Constructive Approximation
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CONSTRUCTIVE APPROXIMATION

Electrostatic Partners and Zeros of Orthogonal and Multiple Orthogonal Polynomials

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Abstract

For a given polynomial P with simple zeros, and a given semiclassical weight w, we present a construction that yields a linear second-order differential equation (ODE), and in consequence, an electrostatic model for zeros of P. The coefficients of this ODE are written in terms of a dual polynomial that we call the electrostatic partner of P. This construction is absolutely general and can be carried out for any polynomial with simple zeros and any semiclassical weight on the complex plane. An additional assumption of quasi-orthogonality of P with respect to w allows us to give more precise bounds on the degree of the electrostatic partner. In the case of orthogonal and quasi-orthogonal polynomials, we recover some of the known results and generalize others. Additionally, for the Hermite–Padé or multiple orthogonal polynomials of type II, this approach yields a system of linear second-order differential equations, from which we derive an electrostatic interpretation of their zeros in terms of a vector equilibrium. More detailed results are obtained in the special cases of Angelesco, Nikishin, and generalized Nikishin systems. We also discuss the discrete-to-continuous transition of these models in the asymptotic regime, as the number of zeros tends to infinity, into the known vector equilibrium problems. Finally, we discuss how the system of obtained

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second-order ODEs yields a third-order differential equation for these polynomials, well described in the literature. We finish the paper by presenting several illustrative examples.

Keywords Orthogonal polynomials · Multiple orthogonal polynomials · Zeros · Electrostatic model · Equilibrium · Linear differential equations

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1 Introduction

Hermite polynomials[1]

$$H_N(x) = N! \sum_{\ell=0}^{[N/2]} \frac{(-1)^\ell (2x)^{N-2\ell}}{\ell!\,(N-2\ell)!} = 2^N x^N + \dots \tag{1.1}$$

are probably the simplest representatives of the family of *classical* orthogonal polynomials. They satisfy the linear differential equation

$$y''(x) - 2xy'(x) + 2Ny(x) = 0 \tag{1.2}$$

and the orthogonality conditions

$$\begin{aligned} &\int_{-\infty}^{+\infty} x^j H_N(x) e^{-x^2}\, dx = 0, \qquad j = 0, 1, \dots, N-1, \\ &\int_{-\infty}^{+\infty} x^N H_N(x) e^{-x^2}\, dx \neq 0. \end{aligned}$$

As a consequence, their zeros are all real and simple. A well-known calculation that goes back to Stieltjes [94] (see also [97, Theorem 6.8] or [98]) shows that there are two equivalent physical interpretations of these zeros:

- As equilibrium positions of equally charged points on the plane in the presence of an external field (background potential); or
- As an appropriately rescaled configuration of vortices on the plane under assumption that they all have same circulations and rotate as a rigid body.

We explain these notions in more detail in Sect. 2. Both models are rooted in the linear second order differential equation (1.2) satisfied by these polynomials. There are several ways of obtaining this equation, all of them relying on a specific feature of the orthogonality weight, namely the fact that its logarithmic derivative is a rational function. This idea allows to extend the classical theory to the so-called *semiclassical*

1 $[\cdot]$ stands for the integer part or the floor function.

orthogonal polynomials, a construction that probably goes back to J. Shohat [90], see also [68, 69]. This generalization preserves several convenient features of classical orthogonal polynomials, such as a Rodrigues-type formula or the existence of raising and lowering operators, see e.g. [49] or [52]; each one of these properties leads to (1.2).

The elegance of the above mentioned model attracted attention of generations of researchers and lead to several generalizations. For instance, we can choose as a starting point a second-order linear differential equation with polynomial coefficients (*generalized Lamé equations* in algebraic form), whose particular cases are the hypergeometric and the Heun differential equation [87], and develop an electrostatic/vortex dynamics interpretation for the zeros of its polynomial solutions (*Heine–Stieltjes polynomials*). This was carried out in the classical works of Bôcher [23], Heine [48] and Van Vleck [105]; for more modern treatment, see e.g. [39, 40, 47, 67], as well as [70–72] for the asymptotic results.

Another approach starts from the orthogonality conditions with respect to a semi-classical (or even a more general) weight, as it was done in the pioneering work of Ismail [49–51], which has been extended in many directions, see e.g. [27, 44, 88, 91, 93, 103, 104], to cite a few. One of such generalizations is the case of *quasi-orthogonal polynomials*, which satisfy "incomplete" orthogonality conditions. As it was shown in [53] in the simplest case of one condition short of full orthogonality, such polynomials also satisfy a linear second order differential equation that can be interpreted in electrostatic terms.

Hermite–Padé or *multiple* orthogonal polynomials (MOP) of type II are defined by distributing the orthogonality conditions among different weights or measures. In the simplest case of two weights w_1, w_2, supported on $\Delta_1 \subset \mathbb{R}$ and $\Delta_2 \subset \mathbb{R}$, respectively, and for a given multi-index $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2_{\geq 0}$, it is a polynomial $P_{\boldsymbol{n}}$ of total degree at most $N = |\boldsymbol{n}| := n_1 + n_2$, such that

$$\int_{\Delta_i} x^j P_{\boldsymbol{n}}(x) w_i(x) dx \begin{cases} = 0, & j \leq n_i - 1, \\ \neq 0, & j = n_i, \end{cases} \qquad i = 1, 2. \tag{1.3}$$

Polynomial $P_{\boldsymbol{n}}$ appears as a common denominator of a pair of rational approximants to Markov functions related to the weights w_i's, see Sect. 5.1. For a more detailed account of the corresponding theory we recommend the monograph [81], as well as the works of Aptekarev, Gonchar, Kuijlaars, Rakhmanov, Stahl, Suetin, Van Assche, Yattselev, and others, see e.g. [4, 5, 7, 12–14, 21, 46, 84–86, 99–102] (the list is far from complete). MOP find applications in number theory, numerical analysis, integrable systems, interacting particle systems and random matrix models [9, 11, 59, 76]. Although the general analytic theory of multiple orthogonal polynomials is in its infancy, their zeros (especially, their asymptotic behavior) have been studied in several particular situations, known as the Angelesco and Nikishin cases, described in Sect. 5.2. However, there is no known electrostatic interpretation of these zeros. Linear differential equations satisfied by multiple orthogonal polynomials have been found for many families of polynomials, see e.g. [7, 12, 32, 43], but in all cases these are equations of order 3 and higher. The problem is that an electrostatic interpretation of the solutions to these ODE is not straightforward.

Our main goal is to present a unified construction that yields an electrostatic model for polynomials related with a (system of) semiclassical weights. As it was mentioned, the known constructions of the differential equations for polynomials use either a Rodrigues formula or the so-called raising and lowering operators that can be combined into a single ODE [32, 43, 52]. Instead of that, we start from a construction that can associate to an arbitrary polynomial P and to a semiclassical weight w supported on a set $\Delta \subset \mathbb{C}$ another polynomial S, that we have called its *electrostatic partner*, see Definition 3.5 and the schematic representation below.

Using S and w we can write a linear second order differential equation with polynomial coefficients whose solutions are P and the corresponding function of the second kind q, defined in Sect. 3. This shows that the zeros of P (assumed simple) are in an electrostatic equilibrium in an external field created by w and by the attracting zeros of S (understanding by this a stationary point of their energy, and not necessarily its local or global minimum). This construction uses only the semiclassical character of w; no orthogonality conditions on P are required. An additional assumption that P is quasi-orthogonal with respect to w (in the complex case, we mean by that a non-hermitian orthogonality, see (4.1)) allows us to make more precise statements about the electrostatic partner of P. Moreover, two alternative representations for S in this case yield a generalization of an identity involving Wronskian and Casorati determinants of P and q, known in the case of classical orthogonal polynomials, see Sect. 4.

Since the definition of type II Hermite–Padé orthogonal polynomials (1.3) boils down to two simultaneous quasi-orthogonality conditions, we can associate with the corresponding MOP $P_{\boldsymbol{n}}$ *two* electrostatic partners, $S_{\boldsymbol{n},1}$ and $S_{\boldsymbol{n},2}$, and a system of two linear differential equations of order 2, whose solution is $P_{\boldsymbol{n}}$. This apparent redundancy can be used to find an electrostatic model for the zeros of $P_{\boldsymbol{n}}$. Namely, by a procedure similar to the definition of an electrostatic partner, we associate with $P_{\boldsymbol{n}}$, w_1 and w_2 a polynomial $R_{\boldsymbol{n}}$:

With this construction, the zeros of $P_{\boldsymbol{n}}$ and the zeros of $S_{\boldsymbol{n},1}$ (or $S_{\boldsymbol{n},2}$) are in a vector equilibrium given by their mutual interaction and by the vector external field created by w_1, w_2 and the zeros of $R_{\boldsymbol{n}}$, see Sect. 5 for details. This model is especially convenient because it is known that the asymptotic distribution of the zeros of $P_{\boldsymbol{n}}$ is usually described by vector equilibria. We discuss this connection and provide some heuristic arguments for this discrete-to-continuous transition in Sect. 6, where several particular configurations are analyzed in detail. So far, both two- and three-component critical vector measures have been used to describe asymptotics in several cases. Our construction suggests that there is one universal two-component vector equilibrium valid for all known configurations corresponding to perfect systems, and that all descriptions constitute just its particular manifestations.

In order to establish another connection with previous literature, we describe in Sect. 7 how to combine the system of ODEs from Sect. 5 into a third order linear differential equation whose solutions are $P_{\boldsymbol{n}}$ and the corresponding functions of the second kind. An additional advantage of this construction is that it is possibly generalized to the case of more than 2 weights and explains the appearance of higher order ODEs (see Remark 7.2 in Sect. 7).

In the last section we discuss several examples of multiple orthogonal polynomials well known in the literature.

We hope that this approach can be applied in some other contexts; in particular, it would be interesting to explore a possible electrostatic interpretation of the zeros of the Type I multiple orthogonal polynomials, see e.g. [81].

Since this paper unfortunately contains a large amount of technical details and auxiliary results (some of them, relegated to Appendices A and B), we finish this introduction with a short navigation guide for the reader interested in the main highlights:

- An electrostatic partner S of a given polynomial P (in a sense, the starting fundamental construction) is introduced in Definition 3.5, whose consistency is justified by Theorem 3.3.
- The second order linear differential equations whose solution is P is introduced in Theorem 3.7, which leads to an electrostatic model (Proposition 3.9) for zeros of P, which are shown to be in equilibrium in a field created in part by the attracting zeros of the electrostatic partner S. Additional properties of S under assumptions that P satisfies some orthogonality conditions are discussed in Sect. 4.
- This construction is extended to a type II Hermite–Padé orthogonal polynomial with respect to two weights, giving us now two second order linear differential equations (Theorem 5.2). Additionally, we get another set of differential equations, now for the electrostatic partners (Theorem 5.7, which is based on a construction from Proposition 5.5).
- As a consequence, we derive a vector equilibrium model for the two sets of point charges: the zeros of the Hermite–Padé orthogonal polynomial and the zeros of its electrostatic partner(s), Theorem 5.10.
- More precise results about the location of the zeros of the electrostatic partners in some widely studied cases of Hermite–Padé orthogonal polynomials are matter of Sect. 5.2. They allow us to discuss in Sect. 6 the discrete-to-continuous transition in the equilibrium model as the degrees tend to ∞, and to compare the resulting models with the description of the asymptotic distribution of zeros in terms of the vector equilibrium, both with 2 and 3 components. In particular, Corollary 6.2 suggests that the universal description can be achieved using a 2-component vector

equilibrium with the interaction matrix

$$\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}.$$

- Since third order linear differential equations associated to the multiple orthogonal polynomials are well known in the literature, we have included in Sect. 7 their derivation from the system of second order ODEs described in Theorem 5.2.
- Last, but definitely not least, we have a set of curious examples in Sect. 8, whose examination poses several interesting questions and suggests possible lines of further research.

2 Electrostatics of Point Charges and Vortex Dynamics

2.1 Identical Point Charges and Vortices

We can associate with N pairwise distinct points ζ_i on the plane ($\zeta_i \neq \zeta_j$ for $i \neq j$) their discrete "counting" measure

$$\mu = \sum_{k=1}^{N} \delta_{\zeta_k}, \tag{2.1}$$

where δ_x is a unit mass (Dirac delta) at x, and define its (discrete) logarithmic energy[2]

$$\mathcal{E}(\mu) := \sum_{i \neq j} \log \frac{1}{|\zeta_i - \zeta_j|} \tag{2.2}$$

(we can extend the notion of the energy to the case when two or more ζ_j's coincide by assuming that then $\mathcal{E}(\mu) = +\infty$). Additionally, given a real-valued function (*external field* or *background potential*) φ, finite at $\operatorname{supp}(\mu)$, we consider the weighted energy

$$\mathcal{E}_\varphi(\mu) := \mathcal{E}(\mu) + 2 \sum_{k=1}^{N} \varphi(\zeta_k). \tag{2.3}$$

For our purposes, it will be sufficient to assume that $\varphi = \operatorname{Re} \Phi$, where Φ is an analytic (in general, multivalued) function in $\mathbb{C}$, excluding its (finite number) of isolated singularities and branch points, with a single-valued derivative Φ'.

Definition 2.1 [71] We say that μ in (2.1) is φ-*critical* or just *critical measure* if $\operatorname{supp}(\mu)$ is disjoint with the set of singularities of φ, and is a stationary point of the

[2] Actually, the magnitude in (2.2) is twice the logarithmic energy, which is not relevant, but explains the factor 2 in (2.3) introduced for consistency.

weighted discrete energy $\mathcal{E}_\varphi(\mu) = \mathcal{E}_\varphi(\zeta_1, \dots, \zeta_N)$ defined in (2.3) :

$$\nabla \mathcal{E}_\varphi(\zeta_1, \dots, \zeta_N) = 0\,, \tag{2.4}$$

or equivalently,

$$\frac{\partial}{\partial z} \mathcal{E}_\varphi(\zeta_1, \dots, z, \dots \zeta_N)\big|_{z=\zeta_k} = 0, \quad k = 1, \dots, N, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

We also say that the configuration of points (or charges) is in *electrostatic equilibrium* in the external field φ. Notice that with $\varphi = \operatorname{Re} \Phi$, we can write explicitly the equilibrium conditions for $\mathcal{E}_\varphi(\zeta_1, \dots, \zeta_N)$ as the system of equations

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{\zeta_j - \zeta_i} - \Phi'\left(\zeta_j\right) = 0, \quad j = 1, \dots, N. \tag{2.5}$$

Let

$$y(z) := \prod_{j=1}^{N} (z - \zeta_j); \tag{2.6}$$

a common terminology is that μ in (2.1) is the *zero counting measure* of the polynomial y, for which we will use the notation

$$\nu(y) := \sum_{j=1}^{n} \delta_{\zeta_j}. \tag{2.7}$$

It is easy to check that

$$y'(z) = y(z) \sum_{j=1}^{N} \frac{1}{z - \zeta_j}, \qquad y''(z) = y(z) \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{(z - \zeta_i)(z - \zeta_j)}, \tag{2.8}$$

from where

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{\zeta_j - \zeta_i} = \frac{y''}{2y'}(\zeta_j).$$

In particular, (2.5) is equivalent to

$$\left(y'' - 2\Phi' y'\right)\left(\zeta_j\right) = 0, \quad j = 1, \dots, N. \tag{2.9}$$

In the case when all zeros ζ_j's are on the real line and the external field is given by $\varphi(x) = x^2/2$, we get from (2.9) that $y'' - 2xy'$ matches y up to a multiplicative constant. Comparing the leading coefficients we conclude that y solves the differential equation (1.2); in other words, the zeros of Hermite polynomials are in electrostatic equilibrium on $\mathbb{R}$ in the external field $\varphi(x) = x^2/2$, as observed by Stieltjes in [94].[3] He also realized that this electrostatic model is easily generalized to all classical families of polynomials (Jacobi, Laguerre and Bessel), see [95, 96], or [97] and [52] for a more modern account.

Another approach to zeros of these polynomials is via vortex dynamics. The notion of a point vortex is a classical approximation in ideal hydrodynamics of planar flow, introduced almost 150 years ago in Helmholtz's classical paper on vortex dynamics [106]. Considering the flow plane to be the complex plane, the equations of motion for N point vortices with circulations $\gamma_i \in \mathbb{R}$ at positions ζ_i, $i = 1, \ldots, N$, in a background flow Ψ, is

$$\overline{\left(\frac{d\zeta_j}{dt}\right)} = \frac{1}{2\pi i} \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{\gamma_i}{\zeta_j - \zeta_i} + \frac{1}{2\pi i} \overline{\Psi(\zeta_j)}, \quad j = 1, 2, \ldots, N. \tag{2.10}$$

In this paper, the overline indicates complex conjugation.

By (2.5), stationary vortices ($d\zeta_j/dt = 0$) correspond to electrostatic equilibrium in the external field $\varphi = \operatorname{Re} \Phi$ if the background flow Ψ satisfies $\overline{\Psi(\zeta_j)} = -\Phi'(\zeta_j)$, $j = 1, \ldots, N$.

Alternatively, if a vortex configuration rotates as a rigid body with angular velocity Ω, then $\overline{d\zeta_j/dt}$ is equal to $\overline{\zeta_j}$ times a purely imaginary constant proportional to the angular velocity. If we assume additionally that all ζ_j's are real and identical (all γ_i's are equal) and there is no external flow field, then after rescaling (2.10) boils down to

$$\zeta_j = \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{1}{\zeta_j - \zeta_i}, \quad j = 1, 2, \ldots, N. \tag{2.11}$$

Let us use again the polynomial y defined in (2.6), known in this field as the *generating polynomial* for the vortex configuration (see [78]). We can rewrite the second identity in (2.8) equivalently as

$$y''(z) = -2y(z) \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{1}{(\zeta_i - \zeta_j)(z - \zeta_j)}, \tag{2.12}$$

which together with (2.11) yields again the differential equation (1.2). Thus, the zeros of the Hermite polynomials give us the positions of vortices on $\mathbb{R}$ such that the configuration rotates like a rigid body. Clearly, these considerations can be extended to more general families of polynomials.

A reader interested in vortex dynamics should check the nice surveys [15] and [30].

[3] As it is pointed out in [98], Stieltjes mentions without proving that the equilibrium configuration is actually the minimum of the energy. The proof can be found for instance in [97, Sect. 6.7].

2.2 Groups of Point Charges and Vortices

We can extend Definition 2.1 to a vector setting (for our purpose, it will be sufficient to consider two-component vectors) that allows us to handle groups of differently charged particles. Given a vector of discrete measures $\vec{\mu} = (\mu_1, \mu_2)$, with $\mathrm{supp}(\mu_1) \cap \mathrm{supp}(\mu_2) = \emptyset$,

$$\mu_1 = \sum_{k=1}^{n_1} \delta_{\zeta_k}, \qquad \mu_2 = \sum_{j=1}^{n_2} \delta_{\xi_j}, \tag{2.13}$$

a real *interaction parameter* $-1 < a < 1$, and a vector external field $\vec{\varphi} = (\varphi_1, \varphi_2)$, both φ_i real-valued and finite at $\mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2)$, the corresponding *weighted vector energy* is

$$\mathcal{E}_{\vec{\varphi},a}(\vec{\mu}) := \mathcal{E}(\mu_1) + 2a \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} \log \frac{1}{|\zeta_k - \xi_j|} + \mathcal{E}(\mu_2) + 2 \sum_{k=1}^{n_1} \varphi_1(\zeta_k) + 2 \sum_{j=1}^{n_2} \varphi_2(\xi_j) \tag{2.14}$$

(see the notation in (2.2)). We can restate this definition using vector notation and the symmetric positive-definite matrix

$$M := \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$$

and say that the weighted vector energy in (2.14) corresponds to the *interaction matrix* M. Moreover, for measures (2.13) we can write alternatively

$$\mathcal{E}_{\vec{\varphi},a}(\zeta_1, \ldots, \zeta_{n_1}, \xi_1, \ldots, \xi_{n_2}) := \mathcal{E}_{\vec{\varphi},a}(\vec{\mu}).$$

Definition 2.2 We say that $\vec{\mu}$ is a *critical vector measure* for $\mathcal{E}_{\vec{\varphi},a}$ if $\mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2)$ is a stationary configuration for $\mathcal{E}_{\vec{\varphi},a}$:

$$\nabla \mathcal{E}_{\vec{\varphi},a}(\zeta_1, \ldots, \zeta_{n_1}, \xi_1, \ldots, \xi_{n_2}) = 0.$$

For any Borel measure μ on $\mathbb{C}$ we can define its *logarithmic potential*,

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, d\mu(t). \tag{2.15}$$

From the expression for $\mathcal{E}_{\vec{\varphi},a}$ it follows that Definition 2.2 is equivalent to simultaneous equilibrium conditions

$$\begin{aligned} &\mu_1 \text{ is } F_1\text{-critical, with } F_1 := a\, U^{\mu_2} + \varphi_1, \\ &\mu_2 \text{ is } F_2\text{-critical, with } F_2 := a\, U^{\mu_1} + \varphi_2. \end{aligned} \tag{2.16}$$

Alternatively, consider the situation when in the absence of the background flow, the circulations γ_i's in (2.10) take only two possible values,

$$\gamma_k = \begin{cases} \gamma > 0, & \text{for} \quad k = 1, 2, \ldots, n_1, \\ -\gamma, & \text{for} \quad k = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 = N. \end{cases}$$

We can rename $\xi_k := \zeta_{n_1+k}$, $k = 1, \ldots, n_2$; in this way, for stationary vortices we have the equations

$$\begin{aligned} \sum_{\substack{i=1 \\ i \neq j}}^{n_1} \frac{1}{\zeta_j - \zeta_i} &= \sum_{k=1}^{n_2} \frac{1}{\zeta_j - \xi_k}, \quad j = 1, 2, \ldots, n_1, \\ \sum_{\substack{i=1 \\ i \neq j}}^{n_2} \frac{1}{\xi_j - \xi_i} &= \sum_{k=1}^{n_1} \frac{1}{\xi_j - \zeta_k}, \quad j = 1, 2, \ldots, n_2. \end{aligned} \tag{2.17}$$

To study these vortex patterns, we define again the generating polynomials

$$y(z) = \prod_{j=1}^{n_1} \left(z - \zeta_j\right), \quad v(z) = \prod_{k=1}^{n_2} (z - \xi_k).$$

Formulas (2.8) and (2.12) show that (2.17) yield the bilinear identity

$$y''v - 2y'v' + yv'' = 0. \tag{2.18}$$

This is currently known as Tkachenko's equation, since it was first derived by Tkachenko in his dissertation in 1964. Polynomial solutions of this equation were studied by Burchnall and Chaundy [25]. Adler and Moser [1] showed that (2.18) is solved by two consecutive polynomials that nowadays are know as Adler-Moser polynomials, see also [36]. Moreover, comparing (2.17) with (2.5) and (2.16) we conclude that the zeros of consecutive Adler-Moser polynomials are stationary configurations (or equivalently, $\vec{\mu} = (\mu_1, \mu_2)$ defined in (2.13) is a critical vector measure) for the vector energy $\mathcal{E}_{\vec{\varphi},a}(\vec{\mu})$ defined in (2.14), with

$$\vec{\varphi} \equiv (0, 0) \quad \text{and} \quad a = -\frac{1}{2},$$

fact that was already observed in [17].

Further generalizations of these ideas, and in particular, their relation to rational solutions of Painlevé equations, can be found in [15, 16, 30, 34, 35, 37, 38, 57, 62, 78], to cite a few.

3 Electrostatics for Semiclassical Weight

In the rest of the paper we will try to adhere to the following notational convention, whenever possible: we will use capital letters to denote polynomials, and small letters to indicate general, usually multivalued, functions. A few exceptions of these rules will be clearly indicated.

3.1 The Semiclassical Weight

We start from a monic polynomial

$$A(z) = z^{\deg(A)} + \text{lower degree terms};$$

we use the notation $\mathbb{A}$ for the set of zeros of A on $\mathbb{C}$ and admit $\mathbb{A} = \emptyset$. Let

$$\Delta := \Gamma_1 \cup \cdots \cup \Gamma_k,$$

where each Γ_j, $j = 1, \ldots, k$, is an oriented Jordan piece-wise analytic arc joining pairs of points from $\mathbb{A} \cup \{\infty\}$ and not containing any other point from $\mathbb{A}$ in its interior $\mathring{\Gamma}_j$. For simplicity, we assume that the interiors of Γ_j's are pairwise disjoint ($\mathring{\Gamma}_i \cap \mathring{\Gamma}_j = \emptyset$, $i \neq j$), but as it will be clear from what follows, this is not an actual restriction.

Given another polynomial, B, we define, up to a normalization constant, (multivalued) analytic functions in $\mathbb{C}$,

$$w(z) := \exp\left(\int^z \frac{B(t)}{A(t)}\, dt\right), \quad v(z) := A(z)w(z), \tag{3.1}$$

with only possible singularities (either isolated or branch points) at $\mathbb{A} \cup \{\infty\}$. Notice that a priori we do not assume that A and B are relatively prime. This implies in particular, that B/A can be analytic at some zero of A, so that not all end-points of the arcs Γ_j's are necessarily branch points, zeros or isolated singularities of w.

The orientation of the arcs Γ_j in Δ defines the left- and right-side boundary values of the function w on Γ_j that we denote by w_+ and w_-, respectively; two different values of w_+, as well as w_+ and w_-, differ by a multiplicative constant. We fix on Δ the *weight* by assuming that on each component Γ_j it coincides, up to a non-zero multiplicative constant, with w_+; for the sake of simplicity of notation, we will be denoting this weight by the same letter w. A fundamental assumption is that such a weight has finite moments:

$$\int_\Delta |z|^m |w(z)||dz| < +\infty, \quad m = 0, 1, 2, \ldots$$

As consequence, for $m = 0, 1, 2, \ldots$,

$$z^m v(z) = 0 \text{ at endpoints of every subinterval of} \Delta. \tag{3.2}$$

Clearly, w is piece-wise differentiable on Δ and

$$\frac{w'(z)}{w(z)} = \frac{B(z)}{A(z)}, \quad z \in \Delta; \tag{3.3}$$

this equality is actually valid in $\mathbb{C} \setminus \mathbb{A}$.

The weight w is known as *semiclassical*, and the value

$$\sigma := \max\{\deg(A) - 2, \deg(B) - 1\} \tag{3.4}$$

is often referred to as its *class*, see e.g. [68, 69]. This is ambiguous in the case when A and B have a common factor, so we prefer to say that σ is the *class of the pair* (A, B) and assume $\sigma \geq 0$. Relation (3.3) can be written in the form

$$(Aw)' - \left(A' + B\right) w = 0,$$

known as the *Pearson differential equation*, see e.g. [29, 97].

Example 3.1 The simplest and well known example is the case of the Jacobi weight, when

$$A(x) = x^2 - 1, \quad B(x) = (\alpha + \beta)x + \alpha - \beta. \tag{3.5}$$

If $\operatorname{Re}\alpha, \operatorname{Re}\beta > -1$ then condition (3.2) is satisfied for $\Delta = [-1, 1]$.

This is an example of a *classical* weight ($\sigma = 0$); here

$$w(z) = (z-1)^{\alpha}(z+1)^{\beta}, \quad v(z) = (z-1)^{\alpha+1}(z+1)^{\beta+1}.$$

Example 3.2 With

$$\begin{aligned} A(x) &= x(x-a)(x-b), \quad b < 0 < a, \quad B(x) \\ &= \alpha x(x-b) + \beta x(x-a) + \gamma(x-a)(x-b), \end{aligned}$$

we have

$$w(x) = (x-a)^{\alpha}(x-b)^{\beta}x^{\gamma}, \quad v(x) = (x-a)^{\alpha+1}(x-b)^{\beta+1}x^{\gamma+1}.$$

If $\alpha, \beta, \gamma > -1$, we may take

$$\Gamma_1 = [b, 0], \quad \Gamma_2 = [0, a], \quad \Delta = \Gamma_1 \cup \Gamma_2,$$

so that condition (3.2) holds. This is a semiclassical weight of class $\sigma = 1$.

3.2 The Electrostatic Partner

In this section we carry out a purely formal construction that will gain content with additional assumptions in Sects. 4 and 5.

For any w-integrable function f on Δ we define its *weighted Cauchy transform*

$$\mathfrak{C}_w[f](z) := \int_\Delta \frac{f(t)w(t)}{t-z}dt, \tag{3.6}$$

holomorphic in $\mathbb{C}\setminus\Delta$. The case of $f \equiv 1$ is particularly important, so we will use a brief notation

$$\widehat{w}(z) := \mathfrak{C}_w[1](z) = \int_\Delta \frac{w(t)}{t-z}dt; \tag{3.7}$$

$\widehat{w}$ is also known as a *Markov function* related to the weight w.

Given a polynomial $P \not\equiv 0$, its *polynomial*[4] *of the second kind* Q is defined as

$$Q(z) := \int_\Delta \frac{P(t)-P(z)}{t-z} w(t)dt. \tag{3.8}$$

Additionally, we call

$$q(z) := \frac{\mathfrak{C}_w[P](z)}{w(z)}, \quad z \in \mathbb{C}\setminus\Delta, \tag{3.9}$$

the corresponding *function of the second kind* of P. They are related by the evident identity

$$P(z)\widehat{w}(z) + Q(z) = \mathfrak{C}_w[P](z) = q(z)w(z). \tag{3.10}$$

Although q is not necessarily single-valued in $\mathbb{C}\setminus\Delta$,

$$wq' = (\mathfrak{C}_w[P])' - \mathfrak{C}_w[P]\,\frac{w'}{w} = (\mathfrak{C}_w[P])' - \mathfrak{C}_w[P]\,\frac{B}{A} \tag{3.11}$$

is meromorphic in $\mathbb{C}\setminus\Delta$, with only possible poles at the poles of B/A.

We denote by $\mathfrak{Wrons}[f_1,\dots,f_k]$ the Wronskian determinant of the functions $f_1,\dots,f_k$, namely

$$\mathfrak{Wrons}[f_1,\dots,f_k] := \det\begin{pmatrix} f_1 & \dots & f_k \\ f_1' & \dots & f_k' \\ \vdots & \dots & \vdots \\ f_1^{(k-1)} & \dots & f_k^{(k-1)} \end{pmatrix}. \tag{3.12}$$

Finally, for a polynomial P, we define the transform

$$\mathfrak{D}_w[P] := \det\begin{pmatrix} P & \mathfrak{C}_w[P] \\ AP' & A\,(\mathfrak{C}_w[P])' - B\mathfrak{C}_w[P] \end{pmatrix} = v\,\mathfrak{Wrons}[P,q], \tag{3.13}$$

[4] Hilbert's Nullstellensatz implies that $P(t)-P(z)$ can be factored as $(t-z)\tilde{P}(t,z)$, where $\tilde{P}$ is a polynomial in its both variables, which implies that Q defined in (3.8) is a polynomial in z.

(with v from (3.1)), a priori holomorphic in $\mathbb{C}\setminus\Delta$.

Theorem 3.3 *If P is a polynomial of degree $N \in \mathbb{N}$ then there exist a polynomial U of degree $\le N-1$ and a polynomial H of degree $\le \sigma$ such that*

$$\mathfrak{D}_w[P] = \det \begin{pmatrix} P & \mathfrak{C}_w[P] \\ U & \mathfrak{C}_w[U] + H \end{pmatrix}. \tag{3.14}$$

Moreover, $\mathfrak{D}_w[P]$ is a polynomial of degree $\le N+\sigma$.

Proof We start with an identity for $\mathfrak{C}_w[P]$: for $z \in \mathbb{C}\setminus\Delta$,

$$\mathfrak{C}_w[AP' + BP](z) = A(z)\left(\mathfrak{C}_w[P]\right)'(z) - D(z), \tag{3.15}$$

where

$$D(z) := \int_\Delta \frac{A(z) - A(x) + A'(x)(x-z)}{(x-z)^2} P(x)w(x)\, dx \tag{3.16}$$

is a polynomial of degree $\le \deg(A) - 2$. In particular, $D \equiv 0$ if $\deg(A) \le 1$.

The identity can be established by direct calculation: integrating by parts and with account of (3.2)–(3.3),

$$\mathfrak{C}_w[AP' + BP](z) = \int_\Delta \frac{A(x)}{x-z}\frac{d\,(Pw)}{dx}\, dx = -\int_\Delta \frac{d}{dx}\left(\frac{A(x)}{x-z}\right) P(x)w(x)dx,$$

so that the left-hand side in (3.15) is equal to

$$\int_\Delta \frac{A(x) - A'(x)(x-z)}{(x-z)^2} P(x)w(x)\, dx,$$

and (3.15) follows.

Denote by E the polynomial part of the expansion of AP'/P at ∞, i.e.

$$A(z)\frac{P'}{P}(z) = E(z) + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty.$$

It is easy to see that E is polynomial of degree $\le \deg(A) - 1$. In this way,

$$U := AP' - EP \tag{3.17}$$

is a polynomial of degree $\le N-1$.

By (3.15),

$$\begin{aligned} \mathfrak{C}_w[U] &= \mathfrak{C}_w[AP' - EP] = \mathfrak{C}_w[AP' + BP] + \mathfrak{C}_w[(-E-B)P] \\ &= A\left(\mathfrak{C}_w[P]\right)' - (B+E)(z)\mathfrak{C}_w[P] - H, \end{aligned}$$

where, with the definition (3.16),

$$H(z) := D(z) + \int_{\Delta} \frac{(E+B)(x) - (E+B)(z)}{x - z} \, p(x) w(x) \, dx \tag{3.18}$$

is a polynomial of degree $\leq \sigma$. In particular (see (3.11)),

$$vq' = Awq' = A\left(\mathfrak{C}_w[P]\right)' - B\mathfrak{C}_w[P] = \mathfrak{C}_w[U] + E\mathfrak{C}_w[P] + H. \tag{3.19}$$

Using (3.17) and (3.19) in (3.13) we conclude that

$$\mathfrak{D}_w[P] = \det \begin{pmatrix} P & \mathfrak{C}_w[P] \\ AP' & \mathfrak{C}_w[U] + E\mathfrak{C}_w[P] + H \end{pmatrix} = \det \begin{pmatrix} P & \mathfrak{C}_w[P] \\ U & \mathfrak{C}_w[U] + H \end{pmatrix},$$

establishing (3.14).

In order to show that $\mathfrak{D}_w[p]$ is a polynomial (of degree at most $N + \sigma$) we use the standard arguments from the Riemann–Hilbert characterization of orthogonal polynomials (see e.g. [33]). Namely, denote by $\boldsymbol{Y}$ the matrix in the right-hand side of (3.14),

$$\boldsymbol{Y}(z) := \begin{pmatrix} P & \mathfrak{C}_w[P] \\ U & \mathfrak{C}_w[U] + H \end{pmatrix},$$

which by definition is holomorphic in $\mathbb{C}\setminus\Delta$. The Sokhotsky-Plemelj formulas imply that $\boldsymbol{Y}$ satisfies that

$$\boldsymbol{Y}_+(x) = \boldsymbol{Y}_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in \Delta, \tag{3.20}$$

where $\boldsymbol{Y}_\pm$ denote the boundary values of $\boldsymbol{Y}$ on Δ from the left/right sides, respectively. Taking determinants in both sides of (3.20) the Morera theorem yields that $\det(\boldsymbol{Y})$ is analytic across Δ. Since the first column of $\boldsymbol{Y}$ is bounded at each finite point of $\mathbb{C}$, and the local behavior of the second column at the end points of Δ essentially matches that of $\widehat{w}$ (see [45, Ch. 1]), it follows that $\det(\boldsymbol{Y})$ is an entire function. Moreover, since H is a polynomial of degree $\leq \sigma$, there exist a constant $c \in \mathbb{C}\setminus\{0\}$ and $s \in \mathbb{N} \cup \{0\}$, $s \leq \sigma$, such that

$$\boldsymbol{Y}(z) = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \left(\boldsymbol{I} + \mathcal{O}\left(\frac{1}{z}\right)\right) \operatorname{diag}\left(z^N, z^s\right), \quad z \to \infty.$$

Liouville's theorem shows that $\det(\boldsymbol{Y}) = \mathfrak{D}_w[P]$ is a polynomial of degree $N + s \leq N + \sigma$, which concludes the proof. $\square$

Remark 3.4 A relation of the form (3.19) was used in [65] as a definition of the semi-classical character of the weight, and in fact, it characterizes (3.3).

Definition 3.5 Let $P \neq 0$ be a polynomial. Then we call the polynomial

$$S := \mathfrak{D}_w[P], \tag{3.21}$$

defined by (3.13), the *electrostatic partner* of P induced by the weight w.

Remark 3.6 The definition (3.21) shows that normalization of S is equivalent to normalization of P: the scaling $P \mapsto \lambda P$, $\lambda \in \mathbb{C}$, is equivalent to $S \mapsto \lambda^2 S$. As it will be clear in the next section (Proposition 3.9), the actual normalization of S is irrelevant to the electrostatic model.

Some properties of the electrostatic partner S and of the function of the second kind q of P are established in the Appendix A.

3.3 The Differential Equation and Electrostatic Model

Theorem 3.7 *Let P be a polynomial, q its function of the second kind defined in (3.9), and S the electrostatic partner defined in (3.21). Then P and q are two solutions of the same second order linear differential equation with polynomial coefficients*

$$ASy'' + (A'S - AS' + BS)y' + Cy = 0. \tag{3.22}$$

If $\deg(A) \leq 1$ *then* $C = \mathfrak{D}_v[P']$, *with* $v = Aw$.

Proof Away from the zeros of A, the formal identity

$$A(z)v(z)\mathfrak{Wrons}[y, P, q](z) = A^2(z)w(z)\det\begin{pmatrix} y & P & q \\ y' & P' & q' \\ y'' & P'' & q'' \end{pmatrix}(z) = 0 \tag{3.23}$$

is clearly satisfied by $y = P$ and $y = q$, and thus, by any linear combination of these two functions. Expanding the determinant along the first column yields the following second order differential equation with respect to y:

$$f(z) = f_2(z)y''(z) + f_1(z)y'(z) + f_0(z)y(z) = 0, \tag{3.24}$$

where (see (3.13)),

$$f_2 = Av\det\begin{pmatrix} P & q \\ P' & q' \end{pmatrix}, \tag{3.25}$$

$$f_1 = -Av\det\begin{pmatrix} P & q \\ P'' & q'' \end{pmatrix}, \tag{3.26}$$

$$f_0 = Av\det\begin{pmatrix} P' & q' \\ P'' & q'' \end{pmatrix}(z) = \det\begin{pmatrix} P' & vq' \\ AP'' & Avq'' \end{pmatrix}. \tag{3.27}$$

By (3.13), $f_2 = AS$, and thus it is a polynomial. Furthermore, differentiating f_2 and using (3.1), (4.7), it is straightforward to deduce that

$$f_1 = A'S - AS' + BS$$

is also a polynomial.

By (3.11) and (3.15),

$$vq' = A\left(\mathfrak{C}_w[P]\right)' - B\mathfrak{C}_w[P] = \mathfrak{C}_w[AP'] + D = \mathfrak{C}_v[P'] + D.$$

Differentiating this identity and using (3.3) we obtain that

$$Avq'' = A\left(\mathfrak{C}_v[P']\right)' - (A' + B)\mathfrak{C}_v[P'] + \left[AD' - A'D - BD\right].$$

Thus,

$$\begin{aligned} f_0 &= \det\begin{pmatrix} P' & vq' \\ AP'' & Avq'' \end{pmatrix} \\ &= \det\begin{pmatrix} P' & \mathfrak{C}_v[P'] \\ AP'' & A\left(\mathfrak{C}_v[P']\right)' - (A' + B)\mathfrak{C}_v[P'] \end{pmatrix} + \det\begin{pmatrix} P' & D \\ AP'' & AD' - A'D - BD \end{pmatrix}. \end{aligned}$$

Observe that the weight $v = Aw$ is also semiclassical, with

$$\frac{v'(z)}{v(z)} = \frac{B_1(z)}{A(z)}, \quad B_1(z) = A'(z) + B(z), \quad z \in \Delta,$$

so that by (3.13),

$$\begin{pmatrix} P' & \mathfrak{C}_v[P'] \\ AP'' & A\left(\mathfrak{C}_v[P']\right)' - (A' + B)\mathfrak{C}_v[P'] \end{pmatrix} = \mathfrak{D}_v[P'],$$

and we get that

$$f_0 = \mathfrak{D}_v[P'] + \det\begin{pmatrix} P' & D \\ AP'' & AD' - A'D - BD \end{pmatrix} := C. \tag{3.28}$$

The first term in the right hand side is the electrostatic partner to P' induced by the semiclassical weight v, while the determinant is clearly a polynomial. Moreover, if $\deg(A) \le 1$, it follows from (3.16) that $D \equiv 0$, which concludes the proof. $\square$

Remark 3.8 Incidentally, we have established the following differential identity:

$$\mathcal{L}[y] := (Av)\,\mathfrak{Wrons}[y, P, q] = ASy'' + (A'S - AS' + BS)y' + Cy, \tag{3.29}$$

for some polynomial C. We will use this later, in the proof of Theorem 7.1.

Moreover, formula (3.28) and Remark 3.6 show that equation (3.22) is independent of normalization of S: a scaling $S \mapsto \lambda^2 S$, $\lambda \in \mathbb{C}$, can be interpreted as $P \mapsto \lambda P$, and thus, $C \mapsto \lambda^2 C$.

Using the notions of Sect. 2, and in particular, characterization (2.9), we see that (3.22) yields that zeros of P are in electrostatic equilibrium in the external field $\varphi(z) = \operatorname{Re}\Phi(z)$, if

$$\Phi'(z) := -\frac{1}{2}\,\frac{A'S - AS' + BS}{AS}(z) = -\frac{1}{2}\left(\frac{A'(z)}{A(z)} + \frac{B(z)}{A(z)} - \frac{S'(z)}{S(z)}\right).$$

Taking into account the definition (3.1), we conclude:

Proposition 3.9 *Assume that the polynomial P of degree N does not vanish at the zeros of AS, where $S = \mathfrak{D}_w[P]$ is its electrostatic partner. Then the discrete zero-counting measure $\nu(P)$ of P is φ-critical for the external field*

$$\varphi(z) = \frac{1}{2}\log\left|\frac{S}{v}\right|(z). \tag{3.30}$$

We can write alternatively that

$$\varphi(z) = \frac{1}{2}\left(U^{\nu(A)}(z) - U^{\nu(S)}(z) + \log\left|\frac{1}{w}\right|(z)\right).$$

In other words, the zeros of P (which under assumptions of Proposition 3.9 are necessarily simple) are in equilibrium in the external field φ induced by the orthogonality weight w, with an additional contribution from point charges of size $1/2$: a *repulsion* from positive charges at $\mathbb{A}$ and an *attraction* from negative charges at the zeros of the electrostatic partner S (whose location is a priori unknown). The presence of attracting "ghost charges" was observed already by M. H. Ismail [49, 50] in his electrostatic interpretation for zeros of orthogonal polynomials with respect to generalized Jacobi weights.

Remark 3.10 The study of polynomial solutions of second order ODE of the form (3.22) is a very classical problem that goes back at least 200 years, see e.g. [71, 72, 89] for some historical references and background. In this context, polynomial coefficients C are known as Van Vleck polynomials, and the solution P is the corresponding Heine–Stieltjes polynomial.

The one-to-one correspondence between Heine–Stieltjes polynomials and the discrete critical measures has been established in [71]. Recently, an additional characterization in terms of non-Hermitian orthogonality satisfied by Heine–Stieltjes polynomials was found in [18]. For the differential equation (3.22), it follows from their work that there exist a set Δ and the constants defining the weight w (see the explanation in Sect. 3.1) such that P satisfies $n + \sigma$ orthogonality conditions with respect to w/S^2. The close connection between orthogonality and electrostatics is well known. In this sense, the electrostatic model in Proposition 3.9, and especially, the form of the external field (3.30), is compatible with the findings of [18].

4 Quasi-Orthogonality

We revisit the facts established in Sect. 3 under an additional assumption on our polynomial P. A (monic) polynomial P_N of degree N is called *quasi-orthogonal*[5] with respect to the weight w on Δ if it satisfies the following (in general, non-Hermitian) orthogonality relations: for $n \in \mathbb{N}$, $n \leq N$,

$$\begin{aligned} &\int_\Delta x^j P_N(x) w(x) dx = 0\,, \qquad j = 0, 1, \ldots, n-1, \\ &m_n := \int_\Delta x^n P_N(x) w(x) dx \neq 0. \end{aligned} \tag{4.1}$$

In particular, if $n = N$, polynomial P_N is the N-th monic *orthogonal polynomial*. Clearly, last condition in (4.1), that is, $m_n \neq 0$, is a constraint on the weight w and orthogonality contour Δ.

Using the notation introduced in (3.6)–(3.9), we denote

$$Q_N(z) := \int_\Delta \frac{P_N(t) - P_N(z)}{t - z} w(t) dt \tag{4.2}$$

and

$$\mathfrak{C}_w[P_N](z) = \int_\Delta \frac{P_N(t) w(t)}{t - z} dt, \quad z \in \mathbb{C} \setminus \Delta. \tag{4.3}$$

It is useful to observe that (4.1) yields that for any polynomial T of degree $\leq n$,

$$T \mathfrak{C}_w[P_N] = \mathfrak{C}_w[T P_N], \quad \text{that is,} \quad T(z) \int_\Delta \frac{P_N(t) w(t)}{t - z} dt = \int_\Delta \frac{T(t) P_N(t) w(t)}{t - z} dt, \tag{4.4}$$

which by (3.10) in particular shows that, as $z \to \infty$,

$$\mathfrak{C}_w[P_N](z) = P_N(z) \widehat{w}(z) + Q_N(z) = -\frac{m_n}{z^{n+1}} + \ldots; \tag{4.5}$$

if Δ is unbounded, we understand the equality above in the asymptotic sense. Notice also that if P_N satisfies a full set of orthogonality conditions ($N = n$), then (4.5) shows that the rational function

$$\pi_N := -\frac{Q_N}{P_N}$$

is the N-th diagonal Padé approximant to $\widehat{w}$ at infinity, fact that is well known.

We denote by q_N, U_N and H_N the functions q, U and H introduced in (3.9), (3.17) and (3.18) for $P = P_N$, and let

$$S_N := \mathfrak{D}_w[P_N] = \det \begin{pmatrix} P_N & \mathfrak{C}_w[P_N] \\ A P_N' & A\left(\mathfrak{C}_w[P_N]\right)' - B \mathfrak{C}_w[P_N] \end{pmatrix} \tag{4.6}$$

[5] A more restrictive notion of quasi-orthogonality was introduced by Chihara [28], where he assumed a condition equivalent to $n = N - 1$; see also [53].

be the electrostatic partner to P_N, induced by the weight w. By Theorem 3.3,

$$S_N = v\,\mathfrak{Wrons}[P_N, q_N] = \det\begin{pmatrix} P_N & \mathfrak{C}_w[P_N] \\ U_N & \mathfrak{C}_w[U_N] + H_N \end{pmatrix}, \tag{4.7}$$

and S_N is a polynomial of degree $\le N+\sigma$. In fact, due to quasi-orthogonality relations we can say more: the upper bound on the degree of S_N is lessened by the number of orthogonality conditions:

Corollary 4.1 *If P_N satisfies (4.1) then for the electrostatic partner S_N,*

$$\begin{aligned} S_N(z) &= m_n\, z^{N-n}\left[(N+n+1)\, z^{\deg(A)-2}\,(1+O(1/z)) + \kappa\, z^{\deg(B)-1}\,(1+O(1/z))\right], \\ & z\to\infty, \end{aligned} \tag{4.8}$$

where κ is the leading coefficient of the polynomial B. In particular, with assumptions (4.1), $\deg(S_N) \le N-n+\sigma$ and can be strictly less only if $\deg(A)=\deg(B)+1$ and $\kappa = -(N+n+1)$.

Moreover, if $n\ge\sigma+1$ then $H_N\equiv 0$, and if $n\ge\sigma+2$, then additionally, U_N (of degree $\le N-1$) is quasi-orthogonal:

$$\int_\Delta x^j U_N(x) w(x)dx = 0, \quad j=0,\dots,n-\sigma-2. \tag{4.9}$$

Proof Using (4.5) in the definition (4.6) we obtain (4.8). Also from (4.5), (3.19) and since $\deg(B+E)\le\sigma+1$, we see that

$$\begin{aligned} \mathfrak{C}_w[U_N](z) + H(z) &= A(z)(\mathfrak{C}_w[P_N])'(z) - (B+E)(z)\mathfrak{C}_w[P_N](z) \\ &= \mathcal{O}\left(\frac{1}{z^{n-\sigma}}\right), \quad z\to\infty. \end{aligned} \tag{4.10}$$

If $n\ge\sigma+1$, (4.10) implies that $H\equiv 0$; with the assumption $n\ge\sigma+2$ we also get the quasi-orthogonality conditions (4.9). □

Remark 4.2 It is interesting to examine the conclusions of Corollary 4.1 in the particular case of polynomials P_N orthogonal on Δ with respect to the weight w (so that $n=N$). Consequently, the degree of S_N is uniformly bounded: $\deg S_N \le \sigma$. Furthermore, $\deg S_N = \sigma$ if either

$$\deg(A) \ne \deg(B)+1 \quad \text{or} \quad \kappa \ne -(N+n+1),$$

see (4.8). If additionally $\sigma = 0$ (i. e. when P_N is basically a classical orthogonal polynomial), (4.9) asserts that U_N is the $(N-1)$-th orthogonal polynomial, and up to normalization, $U_N = P_{N-1}$, in which case (4.7) boils down to

$$S_N = V\det\begin{pmatrix} P_N & q_N \\ P_N' & q_N' \end{pmatrix} = \text{const}\times\det\begin{pmatrix} P_N & \mathfrak{C}_w[P_N] \\ P_{N-1} & \mathfrak{C}_w[P_{N-1}] \end{pmatrix},$$

that is, polynomial S_N is the product of a factor related with the weight and the Wronskian of P_N and q_N, and also is the Casorati determinant associated with P_N; such an identity appears for instance in [52, formula (3.6.13)]. Thus, (4.7) is a generalization of such kind of relations to quasi-orthogonal polynomials with respect to semiclassical weights, a fact that has an independent interest.

With the additional assumption of quasi-orthogonality of P_N, Theorem 3.7 and Proposition 3.9 are still valid, with S replaced by S_N. For instance, P_N satisfying (4.1) and its function of the second kind q_N are solutions of the linear differential equation with polynomial coefficients

$$AS_N y'' + (A'S_N - AS_N' + BS_N)y' + C_N y = 0, \tag{4.11}$$

where S_N is the electrostatic partner defined in (4.6), and C_N is a polynomial. Its degree can be easily estimated by using in (4.11) that $\deg P_N = N$ and $\deg(S_N) \leq N - n + \sigma$, which yields that

$$\deg(C_N) \leq N - n + 2\sigma. \tag{4.12}$$

In some cases, we can be more precise. For instance, if $\deg(A) \leq \deg(B)$ and $\deg(P_N) = N$ then using (4.8) and (4.11) we conclude that

$$\deg(S_N) = N - n + \deg(B) - 1, \quad \deg(C_N) = N - n + 2\deg(B) - 2.$$

Equation (4.11) is known in the literature on semiclassical orthogonal polynomials, see for instance [90, Eq. (20)] or [65, Eq. (18)]. The equation in [65] was obtained for orthogonal polynomials only, when the polynomial S_N (denoted by Θ_n there) has degree $\leq \sigma$. In this sense, (4.11) is an extension of these results. Recall that Magnus uses in [65] an alternative definition for the semiclassical weights, in terms of an identity for the Cauchy transform (3.6), which is equivalent to (3.3), see Remark 3.4.

Example 4.3 Let us return to Jacobi polynomials

$$P_N(x) = P_N^{(\alpha,\beta)}(x) = \frac{1}{2^N N!}(x-1)^{-\alpha}(x+1)^{-\beta}\left[(x-1)^{\alpha+N}(x+1)^{\beta+N}\right]^{(N)}, \tag{4.13}$$

corresponding to the weight considered in Example 3.1, for which

$$A(x) = x^2 - 1, \quad B(x) = (\alpha+\beta)x + \alpha - \beta,$$

and $\sigma = 0$. It is known that $P_N^{(\alpha,\beta)}$ may have a multiple zero at $x = 1$ if $\alpha \in \{-1, \dots, -N\}$, at $x = -1$ if $\beta \in \{-1, \dots, -N\}$ or, even, at $x = \infty$ (which means a degree reduction) if $N + \alpha + \beta \in \{-1, \dots, -N\}$; otherwise, all zeros are simple, see e.g. [58, 97].

If $\alpha, \beta > -1$, we can take $\Delta = [-1, 1]$. By Corollary 4.1, the electrostatic partner S_N is a constant ($\not\equiv 0$ if $\alpha + \beta \neq -2N - 1$). Since

$$v(z) = (z-1)^{\alpha+1}(z+1)^{\beta+1},$$

the zeros of P_N are in equilibrium in the external field

$$\varphi(z) = \frac{1}{2}\log\left|\frac{1}{(z-1)^{\alpha+1}(z+1)^{\beta+1}}\right| = \frac{\alpha+1}{2}U^{\delta_1}(z) + \frac{\beta+1}{2}U^{\delta_{-1}}(z). \quad (4.14)$$

In other cases, considered non-standard, when either $\alpha \le -1$ or $\beta \le -1$, Jacobi polynomials satisfy non-hermitian quasi-orthogonality conditions, see [58, Theorem 4.1].

For instance, if $\alpha, \beta, \alpha+\beta \notin \mathbb{Z}$, and $-N < \alpha < -1$, then all zeros of $P_N = P_N^{(\alpha,\beta)}$ are simple, and $P_N(-1) \neq 0$, see e.g. [97, Ch. IV]. Polynomial P_N satisfies also a quasi-orthogonality relation (4.1) with $n = N - [-\alpha]$, but with a modified weight, $w(z) = (z-1)^{\alpha+[-\alpha]}(z+1)^{\beta}$, so that now

$$v(z) = (z-1)^{\alpha+[-\alpha]+1}(z+1)^{\beta+1}.$$

Moreover, as Δ we can take an arbitrary curve oriented clockwise, connecting $1 - i0$ with $1 + i0$ and lying entirely in $\mathbb{C}\setminus[-1, +\infty)$, except for its endpoints; if $\beta > -1$, then Δ can be deformed into $[-1, 1]$.

By Corollary 4.1, the electrostatic partner S_N is of degree exactly $[-\alpha]$. We will show next that, up to a normalizing constant,

$$S_N(x) = (x-1)^{[-\alpha]}. \quad (4.15)$$

However, notice that by Proposition 3.9, the discrete zero-counting measure $\nu(P_N)$ of $P_N = P_N^{(\alpha,\beta)}$ is φ_N-critical in the external field

$$\varphi(z) = \frac{1}{2}\log\left|\frac{S_N}{v}\right|(z),$$

which coincides with the one given in (4.14). In other words, even with the non-standard values of the parameters α, β, we still get equilibrium in the field (4.14).

Returning to the expression for S_N in the case under consideration, it is known that for all values of the parameters, P_N is a solution of the differential equation

$$Ay'' + (B + A')y' - \lambda_N y = 0, \quad \lambda_N = N(N+\alpha+\beta+1),$$

with A and B given in (3.5), see e.g. [97, Ch. IV] or [82, Sect. 18.8]. At the same time, from our discussion it follows that P_N is a solution of the differential equation (4.11), namely

$$\begin{gathered} AS_N y'' + (A'S_N - AS_N' + B_1 S_N)y' + C_N y = 0, \\ B_1(x) = (\alpha + [-\alpha] + \beta)x + \alpha + [-\alpha] - \beta. \end{gathered}$$

Combining these two equations we obtain the identity

$$\left((B - B_1)S_N + AS_N'\right) P_N' = (\lambda_N S_N + C_N)\, P_N,$$

which using the explicit expressions for A, B and B_1, can be rewritten as

$$(x+1)\left((x-1)S_N'(x)-[-\alpha]S_N(x)\right)P_N'(x)=(\lambda_N S_N(x)+C_N(x))P_N(x). \quad (4.16)$$

Recall that $\deg S_N=[-\alpha]\le N$; a simple argument shows that

$$\deg\left((x-1)S_N'(x)-[-\alpha]S_N(x)\right)<N.$$

Indeed, the assertion is obvious for $\deg S_N<N$; if $\deg S_N=[-\alpha]=N$ then the leading coefficients of $(x-1)S_N'(x)$ and $[-\alpha]S_N(x)$ match, so the assertion follows also in this case.

Since in the situation we are analyzing $P_N(x)$ and $(x+1)P_N'(x)$ are relatively prime (up to a multiplicative constant), by (4.16) we conclude that $(x-1)S_N'(x)-[-\alpha]S_N(x)$ must vanish at all N distinct zeros of P_N, which is possible only if $(x-1)S_N'(x)-[-\alpha]S_N(x)\equiv 0$, which implies (4.15). Incidentally, we also obtain that in this case, $C_N=-\lambda_N S_N$.

We will return to the example of Jacobi polynomials with non-standard values of the parameters in the Sect. 8.1, when we will address multiple orthogonality.

Example 4.4 It is instructive to compare our construction with the results of [53] in the case of quasi-orthogonal Jacobi polynomials. These are polynomials

$$P_N(x)=\widehat{P}_N(x)+c\,\widehat{P}_{N-1}(x),\quad c\in\mathbb{R},$$

where $\widehat{P}_N$ is the N-th orthonormal Jacobi polynomial

$$\widehat{P}_N(x)=\sqrt{\frac{(\alpha+\beta+2N+1)\Gamma(\alpha+\beta+N+1)N!}{2^{\alpha+\beta+1}\Gamma(\alpha+N+1)\Gamma(\beta+N+1)}}\,P_N^{(\alpha,\beta)}(x)$$

and $P_N^{(\alpha,\beta)}$ is defined in (4.13). Obviously, for $\alpha,\beta>-1$, P_N satisfies (4.1) with $n=N-1$ and the weight $w(x)=(x-1)^\alpha(x+1)^\beta$. According to Corollary 4.1, $S_N(x)=x-t_N$ (up to a multiplicative constant), and by Proposition 3.9, the discrete zero-counting measure $\nu(P_N)$ of P_N is φ-critical in the external field

$$\begin{aligned}\varphi(x)&=\frac{\alpha+1}{2}U^{\delta_1}(x)+\frac{\beta+1}{2}U^{\delta_{-1}}(x)-\frac{1}{2}U^{\delta_{t_N}}(x)\\&=\frac{\alpha+1}{2}\log\frac{1}{|x-1|}+\frac{\beta+1}{2}\log\frac{1}{|x+1|}-\frac{1}{2}\log\frac{1}{|x-t_N|}.\end{aligned}$$

Explicit expressions allow to calculate t_N by definition (4.7):

$$t_N=-\frac{(\alpha+\beta+1+2N)+c^2(\alpha+\beta-1+2N)}{(2N+\alpha+\beta)c}a_N+\frac{\beta^2-\alpha^2}{(2N+\alpha+\beta)^2},$$

where

$$a_N = \frac{2}{\alpha+\beta+2N}\sqrt{\frac{N(\alpha+N)(\beta+N)(\alpha+\beta+N)}{(\alpha+\beta-1+2N)(\alpha+\beta+1+2N)}}\,.$$

This expression coincides with the one obtained in [53, Sect. 5.2].

5 Multiple Orthogonality of Type II

5.1 The General Case

We are interested in type II *multiple* or *Hermite–Padé* orthogonal polynomials with respect to semiclassical weights. For the sake of simplicity, we restrict ourselves to two positive weights w_1, w_2, supported on $\Delta_1 \subset \mathbb{R}$ and $\Delta_2 \subset \mathbb{R}$, respectively. Given an ordered pair $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}_{\geq 0}^2$, where $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$, we look for a (monic) polynomial $P_{\boldsymbol{n}}$ of total degree at most $N = |\boldsymbol{n}| := n_1 + n_2$, such that

$$\int_{\Delta_i} x^j P_{\boldsymbol{n}}(x) w_i(x) dx \begin{cases} = 0, & j \leq n_i - 1, \\ \neq 0, & j = n_i, \end{cases} \qquad i = 1, 2. \tag{5.1}$$

In this way, the definition of type II Hermite–Padé orthogonal polynomials (5.1) boils down to two simultaneous quasi-orthogonality conditions.

Additionally, we assume that both weights are semiclassical and belong to the framework discussed in Sect. 3. More precisely, we assume that for each $i = 1, 2$, $\Delta_i \subset \mathbb{R}$ is a finite union of non-overlapping intervals joining real zeros of a real polynomial A_i and eventually ∞, and that each weight w_i is defined on Δ_i in such a way that on each component Γ_j it coincides, up to a non-zero multiplicative constant, with a boundary value $(w_i)_+$ of the function defined below:

$$w_i(z) := \exp\left(\int^z \frac{B_i(t)}{A_i(t)}\,dt\right), \qquad i = 1, 2, \tag{5.2}$$

for some real polynomials B_1, B_2. As before, we use the notation $\mathbb{A}_i$ for the set of zeros of A_i on $\mathbb{C}$, $i = 1, 2$.

We assume also that w_i's have finite moments:

$$\int_{\Delta_i} |x|^m |w_i(x)| dx < +\infty, \quad m = 0, 1, 2, \ldots, \quad i = 1, 2.$$

As a consequence, for $m = 0, 1, 2, \ldots$,

$$z^m v_i(z) = 0 \text{ at endpoints of every subinterval of } \Delta_i, \, i = 1, 2, \tag{5.3}$$

with

$$v_i(z) := A_i(z) w_i(z), \quad i = 1, 2. \tag{5.4}$$

We also have that

$$\frac{w_i'(z)}{w_i(z)} = \frac{B_i(z)}{A_i(z)}, \quad i = 1, 2, \tag{5.5}$$

with the identity taking place away from the singularities of w_i. As in (3.4),

$$\sigma_i := \max\{\deg(A_i) - 2, \deg(B_i) - 1\}, \quad i = 1, 2. \tag{5.6}$$

Remark 5.1 In the special case when $A_1 = A_2 = A$ and $B_1 = B_2 = B$, we will always assume, without loss of generality, that both w_i are normalized in such a way that for every selection of the branch,

$$w_1(z) = w_2(z) = w(z) = \exp\left(\int^z \frac{B(t)}{A(t)}\, dt\right).$$

This situation was considered for instance in [12] (and previously, in [54] and [56]). Several examples of the case when $\Delta_1 = \Delta_2$, $A_1 = A_2$, but $B_1 \neq B_2$ for classical weights ($\sigma_i = 0$) appear in [7].

For $P_{\boldsymbol{n}}$ we define the corresponding polynomials

$$Q_{\boldsymbol{n},i}(z) := \int_{\Delta_i} \frac{P_{\boldsymbol{n}}(t) - P_{\boldsymbol{n}}(z)}{t - z} w_i(t) dt, \quad i = 1, 2, \tag{5.7}$$

and functions of the second kind,

$$q_{\boldsymbol{n},i}(z) := \frac{\mathfrak{C}_{w_i}[P_{\boldsymbol{n}}](z)}{w_i(z)} = \frac{1}{w_i(z)} \int_{\Delta} \frac{P_{\boldsymbol{n}}(t) w_i(t)}{t - z} dt, \quad i = 1, 2. \tag{5.8}$$

By (4.4)–(4.5), for $i = 1, 2$,

$$\mathfrak{C}_{w_i}[P_{\boldsymbol{n}}](z) = P_{\boldsymbol{n}}(z)\widehat{w}_i(z) + Q_{\boldsymbol{n},i}(z) = \mathcal{O}(z^{-n_i-1}), \quad z \to \infty. \tag{5.9}$$

This shows that the pair of rational functions

$$\left(\pi_{\boldsymbol{n},1}, \pi_{\boldsymbol{n},2}\right) := \left(-\frac{Q_{\boldsymbol{n},1}}{P_{\boldsymbol{n}}}, -\frac{Q_{\boldsymbol{n},2}}{P_{\boldsymbol{n}}}\right) \tag{5.10}$$

are the *simultaneous* or *Hermite–Padé approximants of type II* to the pair of functions $(\widehat{w}_1, \widehat{w}_2)$, and that the functions $\mathfrak{C}_{w_i}[P_{\boldsymbol{n}}]$ are the corresponding *residues*, see e.g. [12, 60].

We will impose an additional condition enforcing an independence of both weights, namely we will assume that

$$\mathfrak{Wrons}[P_{\boldsymbol{n}}, q_{\boldsymbol{n},1}, q_{\boldsymbol{n},2}] \not\equiv 0. \tag{5.11}$$

It is equivalent to assuming that $\mathfrak{Wrons}[P_{\boldsymbol{n}}, q_{\boldsymbol{n},1}, q_{\boldsymbol{n},2}] \neq 0$ at some point different from the zeros of $A_1 A_2$. Possibly, condition (5.11) is equivalent to $\boldsymbol{n}$ being a normal index, although this is just a natural conjecture that deserves further study.

By the analysis from Sect. 3, $P_{\boldsymbol{n}}$ has now two electrostatic partners,

$$S_{\boldsymbol{n},i}(z) := \mathfrak{D}_{w_i}[P_{\boldsymbol{n}}] = v_i(z) f_{\boldsymbol{n},i}(z), \quad i = 1, 2, \tag{5.12}$$

where

$$f_{\boldsymbol{n},i} := \mathfrak{Wrons}[P_{\boldsymbol{n}}, q_{\boldsymbol{n},i}], \quad i = 1, 2, \tag{5.13}$$

and $\mathfrak{Wrons}[\cdot, \cdot]$ stands for the Wronskian as defined in (3.12). An equivalent formula that might shed some light on the behavior of these polynomials is

$$S_{\boldsymbol{n},i}(z) = v(z) P_{\boldsymbol{n}}^2(z) \left(\frac{\widehat{w}_i(z) - \pi_{\boldsymbol{n},i}(z)}{w_i(z)} \right)' = v(z) P_{\boldsymbol{n}}^2(z) \left(\frac{\mathfrak{C}_{w_i}[P_{\boldsymbol{n}}](z)}{w_i(z) P_{\boldsymbol{n}}(z)} \right)', \quad i = 1, 2, \tag{5.14}$$

where $\pi_{\boldsymbol{n},i}$ are defined in (5.10); it can be checked using (5.9) by direct computation.

By Theorem 3.7,

Theorem 5.2 *Let $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}_{\geq 0}^2$, and let $P_{\boldsymbol{n}}$ be a monic polynomial of degree $N = n_1 + n_2$ (assuming it exists) that satisfies the multiple orthogonality conditions (5.1). Then $S_{\boldsymbol{n},i}$ are polynomials of degree at most $N - n_i + \sigma_i$, $i = 1, 2$, and there exist polynomials $C_{\boldsymbol{n},i}$,*

$$\deg(C_{\boldsymbol{n},i}) \leq N - n_i + 2\sigma_i, \quad i = 1, 2,$$

such that $P_{\boldsymbol{n}}$ is a solution of the system of linear differential equations

$$A_i S_{\boldsymbol{n},i}\, y'' + (A_i' S_{\boldsymbol{n},i} - A_i S_{\boldsymbol{n},i}' + B_i S_{\boldsymbol{n},i})\, y' + C_{\boldsymbol{n},i}\, y = 0, \quad i = 1, 2. \tag{5.15}$$

Remark 5.3 In the case when $A_1 = A_2$ and $B_1 = B_2$ we can replace in the expressions (5.15) the polynomial $S_{\boldsymbol{n},i}$ by any linear combination

$$a\, S_{\boldsymbol{n},1} + b\, S_{\boldsymbol{n},2}, \quad a, b \in \mathbb{R}.$$

The resulting differential equations still have $P_{\boldsymbol{n}}$ as one of their solutions; see the result of numerical experiments for Appell's polynomials at the end of Sect. 8.6, especially the plots in Fig. 6.

An application of Proposition 3.9 yields:

Corollary 5.4 *Assume that the polynomial $P_{\boldsymbol{n}}$ has no common zeros neither with $A_1 S_{\boldsymbol{n},1}$ nor with $A_2 S_{\boldsymbol{n},2}$. Then the discrete zero-counting measure $\nu(P_{\boldsymbol{n}})$ of $P_{\boldsymbol{n}}$ is φ_i-critical, in the sense of Definition 2.1, for the external field*

$$\varphi_i(z) = \frac{1}{2} \log \left| \frac{S_{\boldsymbol{n},i}}{v_i} \right| (z), \tag{5.16}$$

for both $i = 1, 2$.

Thus, the zeros of the Hermite–Padé polynomial $P_{\boldsymbol{n}}$ are in equilibrium (i.e., their counting measure is critical) in two different external fields, each one induced by the corresponding orthogonality weight w_i, with an addition of attracting charges placed at the zeros of the electrostatic partner $S_{\boldsymbol{n},i}$. This "redundancy" allows us to provide also an electrostatic model for these additional charges forming the external fields φ_i. Formally, it will be a consequence of the differential equations for the electrostatic partners $S_{\boldsymbol{n},i}$ that we establish next, but we need to introduce first another auxiliary polynomial, this time generated by both weights simultaneously.

We define the following differential operators:

$$\mathcal{L}_i[y] := (A_i v_i)\,\mathfrak{Wrons}[y, P_{\boldsymbol{n}}, q_{\boldsymbol{n},i}], \quad i = 1, 2. \tag{5.17}$$

Notice that we omit indicating the dependence of $\mathcal{L}_i$ from $\boldsymbol{n}$ for the sake of brevity of notation.

Proposition 5.5 *Function*

$$R_{\boldsymbol{n}} := A_2 v_2\,\mathcal{L}_1[q_{\boldsymbol{n},2}] = -A_1 v_1\,\mathcal{L}_2[q_{\boldsymbol{n},1}] = (A_1 A_2 v_1 v_2)\,\mathfrak{Wrons}[P_{\boldsymbol{n}}, q_{\boldsymbol{n},1}, q_{\boldsymbol{n},2}] \tag{5.18}$$

is a polynomial of degree $\leq 2\sigma_1 + 2\sigma_2 + 3$.

If $A_1 = A_2 =: A$*, then* A *is a factor of* $R_{\boldsymbol{n}}$*. If in addition* $B_1 = B_2$*, then* A^2 *is a factor of* $R_{\boldsymbol{n}}$*, i.e.*

$$R_{\boldsymbol{n}}^* := \frac{R_{\boldsymbol{n}}}{A^2} = v^2\,\mathfrak{Wrons}[P_{\boldsymbol{n}}, q_{\boldsymbol{n},1}, q_{\boldsymbol{n},2}], \quad v := Aw, \tag{5.19}$$

is a polynomial.

Notice that by our assumption (5.11), $R_{\boldsymbol{n}} \not\equiv 0$.

Proof Let us define

$$R_{\boldsymbol{n}} := \frac{A_1 S_{\boldsymbol{n},1} C_{\boldsymbol{n},2} - A_2 S_{\boldsymbol{n},2} C_{\boldsymbol{n},1}}{P_{\boldsymbol{n}}'}. \tag{5.20}$$

Multiplying the equations in (5.15) evaluated at $y = P_{\boldsymbol{n}}$ by $A_2 S_{\boldsymbol{n},2}/P_{\boldsymbol{n}}$ and $A_1 S_{\boldsymbol{n},1}/P_{\boldsymbol{n}}$, respectively, and subtracting we obtain

$$\begin{aligned} R_{\boldsymbol{n}}\, P_{\boldsymbol{n}} \; &= -\mathfrak{Wrons}[A_1, A_2] S_{\boldsymbol{n},1} S_{\boldsymbol{n},2} + A_1 A_2\,\mathfrak{Wrons}[S_{\boldsymbol{n},1}, S_{\boldsymbol{n},2}] \\ &\quad + (A_2 B_1 - A_1 B_2) S_{\boldsymbol{n},1} S_{\boldsymbol{n},2}. \end{aligned} \tag{5.21}$$

Notice that $R_{\boldsymbol{n}}\, P_{\boldsymbol{n}}$ is a polynomial. An immediate consequence of the statement a) of Proposition A.1 is that $S_{\boldsymbol{n},1}/P_{\boldsymbol{n}}'$ and $S_{\boldsymbol{n},2}/P_{\boldsymbol{n}}'$ are analytic at the zeros of $P_{\boldsymbol{n}}$, which together with the bounds on the degree of $C_{\boldsymbol{n},i}$ from Theorem 5.2 yields that $R_{\boldsymbol{n}}$ is a polynomial of degree at most $2\sigma_1 + 2\sigma_2 + 3$.

On the other hand, straightforward calculation using (5.5) and (5.13) shows also that

$$-\,\mathfrak{Wrons}[A_1, A_2] S_{\boldsymbol{n},1} S_{\boldsymbol{n},2} + A_1 A_2\,\mathfrak{Wrons}[S_{\boldsymbol{n},1}, S_{\boldsymbol{n},2}] + (A_2 B_1 - A_1 B_2) S_{\boldsymbol{n},1} S_{\boldsymbol{n},2}$$

$$= A_1 A_2 v_1 v_2 \,\mathfrak{Wrons}[f_{\boldsymbol{n},1}, f_{\boldsymbol{n},2}],$$

so that (5.20) reduces to

$$A_1 A_2 v_1 v_2 \,\mathfrak{Wrons}[f_{\boldsymbol{n},1}, f_{\boldsymbol{n},2}] = P_n R_{\boldsymbol{n}}. \tag{5.22}$$

Let

$$d_{\boldsymbol{n}} := \mathfrak{Wrons}[f_{\boldsymbol{n},1}, f_{\boldsymbol{n},2}].$$

Since

$$f'_{\boldsymbol{n},i} = \mathfrak{Wrons}[P_{\boldsymbol{n}}, q_{\boldsymbol{n},i}]' = \det \begin{pmatrix} P_{\boldsymbol{n}} & q_{\boldsymbol{n},i} \\ P''_{\boldsymbol{n}} & q''_{\boldsymbol{n},i} \end{pmatrix},$$

this gives us the following identity for $d_{\boldsymbol{n}}$:

$$\begin{aligned} d_{\boldsymbol{n}} &= \det \begin{pmatrix} P_{\boldsymbol{n}} & q_{\boldsymbol{n},1} \\ P'_{\boldsymbol{n}} & q'_{\boldsymbol{n},1} \end{pmatrix} \det \begin{pmatrix} P_{\boldsymbol{n}} & q_{\boldsymbol{n},2} \\ P''_{\boldsymbol{n}} & q''_{\boldsymbol{n},2} \end{pmatrix} - \det \begin{pmatrix} P_{\boldsymbol{n}} & q_{\boldsymbol{n},2} \\ P'_{\boldsymbol{n}} & q'_{\boldsymbol{n},2} \end{pmatrix} \det \begin{pmatrix} P_{\boldsymbol{n}} & q_{\boldsymbol{n},1} \\ P''_{\boldsymbol{n}} & q''_{\boldsymbol{n},1} \end{pmatrix} \\ &= P_{\boldsymbol{n}} \,\mathfrak{Wrons}[P_{\boldsymbol{n}}, q_{\boldsymbol{n},1}, q_{\boldsymbol{n},2}], \end{aligned} \tag{5.23}$$

which together with (5.22) proves the first part of assertion.

If $A_1 = A_2 =: A$ then (5.21) becomes

$$A^2(S_{\boldsymbol{n},1} S'_{\boldsymbol{n},2} - S'_{\boldsymbol{n},1} S_{\boldsymbol{n},2}) + A(B_1 - B_2) S_{\boldsymbol{n},1} S_{\boldsymbol{n},2} = A \frac{R_{\boldsymbol{n}}}{A} P_{\boldsymbol{n}},$$

with

$$\frac{R_{\boldsymbol{n}}}{A} = \frac{S_{\boldsymbol{n},1} C_{\boldsymbol{n},2} - S_{\boldsymbol{n},2} C_{\boldsymbol{n},1}}{P'_{\boldsymbol{n}}}. \tag{5.24}$$

Assume that $A(z_0) = 0$. If also $P_{\boldsymbol{n}}(z_0) = 0$ then by the same argument as before, $S_{\boldsymbol{n},1}/P'_{\boldsymbol{n}}$ and $S_{\boldsymbol{n},2}/P'_{\boldsymbol{n}}$ are analytic at z_0, as well as the right hand side of (5.24). If $P_{\boldsymbol{n}}(z_0) \neq 0$ but $P'_{\boldsymbol{n}}(z_0) = 0$, then by (5.15),

$$C_{\boldsymbol{n},i}(z_0)\, P_{\boldsymbol{n}}(z_0) = -A(z_0) S_{\boldsymbol{n},i}(z_0)\, P''_{\boldsymbol{n}}(z_0), \quad i = 1, 2.$$

In this case, $A P''_{\boldsymbol{n}}/P'_{\boldsymbol{n}}$ is analytic at z_0, which implies again that the expression in the right hand side of (5.24) is analytic at z_0. This proves that $R_{\boldsymbol{n}}/A$ is a polynomial.

If in addition $B_1 = B_2$, then (5.21) reduces to

$$A^2(S_{\boldsymbol{n},1} S'_{\boldsymbol{n},2} - S'_{\boldsymbol{n},1} S_{\boldsymbol{n},2}) = R_{\boldsymbol{n}} P_{\boldsymbol{n}} \qquad \Rightarrow \qquad \frac{R_{\boldsymbol{n}}}{A^2} = \frac{S_{\boldsymbol{n},1} S'_{\boldsymbol{n},2} - S'_{\boldsymbol{n},1} S_{\boldsymbol{n},2}}{P_{\boldsymbol{n}}}. \tag{5.25}$$

Hence, $R_{\boldsymbol{n}}/A^2$ could have poles only at the common roots of A and $P_{\boldsymbol{n}}$. But in this case, by Proposition A.1, $S_{\boldsymbol{n},1}/P_{\boldsymbol{n}}$ and $S_{\boldsymbol{n},2}/P_{\boldsymbol{n}}$ are analytic at the zeros of $P_{\boldsymbol{n}}$. The proof is complete. ◻

Remark 5.6 In the case when

$$A_1 = A_2 = A, \qquad \deg(A) < \deg(B_i) + 1, \quad i = 1, 2,$$

formula (5.21) shows that

$$\deg\left(\frac{R_{\boldsymbol{n}}}{A}\right) = \deg(B_1) + \deg(B_2) + \deg(B_1 - B_2) - 2. \tag{5.26}$$

If on the other hand, $A_1 = A_2 = A$, $B_1 = B_2 = B$,

$$\sigma = \max\{\deg(A) - 2, \deg(B) - 1\} = 1 \quad \text{and} \quad n_1 = n_2,$$

then $R_{\boldsymbol{n}}^*$ is a constant; in other words,

$$R_{\boldsymbol{n}}(x) = \text{const} \times A^2.$$

Indeed, by (5.25),

$$R_{\boldsymbol{n}}^* = \frac{S_{\boldsymbol{n},1} S'_{\boldsymbol{n},2} - S'_{\boldsymbol{n},1} S_{\boldsymbol{n},2}}{P_{\boldsymbol{n}}}$$

is a polynomial. Since $\deg(S_{\boldsymbol{n},1}) = n_2 + 1$ and $\deg(S_{\boldsymbol{n},2}) = n_1 + 1$, $\deg(R_{\boldsymbol{n}}^*) \le 1$. The assumption that $n_1 = n_2$ implies that the leading coefficient of $S_{\boldsymbol{n},1} S'_{\boldsymbol{n},2}$ and $S'_{\boldsymbol{n},1} S_{\boldsymbol{n},2}$ match, which proves that $R_{\boldsymbol{n}}^*$ is a constant.

Now we are ready to produce the promised differential equations satisfied by the electrostatic partners:

Theorem 5.7 *There exist polynomials D_1 and D_2 (in general, dependent on $\boldsymbol{n}$) such that $S_{\boldsymbol{n},1}$ is solution of the linear differential equation*

$$A_1 A_2 P_{\boldsymbol{n}} R_{\boldsymbol{n}}\, y'' + ((2A_1 A_2' + A_1 B_2 - A_2 B_1) P_{\boldsymbol{n}} R_{\boldsymbol{n}} - A_1 A_2 (P_{\boldsymbol{n}} R'_{\boldsymbol{n}} + P'_{\boldsymbol{n}} R_{\boldsymbol{n}})) y' + D_1 y = 0\,, \tag{5.27}$$

and $S_{\boldsymbol{n},2}$ satisfies

$$A_1 A_2 P_{\boldsymbol{n}} R_{\boldsymbol{n}}\, y'' + ((2A_1' A_2 - A_1 B_2 + A_2 B_1) P_{\boldsymbol{n}} R_{\boldsymbol{n}} - A_1 A_2 (P_{\boldsymbol{n}} R'_{\boldsymbol{n}} + P'_{\boldsymbol{n}} R_{\boldsymbol{n}})) y' + D_2 y = 0. \tag{5.28}$$

If $A_1 = A_2$ and $B_1 = B_2$, then the two differential equations coincide, i.e., $S_{\boldsymbol{n},1}$ and $S_{\boldsymbol{n},2}$ are solutions of the same differential equation

$$P_{\boldsymbol{n}} R_{\boldsymbol{n}}^* y'' - (P'_{\boldsymbol{n}} R_{\boldsymbol{n}}^* + P_{\boldsymbol{n}} (R_{\boldsymbol{n}}^*)') y' + D^* y = 0\,,$$

where $R_{\boldsymbol{n}}^*$ was defined in (5.19), and D^* is a certain polynomial, dependent on $\boldsymbol{n}$.

Proof Notice that, away from the zeros of A_1 and A_2, the formal identity

$$\mathfrak{Wrons}[y, f_{\boldsymbol{n},1}, f_{\boldsymbol{n},2}] = \det \begin{pmatrix} y & f_{\boldsymbol{n},1} & f_{\boldsymbol{n},2} \\ y' & f'_{\boldsymbol{n},1} & f'_{\boldsymbol{n},2} \\ y'' & f''_{\boldsymbol{n},1} & f''_{\boldsymbol{n},2} \end{pmatrix} (z) = 0 \tag{5.29}$$

is satisfied by $y = f_{\boldsymbol{n},i}$, $i = 1, 2$. Expanding the determinant along the first column yields

$$u_2(z)y''(z) - u_1(z)y'(z) + u_0(z)y(z) = 0,$$

with

$$u_2 = \mathfrak{Wrons}[f_{\boldsymbol{n},1}, f_{\boldsymbol{n},2}] = d_{\boldsymbol{n}}, \quad u_1 = \det \begin{pmatrix} f_{\boldsymbol{n},1} & f_{\boldsymbol{n},2} \\ f''_{\boldsymbol{n},1} & f''_{\boldsymbol{n},2} \end{pmatrix} (z) = \mathfrak{Wrons}[f_{\boldsymbol{n},1}, f_{\boldsymbol{n},2}]' = d'_{\boldsymbol{n}},$$

and

$$u_0 = \det \begin{pmatrix} f'_{\boldsymbol{n},1} & f'_{\boldsymbol{n},2} \\ f''_{\boldsymbol{n},1} & f''_{\boldsymbol{n},2} \end{pmatrix} (z) = \mathfrak{Wrons}[f'_{\boldsymbol{n},1}, f'_{\boldsymbol{n},2}](z). \tag{5.30}$$

Differentiating (5.12) we get that for $i = 1, 2$,

$$\begin{aligned} A_i V_i f'_{\boldsymbol{n},i} &= A_i S'_{\boldsymbol{n},i} - S_{\boldsymbol{n},i} \left(A'_i + B_i\right), \\ A_i^2 V_i f''_{\boldsymbol{n},i} &= A_i \left(-S_{\boldsymbol{n},i} \left(A''_i + B'_i\right) + A_i S''_{\boldsymbol{n},i} - B_i S'_{\boldsymbol{n},i}\right) \\ &\quad + \left(2A'_i + B_i\right)\left(S_{\boldsymbol{n},i} \left(A'_i + B_i\right) - A_i S'_{\boldsymbol{n},i}\right). \end{aligned}$$

This shows that

$$D := A_1^2 A_2^2 v_1 v_2 u_0 = \det \begin{pmatrix} A_1^2 v_1 f'_{\boldsymbol{n},1} & A_2^2 v_2 f'_{\boldsymbol{n},2} \\ A_1^2 v_1 f''_{\boldsymbol{n},1} & A_2^2 v_2 f''_{\boldsymbol{n},2} \end{pmatrix}$$

is a polynomial.

Thus, we conclude that with this polynomial D, functions $f_{\boldsymbol{n},i}$ are two independent solutions of the linear differential equation

$$d_{\boldsymbol{n}} y'' - d'_{\boldsymbol{n}} y' + \frac{D}{A_1^2 A_2^2 v_1 v_2} y = 0.$$

With the change of variable $y \mapsto y/v_1$ and using the definition (5.12) we see that $S_{\boldsymbol{n},1}$ is a solution of the equation

$$g_2(z)y''(z) + g_1(z)y'(z) + g_0(z)y(z) = 0,$$

with

$$g_2 = \frac{d_{\boldsymbol{n}}}{v_1} = \frac{R_{\boldsymbol{n}} P_{\boldsymbol{n}}}{A_1 A_2 v_1^2 v_2},$$

$$g_1 = \frac{-d_{\boldsymbol{n}}'}{v_1} - \frac{2d_{\boldsymbol{n}} v_1'}{v_1^2} = -\frac{R_{\boldsymbol{n}} P_{\boldsymbol{n}}}{A_1 A_2 v_1^2 v_2} \left(\frac{P_{\boldsymbol{n}}'}{P_{\boldsymbol{n}}} + \frac{R_{\boldsymbol{n}}'}{R_{\boldsymbol{n}}} - \frac{2A_1' + B_1}{A_1} - \frac{2A_2' + B_2}{A_2} + 2\frac{A_1' + B_1}{A_1} \right),$$

$$g_0 = \frac{d_{\boldsymbol{n}}' v_1'}{v_1^2} + \frac{2d_{\boldsymbol{n}} (v_1')^2}{v_1^3} - \frac{d_{\boldsymbol{n}} v_1''}{v_1^2} + \frac{D}{A_1^2 A_2^2 v_1^2 v_2}$$
$$= \frac{R_{\boldsymbol{n}} P_{\boldsymbol{n}}}{A_1^2 A_2 v_1^2 v_2} \left(\left(\frac{P_{\boldsymbol{n}}'}{P_{\boldsymbol{n}}} + \frac{R_{\boldsymbol{n}}'}{R_{\boldsymbol{n}}} - \frac{2A_2' + B_2}{A_2} \right) (A_1' + B_1) - A_1'' - B_1' + \frac{D}{A_2} \right),$$

which shows that each $A_1^2 A_2^2 v_1^2 v_2 g_j$ is a polynomial. The differential equation for $S_{\boldsymbol{n},2}$ is obtained in an analogous way.

Finally, in the case $A_1 = A_2$ and $B_1 = B_2$ (and as explained in Remark 5.1, $w_1 = w_2 = w$), both changes of variable coincide, so we have the same linear differential equation of order 2 with polynomial coefficients for $S_{\boldsymbol{n},1}$ and $S_{\boldsymbol{n},2}$. Indeed

$$\mathfrak{Wrons}[y, S_{\boldsymbol{n},1}, S_{\boldsymbol{n},2}] = \det \begin{pmatrix} y & S_{\boldsymbol{n},1} & S_{\boldsymbol{n},2} \\ y' & S_{\boldsymbol{n},1}' & S_{\boldsymbol{n},2}' \\ y'' & S_{\boldsymbol{n},1}'' & S_{\boldsymbol{n},2}'' \end{pmatrix} = 0 \tag{5.31}$$

is satisfied by $y = S_{\boldsymbol{n},i}$, $i = 1, 2$ and by (5.25), the coefficients of y'', y' and y are

$$\mathfrak{Wrons}[S_{\boldsymbol{n},1}, S_{\boldsymbol{n},2}] = P_{\boldsymbol{n}} R_{\boldsymbol{n}}^*, \qquad -\mathfrak{Wrons}[S_{\boldsymbol{n},1}, S_{\boldsymbol{n},2}]' = -(P_{\boldsymbol{n}} R_{\boldsymbol{n}}^*)',$$
$$\mathfrak{Wrons}[S_{\boldsymbol{n},1}', S_{\boldsymbol{n},2}'] = D^*,$$

respectively. The statement is proved. □

The differential equations (5.27)–(5.28) imply that, respectively,

$$y'' + \left(\frac{2A_2'}{A_2} - \frac{B_1}{A_1} + \frac{B_2}{A_2} - \frac{R_{\boldsymbol{n}}'}{R_{\boldsymbol{n}}} - \frac{P_{\boldsymbol{n}}'}{P_{\boldsymbol{n}}} \right) y' = 0 \quad \text{at the zeros of } S_{\boldsymbol{n},1},$$
$$y'' + \left(\frac{2A_1'}{A_1} + \frac{B_1}{A_1} - \frac{B_2}{A_2} - \frac{R_{\boldsymbol{n}}'}{R_{\boldsymbol{n}}} - \frac{P_{\boldsymbol{n}}'}{P_{\boldsymbol{n}}} \right) y' = 0 \quad \text{at the zeros of } S_{\boldsymbol{n},2},$$

and the characterization (2.9), along with definitions (5.5), yields an electrostatic interpretation of the zeros of its solutions. Recall that $R_{\boldsymbol{n}} \not\equiv 0$ is the polynomial defined in (5.18) (see alternative expressions in (5.20), (5.21)). Then

Corollary 5.8 *Let $i = 1, 2$, and let*

$$\phi_1 := \frac{1}{2} \log \left| \frac{P_{\boldsymbol{n}} R_{\boldsymbol{n}}}{A_1 A_2} \right| + \frac{1}{2} \log \left| \frac{v_1}{v_2} \right|, \quad \phi_2(z) := \frac{1}{2} \log \left| \frac{P_{\boldsymbol{n}} R_{\boldsymbol{n}}}{A_1 A_2} \right| + \frac{1}{2} \log \left| \frac{v_2}{v_1} \right|.$$

If for $i \in \{1, 2\}$, the roots of $S_{\boldsymbol{n},i}$ are simple, then the discrete zero-counting measure $\nu\left(S_{\boldsymbol{n},i}\right)$ of $S_{\boldsymbol{n},i}$ is ϕ_i-critical in the sense of Definition 2.1.

Remark 5.9 As it follows from (5.27)–(5.28), zeros of $S_{\boldsymbol{n},i}$ can be multiple only at zeros of $A_1 A_2 P_{\boldsymbol{n}} R_{\boldsymbol{n}}$. By the definition of $S_{\boldsymbol{n},i}$ via $\mathfrak{D}_{w_i}[P_{\boldsymbol{n}}]$ as in (3.13), if A_i and B_i

share a common root (case not excluded by our assumptions) then this root is also a zero of $S_{\boldsymbol{n},i}$.

Notice that the roots of $S_{\boldsymbol{n},i}$ are in equilibrium in the external field ϕ_i created not only by charges fixed at the zeros of A_2 or A_1, but also by additional masses, all charged as $-1/2$, placed at the roots of $P_{\boldsymbol{n}} R_{\boldsymbol{n}}$. As $N = n_1 + n_2$ grows large, the dominant interaction is provided by the relation between the zeros of $S_{\boldsymbol{n},i}$ and $P_{\boldsymbol{n}}$. This motivates to combine the statements of Corollaries 5.4 and 5.8 into a single electrostatic model that we formulate using the notion of the critical vector measures (see Definition 2.2):

Theorem 5.10 *Let $R_{\boldsymbol{n}} \not\equiv 0$ be the auxiliary polynomial of degree $\leq 2\sigma_1 + 2\sigma_2 + 3$ defined in (5.18). If the roots of $P_{\boldsymbol{n}}$ and of $S_{\boldsymbol{n},1}$ are simple, then the discrete vector measure $\vec{\nu}_1 := (\nu(P_{\boldsymbol{n}}), \nu(S_{\boldsymbol{n},1}))$ is a critical vector measure for the energy functional $\mathcal{E}_{\vec{\varphi},a}$, with $a = -1/2$ and*

$$\vec{\varphi} = \left(\frac{1}{2} \log \left| \frac{1}{v_1} \right|, \frac{1}{2} \log \left| \frac{R_{\boldsymbol{n}}}{A_1 A_2} \right| + \frac{1}{2} \log \left| \frac{v_1}{v_2} \right| \right). \tag{5.32}$$

Analogously, if the roots of $P_{\boldsymbol{n}}$ and of $S_{\boldsymbol{n},2}$ are simple, then the discrete vector measure $\vec{\nu}_2 := (\nu(P_{\boldsymbol{n}}), \nu(S_{\boldsymbol{n},2}))$ is a critical vector measure for the energy functional $\mathcal{E}_{\vec{\varphi},a}$, with $a = -1/2$ and

$$\vec{\varphi} = \left(\frac{1}{2} \log \left| \frac{1}{v_2} \right|, \frac{1}{2} \log \left| \frac{R_{\boldsymbol{n}}}{A_1 A_2} \right| + \frac{1}{2} \log \left| \frac{v_2}{v_1} \right| \right). \tag{5.33}$$

Some observations are in order. First, the negative value of the interaction parameter $a = -1/2$ in the electrostatic model above shows that zeros of $P_{\boldsymbol{n}}$ and zeros of $S_{\boldsymbol{n},i}$ have opposite charges, and thus are mutually attracting. This indicates that in general we should not expect the equilibrium configurations to provide minima for the energy functionals, at least, without additional constraints.

In the case when $S_{\boldsymbol{n},i} \equiv \text{const}$, the assertion of the theorem is still valid taking $\nu(S_{\boldsymbol{n},i}) = 0$.

If $A_1 = A_2 = A$ and $B_1 = B_2$, we have that $v_1 = v_2 = v$, so same external field,

$$\vec{\varphi} = \left(\frac{1}{2} \log \left| \frac{1}{v} \right|, \frac{1}{2} \log \left| R_{\boldsymbol{n}}^* \right| \right) \tag{5.34}$$

is acting both on the zeros of $S_{\boldsymbol{n},1}$ and $S_{\boldsymbol{n},2}$. Moreover, as observed in Remark 5.6 ii), if additionally

$$\sigma = \max\{\deg(A) - 2, \deg(B) - 1\} = 1 \quad \text{and} \quad n_1 = n_2,$$

then $R_{\boldsymbol{n}}^*$ is a constant. In other words, the second component of $\vec{\varphi}$ is zero, so the external field acts only on the zeros of $P_{\boldsymbol{n}}$.

The electrostatic model formulated above becomes meaningful if we complement it with some additional information, such as the localization of the zeros of the participating polynomials (or at least, of the bulk of them). This is impossible in the general case considered so far. Hence, we need to impose additional assumptions on the weights w_i's. This will be carried out in the next section.

5.2 Some Special Cases of Multiple Orthogonal Polynomials

In this section we will try to make our construction more informative by clarifying the location of the zeros of the electrostatic partners $S_{\boldsymbol{n},i}$ of $P_{\boldsymbol{n}}$. Many of these results can be predicted by observing that the roots of $S_{\boldsymbol{n},i}$ are de facto critical points of the error function of the Hermite–Padé approximants, see (5.14).

5.2.1 Angelesco Systems

The best understood situation when the existence and uniqueness of the Hermite–Padé orthogonal polynomial $P_{\boldsymbol{n}}$ satisfying relations (5.1) is assured is the so–called Angelesco case, introduced by Angelesco [2] in 1919, and later studied in [24] and in the works of Aptekarev, Gonchar, Kaliaguin, Nikishin, Rakhmanov and others, see e.g. [8, 46, 55, 81]. We assume now that Δ_1 and Δ_2 are real intervals, and

$$\mathring{\Delta}_1 \cap \mathring{\Delta}_2 = \emptyset, \tag{5.35}$$

and w_1, w_2 are ≥ 0 on their supports. Under these conditions, for every multi-index $\boldsymbol{n} = (n_1, n_2)$, polynomial $P_{\boldsymbol{n}}$ is of maximal degree,

$$\deg P_{\boldsymbol{n}} = n_1 + n_2 = N,$$

(in other words, $\boldsymbol{n}$ is a *normal* index); since this is valid for every $\boldsymbol{n} = (n_1, n_2)$, the system is known as *perfect*, see [66].

Moreover, in the Angelesco case, $P_{\boldsymbol{n}}$ has exactly n_1 and n_2 simple zeros in the interiors of Δ_1 and Δ_2, respectively (see [81, Sect. 5.6]). Additionally, the localization of the majority of the zeros of the polynomials $S_{\boldsymbol{n},i}$ is given in the following result:

Proposition 5.11 *Polynomial $S_{\boldsymbol{n},1}$ (respect. $S_{\boldsymbol{n},2}$) has $n_2 - 1$ (respect. $n_1 - 1$) zeros interlacing with those of $P_{\boldsymbol{n}}$ on Δ_2 (respect. Δ_1).*

Proof Let us write

$$P_{\boldsymbol{n}}(x) = P_{\boldsymbol{n},1}(x)\, P_{\boldsymbol{n},2}(x), \tag{5.36}$$

where $P_{\boldsymbol{n},i}$ is the monic polynomial whose zeros agree with those of $P_{\boldsymbol{n}}$ on Δ_i, $i = 1, 2$. Taking $T = P_{\boldsymbol{n},i}$ in (4.4), we conclude that

$$\mathfrak{C}_{w_i}[P_{\boldsymbol{n}}] = \frac{1}{P_{\boldsymbol{n},i}}\, \mathfrak{C}_{w_i}[P_{\boldsymbol{n},i}\, P_{\boldsymbol{n}}].$$

Since the integrand in $\mathfrak{C}_{w_i}[P_{\boldsymbol{n},i}\, P_{\boldsymbol{n}}]$ preserves sign on Δ_i, it implies that for $i = 1, 2$, the Cauchy transform $\mathfrak{C}_{w_i}[P_{\boldsymbol{n}}]$ has no zeros in $\mathbb{R}\backslash\Delta_i$.

It remains to apply Lemma B.1 with $S = S_{\boldsymbol{n},i}$. □

Recall that by Theorem 5.2, $S_{\boldsymbol{n},i}$ are polynomials of degree at most $N - n_i + \sigma_i$, $i = 1, 2$. Proposition 5.11 shows that all zeros of $S_{\boldsymbol{n},i}$, $i = 1, 2$, except for at most $\sigma_i + 1$ of them (amount only depending on the classes of the weights), are well localized by this interlacing property.

Thus, the electrostatic model for the zeros of $P_{\boldsymbol{n}}$, stated in Corollary 5.4, is as follows. If we consider each one of the n_1 zeros of $P_{\boldsymbol{n}}$ on Δ_1 as a positive unit charge, then they are in equilibrium (or more exactly, they are in critical configuration) in the field generated by:

- The repulsion of the unit positive charges placed at rest of the zeros of $P_{\boldsymbol{n}}$, on Δ_2,
- The attraction of the zeros of $S_{\boldsymbol{n},1}$, this time with charge $-1/2$, all except at most $\sigma_1 + 1$ of them interlacing with the zeros of $P_{\boldsymbol{n}}$ on Δ_2, and
- The background potential from the orthogonality weight w_1 on Δ_1.

A symmetric picture is obviously valid on Δ_2.

Angelesco–Jacobi polynomials constitute an example of an Angelesco system. They are considered in detail in Sect. 8.6.

5.2.2 AT Systems and Generalized Nikishin Systems

We assume now that, unlike the Angelesco case, both weights $w_i \geq 0$, $i = 1, 2$, are supported on the same interval:

$$\Delta_1 = \Delta_2 = \Delta = [a, b] \subset \mathbb{R}. \tag{5.37}$$

These two weights form an algebraic Chebyshev system (or an *AT system*) for the multi-index $\boldsymbol{n} = (n_1, n_2)$ if

$$\left\{ w_1(x), x w_1(x), \ldots, x^{n_1 - 1} w_1(x), w_2(x), x w_2(x), \ldots, x^{n_2 - 1} w_2(x) \right\} \tag{5.38}$$

is a Chebyshev system on Δ, that is, if every non-trivial linear combination of functions from (5.38) with real coefficients has at most $N = n_1 + n_2$ zeros on Δ. For further details, see e.g. [81, Chapter 4, §4] or [52, Sect. 23.1.2].

It is known (see e.g. [52, Theorem 23.1.4]) that if the multi-index $\boldsymbol{n} = (n_1, n_2)$ is such that (w_1, w_2) is an AT system on $[a, b]$ for every index $\boldsymbol{m} = (m_1, m_2)$ for which $m_j \leq n_j$, $j = 1, 2$, then $\boldsymbol{n}$ is normal, and the multiple orthogonal polynomial $P_{\boldsymbol{n}}$ has all its N zeros, all simple, on (a, b).

A construction of an AT system that now is known to be perfect (see [42]) was put forward by E. M. Nikishin [79]; it is called an *MT-system* in [81], but is nowadays known as a *Nikishin system*. Namely, we assume additionally that the ratio w_2/w_1 is the Cauchy transform (also known as a *Markov function*) of a non-negative function on an interval $[c, d] \subset \mathbb{R}$, whose interior is disjoint with Δ. Besides normality for all multi-indices $\boldsymbol{n}$ and that all zeros of $P_{\boldsymbol{n}}$ are simple and belong to the open interval (a, b), this allows localization of zeros of the electrostatic partner, as we show below.

Remark 5.12 Nikishin [79] proved the normality for all indices of the form (n, n) and $(n + 1, n)$, asserting without proof that it holds for any index (n_1, n_2) such that

$n_1 \geq n_2$. He called it a *weakly perfect* system, but a result for Markov functions (see e.g. [92, Lemma 6.3.5]) implies that weak perfectness is equivalent to perfectness of the system. Later, K. Driver and H. Stahl established the normality for any index in the case of Nikishin systems of two functions [41] (see also [26]), and more recently, U. Fidalgo and G. López proved perfectness of a Nikishin system of any order [42].

As before, we restrict our attention to semiclassical weights, but slightly weaken Nikishin's assumptions. Namely, in the situation (5.37) we suppose that w_1, w_2 are non-negative weights on $[a, b]$ such that:

- w_1 is a semiclassical weight on $[a, b]$.
- Weight w_2 is of the form

$$w_2(x) = |\Pi(x)u(x)|w_1(x), \quad x \in [a, b], \tag{5.39}$$

where

$$u(x) = \int_c^d \frac{U(t)}{x-t}\,dt \tag{5.40}$$

is semiclassical, with $(a, b) \cap (c, d) = \emptyset$, U continuous and non-negative on (c, d), and Π is an arbitrary polynomial with real coefficients, non-vanishing on $(a, b) \cup (c, d)$.

Under these assumptions the weight w_2 is also semiclassical. The fact that (w_1, w_2) forms an AT-system for $n_1 \geq n_2 + m$ can be deduced from the fact that the linear form $p + q\Pi u$, for arbitrary polynomials p, q of respective degrees $\leq n_1 - 1, n_2 - 1$, and $n_1 \geq n_2 + m$, has at most $n_1 + n_2 - 1$ zeros in $[c, d]$ (see [79] and [61, p. 1022]).

Example 5.13 It is easy to see that if $c < d$, then for $\gamma, \delta \notin \mathbb{Z}$, $\gamma + \delta < 0$, $\gamma + \delta \in \mathbb{Z}$, function

$$u(x) = (x - c)^\gamma (d - x)^\delta$$

can be represented as the Cauchy integral (5.40). As a consequence, a pair of weights

$$w_1(x) = |x - a|^\alpha\, |x - b|^\beta, \quad w_2(x) = |x - a|^\alpha\, |x - b|^\beta |x - c|^\gamma\, |x - d|^\delta, \quad x \in (a, b),$$

where $(a, b) \cap (c, d) = \emptyset$, and $\alpha, \beta, \gamma, \delta > -1$, $\gamma, \delta \notin \mathbb{Z}$, $\gamma + \delta \in \mathbb{Z}$, constitute an example of a system defined above.

Since we do not assume that the intervals (a, b) and (c, d) are bounded, another example is the pair of weights of the form

$$w_1(x) = \exp(-x^r), \quad w_2(x) = |x - a|^\gamma\, |x - b|^\delta \exp(-x^r), \quad x \in [0, +\infty),$$

with $r \in \mathbb{N}$, $-\infty < a < b < 0$, $\gamma, \delta > -1$, $\gamma, \delta \notin \mathbb{Z}$, and $\gamma + \delta \in \mathbb{Z}$.

With the introduction of the polynomial factor Π in the weight w_2 we can no longer guarantee that all the zeros of $P_{\boldsymbol{n}}$ are in (a, b); however, the following result still holds:

Proposition 5.14 *Under the conditions on the weights w_1 and w_2 stated above, the Hermite–Padé polynomial $P_{\boldsymbol{n}}$, satisfying (5.1), has at least $n_1+\ell+1$ sign changes on (a,b), where*

$$\ell=\min(n_2-1,n_1-m),\quad m=\deg(\Pi),$$

while its Cauchy transform $\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]$ has at least $\ell+1$ sign changes in (c,d).

Proof We basically follow the arguments in [99, Sect. 2.5]. Suppose that $n_1\geq m$. Taking in (4.4) $T=P\Pi$, where P is and arbitrary polynomial of degree $k\leq n_1-m$, we get that

$$P(x)\,\Pi(x)\int_a^b\frac{P_{\boldsymbol{n}}(t)}{t-x}\,w_1(t)dt=\int_a^b\frac{P(t)\,P_{\boldsymbol{n}}(t)\Pi(t)}{t-x}\,w_1(t)dt.$$

Integrating this identity with respect to $d\sigma$ in $[c,d]$, applying Fubini's theorem and using (5.40) yields

$$\int_c^d P(x)\,\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}](x)\,\Pi(x)v(x)\,dx=\int_a^b P(t)\,P_{\boldsymbol{n}}(t)\,w_2(t)dt\;=0\,,$$

as long as the degree k of P is $\leq n_2-1$. Since polynomial $\Pi(x)$ does not vanish in (c,d) it proves that $\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]$ changes sign at least $\ell+1$ times in (c,d).

To prove the first part of the proposition, let P be a polynomial non vanishing on (a,b) and such that $\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]/P$ is analytic in $\mathbb{C}\setminus[a,b]$. By the assertion we just proved, we can take P of degree at least $\ell+1$, so that by (5.9),

$$\frac{\mathfrak{C}_{w_1}}{P}(z)=\mathcal{O}\left(\frac{1}{z^{n_1+\ell+2}}\right),\quad z\to\infty.$$

Let Γ be a positively oriented Jordan contour encircling $[a,b]$ and leaving $[c,d]$ in its exterior. Then for $k=0,1,\ldots,n_1+\ell$,

$$\begin{aligned}0=\frac{1}{2\pi i}\oint_\Gamma z^k\,\frac{\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}](z)}{P(z)}\,dx&=\frac{1}{2\pi i}\oint_\Gamma\frac{z^k}{P(z)}\left(\int_a^b\frac{P_{\boldsymbol{n}}(t)}{t-z}\,w_1(t)dt\right)dz\\&=\int_a^b P_{\boldsymbol{n}}(t)\left(\frac{1}{2\pi i}\oint_\Gamma\frac{z^k}{P(z)}\,\frac{1}{t-z}\,dz\right)w_1(t)\,dt\\&=\int_a^b t^k\,P_{\boldsymbol{n}}(t)\,\frac{w_1(t)dt}{P(t)},\end{aligned}\tag{5.41}$$

where we have used Fubini's and Cauchy's theorems. Since w_1/P has a constant sign on (a,b), $P_{\boldsymbol{n}}$ satisfies quasi–orthogonality conditions (of order at least $n_1+\ell$) there. Standard arguments yield that $P_{\boldsymbol{n}}$ has at least $n_1+\ell+1$ sign changes on (a,b). $\square$

Consider the case when $n_2\leq n_1-m+1$, so that $\ell=n_2-1$. According to Proposition 5.14, $P_{\boldsymbol{n}}$ has exactly $N=n_1+n_2$ zeros, all simple, in (a,b), while

$\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]$ has $\geq n_2$ zeros in (c, d), exactly as in the classical Nikishin setting ($m = 0$). Moreover, if these zeros of $\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]$ are disjoint with the zeros of $A_1 S_{\boldsymbol{n},1}$ then, by the second assertion of Proposition B.1, at least $\ell = n_2 - 1$ zeros of polynomial $S_{\boldsymbol{n},1}$ (out of a total of $\leq n_2 + \sigma_1$ of its zeros) interlace with those of $\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]$.

5.2.3 Generalized Nikishin Systems: Case of Overlapping Supports

Generalized Nikishin systems (GN systems) were introduced in [46] using a rooted tree graph. A particular example of such a system, whose asymptotics was studied in [5], [10] and [84], shares characteristics of both cases described in Sects. 5.2.1 and 5.2.2. Namely, we assume that

$$\Delta_1 \subseteq \Delta_2, \tag{5.42}$$

in addition to the Nikishin relation between the non-negative semiclassical weights w_1 and w_2, given by conditions (5.39)–(5.40), with $\Pi \equiv 1$, and the assumption that the interior of

$$\Delta_3 := [c, d]$$

is disjoint with Δ_2. On one hand, when Δ_1 extends to the whole Δ_2, we obtain the classical Nikishin configuration of Sect. 5.2.2. On the other, if Δ_1 and Δ_2 share an endpoint then redefinition of the support Δ_2 into $\Delta_2 \setminus \Delta_1$ yields the Angelesco setting of Sect. 5.2.1.

Let us study the diagonal case of $n_1 = n_2 = n$, so that $N = 2n$. As before, the key ingredient is the location of the zeros of the Hermite–Padé polynomial $P_{\boldsymbol{n}}$ and its electrostatic partners $S_{\boldsymbol{n},1}$ and $S_{\boldsymbol{n},2}$. It was proved in [84] that *for any n, the zeros of the Hermite–Padé polynomial $P_{\boldsymbol{n}}$, with the possible exception of five of them, are in Δ_2.* Recall that by orthogonality assumptions, at least n of them belong to the subinterval Δ_1. Additionally, we have:

Proposition 5.15 *If $P_{\boldsymbol{n}}$ has $n + k_1 \geq n$ sign changes in Δ_1 and $k_2 \geq 0$ sign changes in $\Delta_2 \setminus \Delta_1$, then $S_{\boldsymbol{n},1}$ has at least* $\max\{k_2 - 2, 0\}$ *zeros in $\Delta_2 \setminus \Delta_1$, which interlace with the zeros of $P_{\boldsymbol{n}}$, and at least* $\max\{k_1 - 3, 0\}$ *zeros in Δ_3, interlacing with the zeros of $\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]$.*

Proof Let us denote by x_i, $i = 1, \ldots, n + k_1$ the points of sign change of $P_{\boldsymbol{n}}$ in Δ_1, and by y_j, $j = 1, \ldots, k_2$ the corresponding points of sign change in $\Delta_2 \setminus \Delta_1$. Using (4.4) with

$$\Pi(x) = \prod_{j=1}^{k_2} (x - y_j),$$

we conclude again that $\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]$ does not change sign in each component of $\Delta_2 \setminus \Delta_1$. Then, the first part of Lemma B.1 asserts that $S_{\boldsymbol{n},1}$ has at least $k_2 - 2$ zeros in $\Delta_2 \setminus \Delta_1$, interlacing with those of $P_{\boldsymbol{n}}$ (observe that $\Delta_2 \backslash \Delta_1$ can have up to two disjoint components).

Notice that $k_1 + k_2 \le n$, so that if $k_2 = n$, our proof is finished. Suppose that $k_2 < n$, and let us see that $\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]$ has at least $k_2 - 2$ sign changes in Δ_3. From (5.1) and (5.39)–(5.40), we have that for any polynomial $P \in \mathbb{P}_{n-1}$,

$$
\begin{aligned}
0 &= \int_{\Delta_1} P(x) P_{\boldsymbol{n}} w_1(x) u(x) dx + \int_{\Delta_2 \backslash \Delta_1} P(x) P_{\boldsymbol{n}} w_2(x) dx \\
&= - \int_{\Delta_3} \pi(t) \mathfrak{C}_{w_1}[P_{\boldsymbol{n}}](t) \sigma(t) dt + \int_{\Delta_2 \backslash \Delta_1} P(x) P_{\boldsymbol{n}} w_2(x) dx \,,
\end{aligned}
\tag{5.43}
$$

where we have used Fubini's theorem for the last identity. Now, denote by z_i , $i = 1, \dots, k_3$ the points where $\mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]$ changes sign in Δ_3. Let us suppose that $k_3 < k_1 - 2$ and define

$$
P(x) := (x - \zeta_1)^{\epsilon_1} (x - \zeta_2)^{\epsilon_2} \prod_{i=1}^{k_2} (x - y_i) \prod_{j=1}^{k_3} (x - z_j) \,,
$$

where $\epsilon_i \in \{0, 1\}$, $i = 1, 2$, $\zeta_1 \in \Delta_1$ and ζ_2 is located in the "gap" between Δ_2 and Δ_3 (which may consist of a single point, since we only require the interiors to be disjoint). We can use the parameters ζ_1, ζ_2 and ϵ_1, ϵ_2 to guarantee that

$$
P(t) \mathfrak{C}_{w_1}[P_{\boldsymbol{n}}](t) \sigma(t) \ge 0, \quad t \in \Delta_3
$$

and

$$
P(x) P_{\boldsymbol{n}}(x) w_2(x) \le 0, \quad x \in \Delta_2.
$$

Since by assumption, $\deg(P) = k_2 + k_3 + 2 < k_1 + k_2 \le n$, so that (5.43) should hold for this particular choice of P. This is possible only if both integrands in the right hand side of (5.43) were identically 0, which is a contradiction. Hence, $k_3 \ge k_1 - 2$, and it remains to use again the second assertion in Lemma B.1 to conclude the proof. $\square$

Thus, as expected from an intermediate case between Angelesco and Nikishin settings, now a part of the zeros of the electrostatic partner $S_{\boldsymbol{n},1}$ lie on Δ_3 (as in the Nikishin case) while the rest are located in $\Delta_2 \backslash \Delta_1$ (Angelesco). Therefore, in this situation, part of the "ghosts" attractive charges interlace with part of the zeros of $P_{\boldsymbol{n}}$ in $\Delta_2 \backslash \Delta_1$, while other part are placed in Δ_3. Of course, depending on the specific case we are dealing with, some of these sets of attractive charges may be empty.

6 Asymptotic Zero Distribution

It is instructive to observe the discrete-to-continuous transition of the electrostatic model described in the previous section, assuming that the total degree $N = n_1 + n_2 \to \infty$.

6.1 Vector Critical Measures

If μ_1, μ_2 are two finite positive Borel measures with compact support then their (continuous) mutual logarithmic energy is

$$\langle \mu_1, \mu_2 \rangle := \iint \log \frac{1}{|x-y|} \, d\mu_1(x) d\mu_2(y); \tag{6.1}$$

and the logarithmic energy of μ_i is

$$E(\mu_i) := \langle \mu_i, \mu_i \rangle, \quad i = 1, 2. \tag{6.2}$$

Given a vector of measures $\vec{\mu} = (\mu_1, \ldots, \mu_k)$, and a symmetric positive-definite *interaction matrix*

$$M = (m_{ij})_{i,j=1}^{k},$$

the vector energy of $\vec{\mu}$ is

$$E_M(\vec{\mu}) := \sum_{i,j=1}^{k} m_{ij} \langle \mu_i, \mu_j \rangle. \tag{6.3}$$

In the particular case of $k = 2$, when

$$M = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}, \quad -1 < a < 1,$$

we call a the *interaction parameter*.

We define the critical vector measures following [71] and [73]. Recall that any smooth complex-valued function h in the closure $\overline{\Omega}$ of a domain Ω generates a local variation of Ω by $z \mapsto z^t = z + t\, h(z)$, $t \in \mathbb{C}$. It is easy to see that $z \mapsto z^t$ is injective for small values of the parameter t. The transformation above induces a variation of sets $e \mapsto e^t := \{z^t : z \in e\}$, and measures: $\mu \mapsto \mu^t$, defined by $\mu^t(e^t) = \mu(e)$; in the differential form, the pullback measure μ^t can be written as $d\mu^t(x^t) = d\mu(x)$. Recall also the notation, introduced in Sect. 5.1, of $\mathbb{A}_i$ for the set of zeros of A_i on $\mathbb{C}$, $i = 1, 2$.

Definition 6.1 *(see* [73]*)* A vector measure $\vec{\mu} = (\mu_1, \dots, \mu_k)$ is a (continuous) *critical vector measure* if for any h smooth in $\mathbb{C}\setminus(\mathbb{A}_1 \cup \mathbb{A}_2)$ such that $h\big|_{\mathbb{A}_1\cup\mathbb{A}_2} \equiv 0$,

$$\frac{d}{dt} E_M(\vec{\mu}^t)\big|_{t=0} = \lim_{t\to 0} \frac{E_M(\vec{\mu}^t) - E_M(\vec{\mu})}{t} = 0. \tag{6.4}$$

As in the discrete case,

$$\mu_i \text{ is } F_i\text{-critical, with } F_i := \sum_{1\le j\le k,\ j\ne i} \frac{m_{ij}}{m_{ii}} U^{\mu_j}, \quad i \in \{1, \dots, k\},$$

which yields the following variational conditions on the components of $\vec{\mu}$: for $x \in \operatorname{supp}(\mu_i)$, with a possible exception of a subset of logarithmic capacity 0,

$$U^{\mu_i}(x) + \sum_{1\le j\le k,\ j\ne i} \frac{m_{ij}}{m_{ii}} U^{\mu_j} = c_i = \text{const}, \quad i \in \{1, \dots, k\}. \tag{6.5}$$

6.2 Asymptotic Electrostatic Model: General Case

Under a general assumption that for each multi-index $\boldsymbol{n} = (n_1, n_2) \in \mathbb{N}^2$ the Hermite–Padé polynomial $P_{\boldsymbol{n}}$ exists and is of degree $N = n_1 + n_2$, as well as that $\deg(S_{\boldsymbol{n},i}) = N - n_i + 1$, we consider the zero-counting measures (see the definition in (2.7)) for $P_{\boldsymbol{n}}$, $S_{\boldsymbol{n},1}$ and $S_{\boldsymbol{n},2}$ in the asymptotic regime

$$N = n_1 + n_2 \to \infty, \quad \lim_{N\to\infty} \frac{n_2}{N} = t \in [0, 1]. \tag{6.6}$$

Let us assume that (perhaps, along a subsequence of multi-indices), weak limits

$$\mu := \lim_{\boldsymbol{n}} \frac{1}{N} \nu(P_{\boldsymbol{n}}), \quad \nu_1 := \lim_{\boldsymbol{n}} \frac{1}{N} \nu(S_{\boldsymbol{n},1}), \quad \nu_2 := \lim_{\boldsymbol{n}} \frac{1}{N} \nu(S_{\boldsymbol{n},2}), \tag{6.7}$$

exist. With our assumptions,

$$\|\mu\| := \int d\mu = 1, \quad \|\nu_1\| := \int d\nu_1 = t, \quad \|\nu_2\| := \int d\nu_2 = 1 - t. \tag{6.8}$$

Since every weak-* limit of a sequence of discrete critical vector measures is critical in the sense of Definition 6.1 (a proof for the scalar case can be found in [71, Theorem 7.1]; it applies to the vector equilibrium without substantial modifications), we obtain:

Corollary 6.2 *With the assumptions and notations above, each vector measure (μ, ν_1) and (μ, ν_2) is critical in the sense of Definition 6.1, with the interaction parameter $a = -1/2$ and constraints (6.8).*

It is worth also discussing a connection with a more traditional model involving 3-component vector measures. Define

$$\Omega = \overline{\{x \in \mathbb{R} : \mu = \nu_1\}}, \quad \lambda_1 := \mu\big|_{\Omega}, \quad \lambda_2 = \mu - \lambda_1, \quad \lambda_3 = \nu_1 - \lambda_1. \tag{6.9}$$

Obviously,

$$\|\lambda_1\| + \|\lambda_2\| = 1, \quad \|\lambda_1\| + \|\lambda_3\| = t. \tag{6.10}$$

Variational conditions (6.5) for (μ, ν_1) with the interaction matrix

$$\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$$

imply that

$$\begin{aligned} U^{\mu}(x) - \frac{1}{2} U^{\nu_1}(x) &= c_1 = \text{const}, \quad x \in \operatorname{supp}(\mu), \\ U^{\nu_1}(x) - \frac{1}{2} U^{\mu}(x) &= c_2 = \text{const}, \quad x \in \operatorname{supp}(\nu_1), \end{aligned}$$

or equivalently,

$$\begin{aligned} \frac{1}{2} U^{\lambda_1}(x) + U^{\lambda_2}(x) - \frac{1}{2} U^{\lambda_3}(x) &= c_1 = \text{const}, \quad x \in \operatorname{supp}(\mu) = \operatorname{supp}(\lambda_1) \cup \operatorname{supp}(\lambda_2), \\ \frac{1}{2} U^{\lambda_1}(x) - \frac{1}{2} U^{\lambda_2}(x) + U^{\lambda_3}(x) &= c_2 = \text{const}, \quad x \in \operatorname{supp}(\nu_1) = \operatorname{supp}(\lambda_1) \cup \operatorname{supp}(\lambda_3). \end{aligned}$$

Additionally, on $\operatorname{supp}(\lambda_1)$, where both identities hold, we have that

$$U^{\lambda_1}(x) + \frac{1}{2} U^{\lambda_2}(x) + \frac{1}{2} U^{\lambda_3}(x) = c_1 + c_2.$$

Corollary 6.3 *With the assumptions and notations above, $(\lambda_1, \lambda_2, \lambda_3)$ is a critical vector measure satisfying the constraints (6.10) and with the interaction matrix*

$$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & -1/2 \\ 1/2 & -1/2 & 1 \end{pmatrix}.$$

This electrostatic model has been used to describe the asymptotics of the zeros of Hermite–Padé polynomials in several situations, see e.g. [5, 11, 73–75, 84]. In particular, a spectral curve for such critical measures was derived in [73], and it was shown that λ_i's are supported on a finite number of analytic arcs, that are trajectories of a quadratic differential globally defined on a three-sheeted Riemann surface.

6.3 Asymptotic Electrostatic Model: Angelesco Case

Using the notation (6.7) and Proposition 5.11, in this case

$$\mathrm{supp}(\mu) \subseteq \Delta_1 \cup \Delta_2, \quad \mathrm{supp}(\nu_1) \subseteq \Delta_2, \quad \mathrm{supp}(\nu_2) \subseteq \Delta_1,$$

and

$$\nu_1 = \mu\big|_{\Delta_2}, \quad \nu_2 = \mu\big|_{\Delta_1},$$

so that in notation (6.9),

$$\lambda_1 = \nu_1, \quad \lambda_2 = \nu_2, \quad \lambda_3 = 0.$$

Thus, in this case the electrostatic vector model of Corollary 6.3 reduces to a 2×2 equilibrium for (ν_1, ν_2), with the interaction matrix

$$M = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

and constraints

$$\mathrm{supp}(\nu_1) \subset \Delta_1, \quad \|\nu_1\| = t, \qquad \text{and} \qquad \mathrm{supp}(\nu_2) \subset \Delta_2, \quad \|\nu_2\| = 1 - t.$$

This is already classical, see e.g. [81, Ch. 5]; actually, a stronger result is valid: the vector measure (ν_1, ν_2) is a *global minimum* for (2.14) and $a = 1/2$. This does not follow directly from our electrostatic model.

Notice that measure $\nu = \nu_1 + \nu_2$ is the limit zero distribution of both $P_{\boldsymbol{n}}$ and $S_{\boldsymbol{n},1} S_{\boldsymbol{n},2}$.

Remark 6.4 Although in the Angelesco case the zeros of $P_{\boldsymbol{n}}$ are confined in $\Delta_1 \cup \Delta_2$, in principle up to $\sigma_i + 1$ of $S_{\boldsymbol{n},i}$, $i = 1, 2$, are out of our control. Same happens with (a bounded number of) zeros of the polynomials $R_{\boldsymbol{n}}$. In order to guarantee weak convergence of the zero-counting measures, it is sufficient to impose an additional assumption: that *the zeros of $S_{\boldsymbol{n},1}$, $S_{\boldsymbol{n},2}$ and $R_{\boldsymbol{n}}$ are uniformly bounded along the double sequence (n_1, n_2).*

6.4 Asymptotic Electrostatic Model: Nikishin Case

Consider the generalized Nikishin system as described in Sect. 5.2.2, in the asymptotic regime (6.6) and with the additional assumption that

$$n_2 \leq n_1 - m + 1, \tag{6.11}$$

so that

$$t = \lim_{N \to \infty} \frac{n_2}{N} \in [0, 1/2]. \tag{6.12}$$

As we have seen, all N zeros of $P_{\boldsymbol{n}}$ live on $[a, b]$; according to Theorem 5.10, each one of them, endowed with a charge $+1$, interacts with zeros of $S_{\boldsymbol{n},1}$, each one with charge $-1/2$. We have seen that the majority of them (at least $\ell = n_2 - 1$ out of $n_2 + \sigma_1$ possible) belongs to $[c, d]$. Imposing the same additional assumption that before, that *the zeros of both $S_{\boldsymbol{n},1}$ and $R_{\boldsymbol{n}}$ are uniformly bounded along the double sequence* (n_1, n_2), we can use again the weak-* compactness of measures.

With the notation (6.7),

$$\operatorname{supp}(\mu) \subseteq \Delta_1 = [a, b], \quad \operatorname{supp}(\nu_1) \subseteq \Delta_2 = [c, d],$$

so that in notation (6.9),

$$\lambda_1 = 0, \quad \lambda_2 = \mu, \quad \lambda_3 = \nu_1.$$

Thus, in this case the electrostatic vector model of Corollary 6.3 reduces to a 2×2 equilibrium for (μ, ν_1), with the interaction matrix

$$M = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$$

and constraints

$$\operatorname{supp}(\mu) \subset \Delta_1, \quad \|\mu\| = 1, \qquad \text{and} \qquad \operatorname{supp}(\nu_1) \subset \Delta_2, \quad \|\nu_2\| = 1 - t.$$

This model was initially put forward by Nikishin himself in [80]. Again, a stronger result is valid: the vector measure (μ, ν_1) is a *global minimum* for the vector energy, which does not follow directly from our electrostatic model.

Observe that the roles of the weights w_1 and w_2 in a Nikishin system are not symmetric, or at least the symmetry is not immediate. An argument that allows to swap w_1 and w_2, and thus break the barrier of $n_2 \leq n_1$, is based on the fact (see e.g. [92, Lemma 6.3.5]) that if u is a Markov function (5.40), then

$$\frac{1}{u(x)} = r(x) - \int \frac{d\tau(t)}{x - t},$$

where r is a polynomial of degree ≤ 1 and τ is a positive measure on $[c, d]$. Standard arguments allow to extend the previous result when $m = 0$ (the classical Nikishin case), see e.g. [61]. Unfortunately, the presence of a non-trivial polynomial factor Π in (5.39) prevents these arguments from going through. Thus, if $m = 0$, in the asymptotic regime (6.6) we can drop the restriction (6.11), and the electrostatic model discussed above is still valid. Another interesting result that sheds light on the roles of w_1 and w_2, allowing to connect the situations of $n_1 \leq n_2$ and $n_1 \geq n_2$, appears in [64].

6.5 Asymptotic Electrostatic Model: Case of the Overlapping Supports

Here we consider the asymptotic electrostatics for the intermediate case studied in [5, 84] and partially analyzed in Sect. 5.2.3. Recall that here we consider $n_1 = n_2 = n$, and thus, we are interested in what happens as $n \to \infty$.

With our notation in the current section, we have that $\operatorname{supp} \mu \subseteq \Delta_2$ and $\operatorname{supp} \nu_1 \subseteq (\Delta_2 \backslash \Delta_1) \cup \Delta_3$, in such a way that now we have,

$$\operatorname{supp} \lambda_1 \subseteq \Delta_2 \backslash \Delta_1 \,, \ \operatorname{supp} \lambda_2 \subseteq \Delta_1 \,, \ \operatorname{supp} \lambda_3 \subseteq \Delta_3 \,,$$

The results in Proposition 5.15 guarantee that

$$\|\lambda_1\| + \|\lambda_2\| = 1 \,, \ \|\lambda_2\| = \|\lambda_3\| + \frac{1}{2} \,.$$

Since in this intermediate case none of the measures λ_i becomes null, the 3×3 interaction matrix in Corollary 6.3 does not reduce to a 2×2 one, as in the previous (extremal) cases. As for the constraints on the size of the measures, in this case we have

$$\|\lambda_1\| = \frac{1}{2} - \theta \,, \ \|\lambda_2\| = \frac{1}{2} + \theta \,, \ \|\lambda_3\| = \theta \,,$$

where $\theta \in [0, 1/2]$ is a parameter which depends on the relative sizes and mutual positions of the three intervals Δ_i , $i = 1, 2, 3$; but especially on the first two ones, as asserted by the author of [84].

Moreover, as in the previous cases, for this "critical" value of the parameter θ, a stronger result is valid. The vector measure $(\lambda_1, \lambda_2, \lambda_3)$ described above is a global minimum of the vector energy (see [84]); but, again, this result does not follow directly from our electrostatic approach.

7 Differential Equation of Order 3

The system of second order linear differential equations in Theorems 5.2 and 5.7 allowed us to derive an electrostatic interpretation of the zeros of the Hermite–Padé polynomials of type II. In this section, we show how these equations can be combined into a single third order homogeneous differential equation, satisfied simultaneously by the polynomial $P_{\boldsymbol{n}}$ and by the functions of the second kind $q_{\boldsymbol{n},1}$ and $q_{\boldsymbol{n},2}$. Notice that if we only cared about a third order ODE solved by $P_{\boldsymbol{n}}$, it would be sufficient to differentiate one of the equations (5.15); thus, it is convenient to stress here that we seek equations whose basis of solutions is precisely $(P_{\boldsymbol{n}}, q_{\boldsymbol{n},1}, q_{\boldsymbol{n},2})$.

As it was mentioned in the introduction, third order homogeneous linear ODE whose solutions are $P_{\boldsymbol{n}}$ have been already described in the literature. For instance, such an equation was found in [56] for the Jacobi–Angelesco multiple orthogonal polynomials, see Sect. 8.6. In [12], Aptekarev et al. considered the case of a semiclassical weight w of class σ and multiple orthogonal polynomials $P_{\boldsymbol{n}}$ of type II with respect to w and

$\sigma + 1$ non-homotopic paths of integration, showing for instance that in the diagonal setting $\boldsymbol{n} = (n, n, ..., n)$, $P_{\boldsymbol{n}}$ satisfies a linear differential equation of order $\sigma + 2$. A kind of opposite case, when distinct classical ($\sigma = 0$) weights w_j are supported on the same contour Γ, was analyzed in [7], where again a linear ODE of order $r + 1$, where r is the number of weights, was derived.

In a clear resemblance to the definition of the polynomial $R_{\boldsymbol{n}}$ in (5.17)–(5.18), let

$$E_{\boldsymbol{n}} := A_1^2 A_2^2 v_1 v_2 \det \begin{pmatrix} P_{\boldsymbol{n}} & q_{\boldsymbol{n},1} & q_{\boldsymbol{n},2} \\ P''_{\boldsymbol{n}} & q''_{\boldsymbol{n},1} & q''_{\boldsymbol{n},2} \\ P'''_{\boldsymbol{n}} & q'''_{\boldsymbol{n},1} & q'''_{\boldsymbol{n},2} \end{pmatrix}, \quad F_{\boldsymbol{n}} := A_1^2 A_2^2 v_1 v_2 \det \begin{pmatrix} P'_{\boldsymbol{n}} & q'_{\boldsymbol{n},1} & q'_{\boldsymbol{n},2} \\ P''_{\boldsymbol{n}} & q''_{\boldsymbol{n},1} & q''_{\boldsymbol{n},2} \\ P'''_{\boldsymbol{n}} & q'''_{\boldsymbol{n},1} & q'''_{\boldsymbol{n},2} \end{pmatrix}. \tag{7.1}$$

In this section we maintain the assumption (5.11).

Theorem 7.1 *(a) Functions $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$, defined above, are polynomials, with*

$$\begin{aligned} \deg(E_{\boldsymbol{n}}) \le \max\{&\deg(A_1) + \sigma_2 + \deg(R_{\boldsymbol{n}}), \deg(A_2) \\ &+ \sigma_1 + \deg(R_{\boldsymbol{n}}), \deg(B_1) + \deg(B_2) + \deg(R_{\boldsymbol{n}})\}, \\ \deg(F_{\boldsymbol{n}}) \le\ & \sigma_1 + \sigma_2 + \deg(R_{\boldsymbol{n}}) + 1, \end{aligned}$$

and σ_i defined in (5.6).

(b) $P_{\boldsymbol{n}}$, $q_{\boldsymbol{n},1}$ and $q_{\boldsymbol{n},2}$ are solutions of the linear differential equation with polynomial coefficients

$$\begin{aligned} &A_1 A_2 R_{\boldsymbol{n}} y''' + \left[(A_1(2A_2' + B_2) + A_2(2A_1' + B_1)) R_{\boldsymbol{n}} - A_1 A_2 R'_{\boldsymbol{n}} \right] y'' \\ &+ E_{\boldsymbol{n}} y' + F_{\boldsymbol{n}} y = 0. \end{aligned} \tag{7.2}$$

(c) In the particular case when $A_1 = A_2 = A$, $B_1 = B_2 = B$ (so that $w_1 = w_2$), and $\sigma = 1$, $n_1 = n_2$, the differential equation (7.2) reduces to

$$A^2 y''' + 2A(A' + B) y'' + E^*_{\boldsymbol{n}} y' + F^*_{\boldsymbol{n}} y = 0, \tag{7.3}$$

where $E^*_{\boldsymbol{n}}$ and $F^*_{\boldsymbol{n}}$ are polynomials of degree at most 4 and 3, respectively.

Proof Recall the second order differential operators introduced in (5.17),

$$\mathcal{L}_i[y] := A_i V_i \,\mathfrak{Wrons}[y, P_{\boldsymbol{n}}, q_{\boldsymbol{n},i}], \quad i = 1, 2.$$

By (3.29) (see Remark 3.8),

$$\mathcal{L}_i[y] = A_i S_{\boldsymbol{n},i}\, y'' + (A_i' S_{\boldsymbol{n},i} - A_i S'_{\boldsymbol{n},i} + B_i S_{\boldsymbol{n},i})\, y' + C_{\boldsymbol{n},i}\, y.$$

Clearly, $\mathcal{L}_i[P_{\boldsymbol{n}}] = \mathcal{L}_i[q_{\boldsymbol{n},i}] = 0$, and by (5.18),

$$\mathcal{L}_1[q_{\boldsymbol{n},2}] = \frac{R_{\boldsymbol{n}}}{A_2 v_2}, \quad \mathcal{L}_2[q_{\boldsymbol{n},1}] = -\frac{R_{\boldsymbol{n}}}{A_1 v_1}. \tag{7.4}$$

Consider the third order linear differential operator

$$\mathcal{M}[y] := \frac{A_2^2 v_2}{S_{\boldsymbol{n},1}} \left(\mathcal{L}_1[q_{\boldsymbol{n},2}] \, (\mathcal{L}_1[y])' - \left(\mathcal{L}_1[q_{\boldsymbol{n},2}]\right)' \mathcal{L}_1[y] \right). \tag{7.5}$$

By construction, the differential equation $\mathcal{M}[y] = 0$ is solved by $P_{\boldsymbol{n}}$, $q_{\boldsymbol{n},1}$ and $q_{\boldsymbol{n},2}$. Using (5.18) and (7.4) we can find the explicit expressions for the coefficients of $\mathcal{M}$. For instance, the coefficient at y''' is

$$\frac{A_2^2 v_2}{S_{\boldsymbol{n},1}} \frac{R_{\boldsymbol{n}}}{A_2 v_2} A_1 S_{\boldsymbol{n},1} = A_1 A_2 R_{\boldsymbol{n}}.$$

The one at y'' is

$$\begin{aligned}
&\frac{A_2^2 v_2}{S_{\boldsymbol{n},1}} \left(\frac{R_{\boldsymbol{n}}}{A_2 v_2} \Big((A_1 S_{\boldsymbol{n},1})' + (-A_1 S'_{\boldsymbol{n},1} + A'_1 S_{\boldsymbol{n},1} + B_1 S_{\boldsymbol{n},1}) \Big) - \left(\frac{R_{\boldsymbol{n}}}{A_2 v_2} \right)' A_1 S_{\boldsymbol{n},1} \right) \\
&= \frac{A_2^2 v_2}{S_{\boldsymbol{n},1}} \left(\frac{R_{\boldsymbol{n}}}{A_2 v_2} \left(2A'_1 S_{\boldsymbol{n},1} + B_1 S_{\boldsymbol{n},1} \right) - \frac{R_{\boldsymbol{n}}}{A_2 v_2} \left(\frac{R'_{\boldsymbol{n}}}{R_{\boldsymbol{n}}} - \frac{A'_2}{A_2} - \frac{v'_2}{v_2} \right) A_1 S_{\boldsymbol{n},1} \right) \\
&= A_2 R_{\boldsymbol{n}} \left(\left(2A'_1 + B_1 \right) - \left(\frac{R'_{\boldsymbol{n}}}{R_{\boldsymbol{n}}} - \frac{A'_2}{A_2} - \frac{v'_2}{v_2} \right) A_1 \right) \\
&= A_2 R_{\boldsymbol{n}} \left(\left(2A'_1 + B_1 \right) - \left(\frac{R'_{\boldsymbol{n}}}{R_{\boldsymbol{n}}} - \frac{A'_2}{A_2} - \frac{A'_2 + B_2}{A_2} \right) A_1 \right) \\
&= A_1 (2A'_2 + B_2) R_{\boldsymbol{n}} + A_2 (2A'_1 + B_1) R_{\boldsymbol{n}} - A_1 A_2 R'_{\boldsymbol{n}}.
\end{aligned}$$

Similar calculations for the rest of the coefficients show that

$$\begin{aligned}
\mathcal{M}[y] \; &= A_1 A_2 R_{\boldsymbol{n}} y''' + (A_1 (2A'_2 + B_2) R_{\boldsymbol{n}} + A_2 (2A'_1 + B_1) R_{\boldsymbol{n}} - A_1 A_2 R'_{\boldsymbol{n}}) y'' \\
&\quad + E_{\boldsymbol{n}} y' + F_{\boldsymbol{n}} y,
\end{aligned}$$

where

$$\begin{aligned}
E_{\boldsymbol{n}} = {}& \frac{-A_1 A_2 R_{\boldsymbol{n}} S''_{\boldsymbol{n},1} + ((-A_1 (2A'_2 + B_2) + A_2 B_1) R_{\boldsymbol{n}} + A_1 A_2 R'_{\boldsymbol{n}}) S'_{\boldsymbol{n},1} + A_2 R_{\boldsymbol{n}} C_{\boldsymbol{n},1}}{S_{\boldsymbol{n},1}} \\
&+ (A''_1 A_2 + A'_1 (2A'_2 + B_2) + 2A'_2 B_1 + A_2 B'_1 + B_1 B_2) R_{\boldsymbol{n}} - A_2 (A'_1 + B_1) R'_{\boldsymbol{n}},
\end{aligned} \tag{7.6}$$

and

$$F_{\boldsymbol{n}} = \frac{(A_2 C'_{\boldsymbol{n},1} + (B_2 + 2A'_2) C_{\boldsymbol{n},1}) R_{\boldsymbol{n}} - A_2 C_{\boldsymbol{n},1} R'_{\boldsymbol{n}}}{S_{\boldsymbol{n},1}}. \tag{7.7}$$

Construction (7.5) can be carried out exchanging the role of the indices $i = 1$ and $i = 2$. It yields a third order linear differential equation with the same coefficient at y'''. Since $P_{\boldsymbol{n}}$, $q_{\boldsymbol{n},1}$ and $q_{\boldsymbol{n},2}$ are linearly independent (assumption (5.11)), we conclude that this is the same ODE. In other words, (7.6) and (7.7) are invariant by exchange of the indices $i = 1$ and $i = 2$.

Moreover, a third order linear differential equation with the same set of solutions is

$$A_1^2 A_2^2 v_1 v_2 \, \mathfrak{Wrons}[P_{\boldsymbol{n}}, q_{\boldsymbol{n},1}, q_{\boldsymbol{n},2}, y] = 0.$$

Again, by (5.18), the coefficient at y''' is $A_1 A_2 R_{\boldsymbol{n}}$, which shows that

$$\mathcal{M}[y] = A_1^2 A_2^2 v_1 v_2 \, \mathfrak{Wrons}[P_{\boldsymbol{n}}, q_{\boldsymbol{n},1}, q_{\boldsymbol{n},2}, y]. \tag{7.8}$$

In particular, functions $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$ in (7.6) and (7.7) coincide with those defined in (7.1).

From (7.6) and (7.7) it follows that $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$ are rational functions. By (7.8) and the definition of $q_{\boldsymbol{n},i}$, $i = 1, 2$, their poles can be located at the zeros of A_1 and A_2 only. However, expressions (7.1) and assertion c) of Proposition A.1 imply that all their singularities are all removable, so that $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$ are polynomials. Since by (3.22), $\deg(C_{\boldsymbol{n},1}) - \deg(S_{\boldsymbol{n},1}) \le \sigma_1$, identities (7.6) and (7.7) imply the claimed upper bounds for the degrees of $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$. This proves a) and b) of the statement of the theorem.

Finally, let $A_1 = A_2 = A$, $B_1 = B_2 = B$ (so that $w_1 = w_2$), and $\sigma = 1$, $n_1 = n_2$. In this situation, $R_{\boldsymbol{n}} = cA^2$, $c \in \mathbb{R}\backslash\{0\}$ (see Remark 5.6), so that by (7.6),

$$E_{\boldsymbol{n}} = cA^2 \left(A \frac{-AS''_{\boldsymbol{n},i} + C_{\boldsymbol{n},i}}{S_{\boldsymbol{n},i}} + AA'' + A'B + AB' + B^2 \right), \quad i = 1, 2, \tag{7.9}$$

while by (7.7),

$$F_{\boldsymbol{n}} = cA^2 \frac{BC_{\boldsymbol{n},i} + AC'_{\boldsymbol{n},i}}{S_{\boldsymbol{n},i}}, \quad i = 1, 2. \tag{7.10}$$

Dividing (7.2) through by cA^2 we get (7.3). □

Remark 7.2 The construction given in this proof can be, apparently, extended to MOP of type II with respect to more than two weights. For instance, for a multiple orthogonal polynomial $P_{\mathbf{n}}$ with respect to semiclassical weightss w_i, $i = 1, 2, \ldots, m \ge 3$, we get the corresponding ODEs

$$\mathcal{L}_i[y] := A_i S_{\mathbf{n},i} y'' + (A'_i S_{\mathbf{n},i} - A_i S'_{\mathbf{n},i} + B_i S_{\mathbf{n},i}) y' + C_{\mathbf{n},i} y = 0, \qquad i = 1, 2, \ldots, m.$$

Then $P_{\mathbf{n}}$ and $q_{\mathbf{n},i}$, $i = 1, 2, \ldots, m$, are solutions of the fourth order ODE

$$\mathcal{M}[y] := \frac{1}{S_{\mathbf{n},1}} \left(\prod_{i=2}^{m} A_i^{m-1} v_i \right) \mathfrak{Wrons}[\mathcal{L}_1[y], \mathcal{L}_1\left[q_{\mathbf{n},2}\right], \ldots, \mathcal{L}_1[q_{\mathbf{n},3}], \ldots, \mathcal{L}_1[q_{\mathbf{n},m}]\right] = 0. \tag{7.11}$$

As in the proof above, one could expect that the coefficients of this ODE are rational functions with only possible poles at the roots of $S_{\mathbf{n},1}$. Analogous ODEs can be obtained by replacing in (7.11) $S_{\mathbf{n},1}$ by $S_{\mathbf{n},i}$ and $\mathcal{L}_1$ by $\mathcal{L}_i$, with $i = 2, \ldots, m$. This will imply again that the poles of the coefficients of the ODE could be only at common roots of the electrostatic partners. On the other hand, these ODEs are equivalent to

$$\left(\prod_{i=1}^{m} A_i^m v_i\right) \mathfrak{Wrons}\left[y, P_{\mathbf{n}}, q_{\mathbf{n},1}, q_{\mathbf{n},2}, \ldots, q_{\mathbf{n},m}\right] = 0.$$

Assertion c) of Proposition A.1 yields again that all the possible poles of the coefficients should be removable. This construction leads to a linear ODE of order $m + 1$ with polynomials coefficients whose degrees depend only on the classes of the weights w_i, $i = 1, 2, \ldots, m$.

8 Further Examples

In this section, we discuss several examples of multiple orthogonal polynomials.

8.1 Jacobi Polynomials with Non-standard Parameters

Les us return to Jacobi polynomials $P_N = P_N^{(\alpha,\beta)}$ in the non-standard situation considered already in Example 4.3, namely, when neither α, β, or $\alpha + \beta$ are integers, $\beta > -1$, and $-N < \alpha < -1$. As it was shown in [58, Theorem 6.1], in this case P_N is a type II multiple orthogonal polynomial. Indeed, on one hand it satisfies

$$\int_{-1}^{1} x^j P_N(x) w_1(x) dx = 0\,, \quad j = 0, 1, \ldots, n_1 - 1,$$

where $\Delta_1 = [-1, 1]$, $n_1 = N - [-\alpha]$, and $w_1(x) = (x - 1)^{\alpha+[-\alpha]}(x + 1)^{\beta}$ (see Example 4.3). On the other,

$$\int_{\Delta_2} z^j P_N(z) w_2(z) dz = 0\,, \quad j = 0, 1, \ldots, n_2 - 1,$$

where Δ_2 is an arbitrary curve oriented clockwise, connecting $-1 - i0$ with $-1 + i0$ and lying entirely in $\mathbb{C}\backslash(-\infty, 1]$, except for its endpoints, $n_2 = [-\alpha]$, and $w_2(z) = (z - 1)^{\alpha}(z + 1)^{\beta}$.

We have established in Example 4.3 that

$$S_{\boldsymbol{n},1}(x) = (x - 1)^{[-\alpha]},$$

as well as that $C_{\mathbf{n},1}(x) = -\lambda_{\mathbf{n},1} S_{\mathbf{n},1}(x)$. As for $S_{\boldsymbol{n},2}$, we lack in this case the second condition in (5.1), that is, we cannot guarantee that

$$\int_{\Delta_2} z^{n_2} P_N(z) w_2(z) dz \neq 0, \tag{8.1}$$

(and in general, this is false), which makes the formula (4.8) of no value.

Reasoning as in Example 4.3 and combining (4.11) and the standard differential equation for the Jacobi polynomials we arrive at the identity

$$(x^2-1)S'_{\mathbf{n},2}(x)P'_N(x) = (\lambda_N S_{\boldsymbol{n},2}(x) + C_{\boldsymbol{n},2}(x))\, P_N(x)\,. \tag{8.2}$$

Again, we have two options: either $S'_{\boldsymbol{n},2} \equiv 0$ and, thus,

$$S_{\boldsymbol{n},2}(x) \equiv \text{ const}\,, \quad C_{\boldsymbol{n},2} = -\lambda_N S_{\boldsymbol{n},2} \equiv \text{ const}\,, \tag{8.3}$$

or, otherwise, the identity

$$\frac{P'_N(x)}{P_N(x)} = \frac{\lambda_N S_{\boldsymbol{n},2}(x) + C_{\boldsymbol{n},2}(x)}{(x+1)\left((x-1)S'_{\boldsymbol{n},2}(x) - [-\alpha]S_{\boldsymbol{n},2}(x)\right)} \tag{8.4}$$

holds. In this case, the facts that P_N and P'_N are relatively prime (P_N has no multiple roots) and that the degree of $S_{\boldsymbol{n},2} \leq N - [-\alpha]$, with $-N < \alpha < -1$, imply that this cannot take place, and thus, (8.3) is the unique possible solution.

We already saw in Example 4.3 that the zeros of $P_{\boldsymbol{n}}$ were in equilibrium (that is, their counting measure was critical) in the external field (4.14). From (4.15), (5.24) and (8.3), and taking into account that in this case $A_1 = A_2 = A$, we have that $R_{\boldsymbol{n}} \equiv 0$. Moreover, the zeros of $S_{\boldsymbol{n},1}$ are obviously not simple, and in such a case we cannot say anything about electrostatics for its zeros.

Obviously, what fails in this case is (8.1), which implies that the index $\boldsymbol{n} = (N - [-\alpha], [-\alpha])$ is not normal.

8.2 Multiple Hermite Polynomials

Multiple Hermite polynomials $\mathfrak{H}_{\boldsymbol{n}}$, $\boldsymbol{n} = (n_1, n_2)$, are type II MOP of degree $\leq N = n_1 + n_2$, defined by

$$\begin{aligned}
&\int_{-\infty}^{\infty} x^k \mathfrak{H}_{\boldsymbol{n}}(x) e^{-x^2+c_1x}\, dx = 0, \quad k = 0, 1, \ldots, n_1 - 1, \\
&\int_{-\infty}^{\infty} x^k \mathfrak{H}_{\boldsymbol{n}}(x) e^{-x^2+c_2x}\, dx = 0, \quad k = 0, 1, \ldots, n_2 - 1.
\end{aligned}$$

If $c_1 \neq c_2$ then the weights $w_i(x) = e^{-x^2+c_ix}$ form an AT-system, see [21], [52, Sect. 23.5] and [100, Sect. 3.4]. These MOP can be obtained using the Rodrigues formula

$$e^{-x^2}\mathfrak{H}_{\boldsymbol{n}}(x) = (-1)^N 2^{-N} \left(\prod_{j=1}^{2} e^{-c_jx} \frac{d^{n_j}}{dx^{n_j}} e^{c_jx} \right) e^{-x^2},$$

which yields the explicit expression

$$\mathfrak{H}_{\boldsymbol{n}}(x) = (-1)^{|\vec{n}|} 2^{-|\vec{n}|} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1}\binom{n_2}{k_2} c_1^{n_1-k_1} c_2^{n_2-k_2} (-1)^{k_1+k_2} H_{k_1+k_2}(x),$$

where $H_n(x) = 2^n x^n + \dots$ is the standard Hermite polynomial (1.1), see [52, Sect. 23.5] or [102].

In our notation,

$$A_1(x) = A_2(x) \equiv 1, \quad B_i(x) = -2x + c_i, \quad \sigma_i = 0, \quad i = 1, 2,$$

with $\Delta_1 = \Delta_2 = \mathbb{R}$. The differential equation (7.2) takes the form

$$R_{\boldsymbol{n}} y''' + \left[(B_1 + B_2) R_{\boldsymbol{n}} - R_{\boldsymbol{n}}'\right] y'' + E_{\boldsymbol{n}} y' + F_{\boldsymbol{n}} y = 0.$$

Formula (5.26) shows that in this case $R_{\boldsymbol{n}}$ is a constant, so that the equation boils down to

$$y''' + (B_1 + B_2) y'' + E_{\boldsymbol{n}} y' + F_{\boldsymbol{n}} y = 0,$$

where by Theorem 7.1, $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$ have degree at most 2 and 1, respectively. These polynomials can be obtained explicitly by taking into account the behavior of the solutions of the equation at the singular points. Indeed, if we write

$$E_{\boldsymbol{n}}(x) = e_0 + e_1 x + e_2 x^2\,, \qquad F_{\boldsymbol{n}}(x) = f_0 + f_1 x\,,$$

and replace the asymptotic behavior of $q_{\boldsymbol{n},1}(x)$ as $x \to \infty$ (for instance, along the imaginary axis),

$$q_{\boldsymbol{n},1}(x) = \text{const} \times e^{x^2 - c_1 x} x^{-n_1 - 1} (1 + \mathcal{O}(1/x)),$$

into the differential equation, we get consecutively

$$\begin{aligned} e_2 = 4, \qquad e_1 = -2c_1 - 2c_2, \qquad e_0 = -2 + c_1 c_2 - \frac{f_1}{2}, \\ f_0 = 2c_2 n_1 - 2c_1 n_1 - \frac{c_1 f_1}{2}. \end{aligned}$$

An analogous procedure for $q_{\boldsymbol{n},2}$, together with the previous identities, yields

$$f_1 = -4n_1 - 4n_2\,.$$

As a consequence, we get explicit expressions for polynomials $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$, which can be expressed as follows:

$$E_{\boldsymbol{n}} = B_1 B_2 + 2(n_1 + n_2 - 1)\,, \qquad F_{\boldsymbol{n}} = 2n_2 B_1 + 2n_1 B_2.$$

Thus, $\mathfrak{H}_{\boldsymbol{n}}$ and the corresponding functions of the second kind $q_{\boldsymbol{n},1}$ and $q_{\boldsymbol{n},2}$ are independent solutions of the equation

$$y''' + (B_1 + B_2)y'' + (B_1 B_2 + 2(n_1 + n_2 - 1))y' + (2n_2 B_1 + 2n_1 B_2)y = 0,$$

which coincides with the one obtained previously in [43, Sect. 5.1].

By Theorem 5.10, the discrete vector measure $\vec{\nu}_1 := (\nu(\mathfrak{H}_{\boldsymbol{n}}), \nu(S_{\boldsymbol{n},1}))$ is a critical vector measure for the energy functional $\mathcal{E}_{\vec{\varphi},a}$, with $a = -1/2$ and

$$\vec{\varphi}(z) = \frac{1}{2}\left(x^2 - c_1 x, (c_1 - c_2)x\right), \quad z = x + iy.$$

Direct computation shows that

$$m_k^{\pm} := \int_{-\infty}^{\infty} x^k e^{-x^2 \pm x} dx = \sqrt{\pi}\sqrt[4]{e}\left(\frac{\pm 1}{2i}\right)^k H_k\left(\frac{i}{2}\right), \quad k = 0, 1, \ldots,$$

and that

$$\int_{-\infty}^{\infty} x^k e^{-x^2 + cx} dx = e^{(c^2-1)/4} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{c-1}{2}\right)^{k-j} m_j^{+}, \quad c \neq 1, \quad k = 0, 1, \ldots.$$

These formulas allow us to obtain (at least, using symbolic computation) the moments of w_i, and in consequence, the asymptotic expansion of $\mathfrak{C}_{w_i}[\mathfrak{H}_{\boldsymbol{n}}]$ at infinity. This yields $S_{\boldsymbol{n},i}$ by formula (5.12).

In the following examples we will consider the symmetric case $n_1 = n_2$ and $c_1 = -c_2 = c$, for which clearly $S_{\boldsymbol{n},1}(-x)$ and $S_{\boldsymbol{n},2}(x)$ coincide up to a multiplicative constant. In this case, explicit formulas for $\mathfrak{H}_{(n,n)}$ and $S_{\boldsymbol{n},1}$ are easily obtained with the help of a computer algebra system, at least for low n's. For instance, for $\boldsymbol{n} = (5, 5)$ and $c = 1$ we have that

$$\mathfrak{H}_{(5,5)}(x) = x^{10} - \frac{95x^8}{4} + \frac{1405x^6}{8} - \frac{14855x^4}{32} + \frac{94325x^2}{256} - \frac{39971}{1024},$$

and up to normalization,

$$S_{\boldsymbol{n},1}(x) = S_{\boldsymbol{n},2}(-x) = 32x^5 + 560x^4 + 4240x^3 + 17560x^2 + 39970x + 39971.$$

All zeros of $\mathfrak{H}_{(5,5)}$ are real and simple, while $S_{\boldsymbol{n},1}$ has one real zero (smaller than the zeros of $\mathfrak{H}_{(5,5)}$) and two pairs of complex conjugate simple zeros.

Asymptotics of sequences of (rescaled) multiple Hermite polynomials

$$p_{(n,n)}(t) = \mathfrak{H}_{(n,n)}\left(\sqrt{n}\, t\right),$$

with $c_1 = -c_2 = c$ proportional to $\sqrt{n}$, has been studied by Aptekarev, Bleher and Kuijlaars in a series of papers [6, 20, 22] in the context of the random matrix theory.

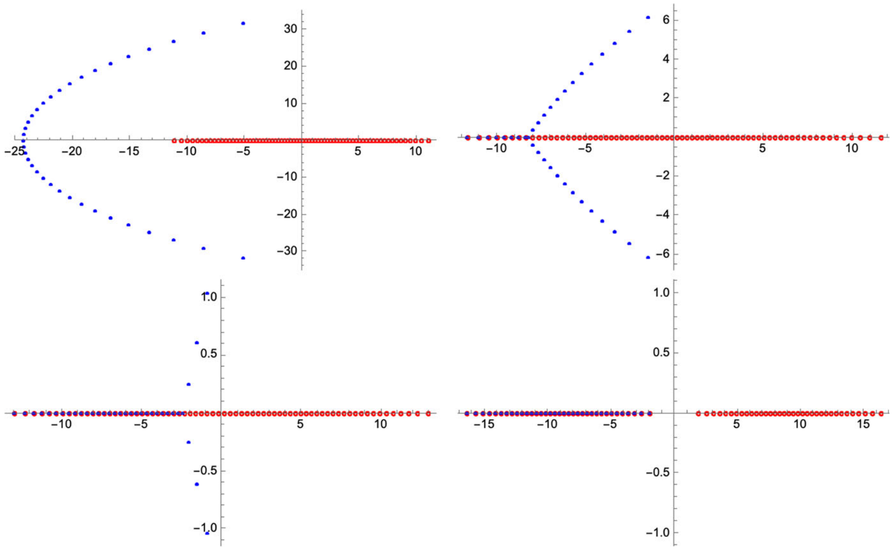

Fig. 1 Zeros of the multiple Hermite polynomial $\mathfrak{H}_{\boldsymbol{n}}$ (indicated by empty circles, all on the real axis) and of $S_{\boldsymbol{n},1}$ (filled circles) for $\boldsymbol{n} = (35, 35)$, $c_1 = -c_2$, for different values of c: $c = 1$ (top left), $c = 4$ (top right), $c = 8$ (bottom left) and $c = 16$ (bottom right). The zeros of $S_{\boldsymbol{n},1}$ that are on the left semi-axis apparently interlace with the zeros of $\mathfrak{H}_{\boldsymbol{n}}$

In particular, they found that the support of the limit of the zero-counting measures $\nu(p_{(n,n)})$ is a single interval, roughly speaking, for $0 \le c \ll 2\sqrt{n}$, and is comprised of two symmetric intervals for $c \gg 2\sqrt{n}$. It is interesting to compare these conclusions with results of the numerical experiments presented in Fig. 1. There, $\boldsymbol{n} = (35, 35)$ and $c_1 = -c_2 = c > 0$, with the phase transition happening around $c \approx 12$. Notice that for $c = 1$, the zeros of $S_{\boldsymbol{n},1}$ are visibly distributed along a curve on the complex plane. As c increases, more and more zeros of $S_{\boldsymbol{n},1}$ migrate to the negative semi-axis, interlacing with the zeros of $\mathfrak{H}_{(n,n)}$, until we get a two-cut situation. In this case, the configuration resembles the relative position of the zeros of $\mathfrak{H}_{(n,n)}$ and $S_{\boldsymbol{n},1}$ for the Angelesco system, described in Sect. 5.2.1, which explains why the description of the asymptotic limit of $\nu(\mathfrak{H}_{\boldsymbol{n}})$ in this case is given in [20] in terms of the Angelesco vector equilibrium problem, see Sect. 6.3.

A generalization of multiple Hermite polynomials to the case of polynomials $P_{\boldsymbol{n}}$, $\boldsymbol{n} = (n, n)$, satisfying the varying orthogonality conditions

$$\int_{-\infty}^{\infty} x^k P_{\boldsymbol{n}}(x) e^{-n(V(x)\pm cx)}\, dx = 0, \quad k = 0, 1, \dots, n-1,$$

where V is a polynomial of even degree and positive leading coefficient, has been carried out in [19], again associated to random matrix models with external source. The limit zero distribution of $P_{\boldsymbol{n}}$'s was described there in terms of a constrained 2-component vector equilibrium problem, with one of the components on the imaginary axis. The two-cut situation in the asymptotics of multiple Hermite polynomials, and thus the reduction to the Angelesco equilibrium, is in this case equivalent to the constraint on the imaginary axis to be not achieved (not "saturated"). The study in [19] has been extended in [75] (see also [11] for the quartic case) to a non-symmetric situation, showing that it can be alternatively characterized in terms of a 3-component critical vector measure with the interaction matrix from Corollary 6.3. The curves outlined by the zeros of $P_{\boldsymbol{n}}$ and $S_{\boldsymbol{n},1}$ in Fig. 1 are consistent with the support of the three components described in [75, Theorem C].

8.3 Multiple Laguerre Polynomials of the First Kind

These polynomials are defined by the orthogonality conditions

$$\begin{aligned} &\int_0^\infty x^k \mathfrak{L}_{\boldsymbol{n}}(x) x^{\alpha_1} e^{-x}\, dx = 0, \quad k = 0, 1, \ldots, n_1 - 1, \\ &\int_0^\infty x^k \mathfrak{L}_{\boldsymbol{n}}(x) x^{\alpha_2} e^{-x}\, \mathrm{d}x = 0, \quad k = 0, 1, \ldots, n_2 - 1, \end{aligned} \qquad \deg \mathfrak{L}_{\boldsymbol{n}} \leq N = n_1 + n_2,$$

where $\alpha_1, \alpha_2 > 0$ and $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, under which condition the weights form an AT-system; see [7], [52, Sect. 23.4.1], and [100, Sect. 3.2]. Not only that, since x^β, for $\beta < 0$ and $x > 0$, can be written as the Cauchy integral of a positive weight on $(-\infty, 0)$, coinciding up to a multiplicative constant with $|x|^\beta$, we conclude that this pair of weight forms a Nikishin system with $\Delta_1 = \Delta_2 = [0, +\infty)$ and $[c, d] = [-\infty, 0]$, see Sect. 5.2.2.

Polynomials $\mathfrak{L}_{\boldsymbol{n}}$ can be obtained by using the Rodrigues formula,

$$(-1)^N e^{-x} \mathfrak{L}_{\boldsymbol{n}}(x) = \prod_{j=1}^{2} \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j+\alpha_j} \right) e^{-x}, \quad N = n_1 + n_2,$$

from which one can find the explicit expression

$$(-1)^N e^{-x} \mathfrak{L}_{\boldsymbol{n}}(x) = (\alpha_1+1)_{n_1} (\alpha_2+1)_{n_2}\, {}_2F_2 \left(\left. \begin{matrix} \alpha_1+n_1+1, \alpha_2+n_2+1 \\ \alpha_1+1, \alpha_2+1 \end{matrix} \right| -x \right).$$

In our notation,

$$A_1(x) = A_2(x) = x, \quad B_i(x) = \alpha_i - x, \quad \sigma_i = 0, \quad i = 1, 2,$$

with $\Delta_1 = \Delta_2 = [0, +\infty)$. These polynomials satisfy the differential equation (7.2), which takes the form

$$\begin{aligned} &x^2 R_{\boldsymbol{n}}(x) y'''(x) + \left[(B_1(x) + B_2(x) + 4) x R_{\boldsymbol{n}}(x) - x^2 R_{\boldsymbol{n}}'(x) \right] y''(x) \\ &\quad + E_{\boldsymbol{n}}(x) y'(x) + F_{\boldsymbol{n}}(x) y(x) = 0. \end{aligned} \tag{8.5}$$

It follows from formula (5.26) that $R_{\boldsymbol{n}}(x)/x$ is a constant. Thus, the equation reduces to

$$x^3 y''' + x^2 [B_1(x) + B_2(x) + 3]\, y'' + E_{\boldsymbol{n}} y' + F_{\boldsymbol{n}} y = 0,$$

where $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$ are of degrees at most 3 and 2, respectively. As in Sect. 8.2, the asymptotics of the corresponding functions of second kind $q_{\boldsymbol{n},1}$ and $q_{\boldsymbol{n},2}$ at ∞ yields some constraints on the coefficients of $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$, which unfortunately are not sufficient to determine the polynomials in this case. We need to make use also of the predicted behavior of the solutions at the origin.

Notice that 0 is a regular singular point (a Fuchsian singularity) of (8.5). The fact that the weights constitute an AT-system on $[0, +\infty)$ implies also that $\mathfrak{L}_{n_1,n_2}(0) \neq 0$. In consequence, $E_{\boldsymbol{n}}(0) = 0$ and $F_{\boldsymbol{n}}(0) = 0$. Expanding the solutions at the origin we conclude that the indicial polynomial must vanish at 0, $-\alpha_1$ and $-\alpha_2$. All this additional information allows us to determine $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$:

$$\begin{aligned} E_{\boldsymbol{n}} &= x(x^2 + (n_1 + n_2 - \alpha_1 - \alpha_2 - 3)x + (1 + \alpha_1)(1 + \alpha_2)\,, \\ F_{\boldsymbol{n}} &= x(-(n_1 + n_2)x + n_1 + n_2 + n_1 n_2 + \alpha_1 n_2 + \alpha_2 n_1)\,. \end{aligned}$$

Canceling the common factor x in the four coefficients of (8.5) yields the equation that appeared already in [7, Sect. 4.3],

$$\begin{aligned} & x^2 y'''(x) + \left(-2x^2 + (\alpha_1 + \alpha_2 - 3)\, x\right) y''(x) + \left(x^2 - x\,(\alpha_1 + \alpha_2 - n_1 - n_2 + 3)\right. \\ & \quad + (\alpha_1 + 1)\,(\alpha_2 + 1))\, y'(x) \\ & \quad - (x(n_1 + n_2) - (n_1 + n_2 + n_1 n_2 + \alpha_1 n_2 + \alpha_2 n_1))\, y(x) = 0. \end{aligned}$$

From the conclusions of Sect. 5.2.2 it follows that the zeros of $\mathfrak{L}_{\boldsymbol{n}}$ are all positive, and those of $S_{\boldsymbol{n},1}$ are negative. By Theorem 5.10, the discrete vector measure $\vec{\nu}_1 := (\nu(\mathfrak{L}_{\boldsymbol{n}}), \nu(S_{\boldsymbol{n},1}))$ is a critical vector measure for the energy functional $\mathcal{E}_{\vec{\varphi},a}$, with $a = -1/2$ and

$$\vec{\varphi}(x) = \frac{1}{2}\,(x - (\alpha_1 + 1)\log(x), (\alpha_1 - \alpha_2 - 1)\log|x|)\,.$$

Direct computations show that

$$\int_0^\infty x^{k+\alpha_j} e^{-x} dx = \begin{cases} \sqrt{\pi}\, 2^{-k-1}(2k+1)!!, & j = 1, \\ (k+1)!, & j = 2, \end{cases} \qquad k = 0, 1, \ldots,$$

which allows to find the moments of w_i, the asymptotic expansion of $\mathfrak{C}_{w_i}[\mathfrak{L}_{\boldsymbol{n}}]$ at infinity, and in consequence, $S_{\boldsymbol{n},i}$ (using formula (5.12)).

Let us consider the particular case of $n_1 = n_2 = n$, with $\alpha_1 = 1/2, \alpha_2 = 1$. Then, for $\boldsymbol{n} = (5, 5)$,

$$\begin{aligned}\mathfrak{L}_{(5,5)}(x) = x^{10} - \frac{165x^9}{2} + \frac{5445x^8}{2} - \frac{186615x^7}{4} + \frac{7224525x^6}{16} \\ - \frac{80613225x^5}{32} + \frac{127182825x^4}{16} \\ - \frac{107120475x^3}{8} + 10758825x^2 - \frac{13253625x}{4} + \frac{467775}{2},\end{aligned}$$

and up to normalization,

$$\begin{aligned}S_{\boldsymbol{n},1}(x) &= 8x^5 + 2720x^4 + 107500x^3 + 1945020x^2 + 46682295x + 1425581520, \\ S_{\boldsymbol{n},2}(x) &= 32x^5 + 2960x^4 + 67424x^3 + 1313480x^2 + 37066290x + 1173966885.\end{aligned}$$

All zeros of $\mathfrak{L}_{(5,5)}$ are positive and simple, while each $S_{\boldsymbol{n},j}$ has one negative and two pairs of complex conjugate simple zeros, all of them simple.

Asymptotics of sequences of (rescaled) multiple Laguerre polynomials of the first kind

$$p_{(n,n)}(t) = \mathfrak{L}_{(n,n)}(2nt)$$

was obtained in [31] and [77]. It was shown that the support of the weak-* limit of the zero-counting measures $\nu(p_{(n,n)})$ is the interval $[0, 27/8]$, with the density presenting the usual square root vanishing at the rightmost endpoint of the support. The expression of the density was derived from the recurrence relation satisfied by polynomials $\mathfrak{L}_{(n_1,n_2)}$ and no equilibrium problem associated to that distribution was given.

Again, it is interesting to compare these conclusions with results of the numerical experiments presented in Fig. 2, where we take $\boldsymbol{n} = (35, 35)$, with $\alpha_1 = 1/2$, and $\alpha_2 = 1$. The largest zero of $\mathfrak{L}_{(35,35)}$ is 217.597, which is consistent with the expected value of

$$\frac{27}{8} \times 70 = 236.25.$$

8.4 Multiple Laguerre Polynomials of the Second Kind

These polynomials are defined by the orthogonality conditions

$$\begin{aligned}\int_0^\infty x^k \mathcal{L}_{\boldsymbol{n}}(x) x^\alpha e^{-c_1 x}\, dx = 0, \quad k = 0, 1, \ldots, n_1 - 1, \\ \int_0^\infty x^k \mathcal{L}_{\boldsymbol{n}}(x) x^\alpha e^{-c_2 x}\, dx = 0, \quad k = 0, 1, \ldots, n_2 - 1,\end{aligned}$$

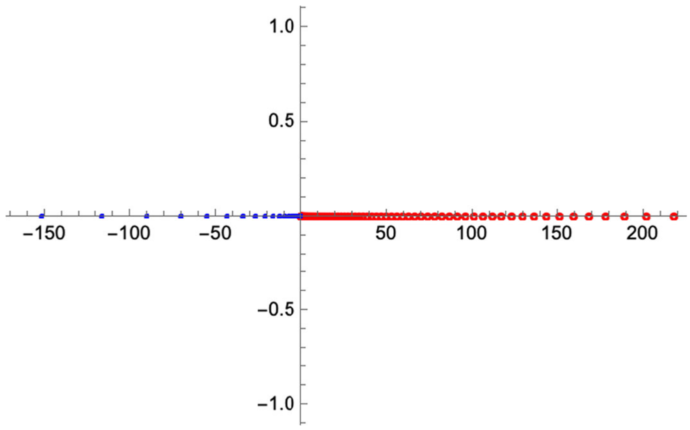

Fig. 2 Zeros of the multiple Laguerre polynomial of the first kind $\mathfrak{L}_{\boldsymbol{n}}$ (indicated by empty circles, all on the positive semiaxis) and of $S_{\boldsymbol{n},1}$ (filled circles, all on the negative semiaxis) for $\boldsymbol{n} = (35, 35)$ and $\alpha_1 = 1/2$, $\alpha_2 = 1$. Nine real zeros of $S_{\boldsymbol{n},1}$ (ranging from $-74{,}000$ to -201.53, are not represented

where we assume that $\alpha > 0$ and $c_1, c_2 > 0$ with $c_1 \neq c_2$, under which condition the weights form an AT-system; see e.g. [21], [63], or [100, Sect. 3.3].

An explicit expression can be found in [7, Sect. 3] or [52, Sect. 23.4]:

$$\mathcal{L}_{\boldsymbol{n}}(x) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} (-1)^{k_1+k_2} \frac{(k_1+k_2)!}{c_1^{k_1} c_2^{k_2}} \binom{n_1}{k_1} \binom{n_2}{k_2} \binom{N+\alpha}{k_1+k_2} x^{N-k_1-k_2}.$$

In our notation,

$$A_1(x) = A_2(x) = x, \quad B_i(x) = \alpha - c_i x, \quad \sigma_i = 0, \quad i = 1, 2,$$

with $\Delta_1 = \Delta_2 = [0, +\infty)$. These polynomials also satisfy the differential equation (8.5), that is,

$$\begin{aligned} &x^2 R_{\boldsymbol{n}}(x) y'''(x) + \Big[(-(c_1+c_2)x^2 + 2(\alpha+2)x) R_{\boldsymbol{n}}(x) - x^2 R_{\boldsymbol{n}}'(x) \Big] y'' \\ &+ E_{\boldsymbol{n}}(x) y'(x) + F_{\boldsymbol{n}}(x) y(x) = 0. \end{aligned}$$

Formula (5.26) shows that now $R_{\boldsymbol{n}}(x)/x$ is a polynomial of degree 1. Furthermore, by (5.21) and since $B_1 - B_2$ vanishes at 0, $R_{\boldsymbol{n}} \mathfrak{L}_{\boldsymbol{n}}$ has a double root at 0. Again, the fact that the weights constitute an AT-system on $[0, +\infty)$ implies also that $\mathfrak{L}_{\boldsymbol{n}}(0) \neq 0$. Hence, $R_{\boldsymbol{n}}(x)/x^2$ is a constant, and the third order differential equation is

$$x^4 y''' - x^3((c_1+c_2)x - \alpha - 2) y'' + E_{\boldsymbol{n}} y' + F_{\boldsymbol{n}} y = 0.$$

Arguments as described in Sects. 8.2 and 8.3, using the asymptotics of the functions of second kind both at 0 and at ∞ and the fact that the indicial polynomial vanishes at 0 and $-\alpha$, yield the expressions for polynomials $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$:

$$\begin{aligned}E_{\boldsymbol{n}} &= c_1c_2x^4 - [(c_1+c_2)(\alpha+1) - n_1c_1 - n_2c_2]x^3 + \alpha(\alpha+1)x^2,\\ F_{\boldsymbol{n}} &= -c_1c_2(n_1+n_2)x^3 + \alpha(c_1n_1+c_2n_2)x^2.\end{aligned}$$

Canceling the common factor x^2 in the differential equation yields

$$\begin{aligned}&x^2y'''(x) - \Big(x^2\,(c_1+c_2) - 2x(\alpha+1)\Big)\,y''(x)\\ &\quad+ \Big(x^2c_1c_2 - x\,[(c_1+c_2)\,(\alpha+1) - n_1c_1 - n_2c_2]\\ &\quad+ \alpha(\alpha+1)\big)y'(x) - (xc_1c_2(n_1+n_2) - \alpha\,(n_1c_1+n_2c_2))\,y(x) = 0,\end{aligned}$$

which matches the equation found in [7, Sect. 4.3] and [43, Sect. 5.2].

By Theorem 5.10, the discrete vector measure $\vec{\nu}_1 := \big(\nu(\mathcal{L}_{\boldsymbol{n}}), \nu(S_{\boldsymbol{n},1})\big)$ is a critical vector measure for the energy functional $\mathcal{E}_{\vec{\varphi},a}$, with $a = -1/2$ and

$$\vec{\varphi}(x) = \frac{1}{2}\,(c_1x - (\alpha+1)\log x, (c_2-c_1)x)\,, \quad z = x+iy.$$

Since for $c > 0$ and $\alpha > -1$,

$$\int_0^\infty x^\alpha e^{-cx}\,dx = c^{-\alpha-1}\Gamma(\alpha+1),$$

we can easily calculate the moments of w_i, the asymptotic expansion of $\mathfrak{C}_{w_i}[\mathcal{L}_{\boldsymbol{n}}]$ at infinity, and in consequence, $S_{\boldsymbol{n},i}$ (using formula (5.12)).

Let us consider the particular case of $n_1 = n_2 = n$, with $\alpha = 1$, $c_1 = 1$, and $c_2 = 2$. Then, for $\boldsymbol{n} = (5,5)$,

$$\begin{aligned}\mathcal{L}_{(5,5)}(x) = x^{10} &- \frac{165x^9}{2} + 2750x^8 - \frac{96525x^7}{2}\\ &+ 487575x^6 - \frac{5831595x^5}{2} + 10239075x^4\\ &- 20270250x^3 + 20790000x^2 - 9355500x + 1247400,\end{aligned}$$

and up to normalization,

$$\begin{aligned}S_{\boldsymbol{n},1}(x) &= 2x^5 + 25x^4 + 605x^3 + 16580x^2 + 506065x + 16197810,\\ S_{\boldsymbol{n},2}(x) &= 4x^5 - 320x^4 + 9975x^3 - 151645x^2 + 1115560x - 2967600.\end{aligned}$$

All zeros of $\mathcal{L}_{(5,5)}$ are positive and simple, $S_{\boldsymbol{n},1}$ has one negative and two pairs of complex conjugate simple zeros, while $S_{\boldsymbol{n},2}$, has one positive and two pairs of complex conjugate roots, all of them simple.

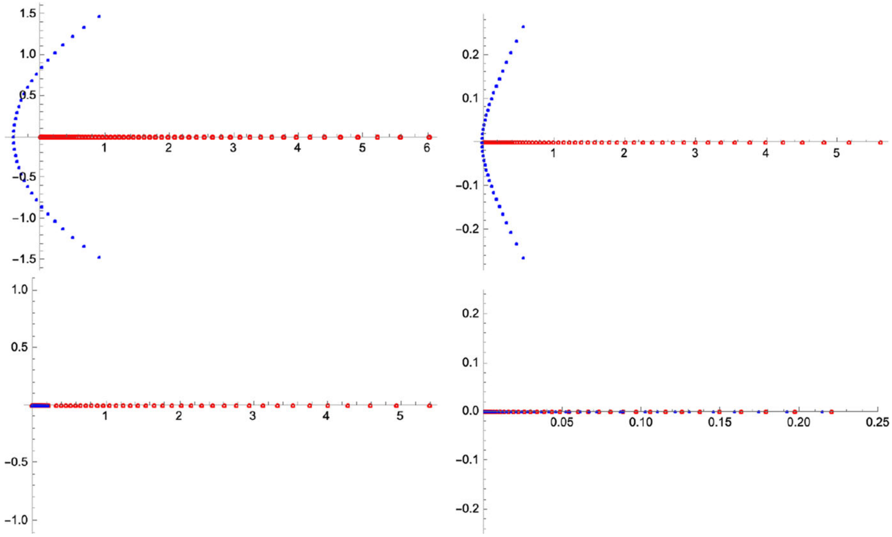

Fig. 3 Zeros of the multiple Laguerre polynomial of the second kind $\mathcal{L}_{\boldsymbol{n}}$ (indicated by empty circles, all on the positive semiaxis) and of $S_{\boldsymbol{n},1}$ (filled circles) for $\boldsymbol{n} = (35, 35)$, and $(c_1, c_2) = (35, 70)$ (top left), $(c_1, c_2) = (35, 140)$ (top right), and $(c_1, c_2) = (35, 525)$ (bottom left). Bottom right: zoom of the interval $(0, 0.25)$ for $(c_1, c_2) = (35, 525)$

Asymptotics of sequences of (rescaled) multiple Laguerre polynomials of the second kind,

$$p_{(n,n)}(t) = \mathcal{L}_{(n,n)}\,(nt)\,,$$

with varying $0 < c_1 < c_2$ proportional to n, has been studied by Lysov and Wielonsky in [63] using the Riemann-Hilbert technique and the analysis of the Riemann surface derived from the differential equation. In particular, they found that there is a critical value $\kappa \approx 12.11\ldots$ such that for $0 < c_2/c_1 < \kappa$, the support of the limit of the zero-counting measures $\nu(p_{(n,n)})$ is a single interval of the form $[0, d]$, $d > 0$, while for $c_2/c_1 > \kappa$, it is comprised of two real intervals $[0, a] \cup [b, d]$, with $0 < a < b < d$. The expression of the density was also derived, but no equilibrium problem associated to that distribution was given. It is interesting to compare these conclusions with results of the numerical experiments presented in Fig. 3, where we take $\boldsymbol{n} = (35, 35)$. We observe that for small values of c_2/c_1 the zeros of $S_{\boldsymbol{n},1}$ sit on a curve on the complex plane. However, for large ratios c_2/c_1, zeros of $\mathcal{L}_{(n,n)}$ split into two groups, and the zeros of $S_{\boldsymbol{n},1}$, all real, approximately interlace with the zeros of $\mathcal{L}_{(n,n)}$ on the leftmost subinterval. This allows us to conjecture that in this case, the asymptotic zero distribution can be described again in terms of the Angelesco-type vector equilibrium, see Sect. 6.3.

8.5 Jacobi–Piñeiro Polynomials

The Jacobi–Piñeiro polynomials are multiple orthogonal polynomials associated with an AT system consisting of Jacobi weights on [0, 1] with different powers at 0 and the same behavior at 1. They are defined by the orthogonality conditions

$$\int_0^1 x^k P_{\boldsymbol{n}}(x) x^{\beta_1} (1-x)^\alpha \, dx = 0, \quad k = 0, 1, \ldots, n_1 - 1,$$
$$\int_0^1 x^k P_{\boldsymbol{n}}(x) x^{\beta_2} (1-x)^\alpha \, dx = 0, \quad k = 0, 1, \ldots, n_2 - 1.$$

In our notation, $\Delta_1 = \Delta_2 = [0, 1]$ and $w_j(x) = x^{\beta_j}(1-x)^\alpha$, $j = 1, 2$, with $\alpha, \beta_1, \beta_2 > -1$ and $\beta_1 - \beta_2 \notin \mathbb{Z}$. We have

$$A_1(x) = A_2(x) = x(x-1), \quad B_i(x) = (\beta_i + \alpha)x - \beta_i, \quad \sigma_i = 0, \quad i = 1, 2.$$

For the same reason mentioned at the beginning of Sect. 8.3, this pair of weight forms a Nikishin system with $\Delta_1 = \Delta_2 = [0, 1]$ and $[c, d] = [-\infty, 0]$, see Sect. 5.2.2.

These polynomials were first studied by Piñeiro [83] when $\alpha = 0$. The general case appears in [81, p. 162]. There is a Rodrigues formula for Jacobi–Piñeiro polynomials $P_{\boldsymbol{n}}$, $\boldsymbol{n} = (n_1, n_2)$, see [52, Sect. 23.3.2]: with $N = n_1 + n_2$, and up to a constant factor,

$$P_{\boldsymbol{n}}(x) = (1-x)^{-\alpha} \prod_{j=1}^{2} \left(x^{-\beta_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j + \beta_j} \right) (1-x)^{\alpha + N}.$$

There is even an explicit expression [100, Sect. 3.1]: again, up to a multiplicative constant,

$$P_{\boldsymbol{n}}(x) = \sum_{k=0}^{n_1} \binom{\beta_1 + n_1}{k} \binom{\alpha + N}{n_1 - k} \sum_{j=0}^{n_2} \binom{\beta_2 + N - k}{j} \binom{\alpha + k + n_2}{n_2 - j} x^{N-k-j} (x-1)^{k+j}.$$

Proposition 5.5 assures that polynomial $R_{\boldsymbol{n}}$ has degree at most 3. By (5.21), $R_{\boldsymbol{n}} P_{\boldsymbol{n}}$ has a double root at 1 (observe that $B_1 - B_2$ also vanish at 1) and a simple one at 0. Since the weights form an AT-system, $P_{\boldsymbol{n}}(0) \neq 0$ and $P_{\boldsymbol{n}}(1) \neq 0$, so that, up to a multiplicative constant, $R_{\boldsymbol{n}}(z) = z(z-1)^2$.

By Theorem 5.10, and taking into account the expression for $R_{\boldsymbol{n}}$, we conclude that the discrete vector measure $\vec{\nu}_1 := (\nu(\mathfrak{L}_{\boldsymbol{n}}), \nu(S_{\boldsymbol{n},1}))$ is a critical vector measure for the energy functional $\mathcal{E}_{\vec{\varphi},a}$, with $a = -1/2$ and

$$\vec{\varphi} = \left(\frac{\beta_1 + 1}{2} \log \frac{1}{|x|} + \frac{\alpha + 1}{2} \log \frac{1}{|x-1|}, \left(\beta_1 - \beta_2 + \frac{1}{2} \right) \log \frac{1}{|z|} \right), \quad z = x + iy.$$

The third order differential equation can be also obtained following the arguments used in Sects. 8.2–8.4, and making use of the asymptotics at ∞, 0, and 1, and of the known roots of the indicial polynomials at both finite points. For instance, in the case $\alpha = 0$ (studied by Piñeiro in [83]), polynomials $E_{\boldsymbol{n}}$ and $F_{\boldsymbol{n}}$ (coefficients of y' and y, respectively) are

$$\begin{aligned} E_{\boldsymbol{n}} &= -x(x-1)^3((1+\beta_1)(1+\beta_2) + x(n_1^2 + n_2^2 + n_1 n_2 + n_1(1+\beta_1) \\ &\quad + n_2(1+\beta_2) - (2+\beta_1)(2+\beta_2))), \\ F_{\boldsymbol{n}} &= -x(x-1)^3(n_1+n_2)(1+n_1+\beta_1)(1+n_2+\beta_2). \end{aligned}$$

Canceling the common factor $x(x-1)^3$ we obtain the differential equation

$$\begin{aligned} & x^2(x-1)y''' + x(x(5+\beta_1+\beta_2) - 3 - \beta_1 - \beta_2)y'' \\ & \quad - (x(n_1^2 + n_2^2 + n_1 n_2 + n_1(1+\beta_1) + n_2(1+\beta_2) \\ & \quad\quad - (2+\beta_1)(2+\beta_2)) - (1+\beta_1)(1+\beta_2))y' \\ & \quad - (n_1+n_2)(1+n_1+\beta_1)(1+n_2+\beta_2)y = 0. \end{aligned}$$

This is the particular case (after canceling the common factor $x-1$) of the equation derived in [7, Sect. 4.3].

As for the electrostatic partners, let us consider the example when $\alpha = \beta_1 = 0$ and $\beta_2 = -1/2$. Direct computation shows that the moments of w_i are

$$\int_0^1 x^k w_j(x)dx = \begin{cases} (k+1)^{-1}, & j = 1, \\ 2/(2j+1), & j = 2, \end{cases} \quad k = 0, 1, \ldots,$$

which allows to find the asymptotic expansion of $\mathfrak{C}_{w_i}[P_{\boldsymbol{n}}]$ at infinity, and in consequence, $S_{\boldsymbol{n},i}$ (using formula (5.12)) by means of symbolic computation. For instance, in the case $\boldsymbol{n} = (5,5)$ we obtain that

$$\begin{aligned} \mathcal{L}_{(5,5)}(x) =& x^{10} - \frac{380x^9}{87} + \frac{1615x^8}{203} - \frac{20672x^7}{2639} + \frac{9044x^6}{2001} - \frac{5168x^5}{3335} + \frac{204x^4}{667} \\ & - \frac{64x^3}{2001} + \frac{x^2}{667} - \frac{4x}{182091} + \frac{1}{30045015}, \end{aligned}$$

and up to normalization,

$$\begin{aligned} S_{\boldsymbol{n},1}(x) &= 882230895x^5 + 4709406975x^4 + 5720142090x^3 + 8795888965x^2 \\ &\quad + 11696347475x + 11645469674, \\ S_{\boldsymbol{n},2}(x) &= 5192762585x^5 + 313459871725x^4 + 662076961780x^3 + 782465377400x^2 \\ &\quad + 1267133219685x + 1386883054197. \end{aligned}$$

All zeros of $P_{(5,5)}$ are positive and simple, both $S_{\boldsymbol{n},j}$ have one negative and two pairs of complex conjugate simple zeros, all of them simple.

Zero asymptotics for sequences of Jacobi–Piñeiro polynomials $P_{(n,n)}$ as $n \to \infty$ and α, β_j's fixed, was obtained in [31] and [77]. Again, the expression of the density,

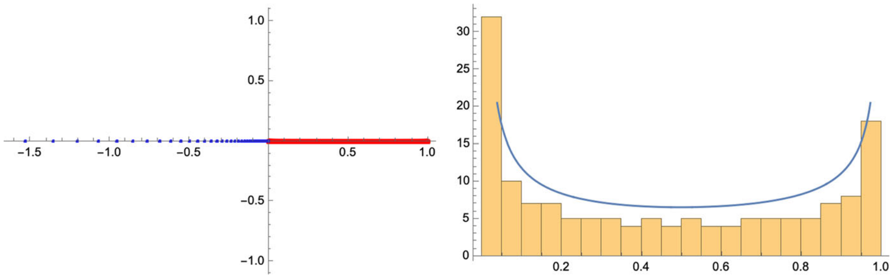

Fig. 4 Left: zeros of the Jacobi–Piñeiro polynomial $P_{\boldsymbol{n}}$ (all on [0, 1]) and of $S_{\boldsymbol{n},1}$ (filled circles, all negative) for $\boldsymbol{n} = (75, 75)$, with $\beta_1 = 0$, $\beta_2 = -1/2$, and $\alpha = 0$; approximately 19 real zeros of $S_{\boldsymbol{n},1}$ (ranging from -771 to -1.74), are not represented. Right: the histogram of the zeros of $P_{\boldsymbol{n}}$ and the plot of the asymptotic density, predicted in [31]

this time on [0, 1], was derived from the recurrence relation satisfied by polynomials $\mathfrak{L}_{(n_1,n_2)}$ and no equilibrium problem associated to that distribution was given. For comparison, results of the numerical experiments are presented in Fig. 4, where we take $\boldsymbol{n} = (75, 75)$, with $\beta_1 = 0$, $\beta_2 = -1/2$, and $\alpha = 0$. According to our discussion in Sect. 5.2.2, the zeros of $S_{\boldsymbol{n},1}$ are real and negative, and the asymptotic zero distribution can be described in terms of the Nikishin-type vector equilibrium, see Sect. 6.4.

8.6 Angelesco–Jacobi Polynomials

These polynomials, known also as Jacobi–Jacobi polynomials (see [12]) are Hermite–Padé polynomials $P_{\boldsymbol{n}}$, $\boldsymbol{n} = (n_1, n_2)$, satisfying orthogonality relations

$$\int_a^0 x^k P_{\boldsymbol{n}}(x)(x-a)^\alpha |x|^\beta (1-x)^\gamma \, dx = 0, \quad k = 0, 1, 2, \ldots, n_1 - 1,$$
$$\int_0^1 x^k P_{\boldsymbol{n}}(x)(x-a)^\alpha x^\beta (1-x)^\gamma x^k \, dx = 0, \quad k = 0, 1, 2, \ldots, n_2 - 1,$$

with $a < 0$. In our notation,

$$A(x) = x(x-a)(x-1), \quad B(x) = \alpha x(x-1) + \beta(x-a)(x-1) + \gamma x(x-a),$$

and

$$w_1(x) = w_2(x) = w(x) = (x-a)^\alpha |x|^\beta (1-x)^\gamma,$$
$$v_1(x) = v_2(x) = v(x) = (x-a)^{\alpha+1} |x|^{\beta+1} (1-x)^{\gamma+1},$$

with $\alpha, \beta, \gamma > -1$ and

$$\Delta_1 = [a, 0], \quad \Delta_2 = [0, 1]$$

(cf. Example 3.2).

This is an Angelesco system of semiclassical weights of class $\sigma = 1$, see (3.4). As it follows from [12, Theorem 2.1], for $n_1 = n_2 = n$, $\boldsymbol{n} = (n, n)$, $n \in \mathbb{N}$, $P_{\boldsymbol{n}}$ can be expressed using a Rodrigues formula,

$$P_{\boldsymbol{n}}(x) = \frac{1}{w(x)} \left(\frac{d}{dx}\right)^n \left[A^n(x)w(x)\right],$$

or explicitly, see [100, Sect. 3.5]: up to a constant factor,

$$P_{\boldsymbol{n}}(x) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-n)_{k+j}(-\alpha - n)_j(-\gamma - n)_k}{(\beta + 1)_{k+j} k! j!} (x - a)^{n-k}(x - 1)^{k+j} x^{n-j}.$$

Kaliaguin [54] studied the case of $a = -1$, with the particular sub-case of $B \equiv 0$ or $w(x) \equiv 1$ going back to the work of Appell [3]. The case of $-1 < a < 0$ was addressed in [56], but see [12] for further historical details.

If $B \equiv 0$ (that is, $\alpha = \beta = \gamma = 0$), definition (3.13) reduces to

$$S_{\boldsymbol{n},1} = A \times \mathfrak{Wrons}\left[P_{\boldsymbol{n}}, \mathfrak{C}_{w_1}[P_{\boldsymbol{n}}]\right], \quad \mathfrak{C}_{w_1}[P_{\boldsymbol{n}}](x) = \int_a^0 \frac{P_{\boldsymbol{n}}(t)}{t - x} dt.$$

Notice that in particular, $x = 1$ is one of the $n + 1$ zeros of $S_{\boldsymbol{n},1}$, $n - 1$ of which interlace with the zeros of $P_{\boldsymbol{n}}$ on $[0, 1)$.

In Appell's case, $w \equiv 1$ and $a = -1$,

$$P_{\boldsymbol{n}}(x) = \left(\frac{d}{dx}\right)^n \left[x(x^2 - 1)\right]^n,$$

and

$$S_{\boldsymbol{n},1}(x) = (-1)^{n+1} S_{\boldsymbol{n},2}(-x), \quad \deg S_{\boldsymbol{n},1} = n + 1.$$

These explicit formulas allow to use symbolic computation (e.g. Mathematica) to find explicit expressions for small values of n. For instance, for $\boldsymbol{n} = (6, 6)$, and up to normalization,

$$P_{\boldsymbol{n}}(x) = x^{12} - \frac{44x^{10}}{17} + \frac{165x^8}{68} - \frac{220x^6}{221} + \frac{75x^4}{442} - \frac{2x^2}{221} + \frac{1}{18564},$$
$$S_{\boldsymbol{n},1}(x) = (x - 1)\left(12288x^6 - 38763x^5 + 47253x^4 - 27822x^3 + 8018x^2 - 991x + 33\right),$$

and their graphs are plotted in Fig. 5. We can clearly observe the interlacing predicted by Proposition 5.11, which in the limit $n \to \infty$ gives the description in therms of the Angelesco equilibrium problem, as described in Sect. 6.3.

However, according to Remark 5.3, the critical configuration for the zeros of $P_{\boldsymbol{n}}$ in this case is not unique: we can use for the second component the zeros of any linear combination of $S_{\boldsymbol{n},1}$ and $S_{\boldsymbol{n},2}$. In Fig. 6 we illustrate the behavior of zeros of

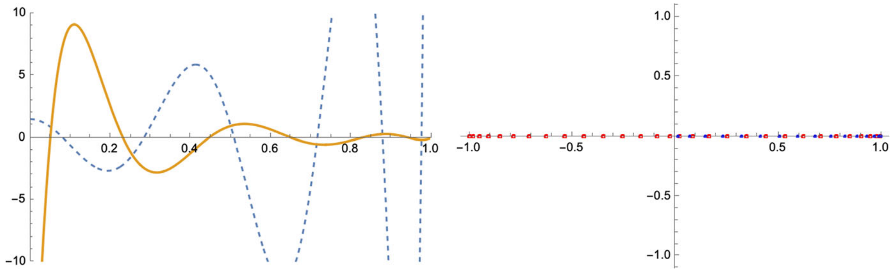

Fig. 5 Appell's polynomials ($\alpha = \beta = \gamma = 0$). Left: graph of $P_{\boldsymbol{n}}$ (dashed line) and $S_{\boldsymbol{n},1}$ (thick line) on $[0, 1]$ for $\boldsymbol{n} = (6, 6)$ in the case. Right: zeros of $P_{\boldsymbol{n}}$ (empty circles, all on $[-1, 1]$) and of $S_{\boldsymbol{n},1}$ (filled circles, all on $[0, 1]$) for $\boldsymbol{n} = (15, 15)$

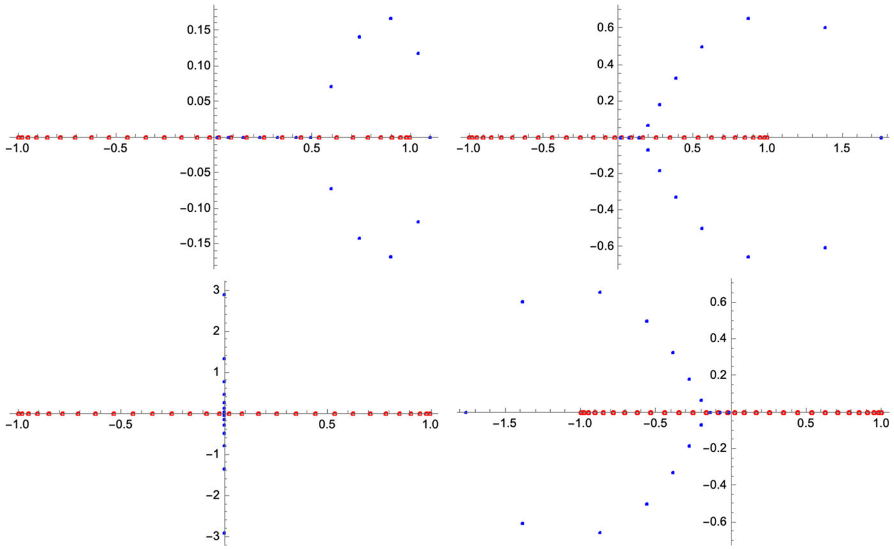

Fig. 6 Appell's polynomials ($\alpha = \beta = \gamma = 0$) and $\boldsymbol{n} = (35, 35)$: zeros of $P_{\boldsymbol{n}}$ (indicated by empty circles, all on $(-1, 1)$) and of $S_{\boldsymbol{n},1} + t S_{\boldsymbol{n},2}$ (filled circles) for $t = 10^{-10}$ (top left), $t = 10^{-5}$ (top right), $t = 1$ (bottom left), and $t = 10^5$ (bottom right)

$S_{\boldsymbol{n},1} + t S_{\boldsymbol{n},2}$, for different values of $0 < t \leq 1$ (notice that the representation of the zeros of $S_{\boldsymbol{n},1}$ in Fig. 5, right, corresponds to $t = 0$).

8.7 Multiple Orthogonal Polynomials for the Cubic Weight

Although we have not discussed the purely complex weights, we finish our presentation with the illustrative example of polynomials $P_{\boldsymbol{n}}$, $\boldsymbol{n} = (n_1, n_2)$, satisfying orthogonality

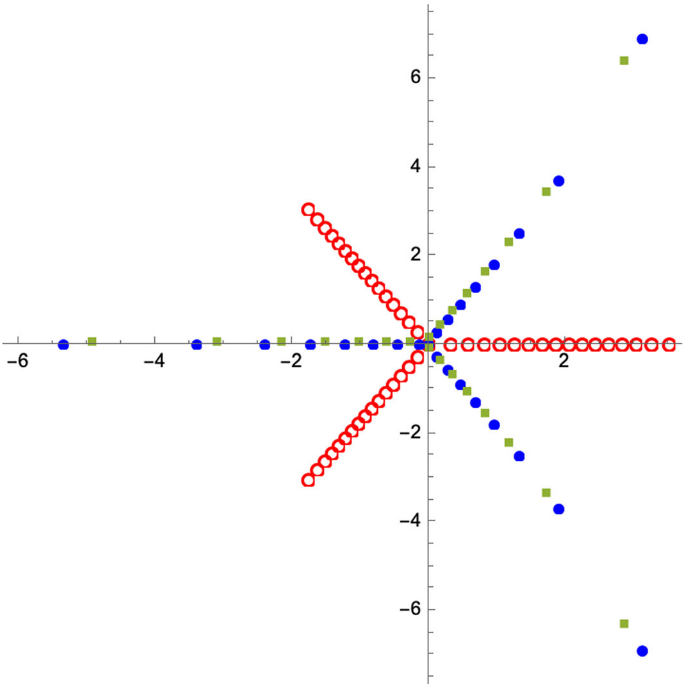

Fig. 7 Zeros of the multiple orthogonal polynomial $P_{\boldsymbol{n}}$ with respect to the cubic weight (indicated by empty circles, part of them on the positive semiaxis, forming a symmetric star) and of $S_{\boldsymbol{n},1}$ (filled circles) for $\boldsymbol{n} = (25, 25)$. For comparison, zeros of type I MOP $\mathfrak{B}_{\boldsymbol{n}}$ (filled squares) are also represented

relations

$$\int_{\Delta_1} z^k P_{\boldsymbol{n}}(z) e^{-z^3} dz = 0, \quad k = 0, 1, 2, \ldots, n_1 - 1,$$

$$\int_{\Delta_2} z^k P_{\boldsymbol{n}}(z) e^{-z^3} dz = 0, \quad k = 0, 1, 2, \ldots, n_2 - 1,$$

where Δ_1 and Δ_2 are contours on the complex plane, extending to ∞ on their two ends along the directions determined by the angles $-2\pi/3$ and 0, and $-2\pi/3$ and $2\pi/3$, respectively. They were introduced in [101] and studied in full generality in [74].

In our notation,

$$A_1(x) = A_2(x) = A(x) = 1, \quad B_1(x) = B_2(x) = B(x) = -3x^2.$$

Detailed explanation of the algorithm of computation of the zeros of $P_{\boldsymbol{n}}$ was given in [74, Sect. 9]. Since the moments of the weight were given explicitly, we use the expansion of $\mathfrak{D}_w[P_{\boldsymbol{n}}]$ at infinity to calculate the expressions for $S_{\boldsymbol{n},1}$ and $S_{\boldsymbol{n},2}$, see Fig. 7.

It is interesting to compare the location of the zeros of $S_{\boldsymbol{n},1}$ with those of *type I* multiple orthogonal polynomials $\mathfrak{A}_{\boldsymbol{n}}$ and $\mathfrak{B}_{\boldsymbol{n}}$, defined by the following conditions:

$$\deg \mathfrak{A}_{\boldsymbol{n}} \leq n-1, \quad \deg \mathfrak{B}_{\boldsymbol{n}} \leq m-1,$$

and

$$\begin{aligned} &\int_{\Delta_1} z^k \mathfrak{A}_{\boldsymbol{n}}(z) e^{-z^3} dz + \int_{\Delta_2} z^k \mathfrak{B}_{\boldsymbol{n}}(z) e^{-z^3} dz = 0, \quad k = 0, \dots, N-2, \\ &\int_{\Delta_1} z^k \mathfrak{A}_{\boldsymbol{n}}(z) e^{-z^3} dz + \int_{\Delta_2} z^k \mathfrak{B}_{\boldsymbol{n}}(z) e^{-z^3} dz = 1, \quad k = N-1, \end{aligned} \tag{8.6}$$

where $N = n + m$. According to Fig. 7, the zeros of $\mathfrak{B}_{\boldsymbol{n}}$ "interlace" with the zeros of $S_{\boldsymbol{n},1}$, which brings up a natural question of a possible connection of these two polynomials.

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Appendix A: Properties of the Electrostatic Partner

Proposition A.1 *If P is a polynomial of degree $N \in \mathbb{N}$, q the function of the second kind (3.9), and S is the electrostatic partner of P defined in (3.21) then:*

(a) If $z_0 \in \mathbb{C}$ is a zero of P of multiplicity $k \geq 1$ then S also has a root at z_0 with multiplicity at least $k-1$. If in addition $A(z_0) = 0$ then the multiplicity of z_0 in S is at least k.

(b) If $z_0 \in \mathbb{C} \setminus \Delta$ is a zero of $\mathfrak{C}_w[P]$ of multiplicity $k \geq 1$ then S also has a root at z_0 with multiplicity at least $k-1$. If in addition $A(z_0) = 0$ then the multiplicity of z_0 in S is at least k.

(c) Let Ω be a simply-connected domain such that $A(z) \neq 0$ for $z \in \Omega$, $\mathfrak{C}_w[P]$ holomorphic in Ω and let w be a holomorphic branch of this function in Ω. If z_0 is a boundary point of Ω, then

$$\lim_{z \to z_0} (z - z_0) A^k(z) w(z) \frac{d^k}{dz^k} q(z) = 0, \qquad k = 0, 1, \dots, \tag{A.1}$$

where we take non-tangential limit with $z \in \Omega$.

Proof Assume $z_0 \notin \Delta$. If $z_0 \in \mathbb{C}$ is a zero of p of multiplicity $k \geq 1$ then the assertion in a) follows directly from the definition (3.13). Same argument works to prove a) if we assume that $z_0 \in \mathbb{C}$ is a zero of $\mathfrak{C}_w[P]$ of multiplicity $k \geq 1$.

On the other hand, from the assumption (3.2) it follows that for $z_0 \in \Delta$,

$$\lim_{z \to z_0} (z - z_0)\, \mathfrak{C}_w[P](z) = 0, \tag{A.2}$$

which proves b) also in this case.

We turn now to c). If $A(z_0) \neq 0$ then both w^{-1} and q are bounded at z_0 and the assertion is obvious. Hence, let $A(z_0) = 0$.

Denote

$$h_k(z) := (z - z_0) A^k(z) w(z) \frac{d^k}{dz^k} q(z), \quad k = 0, 1, 2, \dots$$

Obviously, $h_0(z) = (z - z_0)\, \widehat{p}(z)$, and (A.1) for $k = 0$ is consequence of (A.2).

Using the induction in k, assume that (A.1) is establised for a certain $k \geq 0$, i.e.

$$\lim_{z \to z_0} h_k(z) = 0,$$

where we always take non-tangential limit from Ω. By Lemma A.2 below,

$$\lim_{z \to z_0} (z - z_0) h_k'(z) = 0. \tag{A.3}$$

Thus, using (3.3),

$$\begin{aligned} 0 &= \lim_{z \to z_0} (z - z_0) h_k'(z) = \lim_{z \to z_0} A(z) h_k'(z) \\ &= \lim_{z \to z_0} \left[h_k(z) \left(\frac{A(z)}{z - z_0} + k A'(z) + A(z) \frac{w'(z)}{w(z)} \right) + h_{k+1}(z) \right] \\ &= \lim_{z \to z_0} \left[h_k(z) \left(\frac{A(z)}{z - z_0} + k A'(z) + B(z) \right) + h_{k+1}(z) \right] = \lim_{z \to z_0} h_{k+1}(z), \end{aligned}$$

which proves (A.1). □

Lemma A.2 *Let Ω be a domain, $z_0 \in \partial\Omega$ a boundary point satisfying the following property: there exists a sector $\Gamma := \{z \in \mathbb{C} : |\arg(z - z_0) - \theta_0| < \delta\}$, with certain $\theta_0 \in [0, 2\pi)$, $\delta > 0$, such that for a sufficiently small $r > 0$, $\Gamma \cap \{z \in \mathbb{C} : |z - z_0| < r\} \subset \Omega$. If function f is holomorphic in Ω and*

$$\lim_{z \to z_0,\, z \in \Gamma} \frac{f(z)}{z - z_0} = 0,$$

then

$$\lim_{z \to z_0,\, z \in \Gamma} f'(z) = 0.$$

Proof Without loss of generality we can assume that $z_0 = 0$ and $\theta_0 = 0$.

Each $0 < q < 1$ defines a sub-sector Γ_q of Γ given by

$$\Gamma_q := \{z \in \Gamma : |\tan(\delta - \arg(z))| \geq q\}.$$

Notice that as $q \to 0$, Γ_q exhausts Γ.

Fix a $0 < q < 1$ and an arbitrarily small $\varepsilon > 0$. By assumptions, there exists $0 < r' = r'(\varepsilon) < r$ such that

$$t \in \Gamma,\ |t| < r' \quad \Rightarrow \quad |f(t)| \leq \varepsilon |t|.$$

Let $z \in \Gamma_q$; by construction, the circle $C_z := \{t \in \mathbb{C} : |t - z| = q|z|\} \subset \Gamma$. We assume $|z|$ small enough so that $|t| < r'$ for all $t \in C_z$. Then

$$f'(z) = \frac{1}{2\pi i} \oint_{C_z} \frac{f(t)}{(t - z)^2} dt,$$

and

$$|f'(z)| \leq \frac{1}{q|z|} \max_{t \in C_z} |f(t)| \leq \frac{\varepsilon}{q|z|} \max_{t \in C_z} |t| = \frac{\varepsilon}{q} \frac{|z| + q|z|}{|z|} = \frac{\varepsilon(1 + q)}{q},$$

which proves the assertion. □

Appendix B: Electrostatic Partner in the Real Case

The construction in Sect. 3 is carried out in a very general setup. In this Appendix, we prove a technical result valid in the real case, when A and B have real coefficients,

$$\Delta \subset \mathbb{R} \quad \text{and} \quad w(x) \geq 0 \text{ on } \Delta. \tag{B.1}$$

We assume that $P \not\equiv 0$ is a polynomial with real coefficients, and as before, denote $S = \mathfrak{D}_w[P]$.

Lemma B.1 *Let $\zeta_1 < \zeta_2$ be two consecutive real zeros of P, such that $(\zeta_1, \zeta_2) \cap \Delta = \emptyset$, $S(\zeta_j) \neq 0$ for $j = 1, 2$, and A preserves sign on (ζ_1, ζ_2). Then $S\,\mathfrak{C}_w[P]$ does change sign in (ζ_1, ζ_2).*

Analogously, if $y_1 < y_2$ are two consecutive real zeros of $\mathfrak{C}_w[P]$, such that $(y_1, y_2) \cap \Delta = \emptyset$, $(AS)(y_j) \neq 0$ for $j = 1, 2$, and A preserves sign on (y_1, y_2), then SP changes sign in (y_1, y_2).

Proof Since $S(\zeta_j) \neq 0$, $j = 1, 2$, it follows from b) in Proposition A.1 that these are simple zeros of P, $A(\zeta_j) \neq 0$, $j = 1, 2$, and thus, $P'(\zeta_1)P'(\zeta_2) < 0$. Evaluating in the definition (3.13), we get

$$S(\zeta_j) = \mathfrak{D}_w[P](\zeta_j) = -A(\zeta_j)\mathfrak{C}_w[P](\zeta_j)P'(\zeta_j), \quad j = 1, 2,$$

so that

$$(S\,\mathfrak{C}_w[P])\,(\zeta_j) = -\left(A(\mathfrak{C}_w[P])^2 P'\right)(\zeta_j), \quad j = 1, 2.$$

Since A preserves sign on $[\zeta_1, \zeta_2]$, we get that

$$(S\mathfrak{C}_w[P])\,(\zeta_1)\,(S\mathfrak{C}_w[P])\,(\zeta_2) < 0.$$

This proves the first assertion.

Similarly from (3.13),

$$S(y_j) = \left(A\,(\mathfrak{C}_w[P])'\,P\right)(y_j), \quad j = 1, 2,$$

so that

$$(SP)(y_j) = \left(A\,(\mathfrak{C}_w[P])'\,P^2\right)(y_j), \quad j = 1, 2.$$

If $y_1 < y_2$ do not coincide with the zeros of AS, then as (3.22) shows, these are simple zeros of $\widehat{p}$, so that

$$(\mathfrak{C}_w[P])'\,(y_1)\,(\mathfrak{C}_w[P])'\,(y_2) < 0,$$

and we conclude that

$$(SP)\,(y_1)\,(SP)\,(y_2) < 0.$$

□

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