bar{x}_i^r = \bar{x}_{i-1}^r$, since $s < r$. The first equality implies that, by using the symmetry of the serial rule,

$$\text{Ser}_i(C^r, x^r) = \text{Ser}_{i-1}(C^r, x^r),$$

while the latter inequality implies that, by again using the symmetry of the serial rule,

$$\text{Ser}_i(C^r, \bar{x}^r) = \text{Ser}_{i-1}(C^r, \bar{x}^r).$$

Consequently, Equation (5.19) holds under the assumption that $q^r_{|N\setminus\{1\}} \neq \bar{q}^r$.

In the rest of the proof, we assume that $q^r_{|N\setminus\{1\}} = \bar{q}^r$.

We distinguish between three cases with respect to $t_r - t_{r-1}$:

I) $\displaystyle\sum_{k \in N} q_k^r \leq t_r - t_{r-1};$

II) $\displaystyle\sum_{k \in N\setminus\{1\}} q_k^r < t_r - t_{r-1} < \sum_{k \in N} q_k^r;$

III) $\displaystyle t_r - t_{r-1} \leq \sum_{k \in N\setminus\{1\}} q_k^r.$

Case I) In this case, we assume that $\sum_{k\in N} q_k^r \le t_r - t_{r-1}$. Then also

$$\sum_{k\in N\setminus\{1\}} \bar{q}_k^r = \sum_{k\in N\setminus\{1\}} q_k^r \le \sum_{k\in N} q_k^r \le t_r - t_{r-1}.$$

Hence,

$$\begin{cases} x^r = \mathrm{CEL}(t_r - t_{r-1}, q^r) = q^r; \\ \bar{x}^r = \mathrm{CEL}(t_r - t_{r-1}, \bar{q}^r) = \bar{q}^r = q^r_{|N\setminus\{1\}}. \end{cases}$$

Since $(C^r, q^r) \in \mathcal{CCS}^N$, the fact that the serial rule satisfies MOVASP according to Theorem 5.5 implies that

$$\begin{aligned} \mathrm{Ser}_i(C^r, x^r) - \mathrm{Ser}_i(C^r, \bar{x}^r) &= \mathrm{Ser}_i(C^r, q^r) - \mathrm{Ser}_i(C^r, q^r_{|N\setminus\{1\}}) \\ &\le \mathrm{Ser}_{i-1}(C^r, q^r) - \mathrm{Ser}_{i-1}(C^r, q^r_{|N\setminus\{1\}}) \\ &= \mathrm{Ser}_{i-1}(C^r, x^r) - \mathrm{Ser}_{i-1}(C^r, \bar{x}^r). \end{aligned}$$

Thus, in case I, Equation (5.19) holds.

Case II) In this case, we assume that $\sum_{k\in N\setminus\{1\}} q_k^r < t_r - t_{r-1} < \sum_{k\in N} q_k^r$. Then also

$$\sum_{k\in N\setminus\{1\}} \bar{q}_k^r = \sum_{k\in N\setminus\{1\}} q_k^r < t_r - t_{r-1} < \sum_{k\in N} q_k^r,$$

and thus,

$$\bar{x}_k^r = \mathrm{CEL}_k(t_r - t_{r-1}, \bar{q}^r) = \mathrm{CEL}_k(t_r - t_{r-1}, q^r_{|N\setminus\{1\}}) = q_k^r,$$

for all $k \in N \setminus \{1\}$. Moreover, with

$$\varepsilon = q_1^r - \left(\sum_{h\in N} q_h^r - (t_r - t_{r-1}) \right),$$

we have that $\varepsilon = t_r - t_{r-1} - \sum_{k\in N\setminus\{1\}} q_k^r > 0$ and thus, $q_1^r > \sum_{h\in N} q_h^r - (t_r - t_{r-1})$. Consequently, since

$$q_1^r - \frac{\sum_{h\in N} q_h^r - (t_r - t_{r_1})}{n} > q_1^r - \frac{q_1^r}{n} > 0,$$

we have, due to the nature of the constrained equal losses rule, for all $k \in N$,

$$x_k^r = \mathrm{CEL}_k(t_r - t_{r-1}, q^r) = q_k^r - \frac{\sum_{h\in N} q_h^r - (t_r - t_{r-1})}{n}.$$

Then, for all $\ell \in \{2, \dots, n\}$,

$$s_\ell^r = \sum_{k=1}^{\ell-1} x_k^r + (n - \ell + 1) x_\ell^r$$

$$
\begin{aligned}
&=\sum_{k=1}^{\ell-1}\left(q_k^r-\frac{\sum_{h\in N}q_h^r-(t_r-t_{r-1})}{n}\right)+(n-\ell+1)\left(q_\ell^r-\frac{\sum_{h\in N}q_h^r-(t_r-t_{r-1})}{n}\right)\\
&=(n-\ell+1)q_\ell^r+\sum_{k=1}^{\ell-1}q_k^r-[(n-\ell+1)+(\ell-1)]\left(\frac{\sum_{h\in N}q_h^r-(t_r-t_{r-1})}{n}\right)\\
&=(n-\ell+1)q_\ell^r+\sum_{k=1}^{\ell-1}q_k^r-\left(\sum_{h\in N}q_h^r-(t_r-t_{r-1})\right)\\
&=(n-\ell+1)q_\ell^r+\sum_{k=2}^{\ell-1}q_k^r+q_1^r-\left(\sum_{h\in N}q_h^r-(t_r-t_{r-1})\right)\\
&=(n-\ell+1)\bar{x}_\ell^r+\sum_{k=2}^{\ell-1}\bar{x}_k^r+\varepsilon\\
&=\bar{s}_\ell^r+\varepsilon.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\mathrm{Ser}_i(C^r,x^r)-\mathrm{Ser}_{i-1}(C^r,x^r)-&\left(\mathrm{Ser}_i(C^r,\bar{x}^r)-\mathrm{Ser}_{i-1}(C^r,\bar{x}^r)\right)\\
&=\frac{C^r(s_i^r)-C^r(s_{i-1}^r)}{n-i+1}-\frac{C^r(\bar{s}_i^r)-C^r(\bar{s}_{i-1}^r)}{n-i+1}\\
&=\frac{C^r(\bar{s}_i^r+\varepsilon)-C^r(\bar{s}_{i-1}^r+\varepsilon)}{n-i+1}-\frac{C^r(\bar{s}_i^r)-C^r(\bar{s}_{i-1}^r)}{n-i+1}\\
&=\frac{1}{n-i+1}\left(C^r(\bar{s}_i^r+\varepsilon)-C^r(\bar{s}_i^r)-C^r(\bar{s}_{i-1}^r+\varepsilon)+C^r(\bar{s}_{i-1}^r)\right)\\
&\leq 0,
\end{aligned}
$$

where the inequality follows from Lemma 5.2 part ii), by using that $0<\bar{s}_{i-1}^r\leq\bar{s}_i^r$ and $\varepsilon>0$. Consequently, also in case II, Equation (5.19) holds.

Case III) In this final case, we assume that $t_r-t_{r-1}\leq\sum_{k\in N\backslash\{1\}}q_k^r$. Then also

$$
t_r-t_{r-1}\leq\sum_{k\in N\backslash\{1\}}q_k^r=\sum_{k\in N\backslash\{1\}}\bar{q}_k^r,
$$

and,

$$
\begin{aligned}
\bar{x}_\ell^r&=\mathrm{CEL}_\ell(t_r-t_{r-1},\bar{q}^r)\\
&=\mathrm{CEL}_\ell(t_r-t_{r-1},q^r_{|N\backslash\{1\}})\\
&=\mathrm{CEL}_\ell(t_r-t_{r-1},q^r)+\frac{\mathrm{CEL}_1(t_r-t_{r-1},q^r)}{n-1}
\end{aligned}
$$

$$= x^r_\ell + \frac{x^r_1}{n-1},$$

for all $\ell \in \{2, \ldots, n\}$, where the third equality is a consequence of Lemma 5.3. Hence,

$$\bar{s}^r_2 = (n-1)\bar{x}^r_2 = (n-1)(x^r_2 + \frac{x^r_1}{n-1}) = (n-1)x^r_2 + x^r_1 = s^r_2,$$

and for all $\ell \in \{3, \ldots, n\}$,

$$\begin{aligned}
\bar{s}^r_\ell &= \sum_{k=2}^{\ell-1} \bar{x}^r_k + (n-\ell+1)\bar{x}^r_\ell \\
&= \sum_{k=2}^{\ell-1} (x^r_k + \frac{x^r_1}{n-1}) + (n-\ell+1)(x^r_\ell + \frac{x^r_1}{n-1}) \\
&= (n-\ell+1)x^r_\ell + \sum_{k=2}^{\ell-1} x^r_k + \big((n-\ell+1) + (\ell-2)\big)\frac{x^r_1}{n-1} \\
&= (n-\ell+1)x^r_\ell + \sum_{k=2}^{\ell-1} x^r_k + x^r_1 \\
&= (n-\ell+1)x^r_\ell + \sum_{k=1}^{\ell-1} x^r_k \\
&= s^r_\ell.
\end{aligned}$$

Thus, $\bar{s}^r_\ell = s^r_\ell$ for all $\ell \in \{2, \ldots, n\}$. Consequently, for all $\ell \in \{2, 3, \ldots, n\}$,

$$\begin{aligned}
\mathrm{Ser}_\ell(C^r, x^r) - \mathrm{Ser}_\ell(C^r, \bar{x}^r) &= \sum_{k=1}^{\ell} \frac{C^r(s^r_k) - C^r(s^r_{k-1})}{n-k+1} - \sum_{k=2}^{\ell} \frac{C^r(\bar{s}^r_k) - C^r(\bar{s}^r_{k-1})}{n-k+1} \\
&= \frac{C^r(s^r_1)}{n} + \frac{C^r(s^r_2) - C^r(s^r_1)}{n-1} + \sum_{k=3}^{\ell} \frac{C^r(s^r_k) - C^r(s^r_{k-1})}{n-k+1} \\
&\quad - \left(\frac{C^r(\bar{s}^r_2)}{n-1} + \sum_{k=3}^{\ell} \frac{C^r(\bar{s}^r_k) - C^r(\bar{s}^r_{k-1})}{n-k+1} \right) \\
&= \frac{C^r(s^r_1)}{n} + \frac{C^r(s^r_2) - C^r(s^r_1)}{n-1} - \frac{C^r(\bar{s}^r_2)}{n-1} \\
&= \frac{C^r(s^r_1)}{n} - \frac{C^r(s^r_1)}{n-1},
\end{aligned}$$

is independent of ℓ. Hence,

$$\mathrm{Ser}_i(C^r, x^r) - \mathrm{Ser}_i(C^r, \bar{x}^r) = \mathrm{Ser}_{i-1}(C^r, x^r) - \mathrm{Ser}_{i-1}(C^r, \bar{x}^r),$$

which proves Equation (5.19) in case III with an equality.

We can conclude that Equation (5.19) also holds in case $q^r_{|N\setminus\{1\}} = \bar{q}^r$. Overall, this shows that Ψ^{CEL} satisfies MOVASP on $\mathcal{CCS}^m$ for all $m \in \{1, 2, \ldots\}$. $\square$

5.5 Cost allocations for CRCP-situations

In this chapter, we introduced capacity restricted cooperative purchasing situations and modeled the problem of finding an adequate allocation of the joint purchasing costs as a cost sharing problem. This final section focuses on the class of cost sharing problems that indeed stem from capacity restricted cooperative purchasing situations. This is especially valuable with respect to a possible compatibility result of UCM and MOVASP on this specific domain that is smaller than the whole class of cost sharing problems with piecewise concave cost functions.

We denoted $(C^Z, q) \in \mathcal{CS}^N$ as the cost sharing problem corresponding to the CRCP-situation $Z = (S, q) \in \mathcal{Z}^N$. Let $\mathcal{ZCCS}^N$ denote the class of all such cost sharing problems on fixed N, and $\mathcal{ZCCS}$ the class of all such cost sharing problems on variable but finite N. The cost function C^Z of a cost sharing problem $(C^Z, q) \in \mathcal{ZCCS}^N$ turned out to be piecewise concave with finitely many (maximal) intervals of concavity. Hence,

$$\mathcal{ZCCS}^N \subseteq \bigcup_{m=1}^{\infty} \mathcal{CCS}^{N,m}.$$

Example 5.5 shows that the serial rule does not satisfy UCM on the class of all cost sharing problems with piecewise concave cost functions. In fact, the cost sharing problem as described in Example 5.5 originates from the CRCP-situation as described in Example 5.2. Hence, the serial rule does not satisfy UCM on $\mathcal{ZCCS}^N$ either.

Similarly, the cost sharing problem as described in Example 5.6 originates from the CRCP-situation as described in Example 5.3. Hence, the serial rule does not satisfy MOVASP on $\mathcal{ZCCS}$.

Also the cost sharing problem of Example 5.8, which shows that the PROP-piecewise serial rule does not satisfy MOVASP on $\mathcal{CCS}$, corresponds to a CRCP-situation. It can be readily verified that the cost function C of Equation (5.18) is the restriction of the cost function γ^S of the CRCP-situation $Z = (S, q) \in \mathcal{Z}^N$, with $S = (Q_A, p_A, Q_B, p_B)$ be given by $Q_A = 16$, $Q_B = 18$ and

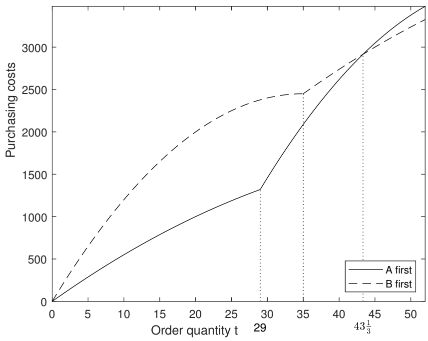
$$\begin{cases} p_A(t) = 20 - \frac{1}{2}t, & \text{for all } t \in [0, 16]; \\ p_B(t) = 18 - \frac{1}{3}t, & \text{for all } t \in [0, 18]. \end{cases}$$

Hence, the PROP-piecewise serial rule does not satisfy MOVASP on $\mathcal{ZCCS}$.

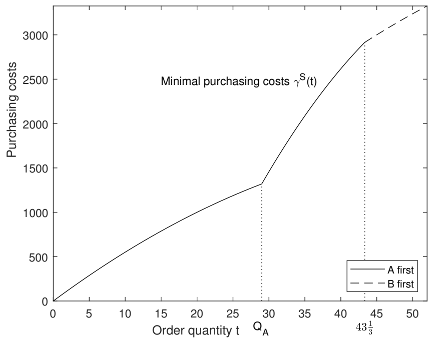
On the other hand, the CEL-piecewise serial rule satisfies MOVASP on $\mathcal{CCS}^m$ according to Theorem 5.8. This implies that the CEL-piecewise serial rule also satisfies MOVASP on $\mathcal{ZCCS}$.

With regard to UCM, Theorem 5.7 shows that the PROP-piecewise serial rule is the only piecewise serial rule based on a claims rule that satisfies UCM on $\mathcal{CCS}^{N,m}$. In particular, this implies that the PROP-piecewise serial rule satisfies UCM on $\mathcal{ZCCS}^N$. More importantly, it implies that the CEL-piecewise serial rule does not satisfy UCM on $\mathcal{CCS}^{N,m}$. The proof of Theorem 5.7 however does not immediately guarantee that this is also the case on $\mathcal{ZCCS}$. In other words, the two properties UCM and MOVASP are possibly compatible on the class of cost sharing problems corresponding to CRCP-situations.

Compatibility can be reached if the CEL-piecewise serial rule satisfies UCM on $\mathcal{ZCCS}$. To investigate whether this is indeed the case, we compute its average unit prices for specific numerical cost sharing problems arising from CRCP-situations using simulation techniques. For comparison reasons, we also compute the average unit prices according to both the PROP-piecewise serial rule and the classical serial rule.



(a) *The two extreme policies.*



(b) *The cost function γ^S.*

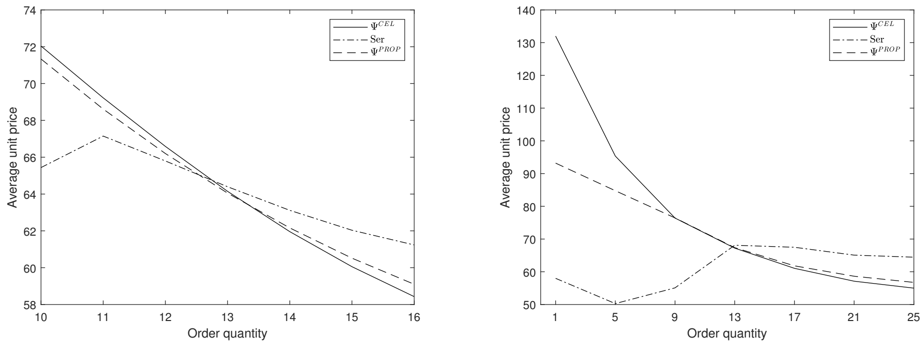
Figure 5.6 – *The two extreme policies and the cost function γ^S for the simulation.*

As input, we take CRCP-situations $Z = (S, q) \in \mathcal{Z}^N$, with $S = (Q_A, p_A, Q_B, p_B)$ be given by $Q_A = 29$, $Q_B = 35$ and

$$\begin{cases} p_A(t) = 60 - \frac{1}{2}t, \text{ for all } t \in [0, 29]; \\ p_B(t) = 140 - 2t, \text{ for all } t \in [0, 35]. \end{cases}$$

We also keep $\sum_{i \in N} q_i = 52$, such that the cost function C^Z of the associated cost sharing problem $(C^Z, q) \in \mathcal{ZCCS}^N$ remains the same for all CRCP-situations. This cost function is depicted in Figure 5.6b and has 2 maximal intervals of concavity: $[0, 29]$ and $[29, 52]$.

To create a CRCP-situation, we set $N = \{1, 2, 3, 4\}$ and randomly generate an integer-valued vector q of order quantities such that the sum of the order quantities equals 52. We compare two separate scenarios: $q_i \in \{10, 11, 12, 13, 14, 15, 16\}$ for all $i \in N$ and $q_i \in \{1, 5, 9, 13, 17, 21, 25\}$ for all $i \in N$, that is, a first scenario with 'small' differences between the possible order quantities and a second scenario with 'big' differences. For each generated instance (C, q), $\Psi^{\text{CEL}}(C, q)$, $\Psi^{\text{PROP}}(C, q)$ and $\text{Ser}(C, q)$ are calculated. Per integer value of the order quantity (independent of the corresponding player), the resulting allocations are stored and, in the end, averaged over the number of times it has occurred in the simulated instances.



(a) *The scenario with 'small' differences.* **(b)** *The scenario with 'big' differences.*

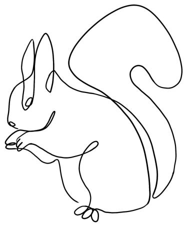
Figure 5.7 – *The average unit prices of the cost sharing rules for two different scenarios.*

In Figure 5.7 we plotted the (average) unit prices according to Ψ^{CEL}, Ψ^{PROP} and Ser for a player $i \in N$ in the scenario with 'small' differences, i.e., with order quantities $q_i \in \{10, 11, 12, 13, 14, 15, 16\}$, and in the scenario with 'big' differences, i.e., with order quantities $q_i \in \{1, 5, 9, 13, 17, 21, 25\}$.

In order to satisfy UCM, the average unit price should be (weakly) decreasing for increasing order quantities. Note that this is not the case for the serial rule, as was already indicated by the observations made above. For example, in Figure 5.7a, the average unit price for a player with an order quantity of 11 is strictly higher than for a player with an order quantity of 10. A similar observation is seen in Figure 5.7b for, e.g., players with order quantities 5 and 9 respectively.

More importantly, both figures clearly point out that both the PROP-piecewise serial rule and the CEL-piecewise serial rule satisfy UCM. The former is no surprise as this is proven in Theorem 5.6. The latter however is remarkable. The data seem to suggest that UCM is in fact satisfied by the CEL-piecewise serial rule for cost sharing problems arising from CRCP-situations.

6 Strategy in claims problems with estate holders



— A squirrel collects and claims nuts everywhere

6.1 Introduction

The claims problem of dividing a remaining estate upon death among the creditors is one that stems from way back. Written guidelines on how to solve these problems include the Babylonian Talmud (O'Neill, 1982; Aumann and Maschler, 1985). According to Moulin (2000) and Thomson (2003), among others, the idea of proportional division is the most well-known solution, which is also used in modern law (Aumann and Maschler, 1985).

Formally, a claims problem consists of a (monetary) estate that has to be divided among the claimants, where each claimant has a non-negative justifiable (monetary) claim on the estate. Initiated by O'Neill (1982) and followed by, among others, Aumann and Maschler (1985), significant research has been done in the study of finding fair allocations of the estate over the players. A wide variety of these so-called claims rules have been proposed, among which the proportional rule, the constrained equal awards rule, the constrained equal losses rule and the (reverse) Talmud rule. For an extensive overview, we refer to Thomson (2003, 2013, 2015).

Besides such a direct approach, O'Neill (1982) already bundles various game theoretic approaches, including a cooperative and a strategic approach. In the cooperative approach, the worth of a coalition is defined in a pessimistic way as the maximum of zero and the amount of money that is not claimed by all non-coalitional members. Such a cooperative game allows for applying standard cooperative game theoretic solution concepts and interpret the resulting vector as an allocation vector. In the strategic approach, the players choose specific parts of the estate to claim and every part that is claimed multiple times is divided equally among the corresponding players. This particular approach leads to, possibly multiple, Nash equilibria of which the payoff vectors are interpreted as allocation vectors.

More strategic approaches are proposed over the years, including various sequential move games, strategic bargaining games and strategic games. For example, Chun (1989) devised a sequential procedure that yields the constrained equal awards solution as unique Nash equilibrium outcome. Herrero (2003) proposed a dual version, leading to the constrained equal losses solution. Serrano (1995) proposed, and Dagan, Serrano, and Volij (1997) extended, a sequential move game that attains any claims rule satisfying estate monotonicity and consistency as unique subgame perfect equilibrium. A strategic game was developed by García-Jurado, González-Díaz, and Villar (2006), who showed that any (acceptable) claims rule can be the unique Nash equilibrium outcome. Finally, Li and Ju (2016), Tsay and Yeh (2019), and Hagiwara and Hanato (2021) proposed different strategic bargaining games. A cooperative approach worth mentioning is the one studied by Calleja, Borm, and Hendrickx (2005), who dealt with multi-issue allocation problems. In such a problem, the claims of the players are multi-dimensional.

In this chapter, we propose a new strategic model: claims problems with estate holders. In the standard claims problem, players have a claim on a single estate. However, in practice, it might happen that this estate is separated into smaller parts. In that case, there is in fact not one single estate that is divided, but multiple estates. In general, the division of each part of the estate does not necessarily have to be according to the same principle. Having multiple estates naturally leads to the assumption that there are also multiple executors or estate holders, who each can apply a particular allocation principle.

Since the total estate is divided into multiple estates with separate estate holders, the players are forced to divide their claims over these estates. Taking into account the various claims rules used by the estate holders, this leads to a strategic approach in which the players seek to maximize their sum of all the awards vectors specified by the claims rules.

As a motivation for our new model, we present two applications: a subsidy system and a taxation model. In a subsidy system, each player wants to launch a project for which the player needs a certain amount of money. To get this money, the players can request a subsidy from several different authorities. One can think of different layers of government: national government, regional government, municipality, and so on. Of course, the players can only request up to their demand for the particular project and thus have to decide how to divide their demand over the several authorities. Each authority has a specified budget available and uses its own way of dividing this budget among the applicants. Alternatively, the traditional claims problem can also be interpreted as a taxation problem in order to assess taxes as a function of incomes, as pointed out by Thomson (2003). With the free movement of workers within, for example, the European Union, it becomes more and more important to determine how much taxes one has to pay in which country. To put it differently, how to divide one's income over the different countries, with possibly different taxation rates, that are eligible for paying taxes?

The focus of this chapter is on the Nash equilibria of the strategic game associated to a claims problem with estate holders. We show that *existence of Nash equilibria* can be guaranteed if each underlying claims rule satisfies a property called *partial concavity*. In order to satisfy partial concavity, the awards of every player specified by a claims rule need to be concave in the claim of this player, given the claims of the other players. Claims rules that satisfy partial concavity include the constrained equal awards rule and the proportional rule. The constrained equal losses rule and the Talmud rule do not, in general, satisfy partial concavity. Using the constrained equal losses rule, we also show that existence of Nash equilibria is indeed not guaranteed.

A subsequent question that arises is one regarding *efficiency of Nash equilibria*. In a standard claims problem, efficiency requires that the whole estate is divided among the players, if their sum total claim is more than the estate. This naturally extends to a claims problem with estate holders for which the total claim is more than the total available estate, in which efficiency for a Nash equilibrium requires that each estate is completely divided among the players. We show that efficiency is guaranteed for all Nash equilibria under a weak condition called *strict marginality*. A claims rule satisfies strict marginality if there exists a player for which a small decrease $\varepsilon > 0$ in this player's claim leads to a decrease in the awards of this player smaller than ε. All common rules satisfy strict marginality.

A consequence of having efficiency for Nash equilibria is the fact that it allows for a direct comparison between Nash equilibrium pay-off vectors and the awards vectors of the underlying claims rules applied to the claims problem in which all estates are consolidated. In particular, by restricting attention to *uniform* claims problems with estate holders, that is, problems in which all underlying claims rules are identical, we can directly compare Nash equilibria pay-off vectors with the awards vector of the common claims rule. Focusing on two specific claims rules, the proportional rule and the constrained equal awards rule, we show that each Nash equilibrium pay-off vector is equal to the awards vector of the corresponding claims rule. For the proportional rule, we also show that this result can be used to determine the full set of Nash equilibria in case there are two estate holders and two players.

This chapter is structured as follows. Section 6.2 introduces the new model of claims problems with estate holders and the associated strategic ceh-games. Section 6.3 studies the existence of Nash equilibria in ceh-games, whereas Section 6.4 focuses on efficiency of these Nash equilibria. Furthermore, Section 6.5 studies uniform claims problems with estate holders and in particular, Section 6.5.1 focuses on the proportional rule as underlying common claims rule, while Section 6.5.2 focuses on the constrained equal awards rule.

6.2 Claims problems with estate holders and associated strategic games

In this section, we introduce the concepts of a claims problem with estate holders and the associated strategic game. Traditionally, a claims problem (O'Neill, 1982) occurs as the result of a bankruptcy and consists of an estate and a claims vector summarizing the claims of the players on this estate. In that case, a single executor or arbiter holds the estate and determines the outcome for the players based on a pre-determined claims rule.

One could however imagine that in practical cases, the estate is separated in multiple parts, each held by a different estate holder. In that case, there are multiple estates summing up to the total available estate. Each estate is held by a particular executor, applying a particular claims rule. This new concept extends the traditional concept of a claims problem and is called a claims problem with estate holders, which is formally defined below.

Definition 6.1 A *claims problem with estate holders (ceh-problem)* is a tuple $(M, N, \{E^k\}_{k\in M}, \{\varphi^k\}_{k\in M}, c)$ where M is a non-empty, finite set of *estate holders*, N is a non-empty, finite set of *players*, $c = (c_i)_{i\in N} \in \mathbb{R}^N_+$ summarizes the outstanding *claims* and for each $k \in M$, there is an *estate* $E^k \in \mathbb{R}_+$ and corresponding *claims rule* $\varphi^k : \mathcal{C}^N \to \mathbb{R}^N_+$.

The total available estate is given by $E = \sum_{k\in M} E^k$.

The set of all claims problems with estate holders M and players N is denoted by $\mathcal{C}^{M,N}$ and a claims problem with estate holders $(M, N, \{E^k\}_{k\in M}, \{\varphi^k\}_{k\in M}, c)$ is also denoted by $(\{E^k\}_{k\in M}, \{\varphi^k\}_{k\in M}, c) \in \mathcal{C}^{M,N}$. ◁

Due to the fact that the total estate is now separated over multiple estates, players are forced to divide their claims over these estates. This naturally leads to a strategic approach where the set of strategies of a player consists of all such divisions of the claim over the estates and the pay-offs are given by the sum of the award vectors of the claims rules. This is formalized in Definition 6.2 below.

Definition 6.2 Let $(\{E^k\}_{k\in M}, \{\varphi^k\}_{k\in M}, c) \in \mathcal{C}^{M,N}$ be a claims problem with estate holders. Then the *(associated) strategic ceh-game* is given by $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ where, for all $i \in N$,

$$X_i = \left\{ x_i \in \mathbb{R}^M \;\middle|\; x_i \geq 0 \quad \text{and} \quad \sum_{k\in M} x_i^k = c_i \right\},$$

and, for all $x = (x_j)_{j\in N} \in \Pi_{j\in N} X_j$,

$$\pi_i(x) = \sum_{k\in M} \varphi_i^k(E^k, (x_j^k)_{j\in N}). \qquad \triangleleft$$

Logically, for a claims problem with estate holders, we are interested in finding the set of Nash equilibria of the associated strategic ceh-game. As a result of the structure of the pay-offs and the fact that claims rules satisfy non-negativity and claims boundedness, we directly see that the pay-off in a Nash equilibrium is also non-negative and bounded from above by the respective claims.

Formally, for a ceh-problem $(\{E^k\}_{k\in M}, \{\varphi^k\}_{k\in M}, c) \in \mathcal{C}^{M,N}$, the associated ceh-game $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ and a Nash equilibrium $\hat{x} \in NE(G)$, we have that, for all $i \in N$,

$$0 \leq \pi_i(\hat{x}) \leq c_i. \tag{6.1}$$

The following example illustrates the strategic aspects that occur in the ceh-game associated to a ceh-problem, as well as the process of finding the set of Nash equilibria.

Example 6.1 Consider the ceh-problem $(\{E^k\}_{k\in M}, \{\varphi^k\}_{k\in M}, c) \in \mathcal{C}^{M,N}$ with $M = \{A, B\}$, $N = \{1, 2\}$, $E^A = E^B = 4$, $\varphi^A = \varphi^B = \text{CEA}$ and $c = (8, 4)$.

Here, the set of strategies for player 1 is given by

$$X_1 = \left\{ x_1 \in \mathbb{R}^{\{A,B\}} \;\middle|\; x_1 \geq 0 \quad \text{and} \quad x_1^A + x_1^B = 8 \right\},$$

whereas the set of strategies for player 2 is given by

$$X_2 = \left\{ x_2 \in \mathbb{R}^{\{A,B\}} \;\middle|\; x_2 \geq 0 \quad \text{and} \quad x_2^A + x_2^B = 4 \right\}.$$

Clearly, both sets can be represented by a line segment[1], which is visualized in Figure 6.1. Here, the numbers on both axes represent the amount that is claimed at estate E^A. For example, a strategy combination $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ with $\bar{x}_1^A = 2$ and $\bar{x}_2^A = 1$ automatically imply that $\bar{x}_1^B = c_1 - \bar{x}_1^A = 8 - 2 = 6$ and $\bar{x}_2^B = c_2 - \bar{x}_2^A = 4 - 1 = 3$.

[1] In general, the dimension of the set of strategies is one less than the number of estate holders.

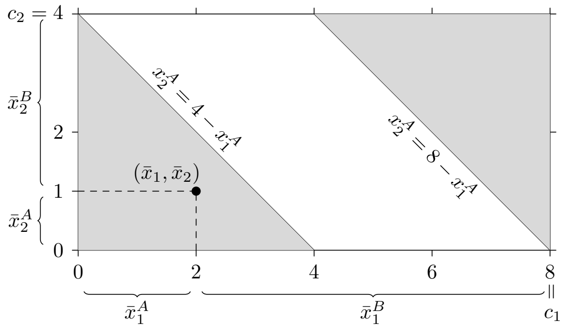
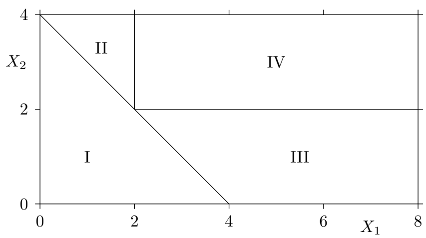


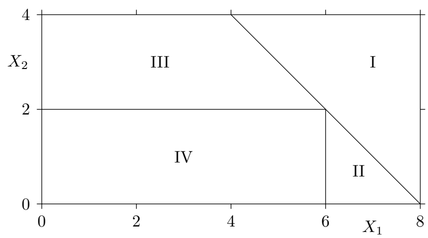
Figure 6.1 – *A visualization of a strategy combination $\bar{x} \in X$ and the two areas where the estate is sufficient to cover the claims of Example 6.1.*

Furthermore, the two gray areas indicate the sets of strategy combinations for which the corresponding estate is sufficient to cover the claims on this estate. More specifically, the left area corresponds to all strategy combinations $x = (x_1, x_2) \in X_1 \times X_2$ for which $x_1^A + x_2^A \leq E^A = 4$, while the right area corresponds to all strategy combinations $x = (x_1, x_2) \in X_1 \times X_2$ for which $x_1^B + x_2^B \leq E^B = 4$.

To determine the sets of best reply strategies, we divide the set of strategy combinations into several areas based on the paths of awards (see Section 2.2) of the claims rule used. For each of these areas, we compute the award specified by the constrained equal awards rule dividing estate E^A and the award specified by the constrained equal awards rule dividing estate E^B. Together, this results in the pay-off for both players in each of these areas.



(a) *For estate E^A.*



(b) *For estate E^B.*

Figure 6.2 – *A division of the set of strategy combinations in four areas for the two estates of Example 6.1.*

First, Figure 6.2a provides an overview of the four different areas. These areas are based on the path of awards for the constrained equal awards rule, as depicted in Figure 2.2a.

Area I) The first area corresponds exactly to the left gray area of Figure 6.1 and is given by all strategy combinations $x \in X$ for which $x_1^A + x_2^A \leq E^A = 4$. Clearly,

$$\text{CEA}\left(E^A, (x_1^A, x_2^A)\right) = \left(x_1^A, x_2^A\right).$$

Area II) The second area is given by all strategy combinations $x \in X$ for which both $0 \leq x_1^A \leq 2$ and $4 - x_1^A \leq x_2^A \leq 4$. Consequently, we have that

$$\text{CEA}\left(E^A, (x_1^A, x_2^A)\right) = \left(x_1^A, 4 - x_1^A\right).$$

Area III) The third area is in some sense the reverse of the second area and is given by all strategy combinations $x \in X$ for which both $0 \leq x_2^A \leq 2$ and $4 - x_2^A \leq x_1^A \leq 4$. Consequently, we have that

$$\text{CEA}\left(E^A, (x_1^A, x_2^A)\right) = \left(4 - x_2^A, x_2^A\right).$$

Area IV) Finally, the fourth area is given by all strategy combinations $x \in X$ for which both $2 \leq x_1^A \leq 8$ and $2 \leq x_2^A \leq 4$. Consequently, we have that

$$\text{CEA}\left(E^A, (x_1^A, x_2^A)\right) = (2, 2).$$

Secondly, Figure 6.2b provides an overview of the four areas for estate E^B. As it is also based on the path of awards for the constrained equal awards rule, the overall structure of the areas is similar to the structure in Figure 6.2a.

Area I) The first area corresponds exactly to the right gray area of Figure 6.1 and is given by all strategy combinations $x \in X$ for which $x_1^B + x_2^B \leq E^B = 4$. Clearly,

$$\text{CEA}\left(E^B, (x_1^B, x_2^B)\right) = \left(x_1^B, x_2^B\right) = \left(8 - x_1^A, 4 - x_2^A\right).$$

Area II) The second area is given by all strategy combinations $x \in X$ for which both $6 \leq x_1^A \leq 8$ (or equivalently, $0 \leq x_1^B \leq 2$) and $0 \leq x_2^A \leq 8 - x_1^A$ (or equivalently, $4 - x_1^B \leq x_2^B \leq 4$). Consequently, we have that

$$\text{CEA}\left(E^B, (x_1^B, x_2^B)\right) = \left(x_1^B, 4 - x_1^B\right) = \left(8 - x_1^A, x_1^A - 4\right).$$

Area III) The third area is in some sense the reverse of the second area and is given by all strategy combinations $x \in X$ for which both $2 \leq x_2^A \leq 4$ (or equivalently, $0 \leq x_2^B \leq 2$) and $0 \leq x_1^A \leq 8 - x_2^A$ (or equivalently, $4 - x_1^B \leq x_1^B \leq 4$). Consequently, we have that

$$\text{CEA}\left(E^B, (x_1^B, x_2^B)\right) = \left(4 - x_2^B, x_2^B\right) = \left(x_2^A, 4 - x_2^A\right).$$

Area IV) Finally, the fourth area is given by all strategy combinations $x \in X$ for which both $0 \leq x_1^A \leq 6$ (or equivalently, $2 \leq x_1^B \leq 8$) and $0 \leq x_2^A \leq 2$ (or equivalently, $2 \leq x_2^B \leq 4$). Consequently, we have that

$$\text{CEA}\left(E^B, (x_1^B, x_2^B)\right) = (2, 2).$$

By combining Figures 6.2a and 6.2b, we can divide the set of all strategy combinations into eight areas for which we can readily compute the corresponding pay-off. This is visualized in Figure 6.3.

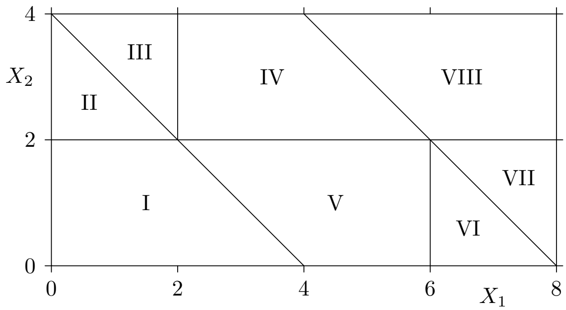


Figure 6.3 – *A division of the set of strategy combinations in Example 6.1.*

Clearly, by adding both awards vectors dividing the estate E^A and E^B, respectively, we obtain the pay-off functions for each of the eight areas. For this, let $x \in X$ and without again specifying the exact conditions on x, the eight possible pay-off functions are listed below for each of the eight areas:

I) $\pi(x_1, x_2) = \left(x_1^A, x_2^A\right) + (2, 2) = \left(2 + x_1^A, 2 + x_2^A\right);$

II) $\pi(x_1, x_2) = \left(x_1^A, x_2^A\right) + \left(x_2^A, 4 - x_2^A\right) = \left(x_1^A + x_2^A, 4\right);$

III) $\pi(x_1, x_2) = \left(x_1^A, 4 - x_1^A\right) + \left(x_2^A, 4 - x_2^A\right) = \left(x_1^A + x_2^A, 8 - x_1^A - x_2^A\right);$

IV) $\pi(x_1, x_2) = (2, 2) + \left(x_2^A, 4 - x_2^A\right) = \left(2 + x_2^A, 6 - x_2^A\right);$

V) $\pi(x_1, x_2) = \left(4 - x_2^A, x_2^A\right) + (2, 2) = \left(6 - x_2^A, 2 + x_2^A\right);$

VI) $\pi(x_1, x_2) = \left(4 - x_2^A, x_2^A\right) + \left(8 - x_1^A, x_1^A - 4\right) = \left(12 - x_1^A - x_2^A, x_1^A + x_2^A - 4\right);$

VII) $\pi(x_1, x_2) = \left(4 - x_2^A, x_2^A\right) + \left(8 - x_1^A, 4 - x_2^A\right) = \left(12 - x_1^A - x_2^A, 4\right);$

VIII) $\pi(x_1, x_2) = (2, 2) + \left(8 - x_1^A, 4 - x_2^A\right) = \left(10 - x_1^A, 6 - x_2^A\right).$

Having this list of possible pay-off functions per area of strategy combinations, it is straightforward to compute the sets of best reply strategies in order to compute the Nash equilibria. The set of best reply strategies for player 1 against a strategy of player 2 is visualized in Figure 6.4a and is, for all $\hat{x}_2 \in X_2$, given by

$$BR_1(\hat{x}_2) = \begin{cases} \left\{x_1 \in X_1 \;\middle|\; 4 - \hat{x}_2^A \leq x_1^A \leq 6\right\}, & \text{if } 0 \leq \hat{x}_2^A \leq 2; \\ \left\{x_1 \in X_1 \;\middle|\; 2 \leq x_1^A \leq 8 - \hat{x}_2^A\right\}, & \text{if } 2 < \hat{x}_2^A \leq 4. \end{cases} \tag{6.2}$$

To show that this is indeed the set of best reply strategies for player 1, let $\hat{x}_2 \in X_2$ and distinguish between the two indicated cases.

First, assume that $0 \leq \hat{x}_2^A \leq 2$. Then the pay-off of player 1 is indeed maximized for all $x_1 \in X_1$ for which $4-\hat{x}_2^A \leq x_1^A \leq 6$, that is, in area V. Note that the corresponding pay-off for these strategy combinations is given by

$$\pi_1(x_1, \hat{x}_2) = 6 - \hat{x}_2^A.$$

Let $x_1 \in X_1$ with $x_1^A < 4 - \hat{x}_2^A$ (that is, a strategy combination in area I). Then it readily follows that

$$\pi_1(x_1, \hat{x}_2) = 2 + x_1^A < 2 + 4 - \hat{x}_2^A = 6 - \hat{x}_2^A.$$

On the other hand, let $x_1 \in X_1$ with $x_1^A > 6$ (that is, a strategy combination in either area VI or VII). Then it readily follows that

$$\pi_1(x_1, \hat{x}_2) = 12 - x_1^A - \hat{x}_2^A < 12 - 6 - \hat{x}_2^A = 6 - \hat{x}_2^A.$$

Secondly, assume that $2 \leq \hat{x}_2^A \leq 4$. Then the pay-off of player 1 is maximized for all $x_1 \in X_1$ for which $2 \leq x_1^A \leq 8 - \hat{x}_2^A$, that is, in area IV, with corresponding pay-off given by

$$\pi_1(x_1, \hat{x}_2) = 2 + x_2^A.$$

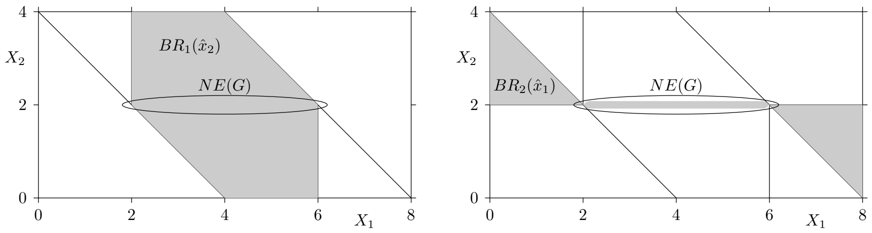
As before, let $x_1 \in X_1$ with $x_1^A < 2$ (that is, a strategy combination in either II or III). Then it readily follows that

$$\pi_1(x_1, \hat{x}_2) = x_1^A + \hat{x}_2^A < 2 + \hat{x}_2^A.$$

Moreover, let $x_1 \in X_1$ with $x_1^A > 8 - \hat{x}_2^A$ (that is, a strategy combination in area VIII). Then it readily follows that

$$\pi_1(x_1, \hat{x}_2) = 10 - x_1^A < 10 - (8 - \hat{x}_2^A) = 2 + \hat{x}_2^A.$$

This shows that Equation (6.2) indeed provides the set of best reply strategies for player 1 against any strategy of player 2.



(a) *For player 1.* **(b)** *For player 2.*

Figure 6.4 – *The sets of best reply strategies and the set of Nash equilibria of the strategic ceh-game of Example 6.1.*

Next, using a similar analysis as before, one can show that the set of best reply strategies for player 2 against a strategy of player 1, which is visualized in Figure 6.4b, is, for all $\hat{x}_1 \in X_1$, given by

$$BR_2(\hat{x}_1) = \begin{cases} \left\{x_2 \in X_2 \;\middle|\; 2 \leq x_2^A \leq 4 - \hat{x}_1^A\right\}, & \text{if } 0 \leq \hat{x}_1^A < 2; \\ \left\{x_2 \in X_2 \;\middle|\; x_2^A = 2\right\}, & \text{if } 2 \leq \hat{x}_1^A \leq 6; \\ \left\{x_2 \in X_2 \;\middle|\; 8 - \hat{x}_1^A \leq x_2^A \leq 2\right\}, & \text{if } 6 < \hat{x}_1^A \leq 8. \end{cases} \tag{6.3}$$

Combining both sets of best reply strategies yield the set of Nash equilibria. In this example, it follows that

$$NE(G) = \left\{(x_1, x_2) \in X_1 \times X_2 \;\middle|\; 2 \leq x_1^A \leq 6 \text{ and } x_2^A = 2\right\},$$

as also indicated in Figure 6.4.

We conclude this example by remarking that, for all $x \in NE(G)$, it holds that

$$\pi(x) = \pi((x_1^A, x_1^B), (2, 2)) = (4, 4),$$

whereas CEA $(E, c) =$ CEA $(8, (8, 4)) = (4, 4)$ too. △

Example 6.1 exposes various features of the newly introduced model of claims problems with estate holders. In the example Nash equilibria exist. This leads to the obvious question whether this is the case for all strategic ceh-games associated to ceh-problems. In Section 6.3, we study this question.

Moreover, in the example all Nash equilibria lead to the same pay-off vector, which is in fact also equal to the awards vector specified by the constrained equal awards rule used to divide both estates. In Section 6.5, it becomes clear that, if either the constrained equal awards rule (Section 6.5.2) or the proportional rule (Section 6.5.1) is used to divide all estates, it is always the case that all Nash equilibria lead to the awards vector specified by the corresponding clams rule.

Furthermore, the unique Nash equilibrium pay-off vector of Example 6.1 is efficient, that is, the sum of the pay-offs of the players is equal to the total available estate. Section 6.4 studies the question whether Nash equilibrium pay-off vectors are efficient in general.

Finally, Example 6.1 also indicates that computing the set of Nash equilibria is not a straightforward task. In fact, the computation of the pay-off vector for every possible strategy combination strongly relies on the (combination of) claims rules that are used to divide the estates. Each claims rule has its own path of awards (see Section 2.2) which ultimately determines the number of convenient areas in which the set of strategy combinations can be divided. Interestingly, for the proportional rule, this path of awards is just one line segment, which allows for a streamlined computation of the set of Nash equilibria in case there are only two estate holders and two players. This is formalized in Section 6.5.1.

6.3 Existence of Nash equilibria in ceh-games

This section studies the question whether Nash equilibria exist for all strategic ceh-games associated to ceh-problems, as was the case in Example 6.1. We formulate a sufficient condition on the underlying claims rules to guarantee the existence of Nash equilibria in ceh-games. We show that there always exists a Nash equilibrium, if all underlying claims rules satisfy a property called partial concavity.

Definition 6.3 Let φ be a claims rule. Then φ satisfies *partial concavity* on $\mathcal{C}^N$ if, for all $(E, c) \in \mathcal{C}^N$ and all $i \in N$, the function $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $f_i(x) = \varphi_i(E, (c_{-i}, x))$ for all $x \in \mathbb{R}_+$ is concave. $\triangleleft$

In other words, a claims rule satisfies partial concavity if for every player it is concave in the claim of this player, given the claims of the other players. Requiring partial concavity is sufficient to show the existence of Nash equilibria in ceh-games, using the (general) existence theorem for Nash equilibria as formulated in Theorem 2.1.

Theorem 6.1 *Let $(\{E^k\}_{k \in M}, \{\varphi^k\}_{k \in M}, c) \in \mathcal{C}^{M,N}$ be a ceh-problem and let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be the associated strategic ceh-game. If, for all $k \in M$, φ^k satisfies partial concavity, then $NE(G) \neq \emptyset$.*

Proof: Assume that, for all $k \in M$, φ^k satisfies partial concavity. We show that $NE(G) \neq \emptyset$ by verifying the four conditions as formulated in Theorem 2.1.

Let $i \in N$. Recall that

$$X_i = \left\{ x_i \in \mathbb{R}^M \;\middle|\; x_i \geq 0 \quad \text{and} \quad \sum_{k \in M} x_i^k = c_i \right\}.$$

Consequently, $X_i \subseteq \mathbb{R}^{m_i}$, with $m_i = |M| \in \mathbb{N}$ fulfilling the first condition. Furthermore, it can be easily seen that X_i is non-empty, convex, closed and bounded, satisfying the second condition.

The third condition, that is, π_i is continuous, follows from the fact π_i is the sum of the awards specified by φ^k for all $k \in M$, according to Definition 6.2, and the (general) assumption that φ_i^k is continuous for all $k \in M$.

Finally, to prove that $g_i : X_i \to \mathbb{R}$, with $g_i(x_i) = \pi_i(x_i, x_{-i})$ is concave for all $x_{-i} \in X_{-i}$, note that

$$\pi_i(x) = \sum_{k \in M} \varphi_i^k(E^k, (x_i^k)_{i \in N}).$$

Then it immediately follows that g_i is concave, since φ_i^k is concave in the claim of player i and the sum of concave functions is again concave. Hence, also the fourth condition is satisfied.

Consequently, by using Theorem 2.1, we can conclude that $NE(G) \neq \emptyset$. □

Theorem 6.1 shows that partial concavity for all the underlying claims rules is a sufficient condition to guarantee the existence of Nash equilibria in ceh-games. Proposition 6.1 shows that both the constrained equal awards rule and the proportional rule satisfy partial concavity, as well as the concede and divide rule on $\mathcal{C}^N$ with $|N| = 2$.

Proposition 6.1 *CEA, PROP and CD satisfy partial concavity.*

Proof: First, we focus on the constrained equal awards rule. Let $(E, c) \in \mathcal{C}^N$ and let $i \in N$. To show that $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_i(x) = \text{CEA}_i\left(E, (c_{-i}, x)\right)$ for all $x \in \mathbb{R}_+$ is concave, we show that

$$\text{CEA}_i\left(E, (c_{-i}, (1-\lambda)x + \lambda y)\right) \geq (1-\lambda)\text{CEA}_i\left(E, (c_{-i}, x)\right) + \lambda \text{CEA}_i\left(E, (c_{-i}, y)\right), \tag{6.4}$$

for all $x, y \in \mathbb{R}_+$ and all $\lambda \in [0,1]$. So, let $x, y \in \mathbb{R}_+$ and $\lambda \in [0,1]$ and assume w.l.o.g. that $x \leq y$, so that $x \leq (1-\lambda)x + \lambda y \leq y$. We distinguish between three cases:

I) $\text{CEA}_i\left(E, (c_{-i}, y)\right) \leq x$;

II) $x < \text{CEA}_i\left(E, (c_{-i}, y)\right) \leq (1-\lambda)x + \lambda y$;

III) $(1-\lambda)x + \lambda y < \text{CEA}_i\left(E, (c_{-i}, y)\right) \leq y$.

Case I) For the first case, where $\text{CEA}_i\left(E, (c_{-i}, y)\right) \leq x$, we immediately have that, due to the nature of the constrained equal awards rule,

$$\text{CEA}_i\left(E, (c_{-i}, x)\right) = \text{CEA}_i\left(E, (c_{-i}, (1-\lambda)x + \lambda y)\right) = \text{CEA}_i\left(E, (c_{-i}, y)\right).$$

Consequently, Equation (6.4) is satisfied with equality.

Case II) In the second case, where $x < \mathrm{CEA}_i\left(E,(c_{-i},y)\right) \leq (1-\lambda)x + \lambda y$, we have that

$$\begin{cases} \mathrm{CEA}_i\left(E,(c_{-i},x)\right) = x; \\ \mathrm{CEA}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) = \mathrm{CEA}_i\left(E,(c_{-i},y)\right). \end{cases}$$

Hence,

$$\begin{aligned} \mathrm{CEA}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) &= (1-\lambda)\mathrm{CEA}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) \\ &\qquad + \lambda\mathrm{CEA}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) \\ &= (1-\lambda)\mathrm{CEA}_i\left(E,(c_{-i},y)\right) + \lambda\mathrm{CEA}_i\left(E,(c_{-i},y)\right) \\ &> (1-\lambda)x + \lambda\mathrm{CEA}_i\left(E,(c_{-i},y)\right) \\ &= (1-\lambda)\mathrm{CEA}_i\left(E,(c_{-i},x)\right) + \lambda\mathrm{CEA}_i\left(E,(c_{-i},y)\right), \end{aligned}$$

proving Equation (6.4).

Case III) Finally, for the third case, where

$$(1-\lambda)x + \lambda y < \mathrm{CEA}_i\left(E,(c_{-i},y)\right) \leq y,$$

we have that

$$\begin{cases} \mathrm{CEA}_i\left(E,(c_{-i},x)\right) = x; \\ \mathrm{CEA}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) = (1-\lambda)x+\lambda y. \end{cases}$$

Hence,

$$\begin{aligned} \mathrm{CEA}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) &= (1-\lambda)x+\lambda y \\ &\geq (1-\lambda)\mathrm{CEA}_i\left(E,(c_{-i},x)\right) + \lambda\mathrm{CEA}_i\left(E,(c_{-i},y)\right), \end{aligned}$$

where the inequality follows from claims boundedness. This proves Equation (6.4).

Consequently, the constrained equal awards rule satisfies partial concavity on $\mathcal{C}^N$.

Secondly, we show that the proportional rule satisfies partial concavity. Let $(E,c) \in \mathcal{C}^N$ and $i \in N$. As before, to show that $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_i(x) = \mathrm{PROP}_i\left(E,(c_{-i},x)\right)$ for all $x \in \mathbb{R}_+$ is concave, we show that

$$\mathrm{PROP}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) \geq (1-\lambda)\mathrm{PROP}_i\left(E,(c_{-i},x)\right) + \lambda\mathrm{PROP}_i\left(E,(c_{-i},y)\right), \tag{6.5}$$

for all $x, y \in \mathbb{R}_+$ and all $\lambda \in [0,1]$. So, let $x, y \in \mathbb{R}_+$ and $\lambda \in [0,1]$ and again assume w.l.o.g. that $x \leq y$, so that $x \leq (1-\lambda)x + \lambda y \leq y$. Here, we distinguish between four cases:

I) $E \leq x + \sum_{j \in N\setminus\{i\}} c_j$;

II) $x + \sum_{j\in N\setminus\{i\}} c_j < E \leq (1-\lambda)x + \lambda y + \sum_{j\in N\setminus\{i\}} c_j$;

III) $(1-\lambda)x + \lambda y + \sum_{j\in N\setminus\{i\}} c_j < E \leq y + \sum_{j\in N\setminus\{i\}} c_j$;

IV) $y + \sum_{j\in N\setminus\{i\}} c_j < E$.

Before dealing with the four cases separately, define $g : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$g(z) = \frac{zE}{z + \sum_{j\in N\setminus\{i\}} c_j},$$

for all $z \in \mathbb{R}_+$. Then we have that g is concave, since

$$g''(z) = \frac{-2E\sum_{j\in N\setminus\{i\}} c_j}{(z + \sum_{j\in N\setminus\{i\}} c_j)^3} \leq 0,$$

for all $z \in \mathbb{R}_+$. In particular, we have that

$$g((1-\lambda)x + \lambda y) \geq (1-\lambda)g(x) + \lambda g(y). \tag{6.6}$$

Case I) In the first case, where $E \leq x + \sum_{j\in N\setminus\{i\}} c_j$, we immediately have that

$$\begin{cases} \mathrm{PROP}_i\left(E, (c_{-i}, x)\right) = g(x); \\ \mathrm{PROP}_i\left(E, (c_{-i}, (1-\lambda)x + \lambda y)\right) = g((1-\lambda)x + \lambda y); \\ \mathrm{PROP}_i\left(E, (c_{-i}, y)\right) = g(y). \end{cases}$$

Hence,

$$\begin{aligned} \mathrm{PROP}_i\left(E, (c_{-i}, (1-\lambda)x + \lambda y)\right) &= g((1-\lambda)x + \lambda y) \\ &\geq (1-\lambda)g(x) + \lambda g(y) \\ &= (1-\lambda)\mathrm{PROP}_i\left(E, (c_{-i}, x)\right) + \lambda\mathrm{PROP}_i\left(E, (c_{-i}, y)\right), \end{aligned}$$

where the inequality follows from Equation (6.6). This shows that Equation (6.5) is satisfied.

Case II) In the second case, we assume that

$$x + \sum_{j\in N\setminus\{i\}} c_j < E \leq (1-\lambda)x + \lambda y + \sum_{j\in N\setminus\{i\}} c_j.$$

Then we have that

$$\begin{cases} \mathrm{PROP}_i\left(E, (c_{-i}, x)\right) = x; \\ \mathrm{PROP}_i\left(E, (c_{-i}, (1-\lambda)x + \lambda y)\right) = g((1-\lambda)x + \lambda y); \\ \mathrm{PROP}_i\left(E, (c_{-i}, y)\right) = g(y). \end{cases}$$

Hence,

$$\begin{aligned}
\mathrm{PROP}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) &= g((1-\lambda)x+\lambda y)\\
&\geq (1-\lambda)g(x)+\lambda g(y)\\
&= (1-\lambda)\frac{xE}{x+\sum_{j\in N\setminus\{i\}} c_j}+\lambda g(y)\\
&> (1-\lambda)x+\lambda g(y)\\
&= (1-\lambda)\mathrm{PROP}_i\left(E,(c_{-i},x)\right)+\lambda \mathrm{PROP}_i\left(E,(c_{-i},y)\right),
\end{aligned}$$

proving Equation (6.5).

Case III) In the third case, we assume that

$$(1-\lambda)x+\lambda y+\sum_{j\in N\setminus\{i\}} c_j < E \leq y+\sum_{j\in N\setminus\{i\}} c_j.$$

Then we have that

$$\begin{cases}
\mathrm{PROP}_i\left(E,(c_{-i},x)\right)=x;\\
\mathrm{PROP}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right)=(1-\lambda)x+\lambda y;\\
\mathrm{PROP}_i\left(E,(c_{-i},y)\right)=g(y).
\end{cases}$$

Hence,

$$\begin{aligned}
\mathrm{PROP}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) &= (1-\lambda)x+\lambda y\\
&\geq (1-\lambda)x+\lambda\frac{yE}{y+\sum_{j\in N\setminus\{i\}} c_j}\\
&= (1-\lambda)x+\lambda g(y)\\
&= (1-\lambda)\mathrm{PROP}_i\left(E,(c_{-i},x)\right)+\lambda \mathrm{PROP}_i\left(E,(c_{-i},y)\right),
\end{aligned}$$

proving Equation (6.5).

Case IV) Finally, in the fourth case, where $y+\sum_{j\in N\setminus\{i\}} c_j < E$, we have that

$$\begin{cases}
\mathrm{PROP}_i\left(E,(c_{-i},x)\right)=x;\\
\mathrm{PROP}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right)=(1-\lambda)x+\lambda y;\\
\mathrm{PROP}_i\left(E,(c_{-i},y)\right)=y.
\end{cases}$$

Hence,

$$\begin{aligned}
\mathrm{PROP}_i\left(E,(c_{-i},(1-\lambda)x+\lambda y)\right) &= (1-\lambda)x+\lambda y\\
&= (1-\lambda)\mathrm{PROP}_i\left(E,(c_{-i},x)\right)+\lambda \mathrm{PROP}_i\left(E,(c_{-i},y)\right),
\end{aligned}$$

proving Equation (6.5).

Together, this shows that the proportional rule satisfies partial concavity on $\mathcal{C}^N$.

For the third and final statement, we show that the concede and divide rule satisfies partial concavity. Let $(E, c) \in \mathcal{C}^N$ with $|N| = 2$, $i \in N$ and $j \in N \setminus \{i\}$. To show that $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_i(x) = \mathrm{CD}_i\left(E, (c_j, x)\right)$ for all $x \in \mathbb{R}_+$ is concave, we provide an explicit expression for f_i and sketch its graph. For this, we distinguish between two cases:

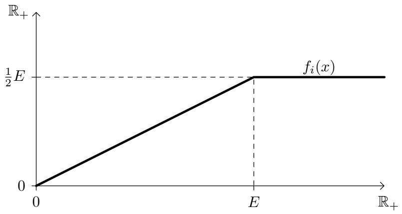
I) $c_j \geq E$;

II) $c_j < E$.

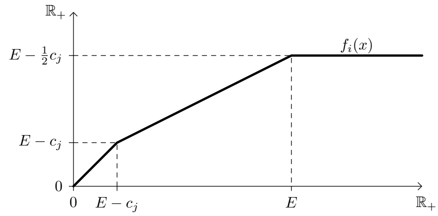
Case I) For the first case, we assume that $c_j \geq E$. Then it readily follows that, for all $x \in \mathbb{R}_+$,

$$f_i(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \leq E; \\ \frac{1}{2}E, & \text{if } x > E, \end{cases}$$

since player j concedes nothing in this case. This is visualized in Figure 6.5a. Clearly, f_i is concave.



(a) *Case I.*



(b) *Case II.*

Figure 6.5 – *The function values $f_i(x)$ for the concede and divide rule in the proof of Proposition 6.1.*

Case II) For the second case, we assume that $c_j < E$. To provide the explicit expression for f_i, we first remark that

$$f_i(x) = x, \tag{6.7}$$

for all $x \leq E - c_j$, because player j concedes enough to player i. Secondly, for all $E - c_j < x \leq E$, we see that

$$\begin{aligned} f_i(x) &= \max\{E - c_j, 0\} + \frac{E - \max\{E - x, 0\} - \max\{E - c_j, 0\}}{2} \\ &= E - c_j + \frac{E - (E - x) - (E - c_j)}{2} \end{aligned}$$

$$= \tfrac{1}{2}\left(E + x - c_j\right). \tag{6.8}$$

Finally, for all $x > E$, we see that

$$\begin{aligned} f_i(x) &= \max\{E - c_j, 0\} + \frac{E - \max\{E - x, 0\} - \max\{E - c_j, 0\}}{2} \\ &= E - c_j + \frac{E - (E - c_j)}{2} \\ &= E - \tfrac{1}{2}c_j. \end{aligned} \tag{6.9}$$

By combining Equations (6.7), (6.8) and (6.9), we thus have that, for all $x \in \mathbb{R}_+$,

$$f_i(x) = \begin{cases} x, & \text{if } x \leq E - c_j; \\ \tfrac{1}{2}\left(E + x - c_j\right), & \text{if } E - c_j < x \leq E; \\ E - \tfrac{1}{2}c_j, & \text{if } x > E. \end{cases}$$

This is visualized in Figure 6.5b. Clearly, $f_i(x)$ is concave.

Consequently, the concede and divide rule satisfies partial concavity on $\mathcal{C}^N$ with $|N| = 2$. □

The following two examples show that, in general, the constrained equal losses rule and the reverse Talmud rule do not satisfy partial concavity.

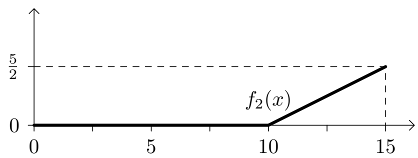


Figure 6.6 – *The function values $f_2(x)$ for the constrained equal losses rule of Example 6.2.*

Example 6.2 Consider the claims problem $(E, c) \in \mathcal{C}^N$ with $N = \{1, 2\}$, $E = 10$ and $c = (20, 15)$. We show that the constrained equal losses rule does not satisfy partial concavity by showing that $f_2(x) = \mathrm{CEL}_2\left(10, (20, x)\right)$ is not concave on $[0, 15]$.

First, note that $f_2(x) = 0$ for all $0 \leq x \leq 10$, due to the nature of the constrained equal losses rule. Secondly, for $10 < x \leq 15$, it can be readily verified that

$$f_2(x) = \mathrm{CEL}_2\left(10, (20, x)\right) = \tfrac{1}{2}x - 5.$$

Together, we have that, for all $x \in [0, 15]$,

$$f_2(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 10; \\ \tfrac{1}{2}x - 5, & \text{if } 10 < x \leq 15, \end{cases}$$

which is visualized in Figure 6.6. Clearly, f_2 is not concave. △

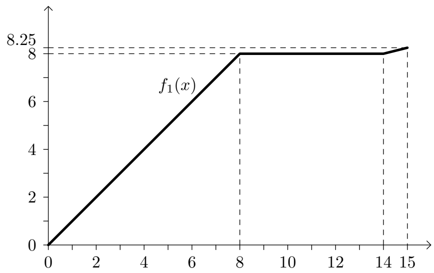


Figure 6.7 – *The function values $f_1(x)$ for the reverse Talmud rule of Example 6.3.*

Example 6.3 Consider the claims problem $(E, c) \in \mathcal{C}^N$ with $N = \{1, 2\}$, $E = 10$ and $c = (15, 2)$. We show that the reverse Talmud rule does not satisfy partial concavity by showing that $f_1(x) = \mathrm{RTAL}_1\,(10, (x, 2))$ is not concave on $[0, 15]$.

Obviously, for $0 \leq x \leq 8$, it holds that $f_1(x) = x$. Furthermore, due to the nature of the reverse Talmud rule, it can be readily verified that

$$f_1(x) = \mathrm{RTAL}_1\,(10, (x, 2))) = 8,$$

for $8 < x \leq 14$ and

$$f_1(x) = \mathrm{RTAL}_1\,(10, (x, 2))) = \tfrac{1}{4}x + \tfrac{9}{2},$$

for $14 < x \leq 15$. Together, we have that, for all $x \in [0, 15]$,

$$f_1(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 8; \\ 8, & \text{if } 8 < x \leq 14; \\ \frac{1}{4}x - \frac{9}{2}, & \text{if } 14 < x \leq 15, \end{cases}$$

which is visualized in Figure 6.7. Clearly, f_1 is not concave. △

For claims problems with only two players, the Talmud rule boils down to the concede and divide rule, which does satisfy partial concavity. However, for claims problems with three (or more) players, partial concavity is no longer guaranteed, as the following example shows.

Example 6.4 Consider the claims problem $(E, c) \in \mathcal{C}^N$ with $N = \{1, 2, 3\}$, $E = 20$ and $c = (12, 10, 7)$. We show that the Talmud rule does not satisfy partial concavity by showing that $f_3(x) = \mathrm{TAL}_3\,(20, (12, 10, x))$ is not concave on $[0, 7]$.

First, note that, for all $x \in [0, 7]$, it holds that $\frac{1}{2}(12 + 10 + x) = 11 + \frac{1}{2}x < 20$. Consequently, for all $x \in [0, 7]$,

$$f_3(x) = \mathrm{TAL}_3\,(20, (12, 10, x)) = x - \mathrm{CEA}_3\left(2 + x, (6, 5, \tfrac{1}{2}x)\right).$$

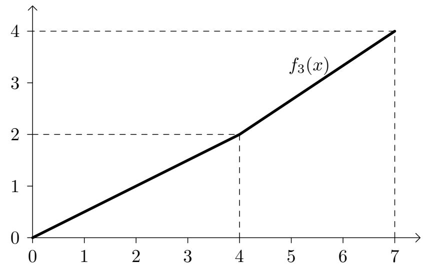


Figure 6.8 – *The function values $f_3(x)$ for the Talmud rule of Example 6.4.*

Focusing on the latter term, we see that, for all $0 \leq x \leq 4$, it holds that $\frac{1}{2}x \leq \frac{1}{3}(2+x)$ and hence, player 3 is awarded his full claim. Consequently, for all $0 \leq x \leq 4$,

$$f_3(x) = x - \text{CEA}_3\left(2 + x, (6, 5, \tfrac{1}{2}x)\right) = x - \tfrac{1}{2}x = \tfrac{1}{2}x.$$

Furthermore, we see that, for all $4 \leq x \leq 7$, it holds that $\frac{1}{3}(2+x) \leq \frac{1}{2}x \leq 5$. Consequently, for all $4 \leq x \leq 7$,

$$f_3(x) = x - \text{CEA}_3\left(2 + x, (6, 5, \tfrac{1}{2}x)\right) = x - \tfrac{1}{3}(2 + x) = \tfrac{2}{3}x - \tfrac{2}{3}.$$

Together, we have that, for all $x \in [0, 7]$,

$$f_3(x) = \begin{cases} \frac{1}{2}x, & \text{if } 0 \leq x \leq 4; \\ \frac{2}{3}x - \frac{2}{3}, & \text{if } 4 < x \leq 7, \end{cases}$$

which is visualized in Figure 6.8. Due to the change in slope at $x = 4$ it can be seen that f_3 is not concave. $\triangle$

To conclude this section, we return to claims problems with estate holders. Theorem 6.1 guarantees the existence of Nash equilibria in case all underlying claims rules satisfy partial concavity, that is, it shows that partial concavity is a sufficient condition. In the three examples above, we showed that, in general, the constrained equal losses rule, the reverse Talmud rule and the Talmud rule do not satisfy partial concavity. This leaves the question whether Nash equilibria also always exist for ceh-problems with underlying claims rules that do not satisfy partial concavity. In Example 6.5 below, we show that the answer to this question is, in general, negative. In the example, we provide a claims problem with two estate holders, two players and the constrained equal losses rule as underlying claims rule for both estates and show that for this ceh-problem, there are no Nash equilibria in the associated ceh-game. In fact, this ceh-problem stems from the claims problem as described in Example 6.2 by separating the estate in two equal parts.

Similarly, by separating the estate of the claims problem as described in Example 6.3, one can construct a ceh-problem with two estate holders, two players and the reverse Talmud rule as underlying claims rule for both estates and show that for this ceh-problem, there are no Nash equilibria in the associated ceh-game. For the Talmud rule however, a ceh-problem with at least three players is needed and in that case, constructing such a ceh-problem with no Nash equilibria in the associated ceh-game is more complex.

For Example 6.5 below, we benefited from the path of awards for the constrained equal losses rule as depicted in Figure 2.2b in Section 2.2.

Example 6.5 Consider the ceh-problem $(\{E^k\}_{k\in M}, \{\varphi^k\}_{k\in M}, c) \in \mathcal{C}^{M,N}$ with $M = \{A, B\}$, $N = \{1, 2\}$, $E^A = E^B = 5$, $\varphi^A = \varphi^B = \text{CEL}$ and $c = (20, 15)$. Moreover, consider its associated strategic ceh-game $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$.

To show that $NE(G) = \emptyset$, we determine, for all $(x_1, x_2) \in X_1 \times X_2$, both $BR_1(x_2)$ and $BR_2(x_1)$.

With regard to set of best reply strategies for player 1, let $x_2 \in X_2$. We show that

$$BR_1(x_2) = \left\{x_1 \in X_1 \;\middle|\; 5 \leq x_1^A + x_2^A \leq 30 \text{ and } x_2^A \leq x_1^A \leq x_2^A + 5\right\}, \qquad (6.10)$$

which is visualized in Figure 6.9.

We start out by showing that $x_1 \in BR_1(x_2)$ implies that $5 \leq x_1^A + x_2^A \leq 30$, which is equivalent to

$$\begin{cases} x_1^A + x_2^A \geq 5; \\ x_1^B + x_2^B \geq 5. \end{cases}$$

Let $x_1 \in BR_1(x_2)$. For the sake of contradiction, we distinguish between two cases: either I) $x_1^A + x_2^A < 5$ or II) $x_1^B + x_2^B < 5$.

Case I) First, we assume that $x_1^A + x_2^A < 5$, that is, the strategy combination (x_1, x_2) lies inside the bottom left light gray area of Figure 6.9. Then it holds that

$$\text{CEL}_1\left(E^A, (x_1^A, x_2^A)\right) = x_1^A,$$

for the estate E^A and similarly, for the estate E^B, it holds that, by using that $x_1^A + x_1^B = c_1 = 20$ and $x_2^A + x_2^B = c_2 = 15$ (see also Figure 6.9),

$$\text{CEL}_1\left(E^B, (x_1^B, x_2^B)\right) = \begin{cases} 5, & \text{if } x_1^A \leq x_2^A; \\ 5 - \frac{1}{2}(x_1^A - x_2^A), & \text{if } x_1^A > x_2^A. \end{cases}$$

Consequently,

$$\pi_1(x_1, x_2) = \begin{cases} x_1^A + 5, & \text{if } x_1^A \leq x_2^A; \\ 5 + \frac{1}{2}(x_1^A + x_2^A), & \text{if } x_1^A > x_2^A. \end{cases}$$

To show that we now have that $x_1 \notin BR_1(x_2)$, define $\bar{x}_1 \in X_1$ as follows:

$$\begin{cases} \bar{x}_1^A = 5 - x_2^A; \\ \bar{x}_1^B = 15 + x_2^A. \end{cases}$$

Then it readily follows that

$$\mathrm{CEL}_1\left(E^A, (\bar{x}_1^A, x_2^A)\right) = \bar{x}_1^A,$$

and

$$\mathrm{CEL}_1\left(E^B, (\bar{x}_1^B, x_2^B)\right) = \begin{cases} \frac{5}{2} + x_2^A, & \text{if } x_2^A \leq \frac{5}{2}; \\ 5, & \text{if } x_2^A > \frac{5}{2}. \end{cases}$$

Hence, it follows that $\pi_1(\bar{x}_1, x_2) > \pi_1(x_1, x_2)$. Indeed, if $x_2^A \leq \frac{5}{2}$ and $x_1^A < x_2^A$, then $x_1^A < \frac{5}{2}$ and consequently,

$$\pi_1(\bar{x}_1, x_2) = \bar{x}_1^A + \tfrac{5}{2} + x_2^A = 5 - x_2^A + \tfrac{5}{2} + x_2^A = \tfrac{15}{2} > x_1^A + 5 = \pi_1(x_1, x_2).$$

Furthermore, if $x_2^A \leq \frac{5}{2}$ and $x_1^A > x_2^A$, and since $x_1^A + x_2^A < 5$, we have that

$$\pi_1(\bar{x}_1, x_2) = \tfrac{15}{2} > 5 + \tfrac{1}{2}(x_1^A + x_2^A) = \pi_1(x_1, x_2).$$

If $x_2^A \leq \frac{5}{2}$ and $x_1^A = x_2^A$, then $x_1^A < \frac{5}{2}$ and $x_2^A < \frac{5}{2}$ due to the fact that $x_1^A + x_2^A < 5$ and consequently,

$$\pi_1(\bar{x}_1, x_2) = \bar{x}_1^A + \tfrac{5}{2} + x_2^A = 5 - x_2^A + \tfrac{5}{2} + x_2^A = \tfrac{15}{2} > x_1^A + 5 = \pi_1(x_1, x_2).$$

Finally, if $x_2^A > \frac{5}{2}$, then automatically $x_1^A < x_2^A$ due to the fact that $x_1^A + x_2^A < 5$ and consequently,

$$\pi_1(\bar{x}_1, x_2) = \bar{x}_1^A + 5 = 10 - x_2^A > x_1^A + 5 = \pi_1(x_1, x_2).$$

Ultimately, this proves that $x_1 \notin BR_1(x_2)$ as $\bar{x}_1 \in X_1$ always yields a strictly higher pay-off. Consequently, $x_1^A + x_2^A \geq 5$.

Case II) Secondly, we assume that $x_1^B + x_2^B < 5$, that is, the strategy combination (x_1, x_2) lies inside the top right light gray area of Figure 6.9. Following a similar (symmetric in fact) line of reasoning as above, we also reach the conclusion that $x_1 \notin BR_1(x_2)$. Consequently, $x_1^B + x_2^B \geq 5$ and thus, $x_1 \in BR_1(x_2)$ implies that

$$\begin{cases} x_1^A + x_2^A \geq 5; \\ x_1^B + x_2^B \geq 5. \end{cases}$$

What is left to prove is the fact that $x_1 \in BR_1(x_2)$ implies that $x_2^A \leq x_1^A \leq x_2^A + 5$. For this, let $x_1 \in BR_1(x_2)$ and note that we can assume that $5 \leq x_1^A + x_2^A \leq 30$. Then it is readily verified that

$$\mathrm{CEL}_1\left(E^A,(x_1^A,x_2^A)\right) = \begin{cases} \max\left\{0, \frac{5}{2} - \frac{1}{2}(x_2^A - x_1^A)\right\}, & \text{if } x_1^A < x_2^A; \\ \frac{5}{2} + \frac{1}{2}(x_1^A - x_2^A), & \text{if } x_2^A \leq x_1^A \leq x_2^A + 5; \\ 5, & \text{if } x_1^A > x_2^A + 5, \end{cases}$$

and, by using that $x_1^A + x_1^B = 20$ and $x_2^A + x_2^B = 15$,

$$\mathrm{CEL}_1\left(E^B,(x_1^B,x_2^B)\right) = \begin{cases} 5, & \text{if } x_1^A < x_2^A; \\ 5 - \frac{1}{2}(x_1^A - x_2^A), & \text{if } x_2^A \leq x_1^A \leq x_2^A + 5; \\ \max\left\{0, 5 - \frac{1}{2}(x_1^A - x_2^A)\right\}, & \text{if } x_1^A > x_2^A + 5. \end{cases}$$

Hence,

$$\pi_1(x_1,x_2) = \begin{cases} \max\left\{0, \frac{5}{2} - \frac{1}{2}(x_2^A - x_1^A)\right\} + 5, & \text{if } x_1^A < x_2^A; \\ \frac{15}{2}, & \text{if } x_2^A \leq x_1^A \leq x_2^A + 5; \\ 5 + \max\left\{0, 5 - \frac{1}{2}(x_1^A - x_2^A)\right\}, & \text{if } x_1^A > x_2^A + 5. \end{cases}$$

To show that for $x_1 \in X_1$ with $x_1^A < x_2^A$ it holds that $x_1 \notin BR_1(x_2)$, assume that $x_1^A < x_2^A$ and define $\bar{x}_1 \in X_1$ as follows:

$$\begin{cases} \bar{x}_1^A = x_2^A; \\ \bar{x}_1^B = 20 - x_2^A. \end{cases}$$

Then it readily follows that

$$\pi_1(\bar{x}_1,x_2) = \frac{15}{2} > \max\left\{0, \frac{5}{2} - \frac{1}{2}(x_2^A - x_1^A)\right\} + 5 = \pi_1(x_1,x_2), \tag{6.11}$$

since $\frac{5}{2} - \frac{1}{2}(x_2^A - x_1^A) < \frac{5}{2}$. Consequently, $x_1 \notin BR_1(x_2)$.

Finally, to show that for $x_1 \in X_1$ with $x_1^A > x_2^A + 5$ it holds that $x_1 \notin BR_1(x_2)$, assume that $x_1^A > x_2^A + 5$ and define $\bar{x}_1 \in X_1$ as follows:

$$\begin{cases} \bar{x}_1^A = x_2^A + 5; \\ \bar{x}_1^B = 15 - x_2^A. \end{cases}$$

Then it readily follows that

$$\pi_1(\bar{x}_1,x_2) = \frac{15}{2} > 5 + \max\left\{0, 5 - \frac{1}{2}(x_1^A - x_2^A)\right\} = \pi_1(x_1,x_2), \tag{6.12}$$

since $5 - \frac{1}{2}(x_1^A - x_2^A) < \frac{5}{2}$. Consequently, $x_1 \notin BR_1(x_2)$.

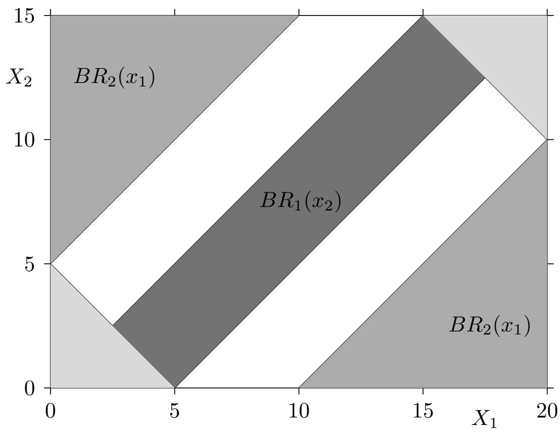


Figure 6.9 – *The sets of best reply strategies for both players of the strategic ceh-game of Example 6.5.*

Together, this shows that $x_2^A \leq x_1^A \leq x_2^A + 5$.

To conclude the proof that Equation (6.10) indeed describes the set of best reply strategies for player 1 against x_2, note that the above analysis proves one inclusion. The reverse inclusion immediately follows from the observation that the pay-off for player 1 within this set is equal to $\frac{15}{2}$ and thus independent of x_2. Together with the inequalities of Equations (6.11) and (6.12), this shows that playing $x_1 \in X_1$ with $5 \leq x_1^A + x_2^A \leq 30$ and $x_2^A \leq x_1^A \leq x_2^A + 5$ is indeed a best reply strategy.

Next, with regard to player 2, it can be verified along similar lines that

$$BR_2(x_1) = \begin{cases} \left\{x_2 \in X_2 \mid x_2^A \geq 5 + x_1^A\right\}, & \text{if } 0 \leq x_1^A < 10; \\ \left\{x_2 \in X_2 \mid x_2^A = 15 \text{ or } x_2^A = 0\right\}, & \text{if } x_1^A = 10; \\ \left\{x_2 \in X_2 \mid x_2^A \leq x_1^A - 10\right\}, & \text{if } 10 < x_1^A \leq 20, \end{cases} \tag{6.13}$$

for all $x_1 \in X_1$, as visualized in Figure 6.9.

Then it follows, as also can be seen from Figure 6.9, that there is no strategy combination $x = (x_1, x_2) \in X$ for which $x_1 \in BR_1(x_2)$ and $x_2 \in BR_2(x_1)$. This immediately implies that $NE(G) = \emptyset$. $\triangle$

6.4 Efficiency of Nash equilibria in ceh-games

This section studies efficiency of Nash equilibria in ceh-games associated to ceh-problems, which means that, in a Nash equilibrium, all estates are completely divided among the players. All Nash equilibrium pay-off vectors of Example 6.1 turned out to be efficient. In fact, Theorem 6.2 shows that efficiency is satisfied if a weak condition on the underlying claims rules is satisfied. This condition is called strict marginality and is formalized in Definition 6.4.

Definition 6.4 Let φ be a claims rule. Then φ satisfies *strict marginality* on $\mathcal{C}^N$ if, for all $(E, c) \in \mathcal{C}^N$ with $\sum_{i \in N} c_i > E$, it holds that there exists a player $j \in N$ and $\varepsilon > 0$ such that

i) $c_j - \varepsilon \geq 0$;

ii) $\sum_{i \in N} c_i - \varepsilon > E$;

iii) for all $0 < \delta \leq \varepsilon$: $\varphi_j(E, c) - \varphi_j(E, (c_{-j}, c_j - \delta)) < \delta$. $\triangleleft$

Strict marginality thus requires that there exists a player for which a small decrease in this player's claim leads to an even smaller decrease in the award of this player. Moreover, it is a weak condition in the sense that basically all common rules satisfy strict marginality. This includes the constrained equal awards rule, the constrained equal losses rule, the proportional rule, the Talmud rule, and the concede and divide rule, for which the proofs are straightforward.

Theorem 6.2 shows that strict marginality is a sufficient condition to guarantee efficiency for all Nash equilibria.

Theorem 6.2 *Let $(\{E^k\}_{k \in M}, \{\varphi^k\}_{k \in M}, c) \in \mathcal{C}^{M,N}$ be a claims problem with estate holders and let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be the associated strategic ceh-game. Moreover, let $\hat{x} \in NE(G)$ be a Nash equilibrium. Then the following two statements hold:*

i) if $\sum_{i \in N} c_i \leq E$, then
$$\pi(\hat{x}) = c;$$

ii) if $\sum_{i \in N} c_i > E$ and, for all $k \in M$, φ^k satisfies strict marginality, then
$$\sum_{i \in N} \pi_i(\hat{x}) = E.$$

Proof: *i)* For the first statement, assume that $\sum_{i \in N} c_i \leq E$. Let $i \in N$ and note that
$$\pi_i(\hat{x}_{-i}, \hat{x}_i) \geq \pi_i(\hat{x}_{-i}, x_i),$$

for all $x_i \in X_i$. The idea of the proof is to pinpoint a specific strategy $y_i \in X_i$ for player i for which $\pi_i(\hat{x}_{-i}, y_i) = c_i$. For this, choose $y_i \in X_i$ such that, for all $k \in M$,

$$0 \leq y_i^k \leq \max\{0, E^k - \sum_{j \in N \setminus \{i\}} \hat{x}_j^k\}. \tag{6.14}$$

Note that such a strategy exists, since, for all $k \in M$,

$$\begin{aligned}
\sum_{k \in M} \max\{0, E^k - \sum_{j \in N \setminus \{i\}} \hat{x}_j^k\} &\geq \sum_{k \in M} E^k - \sum_{k \in M} \sum_{j \in N \setminus \{i\}} \hat{x}_j^k \\
&= E - \sum_{j \in N \setminus \{i\}} \sum_{k \in M} \hat{x}_j^k \\
&= E - \sum_{j \in N \setminus \{i\}} c_j \\
&\geq c_i,
\end{aligned}$$

where the final inequality follows from the fact that $\sum_{i \in N} c_i \leq E$.
To show that $\pi_i(\hat{x}_{-i}, y_i) = c_i$, we first show that, for all $k \in M$,

$$\varphi_i^k(E^k, (\hat{x}_{-i}^k, y_i^k)) = y_i^k.$$

Let $k \in M$. We distinguish between two cases: either I) $E^k - \sum_{j \in N \setminus \{i\}} \hat{x}_j^k \leq 0$ or II) $E^k - \sum_{j \in N \setminus \{i\}} \hat{x}_j^k > 0$.
Case I) In the first case, assume that

$$E^k - \sum_{j \in N \setminus \{i\}} \hat{x}_j^k \leq 0.$$

Then it follows from Equation (6.14) that $y_i^k = 0$. Consequently, by combining non-negativity and claims boundedness,

$$\varphi_i^k(E^k, (\hat{x}_{-i}^k, y_i^k)) = 0 = y_i^k.$$

Case II) In the second case, assume that

$$E^k - \sum_{j \in N \setminus \{i\}} \hat{x}_j^k > 0.$$

Then it follows from Equation (6.14) that

$$y_i^k + \sum_{j \in N \setminus \{i\}} \hat{x}_j^k \leq E^k.$$

Consequently,

$$\varphi_i^k(E^k, (\hat{x}_{-i}^k, y_i^k)) = y_i^k.$$

We may conclude that $\varphi_i^k(E^k, (\hat{x}_{-i}^k, y_i^k)) = y_i^k$ for all $k \in M$. Hence,

$$\pi_i(\hat{x}_{-i}, y_i) = \sum_{k\in M} \varphi_i^k(E^k, (\hat{x}_{-i}^k, y_i^k)) = \sum_{k\in M} y_i^k = c_i.$$

Ultimately, we obtain that

$$\pi(\hat{x}) = \pi_i(\hat{x}_{-i}, \hat{x}_i) \geq \pi_i(\hat{x}_{-i}, y_i) = c_i.$$

Furthermore, due to claims boundedness for the Nash equilibrium pay-off (see Equation (6.1)), we also have that

$$\pi_i(\hat{x}) \leq c_i.$$

Consequently, $\pi_i(\hat{x}) = c_i$.

ii) For the second statement, assume that $\sum_{i\in N} c_i > E$ and that, for all $k \in M$, φ^k satisfies strict marginality. To show that $\sum_{i\in N} \pi_i(\hat{x}) = E$, we first show that, for all $k \in M$,

$$\sum_{i\in N} \hat{x}_i^k \geq E^k. \tag{6.15}$$

Suppose for the sake of contradiction that there exists an $\ell \in M$ for which $\sum_{i\in N} \hat{x}_i^\ell < E^\ell$. Furthermore, since $\sum_{i\in N} c_i > E$ in this case, there also exists a $h \in M$ for which $\sum_{i\in N} \hat{x}_i^h > E^h$.

Since φ^h satisfies strict marginality, we know that there exists a player $j \in N$ and $\varepsilon > 0$ such that $\hat{x}_j^h - \varepsilon \geq 0$, $\sum_{i\in N} \hat{x}_i^h - \varepsilon > E^h$ and, for all $0 < \delta \leq \varepsilon$,

$$\varphi_j^h(E^h, (\hat{x}_{-j}^h, \hat{x}_j^h)) - \varphi_j^h(E^h, (\hat{x}_{-j}^h, \hat{x}_j^h - \delta)) < \delta. \tag{6.16}$$

Let $\delta > 0$ be such that $\delta \leq \varepsilon$ and $\sum_{i\in N} \hat{x}_i^\ell + \delta < E^\ell$. Define $x_j \in X_j$ for all $k \in M$ as follows:

$$x_j^k = \begin{cases} \hat{x}_j^\ell + \delta, & \text{if } k = \ell; \\ \hat{x}_j^h - \delta, & \text{if } k = h; \\ \hat{x}_j^k, & \text{otherwise.} \end{cases}$$

Consequently,[2]

$$\begin{aligned}\pi_j(\hat{x}) - \pi_j(\hat{x}_{-j}, x_j) &= \sum_{k\in M} \varphi_j^k(E^k, (\hat{x}_i^k)_{i\in N}) - \sum_{k\in M} \varphi_j^k(E^k, (\hat{x}_{-j}^k, x_j^k)) \\ &= \varphi_j^\ell(E^\ell, (\hat{x}_i^\ell)_{i\in N}) - \varphi_j^\ell(E^\ell, (\hat{x}_{-j}^\ell, x_j^\ell))\end{aligned}$$

[2]Here, we use the feature that the pay-off functions of the players are separable in the estates.

$$
\begin{aligned}
&+ \varphi_j^h(E^h, (\hat{x}_i^h)_{i\in N}) - \varphi_j^h(E^h, (\hat{x}_{-j}^h, x_j^h))\\
&= \hat{x}_j^\ell - x_j^\ell + \varphi_j^h(E^h, (\hat{x}_i^h)_{i\in N}) - \varphi_j^h(E^h, (\hat{x}_{-j}^h, x_j^h))\\
&= \hat{x}_j^\ell - (\hat{x}_j^\ell + \delta) + \varphi_j^h(E^h, (\hat{x}_i^h)_{i\in N}) - \varphi_j^h(E^h, (\hat{x}_{-j}^h, x_j^h))\\
&< -\delta + \delta = 0.
\end{aligned}
$$

Here, the second equality follows from the fact that

$$\varphi_j^k(E^k, (\hat{x}_i^k)_{i\in N}) = \varphi_j^k(E^k, (\hat{x}_{-j}^k, x_j^k)),$$

for all $k \in M$ with $k \neq \ell, h$. Moreover, for the third and fourth equality, we used that

$$
\begin{cases}
\varphi_j^\ell(E^\ell, (\hat{x}_i^\ell)_{i\in N}) = \hat{x}_j^\ell;\\
\varphi_j^\ell(E^\ell, (\hat{x}_{-j}^\ell, x_j^\ell)) = x_j^\ell = \hat{x}_j^\ell + \delta,
\end{cases}
$$

since both $\sum_{i\in N} \hat{x}_i^\ell < E^\ell$ and $\sum_{i\in N} \hat{x}_i^\ell + \delta < E^\ell$. Finally, the inequality is due to Equation (6.16).

Subsequently, we thus have that $\pi_j(\hat{x}) < \pi_j(\hat{x}_{-j}, x_j)$, contradicting the fact that $\hat{x}$ is a Nash equilibrium. Hence, Equation (6.15) is satisfied, i.e, for all $k \in M$,

$$\sum_{i\in N} \hat{x}_i^k \geq E^k.$$

This immediately implies that, for all $k \in M$,

$$\sum_{i\in N} \varphi_i^k(E^k, (\hat{x}_j^k)_{j\in N}) = E^k.$$

Consequently,

$$
\begin{aligned}
\sum_{i\in N} \pi_i(\hat{x}) &= \sum_{i\in N}\sum_{k\in M} \varphi_i^k(E^k, (\hat{x}_j^k)_{j\in N})\\
&= \sum_{k\in M}\sum_{i\in N} \varphi_i^k(E^k, (\hat{x}_j^k)_{j\in N})\\
&= \sum_{k\in M} E^k = E.
\end{aligned}
$$

□

The proof of Theorem 6.2 indicates that, besides efficiency for Nash equilibria, it also holds that, for each estate, the players claim in total at least the estate in every Nash equilibrium. In other words, a strategy combination for which there exists an estate that is (strictly) sufficient to fulfill all claims on this estate, that is, a strategy combination for which Equation (6.15) is not satisfied for all $k \in M$, is not a Nash equilibrium.

6.5 Uniform claims problems with estate holders

The previous section establishes efficiency for the Nash equilibrium pay-off under the condition of strict marginality for the underlying claims rules. Since claims rules also satisfy efficiency, this allows for a direct comparison between Nash equilibria pay-off vectors in a strategic ceh-game and the awards vectors of the claims rules underlying the associated ceh-problem. Recall that in Example 6.1, all Nash equilibria lead to the same pay-off vector, which is equal to the awards vector of the common underlying claims rule, the constrained equal awards rule.

In this section, we study a special type of ceh-problems: uniform ceh-problems. In a *uniform claims problem with estate holders*, all the underlying claims rules are identical. As a consequence, there is only one awards vector to compare with the Nash equilibria pay-off vectors. In particular, we deal with uniform ceh-problems with the proportional rule and the constrained equal awards rule as common claims rule.

6.5.1 Proportional rule

First, we focus on uniform ceh-problems with the proportional rule as underlying common claims rule. In Theorem 6.3, we show that each Nash equilibrium results in a pay-off vector which is equal to the awards vector of the proportional rule applied to the claims problem in which all estates are consolidated.

Theorem 6.3 *Let* $(\{E^k\}_{k\in M}, \{\varphi^k\}_{k\in M}, c) \in \mathcal{C}^{M,N}$ *be a uniform ceh-problem and let* $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ *be the associated strategic ceh-game. Moreover, let* $\hat{x} \in NE(G)$ *be a Nash equilibrium. If, for all* $k \in M$, $\varphi^k = PROP$*, then*

$$\pi(\hat{x}) = PROP(E, c)\,.$$

Proof: Assume that, for all $k \in M$, $\varphi^k = \mathrm{PROP}$. Note that, if $\sum_{i\in N} c_i \leq E$, then

$$\pi(\hat{x}) = c = \mathrm{PROP}\,(E, c)\,,$$

according to Theorem 6.2.

Therefore, for the remainder of the proof, assume that $\sum_{i\in N} c_i > E$. The idea of the proof is to pinpoint, for all $i \in N$, a strategy $y_i \in X_i$ for which it holds that

$$\pi_i(\hat{x}_{-i}, y_i) \geq \mathrm{PROP}_i\,(E, c)\,. \tag{6.17}$$

If we manage to do so, then it follows that

$$\pi_i(\hat{x}) = \pi_i(\hat{x}_{-i}, \hat{x}_i) \geq \pi_i(\hat{x}_{-i}, y_i) \geq \mathrm{PROP}_i\,(E, c)\,,$$

for all $i \in N$, where the inequality is due to the fact that $\hat{x} \in NE(G)$. Furthermore, since the proportional rule satisfies strict marginality, we have that

$$\sum_{i \in N} \pi_i(\hat{x}) = E = \sum_{i \in N} \mathrm{PROP}_i\left(E, c\right),$$

according to Theorem 6.2. Consequently, for all $i \in N$,

$$\pi_i(\hat{x}) = \mathrm{PROP}_i\left(E, c\right).$$

To pinpoint, for all $i \in N$, a strategy $y_i \in X_i$ for which Equation (6.17) holds, let $i \in N$. For notational convenience, denote for all $k \in M$,

$$\begin{cases} \Sigma \hat{x}^k_{-i} = \sum_{j \in N \setminus \{i\}} \hat{x}^k_j; \\ \Sigma c_{-i} = \sum_{j \in N \setminus \{i\}} c_j. \end{cases}$$

We distinguish between two cases:

I) $\Sigma \hat{x}^k_{-i} + \dfrac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i \geq E^k$ for all $k \in M$;

II) there exists $M' \subsetneq M, M' \neq \emptyset$ for which

$$\Sigma \hat{x}^k_{-i} + \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i < E^k,$$

for all $k \in M'$, while

$$\Sigma \hat{x}^\ell_{-i} + \frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i \geq E^\ell,$$

for all $\ell \in M \setminus M'$.

Case I) In this first case, we assume that

$$\Sigma \hat{x}^k_{-i} + \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i \geq E^k,$$

for all $k \in M$. Choose the strategy $y_i \in X_i$ with, for all $k \in M$,

$$y_i^k = \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i.$$

Note that indeed $y_i \in X_i$, since $y_i^k \geq 0$ for all $k \in M$ and

$$\sum_{k \in M} y_i^k = \sum_{k \in M} \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i = \frac{\sum_{k \in M} \sum_{j \in N \setminus \{i\}} \hat{x}^k_j}{\Sigma c_{-i}} c_i$$

$$= \frac{\sum_{j\in N\setminus\{i\}}\sum_{k\in M}\hat{x}_j^k}{\Sigma c_{-i}} c_i = \frac{\sum_{j\in N\setminus\{i\}} c_j}{\Sigma c_{-i}} c_i = \frac{\Sigma c_{-i}}{\Sigma c_{-i}} c_i = c_i.$$

Then it readily follows that

$$\begin{aligned}
\pi_i(\hat{x}_{-i}, y_i) &= \sum_{k\in M} \mathrm{PROP}_i\left(E^k, (\hat{x}_{-i}^k, y_i^k)\right) \\
&= \sum_{k\in M} \frac{y_i^k}{y_i^k + \Sigma\hat{x}_{-i}^k} E^k \\
&= \sum_{k\in M} \frac{\frac{\Sigma\hat{x}_{-i}^k}{\Sigma c_{-i}} c_i}{\frac{\Sigma\hat{x}_{-i}^k}{\Sigma c_{-i}} c_i + \Sigma\hat{x}_{-i}^k} E^k \\
&= \sum_{k\in M} \frac{c_i}{c_i + \Sigma c_{-i}} E^k \\
&= \frac{c_i}{\sum_{j\in N} c_j} \sum_{k\in M} E^k \\
&= \frac{c_i}{\sum_{j\in N} c_j} E \\
&= \mathrm{PROP}_i\left(E, c\right),
\end{aligned}$$

where the second equality follows from the fact that, for all $k \in M$, $\Sigma\hat{x}_{-i}^k + y_i^k \geq E^k$. Hence, Equation (6.17) is satisfied.

Case II) In the second case, we assume that there exists $M' \subsetneq M, M' \neq \emptyset$ for which

$$\Sigma\hat{x}_{-i}^k + \frac{\Sigma\hat{x}_{-i}^k}{\Sigma c_{-i}} c_i < E^k, \tag{6.18}$$

for all $k \in M'$, while

$$\Sigma\hat{x}_{-i}^\ell + \frac{\Sigma\hat{x}_{-i}^\ell}{\Sigma c_{-i}} c_i \geq E^\ell, \tag{6.19}$$

for all $\ell \in M \setminus M'$. In this case, we can assume that

$$\sum_{k\in M'} \left(E^k - \Sigma\hat{x}_{-i}^k\right) \leq c_i.$$

Indeed, if $\sum_{k\in M'}\left(E^k - \Sigma\hat{x}_{-i}^k\right) > c_i$, then player i is unable to make sure that every estate E^k for $k \in M'$ is fully divided among the players. In other words, there exists $k \in M'$ for which it holds that the estate E^k is large enough to fulfill all claims, i.e.,

$$\hat{x}_i^k + \Sigma\hat{x}_{-i}^k < E^k.$$

Consequently, E^k is (strictly) sufficient to fulfill all the claims, which, as was seen in the proof of Theorem 6.2, leads to a contradiction with the fact that $\hat{x}$ is Nash equilibrium and thus efficient.

Next, we can choose a strategy $y_i \in X_i$ for player i using the following idea: every estate E^k for $k \in M'$ is exactly equal to the sum of the respective claims on that estate. Furthermore, every estate E^ℓ for $\ell \in M \setminus M'$ if sufficient to cover the respective claims on that estate. More formally, choose a strategy $y_i \in X_i$ with, for all $k \in M'$,

$$y_i^k = E^k - \Sigma \hat{x}_{-i}^k,$$

and, for all $\ell \in M \setminus M'$,

$$y_i^\ell = \frac{\Sigma \hat{x}_{-i}^\ell}{\Sigma c_{-i}} c_i - \varepsilon^\ell,$$

where $\varepsilon^\ell \geq 0$ is such that

$$\begin{cases} \varepsilon^\ell \leq \dfrac{\Sigma \hat{x}_{-i}^\ell}{\Sigma c_{-i}} c_i + \Sigma \hat{x}_{-i}^\ell - E^\ell; & (6.20) \\[2ex] \varepsilon^\ell \leq \dfrac{\Sigma \hat{x}_{-i}^\ell}{\Sigma c_{-i}} c_i, & (6.21) \end{cases}$$

for all $\ell \in M \setminus M'$, and

$$\sum_{\ell \in M \setminus M'} \varepsilon^\ell = \sum_{k \in M'} \left(E^k - \Sigma \hat{x}_{-i}^k - \frac{\Sigma \hat{x}_{-i}^k}{\Sigma c_{-i}} c_i \right). \tag{6.22}$$

Note that ε^ℓ exists for all $\ell \in M \setminus M'$, basically due to the fact that $\sum_{i \in N} c_i > E$.

Equation (6.20) makes sure that $y_i^\ell + \Sigma \hat{x}_{-i}^\ell \geq E^\ell$ for all $\ell \in M \setminus M'$. Moreover, to see that indeed $y_i \in X_i$, note that Equation (6.21) guarantees that $y_i^\ell \geq 0$ for all $\ell \in M \setminus M'$. For all $k \in M'$, it follows from Equation (6.18) that $y_i^k \geq 0$. Furthermore, by using Equation (6.22),

$$\begin{aligned} \sum_{k \in M'} y_i^k + \sum_{\ell \in M \setminus M'} y_i^\ell &= \sum_{k \in M'} \left(E^k - \Sigma \hat{x}_{-i}^k \right) + \sum_{\ell \in M \setminus M'} \left(\frac{\Sigma \hat{x}_{-i}^\ell}{\Sigma c_{-i}} c_i - \varepsilon^\ell \right) \\ &= \sum_{k \in M'} \left(E^k - \Sigma \hat{x}_{-i}^k \right) + \sum_{\ell \in M \setminus M'} \left(\frac{\Sigma \hat{x}_{-i}^\ell}{\Sigma c_{-i}} c_i \right) - \sum_{\ell \in M \setminus M'} \varepsilon^\ell \\ &\overset{(6.22)}{=} \sum_{k \in M'} \left(E^k - \Sigma \hat{x}_{-i}^k \right) + \sum_{\ell \in M \setminus M'} \left(\frac{\Sigma \hat{x}_{-i}^\ell}{\Sigma c_{-i}} c_i \right) \\ &\quad - \sum_{k \in M'} \left(E^k - \Sigma \hat{x}_{-i}^k - \frac{\Sigma \hat{x}_{-i}^k}{\Sigma c_{-i}} c_i \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell \in M \setminus M'} \left(\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i \right) + \sum_{k \in M'} \left(\frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i \right) \\
&= \sum_{p \in M} \left(\frac{\Sigma \hat{x}^p_{-i}}{\Sigma c_{-i}} c_i \right) \\
&= \frac{c_i}{\Sigma c_{-i}} \sum_{p \in M} \Sigma \hat{x}^p_{-i} \\
&= \frac{c_i}{\Sigma c_{-i}} \Sigma c_{-i} \\
&= c_i.
\end{aligned}
$$

Together, we see that indeed $y_i \in X_i$.

To show that Equation (6.17) is satisfied in case II, we first show that, for all $\ell \in M \setminus M'$, it holds that

$$
\mathrm{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, y^\ell_i) \right) \geq \mathrm{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, \frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i) \right) - \frac{\Sigma c_{-i}}{\sum_{j \in N} c_j} \varepsilon^\ell. \tag{6.23}
$$

To prove Equation (6.23), let $\ell \in M \setminus M'$. Then it holds that, by using Equation (6.20),

$$
\begin{aligned}
&\mathrm{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, \frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i) \right) - \mathrm{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, y^\ell_i) \right) \\
&\quad = \mathrm{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, \frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i) \right) - \mathrm{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, \frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i - \varepsilon^\ell) \right) \\
&\quad \overset{(6.20)}{=} \frac{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i}{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i}} E^\ell - \frac{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i - \varepsilon^\ell}{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} - \varepsilon^\ell} E^\ell \\
&\quad = \frac{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i \left(\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} - \varepsilon^\ell \right) - \left(\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i - \varepsilon^\ell \right) \cdot \left(\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} \right)}{\left(\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} \right) \cdot \left(\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} - \varepsilon^\ell \right)} E^\ell \\
&\quad = \frac{E^\ell \varepsilon^\ell \Sigma \hat{x}^\ell_{-i}}{\left(\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} \right) \cdot \left(\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} - \varepsilon^\ell \right)} \\
&\quad = \frac{E^\ell}{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} - \varepsilon^\ell} \cdot \frac{\Sigma \hat{x}^\ell_{-i}}{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i}} \varepsilon^\ell \\
&\quad = \frac{E^\ell}{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} - \varepsilon^\ell} \cdot \frac{\Sigma c_{-i}}{c_i + \Sigma c_{-i}} \varepsilon^\ell
\end{aligned}
$$

$$= \frac{E^\ell}{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i} - \varepsilon^\ell} \cdot \frac{\Sigma c_{-i}}{\sum_{j \in N} c_j} \varepsilon^\ell$$
$$\overset{(6.20)}{\leq} \frac{\Sigma c_{-i}}{\sum_{j \in N} c_j} \varepsilon^\ell.$$

Rewriting then gives exactly Equation (6.23).

Subsequently, by using Equations (6.22) and (6.23), we can show that Equation (6.17) also holds in this case:

$$\begin{aligned}
\pi_i(\hat{x}_{-i}, y_i) &= \sum_{k \in M'} \text{PROP}_i \left(E^k, (\hat{x}^k_{-i}, y^k_i)\right) + \sum_{\ell \in M \setminus M'} \text{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, y^\ell_i)\right) \\
&= \sum_{k \in M'} y^k_i + \sum_{\ell \in M \setminus M'} \text{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, y^\ell_i)\right) \\
&= \sum_{k \in M'} \left(E^k - \Sigma \hat{x}^k_{-i}\right) + \sum_{\ell \in M \setminus M'} \text{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, y^\ell_i)\right) \\
&= \sum_{k \in M'} \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i + \sum_{k \in M'} \left(E^k - \Sigma \hat{x}^k_{-i} - \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i\right) + \sum_{\ell \in M \setminus M'} \text{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, y^\ell_i)\right) \\
&\overset{(6.22)}{=} \sum_{k \in M'} \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i + \sum_{\ell \in M \setminus M'} \varepsilon^\ell + \sum_{\ell \in M \setminus M'} \text{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, y^\ell_i)\right) \\
&\overset{(6.23)}{\geq} \sum_{k \in M'} \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i + \sum_{\ell \in M \setminus M'} \varepsilon^\ell \\
&\qquad + \sum_{\ell \in M \setminus M'} \text{PROP}_i \left(E^\ell, (\hat{x}^\ell_{-i}, \frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i)\right) - \sum_{\ell \in M \setminus M'} \frac{\Sigma c_{-i}}{\sum_{j \in N} c_j} \varepsilon^\ell \\
&= \sum_{k \in M'} \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i + \left(1 - \frac{\Sigma c_{-i}}{\sum_{j \in N} c_j}\right) \sum_{\ell \in M \setminus M'} \varepsilon^\ell + \sum_{\ell \in M \setminus M'} \frac{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i}{\frac{\Sigma \hat{x}^\ell_{-i}}{\Sigma c_{-i}} c_i + \Sigma \hat{x}^\ell_{-i}} E^\ell \\
&= \frac{c_i}{\sum_{j \in N} c_j} \sum_{k \in M'} \left(\frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} \sum_{j \in N} c_j\right) + \frac{c_i}{\sum_{j \in N} c_j} \sum_{\ell \in M \setminus M'} \varepsilon^\ell + \sum_{\ell \in M \setminus M'} \frac{c_i}{c_i + \Sigma c_{-i}} E^\ell \\
&\overset{(6.22)}{=} \frac{c_i}{\sum_{j \in N} c_j} \sum_{k \in M'} \left(\frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} \sum_{j \in N} c_j\right) + \frac{c_i}{\sum_{j \in N} c_j} \sum_{k \in M'} \left(E^k - \Sigma \hat{x}^k_{-i} - \frac{\Sigma \hat{x}^k_{-i}}{\Sigma c_{-i}} c_i\right) \\
&\qquad + \sum_{\ell \in M \setminus M'} \frac{c_i}{c_i + \Sigma c_{-i}} E^\ell
\end{aligned}$$

$$
\begin{aligned}
&= \frac{c_i}{\sum_{j\in N} c_j}\left(\sum_{k\in M'} E^k + \sum_{k\in M'}\left(\frac{\Sigma\hat{x}^k_{-i}}{\Sigma c_{-i}}\Sigma c_{-i} + \frac{\Sigma\hat{x}^k_{-i}}{\Sigma c_{-i}}c_i - \Sigma\hat{x}^k_{-i} - \frac{\Sigma\hat{x}^k_{-i}}{\Sigma c_{-i}}c_i\right)\right)\\
&\qquad + \frac{c_i}{\sum_{j\in N} c_j}\sum_{\ell\in M\setminus M'} E^\ell\\
&= \frac{c_i}{\sum_{j\in N} c_j}\sum_{k\in M'} E^k + \frac{c_i}{\sum_{j\in N} c_j}\sum_{\ell\in M\setminus M'} E^\ell\\
&= \frac{c_i}{\sum_{j\in N} c_j}E = \mathrm{PROP}_i\left(E, c\right).
\end{aligned}
$$

Consequently, we also have that Equation (6.17) is satisfied in case II.

This finishes both cases and hence, the proof. □

Interestingly, Theorem 6.3 can be used to determine the full set of Nash equilibria of the strategic ceh-game associated to a uniform claims problem with two estate holders, both using the proportional rule, and with two players. As we have seen in Examples 6.1 and 6.5, computing the set of Nash equilibria is, in general, not straightforward. Generally, this requires a tailor-made approach due to the fact that different combinations of claims rules result in a wide variety of different pay-off vectors.

The following theorem shows that, for a uniform claims problem with two estate holders, both using the proportional rule, and with two players, it holds that the set of Nash equilibria is a singleton and consists of the strategy combination in which the players proportionally divide their claim over the two estates. The intricate proof illustrates once more that computing the set of Nash equilibria is, in general, not straightforward. More importantly, it relies on the fact that the path of awards for the proportional rule, as depicted in Figure 2.2e in Section 2.2, is identified by a line segment. This contrasts the paths of awards for the other well-known claims rules, which all consist of multiple line segments.

Theorem 6.4 *Let* $(\{E^k\}_{k\in M}, \{\varphi^k\}_{k\in M}, c) \in \mathcal{C}^{M,N}$ *be a uniform ceh-problem and let* $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ *be the associated strategic ceh-game. If* $M = \{A, B\}$*,* $N = \{1, 2\}$*,* $\varphi^A = \varphi^B = PROP$ *and* $\sum_{i\in N} c_i > E$*, then*

$$
NE(G) = \left\{\left(\left(\frac{E^A}{E^A+E^B}c_1, \frac{E^B}{E^A+E^B}c_1\right), \left(\frac{E^A}{E^A+E^B}c_2, \frac{E^B}{E^A+E^B}c_2\right)\right)\right\}.
$$

Proof: Assume that $M = \{A, B\}$, $N = \{1, 2\}$, $\varphi^A = \varphi^B = \mathrm{PROP}$ and $\sum_{i\in N} c_i > E$. First note that, since the proportional rule satisfies partial concavity according to Proposition 6.1, it follows from Theorem 6.1 that $NE(G) \neq \emptyset$.

Let $\hat{x} \in NE(G)$ be a Nash equilibrium. Then, by using Theorem 6.2, it follows that

$\sum_{i \in N} \pi_i(\hat{x}) = E$ and more specifically, that

$$\begin{cases} \hat{x}_1^A + \hat{x}_2^A \geq E^A; \\ \hat{x}_1^B + \hat{x}_2^B \geq E^B. \end{cases} \tag{6.24}$$

This is indicated in Figure 6.10 by the gray areas.

Furthermore, by using Theorem 6.3, we have that

$$\pi(\hat{x}) = \text{PROP}\,(E, c)\,.$$

Elaborating on this, we first show that it implies that

$$\hat{x}_2^A = \frac{c_2}{c_1}\hat{x}_1^A \quad \text{ or } \quad \hat{x}_2^A = \frac{E^A}{E^A + E^B}(c_1 + c_2) - \hat{x}_1^A. \tag{6.25}$$

Focusing on player 1 first, we have that, by using Equation (6.24),

$$\begin{aligned}
\pi_1(\hat{x}) - \text{PROP}_1\,(E, c) &= \frac{\hat{x}_1^A}{\hat{x}_1^A + \hat{x}_2^A}E^A + \frac{\hat{x}_1^B}{\hat{x}_1^B + \hat{x}_2^B}E^B - \frac{c_1}{c_1 + c_2}(E^A + E^B) \\
&= \frac{\hat{x}_1^A}{\hat{x}_1^A + \hat{x}_2^A}E^A + \frac{c_1 - \hat{x}_1^A}{c_1 - \hat{x}_1^A + c_2 - \hat{x}_2^A}E^B - \frac{c_1}{c_1 + c_2}(E^A + E^B) \\
&= \frac{\hat{x}_1^A E^A (c_1 + c_2)(c_1 + c_2 - \hat{x}_1^A - \hat{x}_2^A)}{(c_1 + c_2)(\hat{x}_1^A + \hat{x}_2^A)(c_1 + c_2 - \hat{x}_1^A - \hat{x}_2^A)} \\
&\qquad + \frac{(c_1 - \hat{x}_1^A)E^B(c_1 + c_2)(\hat{x}_1^A + \hat{x}_2^A)}{(c_1 + c_2)(\hat{x}_1^A + \hat{x}_2^A)(c_1 + c_2 - \hat{x}_1^A - \hat{x}_2^A)} \\
&\qquad - \frac{c_1(E^A + E^B)(\hat{x}_1^A + \hat{x}_2^A)(c_1 + c_2 - \hat{x}_1^A - \hat{x}_2^A)}{(c_1 + c_2)(\hat{x}_1^A + \hat{x}_2^A)(c_1 + c_2 - \hat{x}_1^A - \hat{x}_2^A)} \\
&= \frac{(c_2\hat{x}_1^A - c_1\hat{x}_2^A)\Big((c_1 + c_2)E^A - (\hat{x}_1^A + \hat{x}_2^A)(E^A + E^B)\Big)}{(c_1 + c_2)(\hat{x}_1^A + \hat{x}_2^A)(c_1 + c_2 - \hat{x}_1^A - \hat{x}_2^A)}.
\end{aligned} \tag{6.26}$$

Here, the last equality follows by rewriting the nominator:

$$\begin{aligned}
&\hat{x}_1^A E^A (c_1 + c_2)(c_1 + c_2 - \hat{x}_1^A - \hat{x}_2^A) + (c_1 - \hat{x}_1^A)E^B(c_1 + c_2)(\hat{x}_1^A + \hat{x}_2^A) \\
&\quad - c_1(E^A + E^B)(\hat{x}_1^A + \hat{x}_2^A)(c_1 + c_2 - \hat{x}_1^A - \hat{x}_2^A) \\
&= c_1c_1\hat{x}_1^A E^A + c_1c_2\hat{x}_1^A E^A - c_1\hat{x}_1^A\hat{x}_1^A E^A + c_1c_2\hat{x}_1^A E^A + c_2c_2\hat{x}_1^A E^A - c_2\hat{x}_1^A\hat{x}_1^A E^A \\
&\qquad - \hat{x}_2^A\big(c_1\hat{x}_1^A E^A + c_2\hat{x}_1^A E^A\big) \\
&\quad + c_1c_1\hat{x}_1^A E^B - c_1\hat{x}_1^A\hat{x}_1^A E^B + c_1c_2\hat{x}_1^A E^B - c_2\hat{x}_1^A\hat{x}_1^A E^B \\
&\qquad + \hat{x}_2^A\big(c_1c_1E^B - c_1\hat{x}_1^A E^B + c_1c_2E^B - c_2\hat{x}_1^A E^B\big) \\
&\quad - c_1c_1\hat{x}_1^A E^A - c_1c_2\hat{x}_1^A E^A + c_1\hat{x}_1^A\hat{x}_1^A E^A - c_1c_1\hat{x}_1^A E^B - c_1c_2\hat{x}_1^A E^B + c_1\hat{x}_1^A\hat{x}_1^A E^B
\end{aligned}$$

$$
\begin{aligned}
&\quad + \hat{x}_2^A\big(c_1\hat{x}_1^A E^A + c_1\hat{x}_1^A E^B - c_1c_1E^A - c_1c_2E^A + c_1\hat{x}_1^A E^A - c_1c_1E^B - c_1c_2E^B + c_1\hat{x}_1^A E^B\big)\\
&\quad + \hat{x}_2^A\hat{x}_2^A(c_1E^A + c_1E^B)\\
&= c_1c_2\hat{x}_1^A E^A + c_2c_2\hat{x}_1^A E^A - c_2\hat{x}_1^A\hat{x}_1^A E^A - c_2\hat{x}_1^A\hat{x}_1^A E^B\\
&\quad + \hat{x}_2^A\big(c_1\hat{x}_1^A E^A + c_1\hat{x}_1^A E^B - c_1c_1E^A - c_1c_2E^A - c_2\hat{x}_1^A E^A - c_2\hat{x}_1^A E^B\big)\\
&\quad + \hat{x}_2^A\hat{x}_2^A\big(c_1E^A + c_1E^B\big)\\
&= (c_2\hat{x}_1^A - c_1\hat{x}_2^A)\big(c_1E^A + c_2E^A - \hat{x}_1^A E^A - \hat{x}_1^A E^B - \hat{x}_2^A E^A - \hat{x}_2^A E^B\big)\\
&= (c_2\hat{x}_1^A - c_1\hat{x}_2^A)\Big((c_1+c_2)E^A - (\hat{x}_1^A + \hat{x}_2^A)(E^A + E^B)\Big).
\end{aligned}
$$

Consequently, $\pi_1(\hat{x}) = \mathrm{PROP}_1\,(E, c)$ implies that

$$(c_2\hat{x}_1^A - c_1\hat{x}_2^A)\Big((c_1+c_2)E^A - (\hat{x}_1^A + \hat{x}_2^A)(E^A + E^B)\Big) = 0,$$

and hence,

$$c_2\hat{x}_1^A - c_1\hat{x}_2^A = 0 \quad \text{or} \quad (c_1+c_2)E^A - (\hat{x}_1^A + \hat{x}_2^A)(E^A + E^B) = 0,$$

or equivalently,

$$\hat{x}_2^A = \frac{c_2}{c_1}\hat{x}_1^A \quad \text{or} \quad \hat{x}_2^A = \frac{E^A}{E^A + E^B}(c_1 + c_2) - \hat{x}_1^A,$$

satisfying Equation (6.25).

Note that, since $\pi_2(\hat{x}) = E - \pi_1(\hat{x})$ and $\mathrm{PROP}_2\,(E, c) = E - \mathrm{PROP}_1\,(E, c)$ due to efficiency, $\pi_1(\hat{x}) = \mathrm{PROP}_1\,(E, c)$ and $\pi_2(\hat{x}) = \mathrm{PROP}_2\,(E, c)$ are equivalent.

Figure 6.10 provides an overview of the set of strategy combinations, including the ones for which Equation (6.25) holds. For this, note that $\sum_{i\in N} c_i > E$ implies that

$$E^A < \frac{E^A}{E^A + E^B}(c_1 + c_2) < c_1 + c_2 - E^B.$$

In other words, all strategy combinations for which the second part of Equation (6.25) holds, lie in between the two lines that identify the efficiency boundary.

The only strategy combination for which both parts of Equation (6.25) hold, is reflected by the intersection point and is given by

$$\Big(\big(\frac{E^A}{E^A + E^B}c_1, \frac{E^B}{E^A + E^B}c_1\big), \big(\frac{E^A}{E^A + E^B}c_2, \frac{E^B}{E^A + E^B}c_2\big)\Big).$$

To show that this strategy combination indeed is the only Nash equilibrium, we show that all other strategy combinations for which Equation (6.25) is satisfied, are not Nash equilibria. Therefore, we distinguish between four cases (corresponding to four line segments in Figure 6.10):

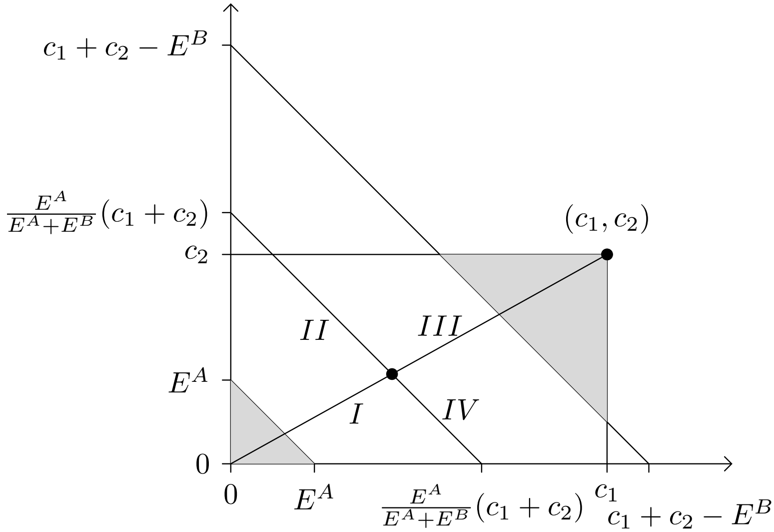


Figure 6.10 – *An overview of the set of strategy combinations in the proof of Theorem 6.4 with four line segments indicated by I, II, III and IV.*

I) $\hat{x}_2^A = \frac{c_2}{c_1}\hat{x}_1^A$ and $\hat{x}_2^A < \frac{E^A}{E^A+E^B}(c_1 + c_2) - \hat{x}_1^A$;

II) $\hat{x}_2^A = \frac{E^A}{E^A+E^B}(c_1 + c_2) - \hat{x}_1^A$ and $\hat{x}_2^A > \frac{c_2}{c_1}\hat{x}_1^A$;

III) $\hat{x}_2^A = \frac{c_2}{c_1}\hat{x}_1^A$ and $\hat{x}_2^A > \frac{E^A}{E^A+E^B}(c_1 + c_2) - \hat{x}_1^A$;

IV) $y\hat{x}_2^A = \frac{E^A}{E^A+E^B}(c_1 + c_2) - \hat{x}_1^A$ and $\hat{x}_2^A < \frac{c_2}{c_1}\hat{x}_1^A$.

Note that, in all cases, we know that Equation (6.24) is still satisfied, that is,

$$\begin{cases} \hat{x}_1^A + \hat{x}_2^A \geq E^A; \\ \hat{x}_1^B + \hat{x}_2^B \geq E^B. \end{cases}$$

Case I) In this case, we assume that $\hat{x}_2^A = \frac{c_2}{c_1}\hat{x}_1^A$ and $\hat{x}_2^A < \frac{E^A}{E^A+E^B}(c_1 + c_2) - \hat{x}_1^A$. In order to show that $\hat{x} \notin NE(G)$, we show that $\hat{x}_1 \notin BR_1(\hat{x}_2)$. Therefore, consider the strategy $x_1 \in X_1$ for player 1, given by

$$\begin{cases} x_1^A = \hat{x}_1^A + \varepsilon; \\ x_1^B = \hat{x}_1^B - \varepsilon, \end{cases}$$

for $\varepsilon > 0$ such that

$$\begin{cases} x_1^A + \hat{x}_2^A \geq E^A; \\ x_1^B + \hat{x}_2^B \geq E^B; \\ \hat{x}_2^A < \frac{E^A}{E^A+E^B}(c_1+c_2) - x_1^A. \end{cases}$$

In Figure 6.10, this means that player 1 is moving to the right a little bit. Then it holds that

$$\frac{c_2}{c_1}x_1^A = \frac{c_2}{c_1}\hat{x}_1^A + \frac{c_2}{c_1}\varepsilon > \hat{x}_2^A \quad \text{and} \quad \hat{x}_2^A < \frac{E^A}{E^A+E^B}(c_1+c_2) - x_1^A,$$

and consequently,

$$c_2 x_1^A - c_1\hat{x}_2^A > 0 \quad \text{and} \quad (c_1+c_2)E^A - (x_1^A + \hat{x}_2^A)(E^A+E^B) > 0.$$

By using Equation (6.26), this implies that

$$\pi_1(x_1, \hat{x}_2) > \text{PROP}_1(E, c) = \pi_1(\hat{x}_1, \hat{x}_2).$$

Hence, $\hat{x}_1 \notin BR_1(\hat{x}_2)$. Consequently, $\hat{x} \notin NE(G)$ in this case.

Case II) In the second case, we assume that $\hat{x}_2^A = \frac{E^A}{E^A+E^B}(c_1+c_2) - \hat{x}_1^A$ and $\hat{x}_2^A > \frac{c_2}{c_1}\hat{x}_1^A$. Again, we show that $\hat{x}_1 \notin BR_1(\hat{x}_2)$. Therefore, consider the strategy $x_1 \in X_1$ for player 1, given by

$$\begin{cases} x_1^A = \hat{x}_1^A + \varepsilon; \\ x_1^B = \hat{x}_1^B - \varepsilon, \end{cases}$$

for $\varepsilon > 0$ such that

$$\begin{cases} x_1^A + \hat{x}_2^A \geq E^A; \\ x_1^B + \hat{x}_2^B \geq E^B; \\ \hat{x}_2^A > \frac{c_2}{c_1}x_1^A. \end{cases}$$

In Figure 6.10, this means that player 1 is moving to the right a little bit. Then it holds that

$$\hat{x}_2^A > \frac{c_2}{c_1}x_1^A \quad \text{and} \quad \frac{E^A}{E^A+E^B}(c_1+c_2) - x_1^A = \frac{E^A}{E^A+E^B}(c_1+c_2) - \hat{x}_1^A - \varepsilon < \hat{x}_2^A,$$

and consequently,

$$c_2 x_1^A - c_1\hat{x}_2^A < 0 \quad \text{and} \quad (c_1+c_2)E^A - (x_1^A + \hat{x}_2^A)(E^A+E^B) < 0.$$

By using Equation (6.26), this implies that

$$\pi_1(x_1, \hat{x}_2) > \text{PROP}_1(E, c) = \pi_1(\hat{x}_1, \hat{x}_2).$$

Hence, $\hat{x}_1 \notin BR_1(\hat{x}_2)$. Consequently, $\hat{x} \notin NE(G)$ also in the second case.

Case III) In the third case, we assume that $\hat{x}_2^A = \frac{c_2}{c_1}\hat{x}_1^A$ and $\hat{x}_2^A > \frac{E^A}{E^A+E^B}(c_1+c_2)-\hat{x}_1^A$. Here, consider the strategy $x_1 \in X_1$ for player 1, given by

$$\begin{cases} x_1^A = \hat{x}_1^A - \varepsilon; \\ x_1^B = \hat{x}_1^B + \varepsilon, \end{cases}$$

for $\varepsilon > 0$ such that

$$\begin{cases} x_1^A + \hat{x}_2^A \geq E^A; \\ x_1^B + \hat{x}_2^B \geq E^B; \\ \hat{x}_2^A > \frac{E^A}{E^A+E^B}(c_1 + c_2) - x_1^A. \end{cases}$$

In Figure 6.10, this means that player 1 is moving to the left a little bit. Then it holds that

$$\frac{c_2}{c_1}x_1^A = \frac{c_2}{c_1}\hat{x}_1^A - \frac{c_2}{c_1}\varepsilon < \hat{x}_2^A \quad \text{and} \quad \hat{x}_2^A > \frac{E^A}{E^A + E^B}(c_1 + c_2) - x_1^A,$$

and consequently,

$$c_2 x_1^A - c_1\hat{x}_2^A < 0 \quad \text{and} \quad (c_1 + c_2)E^A - (x_1^A + \hat{x}_2^A)(E^A + E^B) < 0.$$

By using Equation (6.26), this implies that

$$\pi_1(x_1, \hat{x}_2) > \text{PROP}_1\,(E, c) = \pi_1(\hat{x}_1, \hat{x}_2).$$

Hence, $\hat{x}_1 \notin BR_1(\hat{x}_2)$. Consequently, $\hat{x} \notin NE(G)$ in this case.

Case IV) Finally, in the fourth case, we assume that $y\hat{x}_2^A = \frac{E^A}{E^A+E^B}(c_1 + c_2) - \hat{x}_1^A$ and $\hat{x}_2^A < \frac{c_2}{c_1}\hat{x}_1^A$. Consider the strategy $x_1 \in X_1$ for player 1, given by

$$\begin{cases} x_1^A = \hat{x}_1^A - \varepsilon; \\ x_1^B = \hat{x}_1^B + \varepsilon, \end{cases}$$

for $\varepsilon > 0$ such that

$$\begin{cases} x_1^A + \hat{x}_2^A \geq E^A; \\ x_1^B + \hat{x}_2^B \geq E^B; \\ \hat{x}_2^A < \frac{c_2}{c_1}x_1^A. \end{cases}$$

In Figure 6.10, this means that player 1 is moving to the left a little bit. Then it holds that

$$\hat{x}_2^A < \frac{c_2}{c_1}x_1^A \quad \text{and} \quad \frac{E^A}{E^A + E^B}(c_1 + c_2) - x_1^A = \frac{E^A}{E^A + E^B}(c_1 + c_2) - \hat{x}_1^A + \varepsilon > \hat{x}_2^A,$$

and consequently,

$$c_2 x_1^A - c_1\hat{x}_2^A > 0 \quad \text{and} \quad (c_1 + c_2)E^A - (x_1^A + \hat{x}_2^A)(E^A + E^B) > 0.$$

By using Equation (6.26), this implies that

$$\pi_1(x_1, \hat{x}_2) > \mathrm{PROP}_1\,(E, c) = \pi_1(\hat{x}_1, \hat{x}_2).$$

Hence, $\hat{x}_1 \notin BR_1(\hat{x}_2)$. Consequently, $\hat{x} \notin NE(G)$ also in the fourth case.

Consequently, we see that $\hat{x} \notin NE(G)$ in all four cases. We can thus conclude that $\hat{x} \in NE(G)$ implies that

$$\hat{x} = \left((\frac{E^A}{E^A + E^B} c_1, \frac{E^B}{E^A + E^B} c_1), (\frac{E^A}{E^A + E^B} c_2, \frac{E^B}{E^A + E^B} c_2) \right).$$

Since $NE(G) \neq \emptyset$, it immediately follows that

$$NE(G) = \left\{ \left((\frac{E^A}{E^A + E^B} c_1, \frac{E^B}{E^A + E^B} c_1), (\frac{E^A}{E^A + E^B} c_2, \frac{E^B}{E^A + E^B} c_2) \right) \right\}.$$

This completes the proof. □

6.5.2 Constrained equal awards rule

Secondly, we focus on the constrained equal awards rule. Also for the constrained equal awards rule, each Nash equilibrium pay-off vector of the strategic ceh-game associated to the uniform ceh-problem is equal to the awards vector specified by the constrained equal awards rule. The proof of this statement follows a similar structure as the proof of Theorem 6.3 in the sense that, for each player, a strategy is pinpointed for which the pay-off is at least the award for this player specified by the constrained equal awards rule. In this proof, we use the following lemma.

Lemma 6.1 *Let $(E, c) \in \mathcal{C}^N$ be a claims problem and let $i \in N$. Moreover, let $N' \subseteq N \setminus \{i\}$. Then the following two statements hold:*

i) if $c_i \leq \frac{1}{|N|-|N'|}(E - \sum_{j \in N'} c_j)$, then

$$CEA_i\,(E, c) = c_i;$$

ii) if $c_i \geq \frac{1}{|N|-|N'|}(E - \sum_{j \in N'} c_j)$, then

$$CEA_i\,(E, c) \geq \frac{1}{|N| - |N'|}(E - \sum_{j \in N'} c_j).$$

Proof: *i)* For the first statement, assume that $c_i \leq \frac{1}{|N|-|N'|}(E - \sum_{j \in N'} c_j)$. Then it holds that

$$\mathrm{CEA}_i\,(E, c) = \mathrm{CEA}_i \left(\sum_{j \in N \setminus N'} \mathrm{CEA}_j\,(E, c), (c_j)_{j \in N \setminus N'} \right)$$

$$
\begin{aligned}
&\geq \mathrm{CEA}_i \left(E - \sum_{j \in N'} c_j, (c_j)_{j \in N \setminus N'} \right) \\
&= c_i.
\end{aligned}
$$

Here, we used that the constrained equal awards rule satisfies (see Section 2.2) consistency for the first equality, estate monotonicity for the inequality and exemption for the final equality. Note that estate monotonicity applies, since, by using efficiency and claims boundedness,

$$
\sum_{j \in N \setminus N'} \mathrm{CEA}_j\,(E, c) = E - \sum_{j \in N'} \mathrm{CEA}_j\,(E, c) \geq E - \sum_{j \in N'} c_j.
$$

Moreover, note that exemption is applied to the claims problem

$$
\left(E - \sum_{j \in N'} c_j, (c_j)_{j \in N \setminus N'} \right) \in \mathcal{C}^{N \setminus N'}.
$$

This completes the proof of the first statement.

ii) For the second statement, assume that $c_i \geq \frac{1}{|N|-|N'|}(E - \sum_{j \in N'} c_j)$. Then, by consecutively applying consistency, estate monotonicity, claims monotonicity and exemption, we have that

$$
\begin{aligned}
\mathrm{CEA}_i\,(E, c) &= \mathrm{CEA}_i \left(\sum_{j \in N \setminus N'} \mathrm{CEA}_j\,(E, c)\,, (c_j)_{j \in N \setminus N'} \right) \\
&\geq \mathrm{CEA}_i \left(E - \sum_{j \in N'} c_j, (c_j)_{j \in N \setminus N'} \right) \\
&\geq \mathrm{CEA}_i \left(E - \sum_{j \in N'} c_j, ((c_j)_{j \in N \setminus (N' \cup \{i\})}, \frac{1}{|N| - |N'|}(E - \sum_{j \in N'} c_j)) \right) \\
&= \frac{1}{|N| - |N'|}(E - \sum_{j \in N'} c_j).
\end{aligned}
$$

Here, claims monotonicity applies, since $c_i \geq \frac{1}{|N|-|N'|}(E - \sum_{j \in N'} c_j)$. Hence, by replacing player i's claim, we can use exemption for the final equality. This completes the proof of the second statement. $\square$

Theorem 6.5 *Let $(\{E^k\}_{k \in M}, \{\varphi^k\}_{k \in M}, c) \in \mathcal{C}^{M,N}$ be a uniform ceh-problem and let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be the associated strategic ceh-game. Moreover, let $\hat{x} \in NE(G)$ be a Nash equilibrium. If, for all $k \in M$, $\varphi^k = CEA$, then*

$$
\pi(\hat{x}) = CEA\,(E, c)\,.
$$

Proof: Assume that, for all $k \in M$, $\varphi^k = \text{CEA}$. Note that, if $\sum_{i \in N} c_i \leq E$, then

$$\pi(\hat{x}) = c = \text{CEA}\,(E, c)\,,$$

according to Theorem 6.2.

Therefore, for the remainder of the proof, assume that $\sum_{i \in N} c_i > E$. Furthermore, set $N = \{1, 2, \ldots, n\}$ and assume w.l.o.g. that $c_1 \leq c_2 \leq \ldots \leq c_n$.

Next, we distinguish between the following cases:

I) $c_1 \geq \frac{1}{n} E$;

II) $c_2 \geq \frac{1}{n-1}(E - c_1)$ and $c_1 < \frac{1}{n} E$,

and, proceeding recursively, for $h \in \{3, \ldots, n\}$,

III) $c_h \geq \frac{1}{n-h+1}(E - c_1 - \ldots - c_{h-1})$ and

$$\begin{cases} c_1 < \frac{1}{n} E; \\ c_2 < \frac{1}{n-1}(E - c_1); \\ \vdots \\ c_{h-1} < \frac{1}{n-h+2}(E - c_1 - \ldots - c_{h-2}). \end{cases}$$

In each case, we pinpoint, for all $i \in N$, a specific strategy $y_i \in X_i$ such that

$$\pi_i(\hat{x}_{-i}, y_i) \geq \text{CEA}_i\,(E, c)\,. \tag{6.27}$$

Similar as in the proof of Theorem 6.3, it then follows that

$$\pi_i(\hat{x}) = \pi_i(\hat{x}_{-i}, \hat{x}_i) \geq \pi_i(\hat{x}_{-i}, y_i) \geq \text{CEA}_i\,(E, c)\,,$$

for all $i \in N$. Moreover, due to the fact that the constrained equal awards rule satisfies strict marginality, we have that

$$\sum_{i \in N} \pi_i(\hat{x}) = E = \sum_{i \in N} \text{CEA}_i\,(E, c)\,,$$

according to Theorem 6.2. Consequently, for all $i \in N$,

$$\pi_i(\hat{x}) = \text{CEA}_i\,(E, c)\,.$$

Case I) In the first case, we assume that $c_1 \geq \frac{1}{n} E$. This implies that $c_i \geq \frac{1}{n} E$ for all $i \in N$, which in turn implies that, for all $i \in N$,

$$\text{CEA}_i\,(E, c) = \frac{1}{n} E,$$

according to Lemma 6.1 together with efficiency. To pinpoint, for all $i \in N$, a strategy $y_i \in X_i$ for which Equation (6.27) holds, let $\in N$. Choose a strategy $y_i \in X_i$ such that, for all $k \in M$,

$$y_i^k \geq \frac{1}{n} E^k.$$

Note that such a strategy exists, since $c_i = \sum_{k \in M} y_i^k \geq \sum_{k \in M} \frac{1}{n} E^k = \frac{1}{n} E$. Consequently,

$$\pi_i(\hat{x}_{-i}, y_i) = \sum_{k \in M} \mathrm{CEA}_i \left(E^k, (\hat{x}_{-i}^k, y_i^k) \right) \geq \sum_{k \in M} \frac{1}{n} E^k = \frac{1}{n} E = \mathrm{CEA}_i \left(E, c \right),$$

where we used part ii) of Lemma 6.1 with $N' = \emptyset$ for the inequality. This shows that Equation (6.27) is satisfied in this first case.

Case II) In the second case, we assume that both $c_1 < \frac{1}{n} E$ and $c_2 \geq \frac{1}{n-1}(E - c_1)$. This implies that $c_i \geq \frac{1}{n-1}(E - c_1)$ for all $i \in N \setminus \{1\}$, which in turn implies that, for all $i \in N$,

$$\mathrm{CEA}_i \left(E, c \right) = \begin{cases} c_1, & \text{if } i = 1; \\ \frac{1}{n-1}(E - c_1), & \text{otherwise,} \end{cases}$$

according to Lemma 6.1 together with efficiency. Next, we pinpoint, for all $i \in N$, a strategy $y_i \in X_i$ for which Equation (6.27) holds.

First, for player 1, choose $y_1 \in X_1$ such that, for all $k \in M$,

$$y_1^k < \frac{1}{n} E^k.$$

Then it holds that

$$\pi_1(\hat{x}_{-1}, y_1) = \sum_{k \in M} \mathrm{CEA}_1 \left(E^k, (\hat{x}_{-1}^k, y_1^k) \right) = \sum_{k \in M} y_1^k = c_1 = \mathrm{CEA}_1 \left(E, c \right),$$

where we used part i) of Lemma 6.1 with $N' = \emptyset$ for the second equality. Hence, Equation (6.27) is satisfied for player 1.

Secondly, let $i \in N \setminus \{1\}$ and choose $y_i \in X_i$ such that, for all $k \in M$,

$$y_i^k \geq \frac{1}{n-1}(E^k - \hat{x}_1^k).$$

As before, such a strategy exists, since $c_i \geq \frac{1}{n-1}(E - c_1)$. Consequently,

$$\pi_i(\hat{x}_{-i}, y_i) = \sum_{k \in M} \mathrm{CEA}_i \left(E^k, (\hat{x}_{-i}^k, y_i^k) \right)$$

$$
\begin{aligned}
&\geq \sum_{k\in M} \frac{1}{n-1}(E^k - \hat{x}_1^k)\\
&= \frac{1}{n-1}(E - c_1) = \mathrm{CEA}_i\,(E,c)\,,
\end{aligned}
$$

where we used part ii) of Lemma 6.1 with $N' = \{1\}$ for the inequality. Hence, Equation (6.27) is also satisfied for all $i \in N \setminus \{1\}$.

Case III) Proceeding recursively, let $h \in \{3, \ldots, n\}$ and assume that $c_h \geq \frac{1}{n-h+1}(E - c_1 - \ldots - c_{h-1})$ and

$$
\begin{cases}
c_1 < \frac{1}{n}E;\\
c_2 < \frac{1}{n-1}(E - c_1);\\
\vdots\\
c_{h-1} < \frac{1}{n-h+2}(E - c_1 - \ldots - c_{h-2}).
\end{cases}
$$

This implies that $c_i \geq \frac{1}{n-h+1}(E - c_1 - \ldots - c_{h-1})$ for all $i \in N \setminus \{1, 2, \ldots, h-1\}$, which in turn implies that, for all $i \in N$,

$$
\mathrm{CEA}_i\,(E,c) = \begin{cases}
c_i, & \text{if } i \in \{1,2,\ldots,h-1\};\\
\frac{1}{n-h+1}(E - c_1 - \ldots - c_{h-1}), & \text{otherwise,}
\end{cases}
$$

according to Lemma 6.1 together with efficiency. To pinpoint, for all $i \in N$, a strategy $y_i \in X_i$ for which Equation (6.27) holds, we first focus on the players in $\{1, 2, \ldots, h-1\}$. For each of them, we separately choose a strategy such that, for all $k \in M$,

$$
\begin{cases}
y_1^k < \frac{1}{n}E^k;\\
y_2^k < \frac{1}{n-1}(E^k - \hat{x}_1^k);\\
\vdots\\
y_{h-1}^k < \frac{1}{n-h+2}(E - \hat{x}_1^k - \ldots - \hat{x}_{h-2}^k).
\end{cases}
$$

As before, it follows that, by using part i) of Lemma 6.1,

$$
\begin{cases}
\pi_1(\hat{x}_{-1}, y_1) = c_1 = \mathrm{CEA}_1\,(E,c)\,;\\
\pi_2(\hat{x}_{-2}, y_2) = c_2 = \mathrm{CEA}_2\,(E,c)\,;\\
\vdots\\
\pi_{h-1}(\hat{x}_{-(h-1)}, y_{h-1}) = c_{h-1} = \mathrm{CEA}_{h-1}\,(E,c)\,.
\end{cases}
$$

Hence, Equation (6.27) is satisfied for these players.

Secondly, let $i \in N \setminus \{1, 2, \ldots, h-1\}$ and choose $y_i \in X_i$ such that, for all $k \in M$,

$$
y_i^k \geq \frac{1}{n-h+1}(E^k - \hat{x}_1^k - \ldots - \hat{x}_{h-1}^k).
$$

Again, such a strategy clearly exists. Consequently

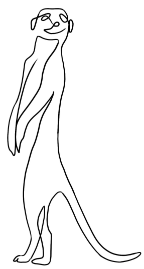
$$\begin{aligned}
\pi_i(\hat{x}_{-i}, y_i) &= \sum_{k\in M} \mathrm{CEA}_i\left(E^k, (\hat{x}^k_{-i}, y^k_i)\right) \\
&\geq \sum_{k\in M} \frac{1}{n-h+1}(E^k - \hat{x}^k_1 - \ldots - \hat{x}^k_{h-1}) \\
&= \frac{1}{n-h+1}(E - c_1 - \ldots - c_{h-1}) = \mathrm{CEA}_i\left(E, c\right),
\end{aligned}$$

where we used part ii) of Lemma 6.1 with $N' = \{1, 2, \ldots, h-1\}$ for the inequality. Hence, Equation (6.27) is also satisfied for these players.

Ultimately, this finishes all cases and hence, the proof. □

For the proportional rule, Theorem 6.4 shows that the set of Nash equilibria of the strategic ceh-game associated to a uniform ceh-problem with two estate holders, both using the proportional rule, and two players is a singleton. In contrast, for the constrained equal awards rule, Example 6.1 already indicates that the set of Nash equilibria of the strategic ceh-game associated to a uniform ceh-problem using the constrained equal awards rule is much larger. In fact, one can show that different structures are possible for the set of Nash equilibria for various uniform ceh-problems using the constrained equal awards rule, depending on the sizes of both the claims and the estates.

7 Unilateral support equilibria



— A meerkat supports the pack by keeping watch

7.1 Introduction

An essential issue within the field of strategic game theory is to provide equilibrium concepts to somehow solve situations of interaction and conflicts between players. The standard equilibrium concept of Nash (1950, 1951) is based on the fact that no player should have an incentive to unilaterally deviate from an equilibrium strategy. More specifically, a *Nash equilibrium* is a strategy combination in which every player maximizes his own pay-off by playing the equilibrium strategy, given the equilibrium strategy combination of the other players. In other words, for every player there is no strategy that, given the Nash equilibrium strategy combination of the other players results in a strictly higher pay-off for this player than the Nash equilibrium strategy.

In contrast to the fully selfish behavior in a Nash equilibrium, an alternative equilibrium concept based on fully altruistic behavior is proposed by Berge (1957). In a so-called *Berge equilibrium*, players are not maximizing their own pay-offs, but maximize the other players' pay-offs instead. More precisely, a Berge equilibrium is a strategy combination in which the group of all players except one player maximizes the pay-off of this one player by playing the Berge equilibrium strategy combination, given the equilibrium strategy of this one player. In other words, every player's payoff is maximized by the group of all other players, that is, every player is supported by all other players together. Therefore, a Berge equilibrium is also called a mutual support equilibrium (Colman, Körner, Musy, and Tazdaït, 2011).

The study of Berge equilibria has focused on several aspects. For example, Radjef (1988), Abalo and Kostreva (2004), Nessah, Larbani, and Tazdaït (2007) and Larbani and Nessah (2008) focused on finding existence theorems for general classes of strategic games. Specific classes of games (in particular, mixed extensions of finite games) are studied by Colman et al. (2011), Musy, Pottier, and Tazdaït (2012), Corley and Kwain (2014) and Corley (2015). Both Colman et al. (2011) and Musy et al. (2012) also pay special attention to experimental results. Algorithms to find Berge equilibria are studied in Corley and Kwain (2015) and Sawicki, Pykacz, and Bytner (2020). Moreover, Abalo and Kostreva (1996), Colman et al. (2011), Corley (2015) and Courtois, Nessah, and Tazdaït (2017) studied the relation between the Nash equilibrium concept and the Berge equilibrium concept.

In this chapter, based on Schouten, Borm, and Hendrickx (2019), we focus on the essence of the concept of a Berge equilibrium, which is supportive behavior. It aims to provide more insight into the idea of supportive behavior, which then can be used in the study to both the Nash and Berge equilibrium concepts. In a Berge equilibrium, every player is supported by the group of all other players together and in that sense, a Berge equilibrium reflects the idea of mutual support. To quote Colman et al. (2011):

> "A Berge equilibrium can be viewed as an implication of the altruistic social value orientation of interdependence theory, just as Nash equilibrium is an implication of the individualistic orientation."

In other words, in a Nash equilibrium players choose for themselves and behave selfishly, whereas in a Berge equilibrium players collectively support the other players, sometimes at the cost of themselves. According to both Larbani and Nessah (2008) and Corley (2015), this altruistic behavior follows the idea of 'one for all, and all for one'. Indeed, every player supports (as part of a larger group) every other player and all other players support every single player. However, the support relation in a Berge equilibrium is restricted to group support: the group of all players except for one single player supports the single player in the best way possible. To do so, they have to coordinate their actions, which can cause coordination issues for the players. To avoid these rather complex coordination issues, we consider individual support rather than group support.

The main contribution of this chapter is to introduce a new equilibrium concept for strategic games, which is based on individual support only. For that reason, we introduce support relations between the individual players, which can be modeled by using a special type of bijections, called derangements. The interpretation of such a derangement is that every player supports exactly one other player and every player is supported by exactly one other player. Subsequently, we define a new equilibrium concept, called a *unilateral support equilibrium*, which is unilaterally supportive with respect to every possible derangement. More specifically, the set of unilateral support equilibria is equal to the intersection of the sets of unilaterally supportive strategy combinations with respect to *all* derangements.

From a computational perspective, it might be a hard task to compute all sets of unilaterally supportive strategy combinations with respect to a derangement, since for a large number of players, there are many derangements. However, it is shown that it is sufficient to only consider cyclic derangements, a special type of derangements: every strategy combination that is unilaterally supportive with respect to every possible cyclic derangement is a unilateral support equilibrium. This leads to a drastic reduction of the number of derangements that have to be considered.

The idea of using individual support only is elegantly reflected by our main result in Theorem 7.1: in a unilateral support equilibrium, every player is supported by every other player individually, whereas in a Berge equilibrium, every player is supported by the group of all other players. This shows the key difference in the underlying support relations: a unilateral support equilibrium is based on everybody's individual support, while a Berge equilibrium is based on group support. Moreover, Theorem 7.1 provides an alternative formulation of the set of unilateral support equilibria directly in terms of pay-off functions, instead of using derangements.

Another consequence of using individual support rather than group support is the fact that group support directly implies individual support. Subsequently, every Berge equilibrium is also a unilateral support equilibrium. In that sense, this new equilibrium concept extends the concept of Berge equilibria. For any two-person strategic game, the set of unilateral support equilibria coincides with the set of Berge equilibria. Moreover, we see that in an example of Corley (2015) without Berge equilibria, the set of unilateral support equilibria is non-empty, which shows that the set of Berge equilibria is strictly included in the set of unilateral support equilibria. However, existence of unilateral support equilibria is not guaranteed. We provide an example of a trimatrix game in which there is no unilateral support equilibrium.

The fact that Berge equilibria are unilateral support equilibria can also be used to exploit the existence theorems for Berge equilibria to guarantee the existence of unilateral support equilibra, as was recently pointed out by Crettez and Nessah (2020). Besides, Crettez and Nessah (2020) studied, as a follow-up of Schouten et al. (2019), specific existence theorems for unilateral support equilibria.

Finally, we explore the relation between the set of unilateral support equilibria and the set of Nash equilibria. We show that the intersection between these two sets coincides with the intersection of the sets of unilaterally supportive strategy combinations with respect to every possible bijection, not only with respect to every possible derangement. Consequently, a strategy combination is both a unilateral support equilibrium and a Nash equilibrium if and only if it is a Nash equilibrium of all coordination games in which all players face the pay-off function of a single player.

This chapter is structured in the following way. Section 7.2 studies the concept of Berge equilibria. Section 7.3 introduces and analyzes the set of unilateral support equilibria. In particular, it contains a characterization of unilateral support equilibria in terms of pay-off functions. Section 7.4 studies the set of strategy combinations that are both a unilateral support equilibrium and a Nash equilibrium.

7.2 Berge equilibria

For a strategic game $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$, a strategy combination $\hat{x} \in X$ is called a *Berge equilibrium* (cf. Berge, 1957) if, for all $i \in N$, it holds that

$$\pi_i(\hat{x}_i, \hat{x}_{-i}) \geq \pi_i(\hat{x}_i, x_{-i}),$$

for all $x_{-i} \in X_{-i}$. The set of Berge equilibria for G is denoted by $BE(G)$. In a Berge equilibrium it thus holds that for every player there is no strategy combination of all other players that, given the Berge equilibrium strategy of this particular player results in a strictly higher pay-off for this player than the Berge equilibrium strategy combination of the other players. In other words, a Berge equilibrium is a strategy combination in which the group of all players except one player maximizes the pay-off of this one player by playing the equilibrium strategy combination, given the equilibrium strategy of this one player.

Clearly, with

$$BS_{-i}(x_i) = \left\{x_{-i} \in X_{-i} \;\middle|\; \pi_i(x_i, x_{-i}) \geq \pi_i(x_i, x'_{-i}) \quad \text{for all } x'_{-i} \in X_{-i}\right\},$$

for all $x_i \in X_i$ and all $i \in N$, denoting the *set of best support strategy combinations against* x_i (cf. Musy et al., 2012), we have that $\hat{x} \in BE(G)$ if and only if $\hat{x}_{-i} \in BS_{-i}(\hat{x}_i)$ for all $i \in N$.

Berge equilibria always exist for bimatrix games. This readily follows from the observation that, for all bimatrix games (A, B), it holds that

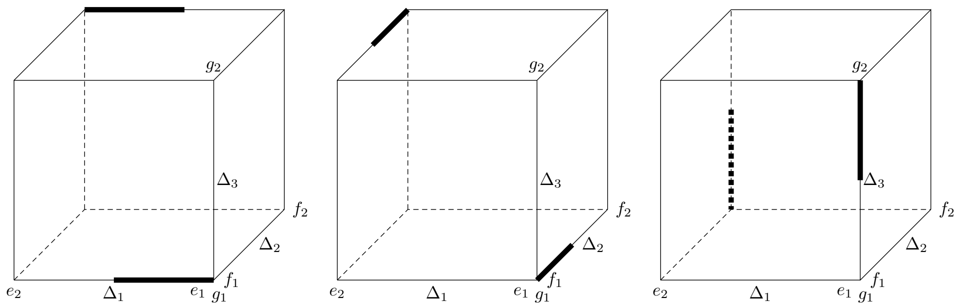
$$BE(A, B) = NE(B, A),$$

and the fact that Nash equilibria always exist for bimatrix games. However, even for the class of trimatrix games, existence of Berge equilibria is no longer guaranteed, as is seen in the following counterexample provided by Corley (2015).

Example 7.1 [cf. Corley, 2015] Consider the following trimatrix game $G = (A, B, C)$:

$$G = (A, B, C) = \begin{array}{c} \\ e_1 \\ e_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_1 & f_2 \end{array} \\ \left[\begin{array}{cc|cc} (1,1,0) & (0,0,0) & (0,0,1) & (0,0,0) \\ (0,0,0) & (0,0,1) & (0,0,0) & (1,1,0) \end{array}\right] \\ \begin{array}{cc} g_1 & g_2 \end{array} \end{array}.$$

Here, the first coordinate represents the entries of matrix A, the second coordinate the entries of B and the third coordinate the entries of C.



(a) *For player 1.* **(b)** *For player 2.* **(c)** *For player 3.*

Figure 7.1 – *The three sets of best support strategy combinations corresponding to $BE(G)$.*

The sets of best support strategy combinations are presented in Figure 7.1.

For example, Figure 7.1a shows the set of best support strategy combinations for player 1. For each strategy $p = p_1 e_1 + (1 - p_1)e_2 \in \Delta_1$ of player 1, the corresponding pay-offs of player 1 are shown below:

$$p_1 e_1 + (1 - p_1)e_2 \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_1 & f_2 \end{array} \\ \left[\begin{array}{cc|cc} p_1 & 0 & 0 & 1 - p_1 \end{array}\right] \\ \begin{array}{cc} g_1 & g_2 \end{array} \end{array}.$$

For $p_1 \in [0, \frac{1}{2})$, it clearly holds that $1 - p_1 > p_1$, such that players 2 and 3 maximize player 1's pay-off by choosing the strategies f_2 and g_2, respectively. In other words, the best support strategy combination against the strategy $p = p_1 e_1 + (1 - p_1)e_2$ of player 1 with $p_1 \in [0, \frac{1}{2})$ is given by $(f_2, g_2) \in \Delta_2 \times \Delta_3$. In Figure 7.1a, this is visualized as the upper bold part.

Next, it is readily seen that the best support strategy combination against the strategy $p = p_1 e_1 + (1 - p_1)e_2$ with $p_1 \in (\frac{1}{2}, 1]$ is given by $(f_1, g_1) \in \Delta_2 \times \Delta_3$. This is the bottom bold part of Figure 7.1a.

Finally, the set of best support strategy combinations against the strategy $p = \frac{1}{2}e_1 + \frac{1}{2}e_2$ consists of both $(f_1, g_1) \in \Delta_2 \times \Delta_3$ and $(f_2, g_2) \in \Delta_2 \times \Delta_3$.

In a similar way, we can derive the sets of best support strategy combinations of players 2 and 3, by using the pay-off function of player 2 for each strategy $q = q_1 f_1 + (1 - q_1)f_2 \in \Delta_2$,

$$\begin{array}{c} \\ e_1 \\ e_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} q_1 f_1 + (1-q_1) f_2 & q_1 f_1 + (1-q_1) f_2 \end{array} \\ \left[\begin{array}{c|c} q_1 & 0 \\ 0 & 1 - q_1 \end{array}\right] \\ \begin{array}{cc} g_1 & g_2 \end{array} \end{array},$$

and the pay-off function of player 3 for each strategy $r = r_1 g_1 + (1 - r_1)g_2 \in \Delta_3$,

$$\begin{array}{c} \\ e_1 \\ e_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{bmatrix} 1 - r_1 & 0 \\ 0 & r_1 \end{bmatrix} \\ r_1 g_1 + (1 - r_1) g_2 \end{array},$$

respectively. The results are shown in Figures 7.1b and 7.1c.

Using Figure 7.1, it can be readily seen that the intersection between the three sets of best support strategy combinations is empty. Hence, $BE(G) = \emptyset$. The reason for this is that, for example, player 1 is cleaved in simultaneously supporting both players 2 and 3 as part of a larger group. To support player 2, player 1 has to choose e_1 and player 3 has to choose g_1 if player 2 chooses f_1. However, if player 3 chooses g_1, then player 1 has to choose e_2 (and player 2 f_2) to support player 3. $\triangle$

7.3 Unilateral support equilibria

Berge equilibria are based on mutually supportive behavior, which means that every player is supported by the group of all other players together. This mutually supportive behavior however could create coordination issues, as seen in Example 7.1. Our new concept of a unilateral support equilibrium will retain supportive behavior, but eliminates the coordination issues by narrowing down supportive behavior to individual support.

For this, we first specify the exact meaning of a support relation. To do so, set $N = \{1, 2, \ldots, n\}$ throughout this chapter. A *bijection* $\sigma : N \to N$ is a transformation of N where each player is mapped to exactly one player and there are no other players mapped to this one player. We denote such a bijection by $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$. The set of all such bijections is denoted by $\Pi(N)$.[1] The identity bijection is denoted by σ_{id}, i.e., $\sigma_{id} = (1, 2, \ldots, n)$. In the context of supportive behavior, a bijection $\sigma \in \Pi(N)$, for a player set N, should be interpreted as follows: player $i \in N$ supports player $\sigma(i) \in N$.

The set of *derangements* is given by

$$D(N) = \{\delta \in \Pi(N) \mid \delta(i) \neq i \quad \text{for all } i \in N\}.$$

In a derangement, no player supports himself. Finally, we introduce the set $C(N)$ of *cyclic derangements*, given by

$$\begin{aligned} C(N) = \{\gamma \in D(N) \mid{} & \text{there exists a number } \alpha \in \{1, 2, \ldots, n-1\} \\ & \text{such that for all } i \in N: \quad \gamma(i) = (i + \alpha) \bmod n\}.^2 \end{aligned}$$

In a cyclic derangement, every player supports the player that is a fixed number of shifts away from himself. The number $\alpha \in \{1, \ldots, n-1\}$ represents this number of shifts. Derangements and cyclic derangements are illustrated in the following example.

Example 7.2 Consider a player set with four players, $N = \{1, 2, 3, 4\}$. Then the identity bijection is given by $\sigma_{id} = (1, 2, 3, 4)$. In a derangement, the players cannot be mapped to themselves. The set of all derangements is thus given by

$$\begin{aligned} D(N) = \{&(2,1,4,3), (2,3,4,1), (2,4,1,3), \\ &(3,1,4,2), (3,4,1,2), (3,4,2,1), \\ &(4,1,2,3), (4,3,1,2), (4,3,2,1)\}. \end{aligned}$$

There are three derangements that are cyclic: for $\alpha = 1$, we obtain $(2, 3, 4, 1)$, while $\alpha = 2$ results in $(3, 4, 1, 2)$ and $\alpha = 3$ gives the cyclic derangement $(4, 1, 2, 3)$. Thus, the set of cyclic derangements is given by

$$C(N) = \{(2,3,4,1), (3,4,1,2), (4,1,2,3)\}. \qquad \triangle$$

We start by introducing unilaterally supportive strategy combinations with respect to a bijection. Afterwards, this is generalized to the definition of a unilateral support equilibrium, where the dependence on a certain bijection is removed.

Definition 7.1 Let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be a strategic game. Moreover, let $\sigma \in \Pi(N)$ be a bijection on the set of players. A strategy combination $\hat{x} \in X$ is called

[1] In this chapter, $\Pi(N)$ denotes a slightly different set compared to Chapter 4 where it denotes the set of all processing orders. However, in essence, both sets are identical.

[2] Here, $(i + \alpha) \bmod n$ is the unique player $j \in N$ for which there exists a $k \in \mathbb{Z}$ such that $j = i + \alpha + kn$.

unilaterally supportive with respect to σ if, for all $i \in N$, it holds that

$$\pi_{\sigma(i)}(\hat{x}_{-i}, \hat{x}_i) \geq \pi_{\sigma(i)}(\hat{x}_{-i}, x_i),$$

for all $x_i \in X_i$. The set of all such strategy combinations is denoted by $USE_\sigma(G)$. $\triangleleft$

A unilaterally supportive strategy combination with respect to a bijection $\sigma \in \Pi(N)$ is thus a strategy combination in which every player $i \in N$ maximizes the pay-off of player $\sigma(i) \in N$ by playing the prescribed strategy, given the prescribed strategy combination of the other players. This also applies to the identity bijection, which implies that every player maximizes his own pay-off in a unilaterally supportive strategy combination with respect to the identity bijection. Consequently, for a strategic game $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$,

$$USE_{\sigma_{id}}(G) = NE(G).$$

More generally, we see that the notion of unilaterally supportive strategy combinations with respect to a bijection is closely related to the notion of a Nash equilibrium: every player maximizes the pay-off of a pre-specified (by the bijection) player. This results in the proposition below, which shows that every set of unilaterally supportive strategy combinations with respect to a bijection σ coincides with the set of Nash equilibria of the game with twisted pay-off functions, in which player i's pay-off function is replaced by the pay-off function of player $\sigma(i)$.

Formally, for a strategic game $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ and a bijection $\sigma \in \Pi(N)$, the game with twisted pay-off functions is given by $G_\sigma = \left(N, \{X_i\}_{i\in N}, \{\pi_{\sigma(i)}\}_{i\in N}\right)$. The proof of the proposition is straightforward and therefore omitted.

Proposition 7.1 *Let* $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ *be a strategic game and let* $\sigma \in \Pi(N)$ *be a bijection. Then it holds that*

$$USE_\sigma(G) = NE(G_\sigma).$$

For each bijection, the corresponding set of unilaterally supportive strategy combination with respect to that bijection can be defined. However, only derangements truly reflect the idea of supportive behavior. If a player is mapped to himself, then this player does not support another player. The set of unilaterally supportive strategy combinations with respect to a derangement has the disadvantage that it is not anonymous in the sense that it relies on the predetermined support relations given by the derangement. For this reason, in order to define the set of unilateral support equilibria, we consider the set of unilaterally supportive strategy combinations with respect to all derangements.

Definition 7.2 Let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be a strategic game. Then the set of *unilateral support equilibria* is defined as

$$USE(G) = \bigcap_{\delta \in D(N)} USE_\delta(G). \qquad \triangleleft$$

The set of unilateral support equilibria is thus equal to the intersection of all sets of unilaterally supportive strategy combinations with respect to a derangement. For bimatrix games, this boils down to just one set, which is, according to Proposition 7.1, the set of Nash equilibria with twisted pay-off functions. So, for a bimatrix game (A, B), we have that

$$USE(A, B) = NE(B, A) = BE(A, B).$$

In fact, a similar reasoning applies for any strategic game with two players:

$$USE(G) = NE(G_{(2,1)}) = BE(G),$$

for all strategic games $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ with $N = \{1, 2\}$. Here, $(2, 1)$ denotes the only possible derangement in this situation.

For strategic games with more than two players, computing the set of unilateral support equilibria is more involved. The following example illustrates how one can use Proposition 7.1 to facilitate the process.

Example 7.3 Consider the following trimatrix game $G = (A, B, C)$:

$$G = (A, B, C) = \begin{array}{c} \\ e_1 \\ e_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc|cc} f_1 & f_2 & f_1 & f_2 \end{array} \\ \left[\begin{array}{cc|cc} (1,1,1) & (0,0,0) & (0,0,0) & (0,0,0) \\ (2,0,0) & (0,0,0) & (0,0,0) & (0,0,0) \end{array}\right] \\ \begin{array}{cc} g_1 \qquad\qquad & \qquad\qquad g_2 \end{array} \end{array}.$$

Clearly, for $N = \{1, 2, 3\}$, there are only two derangements, $(2, 3, 1)$ and $(3, 1, 2)$ respectively. Thus, to compute $USE(G)$, we compute $USE_{(2,3,1)}(A, B, C)$ and $USE_{(3,1,2)}(A, B, C)$ separately and take the intersection.

First, consider $\delta = (2, 3, 1)$. Using Proposition 7.1, we see that

$$USE_{(2,3,1)}(A, B, C) = NE(B, C, A).$$

In other words, the pay-off function of player 1 in the game (B, C, A) is provided by matrix B, the original pay-off function of player 2 (since player 1 supports player 2 in δ), and is given by

$$\begin{array}{cc} & \begin{array}{cccc} f_1 & f_2 & f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \\ & \begin{array}{cc} g_1 & \quad g_2 \end{array} \end{array}$$

or equivalently, with obvious notation,

$$\begin{cases} \tilde{\pi}_1^{BCA}(e_1, q, r) = q_1 r_1; \\ \tilde{\pi}_1^{BCA}(e_2, q, r) = 0, \end{cases}$$

for all $(q, r) \in \Delta_2 \times \Delta_3$. The set of best reply strategies against (q, r) for player 1 consists of e_1 if both $q_1 > 0$ and $r_1 > 0$ and equals Δ_1 if either $q_1 = 0$ (that is, player 2 chooses f_2) or $r_1 = 0$ (that is, player 3 chooses g_2). This is visualized in Figure 7.2a.

Similarly, the pay-off function of player 2 in (B, C, A) is provided by matrix C and thus given by

$$\begin{cases} \tilde{\pi}_2^{BCA}(p, f_1, r) = p_1 r_1; \\ \tilde{\pi}_2^{BCA}(p, f_2, r) = 0, \end{cases}$$

for all $(p, r) \in \Delta_1 \times \Delta_3$. Consequently, the set of best reply strategies against (p, r) for player 2 consists of f_1 if both $p_1 > 0$ and $r_1 > 0$ and equals Δ_2 if either $p_1 = 0$ or $r_1 = 0$. This is visualized in Figure 7.2b.

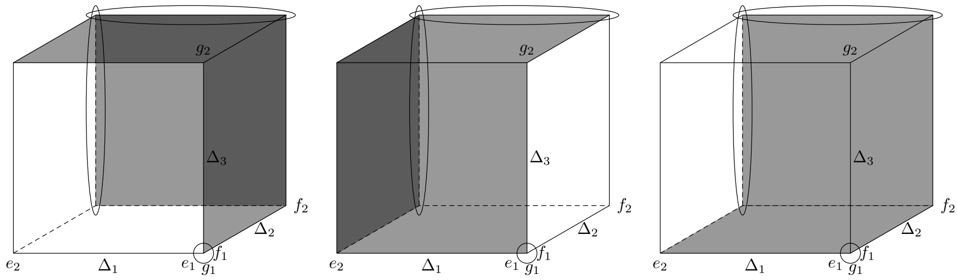
Finally, the pay-off function of player 3 in (B, C, A) is provided by A and thus given by

$$\begin{cases} \tilde{\pi}_3^{BCA}(p, q, g_1) = p_1 q_1 + 2q_1(1 - p_1) = q_1(2 - p_1); \\ \tilde{\pi}_3^{BCA}(p, q, g_2) = 0, \end{cases}$$

for all $(p, q) \in \Delta_1 \times \Delta_2$. Consequently, the set of best reply strategies against (p, q) for player 3 consists of g_1 in almost all situations except if $q_1 = 0$ (that is, if player 2 chooses f_2), in which case the set of best reply strategies equals Δ_3. This is visualized in Figure 7.2c.

The intersection of the three sets of best reply strategies corresponding to $NE(B, C, A)$ as visualized in Figure 7.2 yields the set of Nash equilibria $NE(B, C, A)$ and thus the set of unilaterally supportive strategy combinations with respect to the derangement $(2, 3, 1)$:

$$\begin{aligned} USE_{(2,3,1)}(G) = \big\{(e_1, f_1, g_1)\big\} &\cup \mathrm{Conv}\big\{(e_2, f_2, g_1), (e_2, f_2, g_2)\big\} \\ &\cup \mathrm{Conv}\big\{(e_2, f_2, g_2), (e_1, f_2, g_2)\big\}. \end{aligned}$$



(a) *For player 1.* **(b)** *For player 2.* **(c)** *For player 3.*

Figure 7.2 – *The three sets of best reply strategies corresponding to* $USE_{(2,3,1)}(G)$.

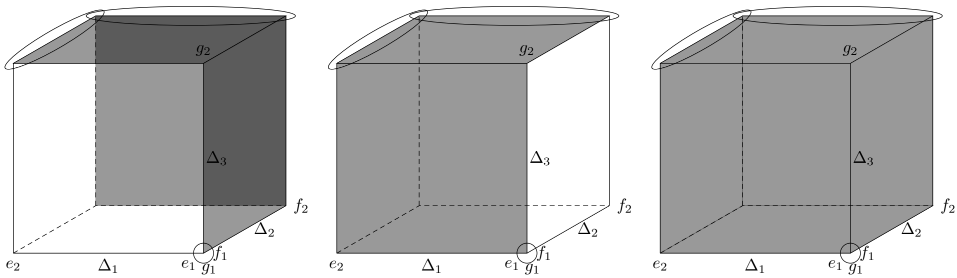
Secondly, consider $\delta = (3,1,2)$. Using Proposition 7.1, we see that

$$USE_{(3,1,2)}(A,B,C) = NE(C,A,B).$$

In (C,A,B), player 1 is thus facing pay-off matrix C, which can be rewritten to

$$\begin{cases} \tilde{\pi}_1^{CAB}(e_1,q,r) = q_1 r_1; \\ \tilde{\pi}_1^{CAB}(e_2,q,r) = 0, \end{cases}$$

for all $(q,r) \in \Delta_2 \times \Delta_3$. The corresponding set of best reply strategies for player 1 is in fact equal to the one corresponding to the game (B,C,A) and is again visualized in Figure 7.3a.



(a) *For player 1.* **(b)** *For player 2.* **(c)** *For player 3.*

Figure 7.3 – *The three sets of best reply strategies corresponding to* $USE_{(3,1,2)}(G)$.

The pay-off function of player 2 is provided by matrix A and given by

$$\begin{cases} \tilde{\pi}_2^{CAB}(p,f_1,r) = p_1 r_1 + 2r_1(1-p_2) = r_1(2-p_1); \\ \tilde{\pi}_2^{CAB}(p,f_2,r) = 0, \end{cases}$$

for all $(p, r) \in \Delta_1 \times \Delta_3$. Consequently, the set of best reply strategies against (p, r) for player 2 consists of f_1 if $r_1 > 0$ and is equal to Δ_2 if $r_1 = 0$ (that is, if player 3 chooses g_2). This is visualized in Figure 7.3b.

Finally, for player 3, the pay-off function in (C, A, B) is given by

$$\begin{cases} \tilde{\pi}_3^{CAB}(p, q, g_1) = p_1 q_1; \\ \tilde{\pi}_3^{CAB}(p, q, g_2) = 0, \end{cases}$$

for all $(p, q) \in \Delta_1 \times \Delta_2$. Consequently, the set of best reply strategies against (p, q) for player 3 consists of g_1 if both $p_1 > 0$ and $q_1 > 0$ and equals Δ_3 if either $p_1 = 0$ (that is, if player 1 chooses e_2) or $q_1 = 0$ (that is, if player 2 chooses f_2). This is visualized in Figure 7.3c (left side, back side and bottom side).

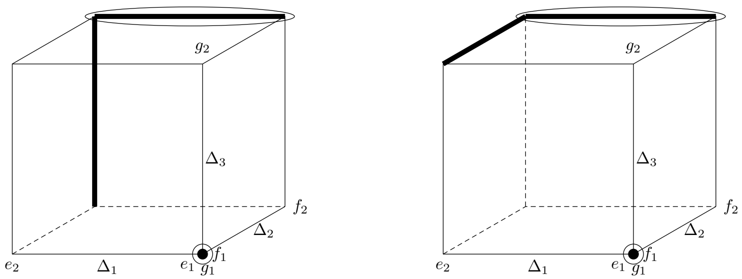
As before, the intersection of the three sets of best reply strategies yields $NE(C, A, B)$, is visualized in Figure 7.3 and is equal to the set of unilaterally supportive strategy combinations with respect to the derangement $(3, 1, 2)$:

$$\begin{aligned} USE_{(3,1,2)}(G) = \big\{(e_1, f_1, g_1)\big\} &\cup \mathrm{Conv}\big\{(e_2, f_1, g_2), (e_2, f_2, g_2)\big\} \\ &\cup \mathrm{Conv}\big\{(e_2, f_2, g_2), (e_1, f_2, g_2)\big\}. \end{aligned}$$

Using Definition 7.2, we thus have that

$$\begin{aligned} USE(G) &= USE_{(2,3,1)}(G) \cap USE_{(3,1,2)}(G) \\ &= \big\{(e_1, f_1, g_1)\big\} \cup \mathrm{Conv}\big\{(e_2, f_2, g_2), (e_1, f_2, g_2)\big\}. \end{aligned}$$

This can be readily seen from Figures 7.4a and 7.4b.



(a) $USE_{(2,3,1)}(G)$. **(b)** $USE_{(3,1,2)}(G)$.

Figure 7.4 – *The two sets $USE_{(2,3,1)}(G)$ and $USE_{(3,1,2)}(G)$ and the intersection $USE(G)$.*

△

Example 7.3 highlights the importance of the use of best reply strategies in the computation of unilateral support equilibria. The following theorem provides an alternative formulation of the set of unilateral support equilibria directly in terms of the payoff functions. This contrasts the definition, which is formulated as the intersection of sets of unilaterally supportive strategy combinations. The theorem clearly highlights the underlying feature of unilaterally supportive behavior, that is, it shows that in a unilateral support equilibrium, every player is supported by every other player individually.

Theorem 7.1 *Let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be a strategic game and let $\hat{x} \in X$ a strategy combination. Then $\hat{x} \in USE(G)$ if and only if, for all $i \in N$ and all $j \in N \setminus \{i\}$, it holds that*

$$\pi_i(\hat{x}_{-j}, \hat{x}_j) \geq \pi_i(\hat{x}_{-j}, x_j) \quad \textit{for all } x_j \in X_j. \tag{7.1}$$

Proof: Let $\hat{x} \in USE(G)$ and let $i \in N, j \in N \setminus \{i\}$ and $x_j \in X_j$. Now, define a bijection $\sigma \in \Pi(N)$ as follows: $\sigma(k) = (k + i - j) \bmod n$ for all $k \in N$. Since $i \neq j$, it follows that $\sigma \in D(N)$. Moreover, $\sigma(j) = i$. Then it holds that

$$\pi_i(\hat{x}_{-j}, \hat{x}_j) = \pi_{\sigma(j)}(\hat{x}_{-j}, \hat{x}_j) \geq \pi_{\sigma(j)}(\hat{x}_{-j}, x_j) = \pi_i(\hat{x}_{-j}, x_j),$$

where the inequality follows from the fact that $\hat{x} \in USE_\sigma(G)$.

For the reverse implication, assume that for all $i \in N$ and all $j \in N \setminus \{i\}$ it holds that

$$\pi_i(\hat{x}_{-j}, \hat{x}_j) \geq \pi_i(\hat{x}_{-j}, x_j) \quad \text{for all } x_j \in X_j. \tag{7.2}$$

Suppose for the sake of contradiction that $\hat{x} \notin USE(G)$. Then there is a derangement $\delta \in D(N)$ such that $\hat{x} \notin USE_\delta(G)$. Accordingly, there is a player $k \in N$ and a strategy $x_k \in X_k$ such that

$$\pi_{\delta(k)}(\hat{x}_{-k}, \hat{x}_k) < \pi_{\delta(k)}(\hat{x}_{-k}, x_k).$$

However, this contradicts Equation (7.2), thus proving that $\hat{x} \in USE(G)$. $\square$

Theorem 7.1 also captures the main difference between a unilateral support equilibrium and a Berge equilibrium: the former is based on individual support, while the latter is based on group support. Group support might cause coordination issues, as was seen in Example 7.1, in which there were no Berge equilibrium. However, the following example shows that unilateral support equilibria do exist in that case. Furthermore, it illustrates the use of Theorem 7.1 to find unilateral support equilibria.

Example 7.4 Reconsider the following trimatrix game $G = (A, B, C)$, as described in Example 7.1:

$$G = (A, B, C) = \begin{array}{c} \\ e_1 \\ e_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_1 & f_2 \end{array} \\ \left[\begin{array}{cc|cc} (1,1,0) & (0,0,0) & (0,0,1) & (0,0,0) \\ (0,0,0) & (0,0,1) & (0,0,0) & (1,1,0) \end{array}\right] \\ \begin{array}{cc} g_1 & g_2 \end{array} \end{array}.$$

As noted before, $BE(G) = \emptyset$. It holds that $(e_1, f_1, g_1) \in USE(G)$. Using the characterization provided in Theorem 7.1, this can be seen from the following six inequalities:

$$i = 1 : \begin{cases} \pi_1(e_1, f_1, g_1) = 1 \geq q_1 = \pi_1(e_1, q, g_1), & \text{for all } q \in \Delta_2; \\ \pi_1(e_1, f_1, g_1) = 1 \geq r_1 = \pi_1(e_1, f_1, r), & \text{for all } r \in \Delta_3, \end{cases}$$

$$i = 2 : \begin{cases} \pi_2(e_1, f_1, g_1) = 1 \geq p_1 = \pi_2(p, f_1, g_1), & \text{for all } p \in \Delta_1; \\ \pi_2(e_1, f_1, g_1) = 1 \geq r_1 = \pi_2(e_1, f_1, r), & \text{for all } r \in \Delta_3, \end{cases}$$

$$i = 3 : \begin{cases} \pi_3(e_1, f_1, g_1) = 0 = \pi_3(p, f_1, g_1), & \text{for all } p \in \Delta_1; \\ \pi_3(e_1, f_1, g_1) = 0 = \pi_3(e_1, q, g_1), & \text{for all } q \in \Delta_2. \end{cases}$$

For the sake of completeness, we show that the set of all unilateral support equilibria is given by

$$USE(G) = \left\{(e_1, f_1, g_1), (e_2, f_2, g_2), \left(\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2, \tfrac{1}{2}f_1 + \tfrac{1}{2}f_2, \tfrac{1}{2}g_1 + \tfrac{1}{2}g_2\right)\right\}.$$

This can be checked by using Theorem 7.1: first, let $i = 1$ and $j = 2$. For all strategy combinations $(p, r) \in \Delta_1 \times \Delta_3$ of players 1 and 3, the pay-off function of player 1 is given by

$$\begin{cases} \pi_1(p, f_1, r) = p_1 r_1; \\ \pi_1(p, f_2, r) = (1 - p_1)(1 - r_1). \end{cases}$$

Consequently, if $p_1 r_1 > (1 - p_1)(1 - r_1)$, or equivalently $p_1 + r_1 > 1$, then player 2 maximizes the pay-off of player 1 by choosing f_1. On the other hand, if $p_1 + r_1 < 1$, then player 2 should choose f_2. Finally, if $p_1 + r_1 = 1$, then player 2 is indifferent and can choose any strategy $q \in \Delta_2$ to maximize the pay-off of player 1. These best choices for player 2 are visualized in Figure 7.5a.

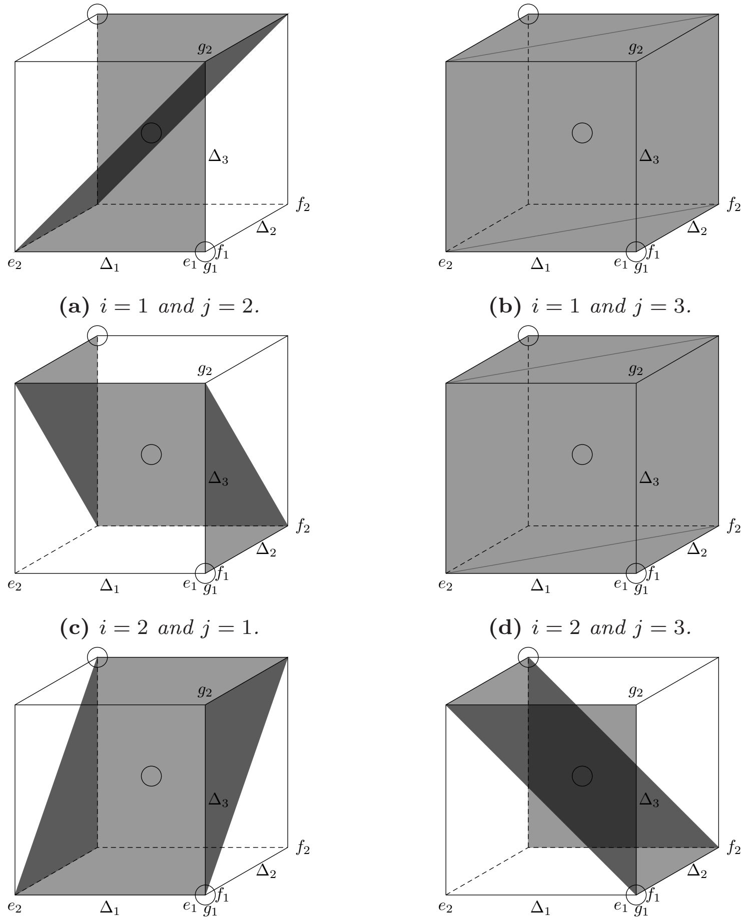
Similarly, for $i = 1$ and $j = 3$, it is seen that the pay-off function of player 1 is given by

$$\begin{cases} \pi_1(p, q, g_1) = p_1 q_1; \\ \pi_1(p, q, g_2) = (1 - p_1)(1 - q_1), \end{cases}$$

for all $(p, q) \in \Delta_1 \times \Delta_2$. Consequently, following a similar reasoning as above, Figure 7.5b visualizes the best choices for player 3 to support player 1.

Proceeding in a similar way, one can obtain the remaining four support relations, as visualized in Figure 7.5. Since according to Theorem 7.1, the set of unilateral support equilibria is equal to the intersection of the depicted sets in Figure 7.5, the result follows. $\triangle$

Example 7.4 shows how one can use the characterization of a unilateral support equilibrium in terms of pay-off functions of Theorem 7.1 to compute the set of unilateral



(a) $i = 1$ *and* $j = 2$.

(b) $i = 1$ *and* $j = 3$.

(c) $i = 2$ *and* $j = 1$.

(d) $i = 2$ *and* $j = 3$.

(e) $i = 3$ *and* $j = 1$.

(f) $i = 3$ *and* $j = 2$

Figure 7.5 – *A visualization of the use of Theorem 7.1 in Example 7.4.*

support equilibria. In fact, for a trimatrix game, this boils down to the exact same analysis as in Example 7.3, where it was shown how Proposition 7.1 can facilitate the process to determine the set of unilateral support equilibria using Definition 7.2. More precisely, both examples evaluated exactly six support relations: in Example 7.4, each of the three players supports exactly each of the two other players, whereas in Example 7.3, two derangements with each three players also lead to the evaluation of six support relations.

However, for strategic games with more than three players, there is a difference in the aforementioned analyses: for Definition 7.2 and the helpful Proposition 7.1, it involves computing the set of Nash equilibria (i.e. the intersection of sets of best reply strategies) for each derangement separately. In doing so, it might occur that the support of a certain player for another player is evaluated multiple times. For example, for four players there are already nine derangements, leading to the evaluation of 36 support relations.

On the other hand, if one uses Theorem 7.1, every support relation is evaluated exactly once: each player supports each of the other players individually. For four players, this leads to the evaluation of only 12 support relations.

The following theorem brings these analyses together. We show that in order to compute the set of unilateral support equilibria, one can restrict to the intersection of the sets of unilaterally supportive strategy combinations with respect to cyclic derangements only.

Theorem 7.2 *Let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be a strategic game. Then it holds that*

$$USE(G) = \bigcap_{\gamma \in C(N)} USE_\gamma(G).$$

Proof: Obviously, $C(N) \subseteq D(N)$. Hence,

$$USE(G) = \bigcap_{\delta \in D(N)} USE_\delta(G) \subseteq \bigcap_{\gamma \in C(N)} USE_\gamma(G).$$

To prove that $\bigcap_{\gamma \in C(N)} USE_\gamma(G) \subseteq USE(G)$, let $\hat{x} \in \bigcap_{\gamma \in C(N)} USE_\gamma(G)$. Using Theorem 7.1, it suffices to show that for all $i \in N$ and all $j \in N \setminus \{i\}$ it holds that

$$\pi_i(\hat{x}_{-j}, \hat{x}_j) \geq \pi_i(\hat{x}_{-j}, x_j) \quad \text{for all } x_j \in X_j.$$

Let $i \in N, j \in N \setminus \{i\}$ and $x_j \in X_j$. Define $\sigma \in \Pi(N)$ in the following cyclic way: $\sigma(k) = (k + i - j) \bmod n$ for all $k \in N$. Clearly, $\sigma(j) = i$ and $\sigma \in C(N)$. Then,

$$\pi_i(\hat{x}_{-j}, \hat{x}_j) = \pi_{\sigma(j)}(\hat{x}_{-j}, \hat{x}_j) \geq \pi_{\sigma(j)}(\hat{x}_{-j}, x_j) = \pi_i(\hat{x}_{-j}, x_j),$$

where the inequality follows from the fact that $\hat{x} \in USE_\sigma(G)$. □

Table 7.1 gives an overview of the number of bijections, the number of derangements and the number of cyclic derangements for a given number of players. It shows that Theorem 7.2 leads to a drastic reduction of the number of sets of unilaterally supportive strategy combinations that have to be computed in order to compute the set of unilateral support equilibria. Moreover, since there are $|N|-1$ cyclic derangements for a player set N, every cyclic derangement is responsible for exactly one support relation for every player. In other words, every player supports another player due to exactly one cyclic derangement.

$\lvert N\rvert$	$\lvert\Pi(N)\rvert$	$\lvert D(N)\rvert$	$\lvert C(N)\rvert$
2	2	1	1
3	6	2	2
4	24	9	3
5	120	44	4
6	720	265	5
7	5040	1854	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lvert N\rvert$	$\lvert N\rvert!$	$\lvert N\rvert! \cdot \sum_{k=0}^{\lvert N\rvert} \frac{(-1)^k}{k!}$	$\lvert N\rvert - 1$

Table 7.1 – *The number of bijections, derangements and cyclic derangements for player set N.*

We conclude this section with several remarks regarding the existence of unilateral support equilibria. First, we provide an example of a trimatrix game in which no unilateral support equilibria exist. This example is inspired by the trimatrix game as discussed in Example 7.1.

Example 7.5 Consider the following trimatrix game $G = (A, B, C)$:

$$G = (A,B,C) = \begin{array}{c} \\ e_1 \\ e_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_1 & f_2 \end{array} \\ \left[\begin{array}{cc|cc} (0,1,1) & (0,0,0) & (1,0,0) & (1,0,0) \\ (0,0,0) & (1,0,0) & (0,0,0) & (0,1,1) \end{array}\right] \\ \begin{array}{cc} g_1 & g_2 \end{array} \end{array}.$$

To show that $USE(G) = \emptyset$, we show that $USE_{(2,3,1)}(G) \cap USE_{(3,1,2)}(G) = \emptyset$. First, consider $\gamma = (2, 3, 1)$. Using Proposition 7.1, we see that

$$USE_{(2,3,1)}(A, B, C) = NE(B, C, A).$$

The pay-off function of player 1 in the game (B, C, A) is thus given by

$$\begin{cases} \tilde{\pi}_1^{BCA}(e_1, q, r) = q_1 r_1; \\ \tilde{\pi}_1^{BCA}(e_2, q, r) = (1 - q_1)(1 - r_1), \end{cases}$$

for all $(q, r) \in \Delta_2 \times \Delta_3$. The set of best reply strategies against (q, r) for player 1 consists of e_1 if $q_1 + r_1 > 1$, while it consists of e_2 if $q_1 + r_1 < 1$. If $q_1 + r_1 = 1$, then the set of best reply strategies against (q, r) for player 1 equals Δ_1. This is visualized in Figure 7.6a.

The pay-off function of player 2 in (B, C, A) is provided by matrix C and thus given by

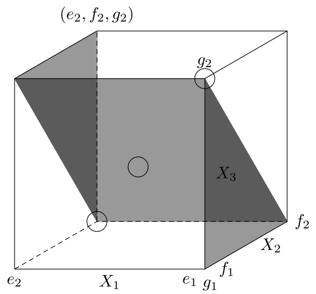
$$\begin{cases} \tilde{\pi}_2^{BCA}(p, f_1, r) = p_1 r_1; \\ \tilde{\pi}_2^{BCA}(p, f_2, r) = (1 - p_1)(1 - r_1), \end{cases}$$

for all $(p, r) \in \Delta_1 \times \Delta_3$. Consequently, the set of best reply strategies against (p, r) for player 2 consists of f_1 if $p_1 + r_1 > 1$, it consists of f_2 if $p_1 + r_1 < 1$ and it equals Δ_2 if $p_1 + r_1 = 1$. This is visualized in Figure 7.6b.

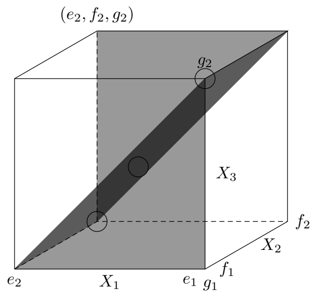
Player 3 supports player 1 in γ such that the pay-off functions of player 3 is given by

$$\begin{cases} \tilde{\pi}_3^{BCA}(p, q, g_1) = (1 - p_1)(1 - q_1); \\ \tilde{\pi}_3^{BCA}(p, q, g_2) = p_1 q_1 + p_1(1 - q_1) = p_1, \end{cases}$$

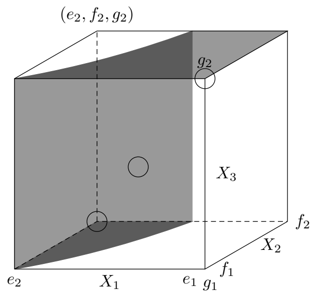
for all $(p, q) \in \Delta_1 \times \Delta_2$. Consequently, the set of best reply strategies against (p, q) for player 3 consists of g_1 if $p_1 < \frac{1-q_1}{2-q_1}$, it consists of g_2 if $p_1 > \frac{1-q_1}{2-q_1}$ and it equals Δ_3 if $p_1 = \frac{1-q_1}{2-q_1}$. This is visualized in Figure 7.6c.



(a) *For player 1.*



(b) *For player 2.*



(c) *For player 3.*

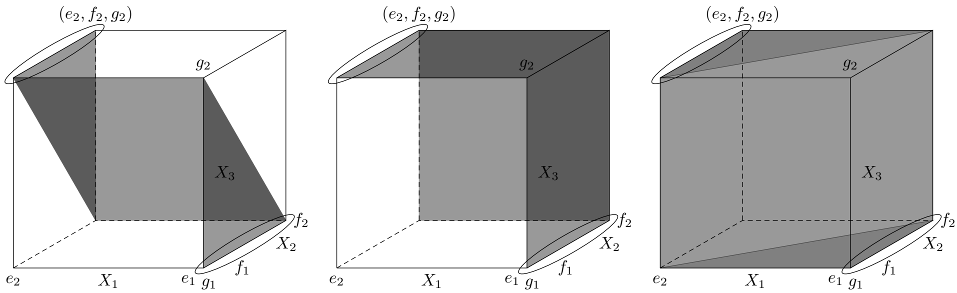
Figure 7.6 – *The three sets of best reply strategies corresponding to $USE_{(2,3,1)}(G)$.*

By intersecting the three sets of best reply strategies as shown in Figure 7.6, we obtain the set of unilaterally supportive strategy combinations with respect to the derangement $(2, 3, 1)$:

$$\begin{aligned} USE_{(2,3,1)}(G) = \big\{&(e_1, f_1, g_2), (e_2, f_2, g_1), \\ &(\alpha e_1 + (1 - \alpha) e_2, \alpha f_1 + (1 - \alpha) f_2, (1 - \alpha) g_1 + \alpha g_2)\big\}, \end{aligned}$$

with $\alpha = \frac{3-\sqrt{5}}{2}$.[3]

[3]This α can be obtained by solving the system of $q_1 + r_1 = 1$, $p_1 + r_1 = 1$ and $p_1 = \frac{1-q_1}{2-q_1}$ for p_1.



(a) *For player 1.* **(b)** *For player 2.* **(c)** *For player 3.*

Figure 7.7 – *The three sets of best reply strategies corresponding to $USE_{(3,1,2)}(G)$.*

Figure 7.7 summarizes a similar analysis when one considers $\gamma = (3, 1, 2)$. Consequently,

$$USE_{(3,1,2)}(G) = \mathrm{Conv}\big\{(e_1, f_1, g_1), (e_1, f_2, g_1)\big\} \cup \mathrm{Conv}\big\{(e_2, f_1, g_2), (e_2, f_2, g_2)\big\}.$$

Hence,

$$USE(G) = USE_{(2,3,1)}(G) \cap USE_{(3,1,2)}(G) = \emptyset. \qquad \triangle$$

Secondly, we show that every Berge equilibrium is also a unilateral support equilibrium.

Theorem 7.3 *Let $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ be a strategic game. Then it holds that $BE(G) \subseteq USE(G)$.*

Proof: Let $\hat{x} \in BE(G)$. Moreover, let $\delta \in D(N)$, $i \in N$ and $x_i \in X_i$. Then it holds that[4]

$$\begin{aligned}\pi_{\delta(i)}(\hat{x}_{-i}, \hat{x}_i) &= \pi_{\delta(i)}(\hat{x}_{\delta(i)}, \hat{x}_{-i,-\delta(i)}, \hat{x}_i) \\ &\geq \pi_{\delta(i)}(\hat{x}_{\delta(i)}, \hat{x}_{-i,-\delta(i)}, x_i) = \pi_{\delta(i)}(\hat{x}_{-i}, x_i),\end{aligned}$$

where the inequality follows from the fact that $\hat{x} \in BE(G)$, since the group of all players except $\delta(i)$ support player $\delta(i)$. Hence, $\hat{x} \in USE_\delta(G)$ for all $\delta \in D(N)$ and consequently, $\hat{x} \in USE(G)$. $\square$

Theorem 7.3 allows for the use of existence theorems for Berge equilibria to guarantee the existence of unilateral support equilibria, as also pointed out by Crettez and Nessah (2020). For example, Radjef (1988), Abalo and Kostreva (2004), Nessah et al.

[4] Here, $\hat{x}_{-i,-\delta(i)}$ is the notation for the strategy combination induced by $\hat{x}$ for the players in $N \setminus \{i, \delta(i)\}$. Consequently, $(\hat{x}_{-i,-\delta(i)}, x_i) \in X_{-\delta(i)}$.

(2007) and Larbani and Nessah (2008), among others, focused on finding existence theorems for Berge equilibria.

More recently, Crettez and Nessah (2020) studied, as a follow-up of Schouten et al. (2019), specific existence theorems for unilateral support equilibria. Theorem 7.4 below formulates the sufficient conditions to guarantee a unilateral support equilibrium.

Formally, for a strategic game $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ and a strategy combination $x \in X$, the so-called *set of unilateral best reply strategy combinations* (cf. Crettez and Nessah, 2020) is given by

$$\begin{aligned} C(x) = \{y \in X \mid \text{ for all } i \in N \text{ and all } j \in N \setminus \{i\} : \\ \pi_i(y_j, x_{-j}) \geq \pi_i(z_j, x_{-j}) \text{ for all } z_j \in X_j\}. \end{aligned} \quad (7.3)$$

Theorem 7.4 [cf. Crettez and Nessah, 2020] *Let $G = (N, \{X_i\}_{i\in N}, \{\pi_i\}_{i\in N})$ be a strategic game. If the following four conditions hold:*

i) for all $i \in N$, X_i is closed and bounded;

ii) for all $i \in N$, $\pi_i : X \to \mathbb{R}$ is continuous;

iii) for all $i \in N$ and all $j \in N \setminus \{i\}$, $g_j : X_j \to \mathbb{R}$ defined by $g_j(x_j) = \pi_i(x_j, x_{-j})$ is quasiconcave[5] for all $x_{-j} \in X_{-j}$;

iv) for all $x \in X$, $C(x) \neq \emptyset$,

then $USE(G) \neq \emptyset$.

Recall that in Example 7.5, no unilateral support equilibria exist. Indeed, the trimatrix as discussed in Example 7.5 does not satisfy the fourth condition, as is seen in the following example.

Example 7.6 Reconsider the following trimatrix game $G = (A, B, C)$, as described in Example 7.5:

$$G = (A,B,C) = \begin{array}{c} \\ e_1 \\ e_2 \\ \\ \end{array} \begin{array}{cc|cc} f_1 & f_2 & f_1 & f_2 \\ \left[\begin{matrix} (0,1,1) \\ (0,0,0) \end{matrix}\right. & \begin{matrix} (0,0,0) \\ (1,0,0) \end{matrix} & \begin{matrix} (1,0,0) \\ (0,0,0) \end{matrix} & \left.\begin{matrix} (1,0,0) \\ (0,1,1) \end{matrix}\right] \\ \multicolumn{2}{c}{g_1} & \multicolumn{2}{c}{g_2} \end{array}.$$

We show that $C\big((e_1, f_1, g_1)\big) = \emptyset$.

[5] A function $f : S \to \mathbb{R}$ is called quasiconcave if S is convex and, for all $x, y \in S$ and all $\lambda \in [0, 1]$ it holds that $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$.

Suppose for the sake of contradiction that $y \in C\big((e_1, f_1, g_1)\big)$. Following the quantifiers of Equation (7.3), consider $i = 1$ and $j = 3$. Then it follows that, for all $r \in \Delta_3$,

$$\pi_1(e_1, f_1, y_3) \geq \pi_1(e_1, f_1, r).$$

This implies that $y_3 = g_2$, since $\pi_1(e_1, f_1, g_2) = 1$, while $\pi_1(e_1, f_1, g_1) = 0$.

On the other hand, consider $i = 2$ and $j = 3$. Then it follows that, for all $r \in \Delta_3$,

$$\pi_2(e_1, f_1, y_3) \geq \pi_2(e_1, f_1, r).$$

This implies that $y_3 = g_1$, since $\pi_2(e_1, f_1, g_1) = 1$, while $\pi_2(e_1, f_1, g_2) = 0$. This clearly yields a contradiction. $\triangle$

7.4 Unilateral support equilibria and Nash equilibria

In this last section of this chapter, we study the relation between unilateral support equilibria and Nash equilibria. In particular, we focus on strategy combinations that are both a unilateral support equilibrium and a Nash equilibrium.

In a unilateral support equilibrium, every player is supported by every other player individually. In a Nash equilibrium, which is a strategy combination that is unilaterally supportive with respect to the identity, every player supports himself. Together this implies that if a strategy combination is both a unilateral support equilibrium and a Nash equilibrium, every player is supported by every player, including himself. This is captured in the following theorem.

Theorem 7.5 *Let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be a strategic game. Then it holds that*

$$USE(G) \cap NE(G) = \bigcap_{\sigma \in \Pi(N)} USE_\sigma(G).$$

Proof: First, let $N = \{1, 2\}$. As seen before, $USE(G) = USE_{(2,1)}(G)$ and $NE(G) = USE_{(1,2)}(G)$. Consequently,

$$USE(G) \cap NE(G) = USE_{(2,1)}(G) \cap USE_{(1,2)}(G) = \bigcap_{\sigma \in \Pi(N)} USE_\sigma(G).$$

Secondly, let $|N| \geq 3$. For the first inclusion, let $\hat{x} \in \bigcap_{\sigma \in \Pi(N)} USE_\sigma(G)$. Then it holds that $\hat{x} \in USE_\sigma(G)$ for all $\sigma \in D(N)$ and $\hat{x} \in USE_{\sigma_{id}}(G)$. The former implies that $\hat{x} \in USE(G)$, while the latter implies that $\hat{x} \in NE(G)$. Consequently, $\bigcap_{\sigma \in \Pi(N)} USE_\sigma(G) \subseteq USE(G) \cap NE(G)$.

For the reverse inclusion, let $\hat{x} \in USE(G) \cap NE(G)$. Since $\hat{x} \in USE(G)$, it holds that $\pi_i(\hat{x}_{-j}, \hat{x}_j) \geq \pi_i(\hat{x}_{-j}, x_j)$ for all $i \in N$, all $j \in N \setminus \{i\}$ and all $x_j \in X_j$, according to Theorem 7.1. Moreover, since $\hat{x} \in NE(G)$, it holds that $\pi_i(\hat{x}_{-i}, \hat{x}_i) \geq \pi_i(\hat{x}_{-i}, x_i)$ for all $i \in N$ and all $x_i \in X_i$, according to the definition of a Nash equilibrium. Hence,

$$\pi_i(\hat{x}_{-j}, \hat{x}_j) \geq \pi_i(\hat{x}_{-j}, x_j) \quad \text{for all } i, j \in N \text{ and all } x_j \in X_j. \tag{7.4}$$

It suffices to prove that $\hat{x} \in USE_\sigma(G)$ for all $\sigma \in \Pi(N) \setminus D(N)$ with $\sigma \neq \sigma_{id}$. Let $\sigma \in \Pi(N) \setminus D(N), \sigma \neq \sigma_{id}$. As a direct consequence of Equation (7.4), we obtain

$$\pi_{\sigma(i)}(\hat{x}_{-i}, \hat{x}_i) \geq \pi_{\sigma(i)}(\hat{x}_{-i}, x_i),$$

for all $i \in N$ and all $x_i \in X_i$. Hence, $\hat{x} \in USE_\sigma(G)$. $\square$

An alternative characterization of a strategy combination that is both a unilateral support equilibrium and a Nash equilibrium is based on Nash equilibria of so-called coordination games.

Formally, a strategic game $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ is a *coordination game* if $\pi_i = \pi_j$ for all $i, j \in N$. In the light of supportive behavior, this means that if a player supports another player, he also supports himself. Interestingly, this fact can be used to describe the intersection between the set of unilateral support equilibria and the set of Nash equilibria for general games. First, we show that for a coordination game, the set of unilateral support equilibria coincides with the set of Nash equilibria.

Proposition 7.2 *Let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be a coordination game. Then, for all $\sigma \in \Pi(N)$, it holds that*

$$USE_\sigma(G) = NE(G).$$

Consequently, $USE(G) = NE(G)$.

Proof: Let $\sigma \in \Pi(N)$. Proposition 7.1 implies that $USE_\sigma(G) = NE(G_\sigma)$, where G_σ is the game with twisted pay-off functions. Since G is a coordination game, $G_\sigma = G$. Consequently, $USE_\sigma(G) = NE(G_\sigma) = NE(G)$. $\square$

Next, we reformulate Theorem 7.5 in terms of coordination games. More specifically, it turns out that every strategy combination that is both a unilateral support equilibrium and a Nash equilibrium corresponds to a Nash equilibrium of all coordination games in which all players face the pay-off function of a single player.

For a given strategic game $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ and a player $k \in N$, we can define the coordination game $G_k = (N, \{X_i\}_{i \in N}, \{\pi_k\}_{i \in N})$, in which every player faces the pay-off function of player k.

Theorem 7.6 *Let $G = (N, \{X_i\}_{i \in N}, \{\pi_i\}_{i \in N})$ be a strategic game. Then it holds that*

$$USE(G) \cap NE(G) = \bigcap_{k \in N} NE(G_k).$$

Proof: First, similar to Equation (7.4) in the proof of Theorem 7.5, note that for a strategy combination $\hat{x} \in X$ it holds that $\hat{x} \in USE(G) \cap NE(G)$ if and only if

$$\pi_i(\hat{x}_{-j}, \hat{x}_j) \geq \pi_i(\hat{x}_{-j}, x_j) \quad \text{for all } i, j \in N \text{ and all } x_j \in X_j. \tag{7.5}$$

For the first inclusion, let $\hat{x} \in USE(G) \cap NE(G)$ and let $k \in N$. As a direct consequence of Equation (7.5), we obtain

$$\pi_k(\hat{x}_{-i}, \hat{x}_i) \geq \pi_k(\hat{x}_{-i}, x_i),$$

for all $i \in N$ and all $x_i \in X_i$. This implies that $\hat{x} \in NE(G_k)$.

For the reverse inclusion, let $\hat{x} \in \bigcap_{k \in N} NE(G_k)$. It suffices to prove Equation (7.5). However, this follows immediately from the fact that $\hat{x} \in NE(G_i)$ for all $i \in N$. $\square$

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Jop Schouten (Hengelo, The Netherlands, 1993) received both his Bachelor's degree and Master's degree in Mathematics from Radboud University Nijmegen in 2014 and 2016, respectively, followed by a Research Master's degree in Operations Research from Tilburg University in 2017. In September 2017, he started his PhD in Operations Research at the department of Econometrics and Operations Research at Tilburg University.

Game theory is the mathematical theory to analyze the behavior of rational decisionmakers in both cooperative and strategic interactive situations. It aims to resolve these situations by developing mathematical models and applying mathematical tools to provide insights in the interactive decision-making process. This dissertation studies the theoretical model of a transferable utility game with limited cooperation possibilities as well as altruistic equilibrium concepts for the model of a strategic game. Furthermore, this dissertation deals with several interactive allocation and operations research problems related to claims, sequencing and purchasing situations in which both cooperative and strategic approaches play a role.

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