

Coloring Mixed and Directional Interval Graphs^{*}

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Abstract. A *mixed graph* has a set of vertices, a set of undirected edges, and a set of directed arcs. A *proper coloring* of a mixed graph G is a function c that assigns to each vertex in G a positive integer such that, for each edge $\{u, v\}$ in G , $c(u) \neq c(v)$ and, for each arc (u, v) in G , $c(u) < c(v)$. For a mixed graph G , the *chromatic number* $\chi(G)$ is the smallest number of colors in any proper coloring of G . A *directional interval graph* is a mixed graph whose vertices correspond to intervals on the real line. Such a graph has an edge between every two intervals where one is contained in the other and an arc between every two overlapping intervals, directed towards the interval that starts and ends to the right.

Coloring such graphs has applications in routing edges in layered orthogonal graph drawing according to the Sugiyama framework; the colors correspond to the tracks for routing the edges. We show how to recognize directional interval graphs, and how to compute their chromatic number efficiently. On the other hand, for *mixed interval graphs*, i.e., graphs where two intersecting intervals can be connected by an edge or by an arc in either direction arbitrarily, we prove that computing the chromatic number is NP-hard.

Keywords: Mixed graphs · mixed interval graphs · directed interval graphs · recognition · proper coloring

1 Introduction

A *mixed graph* is a graph that contains both undirected edges and directed arcs. Formally, a mixed graph G is a tuple (V, E, A) where $V = V(G)$ is the set of vertices, $E = E(G)$ is the set of edges, and $A = A(G)$ is the set of arcs. We require that any two vertices are connected by at most one edge or arc. For a mixed graph G , let $U(G) = (V(G), E')$ denote the *underlying undirected graph*, where $E' = E(G) \cup \{\{u, v\} : (u, v) \in A(G) \text{ or } (v, u) \in A(G)\}$.

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A *proper coloring* of a mixed graph G is a function c that assigns a positive integer to every vertex in G , satisfying $c(u) \neq c(v)$ for every edge $\{u, v\}$ in G , and $c(u) < c(v)$ for every arc (u, v) in G . It is easy to see that a mixed graph admits a proper coloring if and only if the arcs of G do not induce a directed circuit. For a mixed graph G with no directed circuit, we define the chromatic number $\chi(G)$ as the smallest number of colors in any proper coloring of G .

The concept of mixed graphs was introduced by Sotskov and Tanaev [16] and reintroduced by Hansen, Kuplinsky, and de Werra [8] in the context of proper colorings of mixed graphs. Coloring of mixed graphs was used to model problems in scheduling with precedence constraints [15]. It is NP-hard in general, and it was considered for some restricted graph classes, e.g., when the underlying graph is a tree, a series-parallel graph, a graph of bounded tree-width, or a bipartite graph [5,6,14]. Mixed graphs have also been studied in the context of (quasi-) upward planar drawings [2,3,4], and extensions of partial orientations [1,9].

Let \mathcal{I} be a set of closed non-degenerate intervals on the real line. The *intersection graph* of \mathcal{I} is the graph with vertex set \mathcal{I} where two vertices are adjacent if the corresponding intervals intersect. An *interval graph* is a graph G that is isomorphic to the intersection graph of some set \mathcal{I} of intervals. We call \mathcal{I} an *interval representation* of G , and for a vertex v in G , we write $\mathcal{I}(v)$ to denote the interval that represents v . A *mixed interval graph* is a mixed graph G whose underlying graph $U(G)$ is an interval graph.

For a set \mathcal{I} of closed non-degenerate intervals on the real line, the *directional intersection graph* of \mathcal{I} is a mixed graph G with vertex set \mathcal{I} where, for every two vertices $u = [l_u, r_u]$, $v = [l_v, r_v]$ with u starting to the left of v , i.e., $l_u \leq l_v$, exactly one of the following conditions holds:

- u and v are disjoint, i.e., $r_u < l_v \iff u$ and v are independent in G ,
- u and v overlap, i.e., $l_u < l_v \leq r_u < r_v \iff \text{arc } (u, v)$ is in G ,
- u contains v , i.e., $r_v \leq r_u \iff \text{edge } \{u, v\}$ is in G .

A *directional interval graph* is a mixed graph G that is isomorphic to the directional intersection graph of some set \mathcal{I} of intervals. We call \mathcal{I} a *directional representation* of G . Similarly to interval graphs, a directional interval graph may have several different directional representations. As there is no directed circuit in a directional interval graph G , $\chi(G)$ is well defined. Observe that the endpoints in any directional representation can be perturbed so that every endpoint is unique, and the modified intervals represent the same graph.

Further, we generalize directional interval graphs and directional representations to *bidirectional interval graphs* and *bidirectional representations*. There, we assume that we have two types of intervals, which we call *left-going* and *right-going*. For left-going intervals, the edges and arcs are defined as in directional intersection graphs. For right-going intervals, the symmetric definition applies, that is, we have an arc (u, v) if and only if $l_v < l_u \leq r_v < r_u$. Moreover, there is an edge for every pair of a left-going and a right-going interval that intersect.

Interval graphs are a classic subject of algorithmic graph theory whose applications range from scheduling problems to analysis of genomes [7]. Many prob-

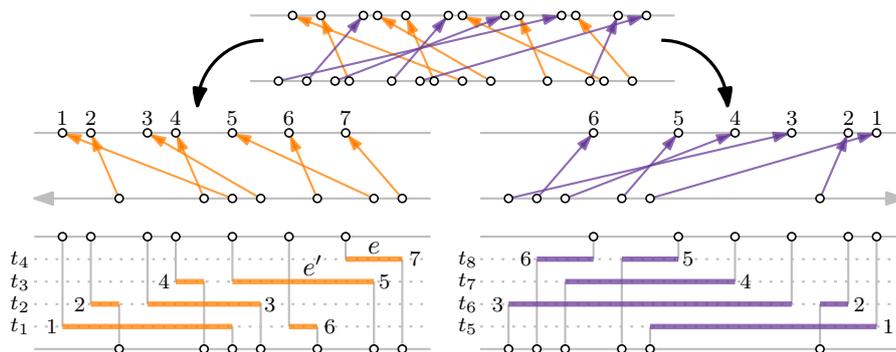


Fig. 1: Separate greedy assignment of left-going and right-going edges to tracks.

lems that are NP-hard for general graphs can be solved efficiently for interval graphs. In particular, the chromatic number of (undirected) interval graphs [7] and directed acyclic graphs [8] can be computed in linear time.

In this paper we combine the research directions of coloring geometric intersection graphs and of coloring mixed graphs, by studying the coloring of mixed interval graphs. Our research is also motivated by the following application.

A subproblem that occurs when drawing layered graphs according to the Sugiyama framework [17] is the edge routing step. This step is applied to every pair of consecutive layers. Zink et al. [18] formalize this for orthogonal edges as follows. Given a set of points on two horizontal lines (corresponding to the vertices on two consecutive layers) and a perfect matching between the points on the lower and those on the upper line, connect the matched pairs of points by x - and y -monotone rectilinear paths. Since we can assume that no two points have the same x -coordinate, each pair of points can be connected by a path that consists of three axis-aligned line segments; a vertical, a horizontal, and another vertical one; see Fig. 1. We refer to the interval that corresponds to the vertical projection of an edge to the x -axis as the *span* of that edge. We direct all edges upward. This allows us to classify the edges into *left-* vs. *right-going*.

Now the task is to map the horizontal pieces to horizontal “tracks” between the two layers such that no two such pieces overlap and no two edges cross twice. This implies that any two edges whose spans intersect must be mapped to different tracks. If there is a left-going edge e whose span overlaps that of another left-going edge e' that lies further to the left (see Fig. 1), then e must be mapped to a higher track than e' to avoid crossings. The symmetric statement holds for pairs of right-going edges. The aim is to minimize the number of tracks in order to get a compact layered drawing of the original graph. This corresponds to minimizing the number of colors in a proper coloring of a bidirectional interval graph. Zink et al. solve this combinatorial problem heuristically. They greedily construct two colorings (of left-going edges and of right-going edges) and combine the colorings by assigning separate tracks to the two directions; see Fig. 1.

Our contribution. We first show that the above-mentioned greedy algorithm of Zink et al. [18] colors directional interval graphs with the minimum number of colors; see Sect. 2. This yields a simple 2-approximation algorithm for the bidirectional case. Then, we prove that computing the chromatic number of a mixed interval graph is NP-hard; see Sect. 3. This result extends to proper interval graphs; see App. B. Finally, we present an efficient algorithm that recognizes directional interval graphs; see Sect. 4. Our algorithm is based on PQ-trees and the recognition of two-dimensional posets. It can construct a directional interval representation of a yes-instance in quadratic time.

We postpone the proofs of statements with a (clickable) “★” to the appendix.

2 Coloring Directional Interval Graphs

We prove that the greedy algorithm of Zink et al. [18] computes an optimal coloring for a given directional interval representation of G . If we are not given a representation (i.e., a set of intervals) but only the graph, we obtain a representation in quadratic time by Theorem 3. The greedy algorithm proceeds analogously to the classic greedy coloring algorithm for (undirected) interval graphs. Also our optimality proof follows, on a high level, the strategy of relating the coloring to a large clique. In our setting, however, the underlying geometry is more intricate, which makes the optimality proof as well as a fast implementation more involved. The algorithm works as follows; see Fig. 1 (left) for an example.

GREEDY ALGORITHM. Iterate over the given intervals in increasing order of their left endpoints. For each interval v , assign v the smallest available color $c(v)$. A color k is *available* for v if, for any interval u that has already been colored, $k \neq c(u)$ if u contains v and $k > c(u)$ if u overlaps v .

A naive implementation of the greedy algorithm runs in quadratic time. Using augmented binary search trees, we can speed it up to optimal $O(n \log n)$ time.

Lemma 1 (★). *The greedy algorithm can be implemented to color n intervals in $O(n \log n)$ time, which is optimal assuming the comparison-based model.*

Next we show that the greedy algorithm computes an optimal proper coloring. This also yields a simple 2-approximation for the bidirectional case.

Theorem 1. *Given a directional representation of a directional interval graph G , the greedy algorithm computes a proper coloring of G with $\chi(G)$ many colors.*

Proof. The *transitive closure* G^+ of G is the graph that we obtain by exhaustively adding transitive arcs, i.e., if there are arcs (u, v) and (v, w) , we add the arc (u, w) if absent. Clearly, no pair of adjacent intervals in the underlying undirected graph $U(G^+)$ of G^+ can have the same color in a proper coloring of G . Therefore, $\omega(U(G^+)) \leq \chi(G)$ where $\omega(U(G^+))$ denotes the size of a largest clique in $U(G^+)$. We show below that the greedy algorithm computes a coloring with at most $\omega(U(G^+))$ many colors, which must therefore be optimal. For $v \in V$ let $\mathcal{I}_{\text{in}}(v)$ be the set of intervals having an arc to v in G .

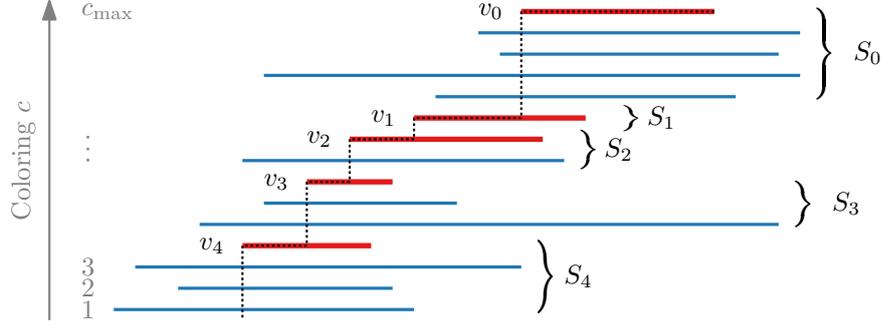


Fig. 2: A staircase and its intermediate intervals, which form a clique in $U(G^+)$.

Let c be the coloring computed by our greedy coloring algorithm. Since we always pick an available color, c is a proper coloring. To prove optimality of c , we show the existence of a clique in $U(G^+)$ of cardinality $c_{\max} = \max_{v \in V} c(v)$.

Consider an interval $v_0 = [l_0, r_0]$ of color c_{\max} . Among $\mathcal{I}_{\text{in}}(v_0)$, let v_1 be the unique interval with the largest color (all intervals in $\mathcal{I}_{\text{in}}(v_0)$ have different colors as they share the point l_0). We call v_1 the *step below* v_0 . We repeat this argument to find the step v_2 below v_1 and so on. For some $t \geq 0$, there is a v_t without a step below it, namely where $\mathcal{I}_{\text{in}}(v_t) = \emptyset$. We call the sequence v_0, v_1, \dots, v_t a *staircase* and each of its intervals a *step*; see Fig. 2. Clearly, (v_j, v_i) is an arc of G^+ for $0 \leq i < j \leq t$. In particular, the staircase is a clique of size $t + 1$ in $U(G^+)$. Next we argue about the intervals with colors in-between the steps.

For a step $v_i = [l_i, r_i]$, $i \in \{0, \dots, t\}$, let S_i denote the set of intervals that contain the point l_i and have a color $x \in \{c(v_{i+1}) + 1, c(v_{i+1}) + 2, \dots, c(v_i)\}$; see Fig. 2. Note that $v_i \in S_i$ and, by the definition of steps, each interval in S_i contains v_i . Observe that $|S_i| = c(v_i) - c(v_{i+1})$, as otherwise the greedy algorithm would have assigned a smaller color to v_i . It follows that $c_{\max} = \sum_{i=0}^t |S_i|$.

We claim that $S = \bigcup_{i=0}^t S_i$ is a clique in $U(G^+)$. Let $u \in S_i$, $v \in S_l$ such that $u \cap v = \emptyset$ (otherwise they are clearly adjacent in $U(G^+)$). Assume without loss of generality that $i < l$. Let j, k be the largest and smallest index so that $v_j \cap u \neq \emptyset$ and $v_k \cap v \neq \emptyset$, respectively. Observe that $u \cap v = \emptyset$, $u \cap v_{i+1} \neq \emptyset$, and $v \cap v_{l-1} \neq \emptyset$ imply $i < j < l$ and $i < k < l$. Since u does not intersect v_{j+1} , it overlaps with v_j , i.e., G contains the arc (v_j, u) and likewise, since v does not intersect v_{k-1} , it overlaps with v_k , i.e., G contains the arc (v, v_k) .

If $j < k$, then G^+ contains (v, v_k) and (v_k, v_j) , and therefore (v, v_j) . If $j \geq k$, then v_j is adjacent to both u and v , and since u, v are disjoint, v_j overlaps with u and v , i.e., G contains (v, v_j) . In either case, the presence of (v, v_j) and (v_j, u) implies that G^+ contains (v, u) . It follows that S forms a clique in $U(G^+)$.

Corollary 1 (\star). *There is an $O(n \log n)$ -time algorithm that, given a bidirectional interval representation, computes a 2-approximation of an optimal proper coloring of the corresponding bidirectional interval graph.*

3 Coloring Mixed Interval Graphs

In this section, we show that computing the chromatic number of a mixed interval graph is NP-hard. Recall that the chromatic number can be computed efficiently for interval graphs [7], directed acyclic graphs [8], and directional interval graphs (Theorem 1). In other words, coloring interval graphs becomes NP-hard only if edges and arcs are combined in a non-directional way.

Theorem 2. *Given a mixed interval graph G and a number k , it is NP-complete to decide whether G admits a proper coloring with at most k colors.*

Proof. Containment in NP is clear since a specific coloring with k colors serves as a certificate of polynomial size. We prove NP-hardness by a polynomial-time reduction from 3-SAT. The high-level idea is as follows. We are given a 3-SAT formula Φ with variables v_1, v_2, \dots, v_n , and clauses c_1, c_2, \dots, c_m , where each clause contains at most three literals. A literal is a variable or a negated variable – we refer to them as a *positive* or a *negative* occurrence of that variable. From Φ , we construct in polynomial time a mixed interval graph G_Φ with the property that Φ is satisfiable if and only if G_Φ admits a proper coloring with $6n$ colors.

To prove that G_Φ is a mixed interval graph, we present an interval representation of $U(G_\Phi)$ and specify which pairs of intersecting intervals are connected by a directed arc, assuming that all other pairs of intersecting intervals are connected by an edge. The graph G_Φ has the property that the color of many of the intervals is fixed in every proper coloring with $6n$ colors. In our figures, the x-dimension corresponds to the real line that contains the interval, whereas we indicate its color by its position in the y-dimension – thus, we also refer to a color as a *layer*. In this model, our reduction has the property that Φ is satisfiable if and only if the intervals of G_Φ admit a drawing that fits into $6n$ layers.

Our construction consists of a *frame* and n *variable gadgets* and m *clause gadgets*. Each variable gadget is contained in a horizontal strip of height 6 that spans the whole construction, and each clause gadget is contained in a vertical strip of width 4 and height $6n$. The strips of the variable gadgets are pairwise disjoint, and likewise the strips of the clause gadgets are pairwise disjoint.

Frame. See Fig. 3c. The frame consists of six intervals $f_i^1, f_i^2, \dots, f_i^6$ for each of the variables v_i , $i = 1, \dots, n$. All of these intervals start at position 0 and extend from the left into the construction. The intervals f_i^2, f_i^4, f_i^6 end at position 1. The intervals f_i^1 and f_i^5 extend to the very right of the construction. Interval f_i^3 ends at position 3. Further, there are arcs (f_i^j, f_i^{j+1}) for $j = 1, \dots, 5$ and (f_i^6, f_{i+1}^1) for $i = 1, \dots, n - 1$. This structure guarantees that any proper coloring with colors $\{1, 2, \dots, 6n\}$ assigns color $6(i - 1) + j$ to interval f_i^j .

Variable Gadget. See Figs. 3a and 3b. For each variable v_i , $i = 1, \dots, n$, we have two intervals v_i^{false} and v_i^{true} , which start at position 2 and extend to the very right of the construction. Moreover, they both have an incoming arc from f_i^1 and an outgoing arc to f_i^5 . This guarantees that they are drawn in the layers

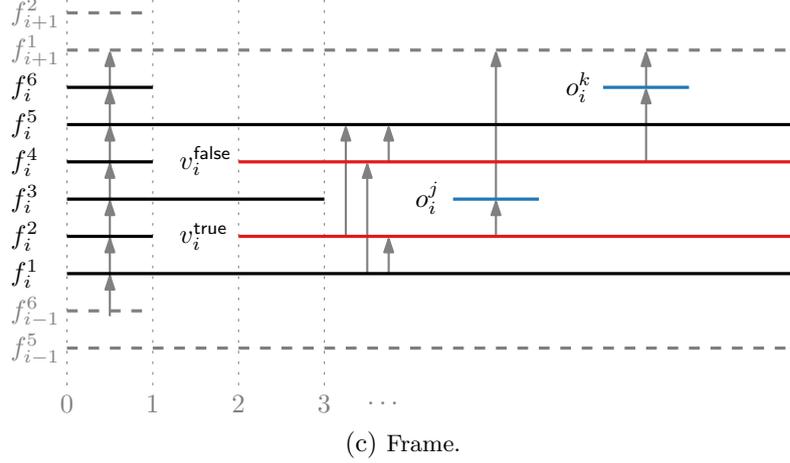
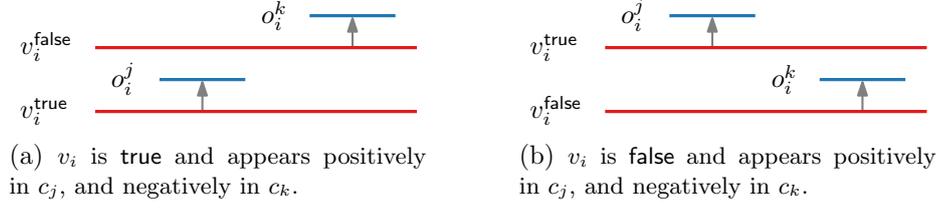


Fig. 3: A variable gadget for a variable v_i .

of f_i^2 and f_i^4 , however their ordering can be chosen freely. We say that v_i is set to true if v_i^{true} is below v_i^{false} , and v_i is set to false otherwise.

For each occurrence of v_i in a clause c_j , $j = 1, \dots, m$, we create an interval o_i^j within the clause gadget of c_j . There is an arc $(v_i^{\text{true}}, o_i^j)$ for a positive occurrence and an arc $(v_i^{\text{false}}, o_i^j)$ for a negative occurrence as well as an arc (o_i^j, f_{i+1}^1) if $i < n$. This structure guarantees that o_i^j is drawn either in the same layer as f_i^3 or as f_i^6 . However, drawing o_i^j in the layer of f_i^3 (which lies between v_i^{true} and v_i^{false}) is possible if and only if the chosen truth assignment of v_i satisfies c_j .

Clause Gadget. See Fig. 4. Our clause gadget starts at position $4j$, relative to which we describe the following positions. Consider a fixed clause c_j that contains variables v_i, v_k, v_ℓ . We create an interval s_j of length 3 starting at position 1. The key idea is that s_j can be drawn in the layer of f_i^6, f_k^6 or f_ℓ^6 , but only if o_i^j, o_k^j or o_ℓ^j , each of which has length 1 and starts at position 3, is not drawn there. This is possible iff the corresponding variable satisfies the clause.

To ensure that s_j does not occupy any other layer, we block all the other layers. More precisely, for each variable v_z with $z \notin \{i, k, \ell\}$, we create *dummy intervals* d_z^j, e_z^j of length 3 starting at position 1 that have arcs from f_z^1 and

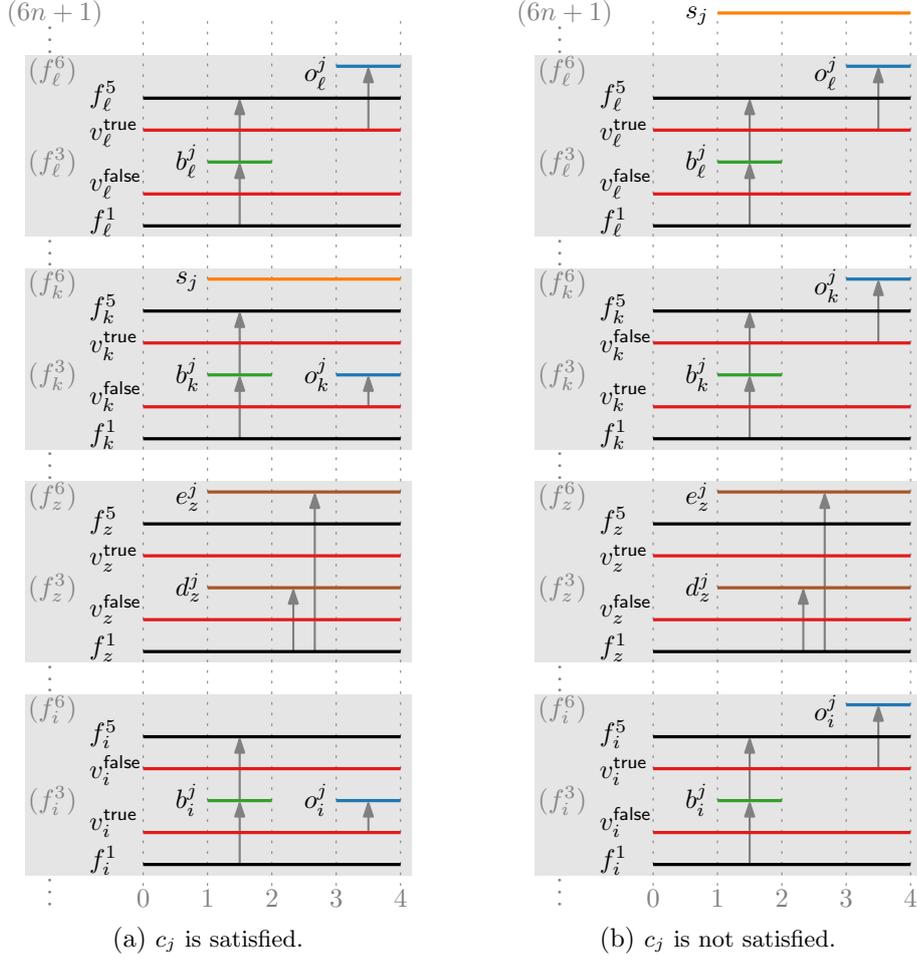


Fig. 4: A clause gadget for a clause $c_j = v_i \vee \neg v_k \vee v_l$, where $z \notin \{i, k, \ell\}$.

to f_{z+1}^1 . These arcs force d_z^j, e_z^j to be drawn in the layers of f_z^3 and f_z^6 , thereby ensuring that s_j is not placed in any layer associated with the variable z .

Similarly, for each $z \in \{i, k, \ell\}$, we create a blocker b_z^j of length 1 starting at position 1 that has arcs from f_z^1 and to f_z^5 . This fixes b_z^j to the layer of f_z^3 (since the layers of f_z^2 and f_z^4 are occupied by v_z^{true} and v_z^{false}), thereby ensuring that, among all layers associated with v_z , s_j can only be drawn in the layer of f_z^6 .

Correctness. Consider for each clause c_j with variables v_i, v_k , and v_l the corresponding clause gadget. To achieve a total height of at most $6n$, s_j needs to be drawn in the same layer as some interval of the frame. Due to the presence of the dummy intervals, the only available layers are the ones of f_z^6 for $z \in \{i, k, \ell\}$. However, the layer of f_z^6 is only free if o_z^j is not there, which is the case if and

only if o_z^j is drawn in the layer of f_z^3 . By construction, this is possible if and only if the variable v_z is in the state that satisfies clause j . Otherwise we need an extra $(6n+1)$ -th layer. Both situations are illustrated in Fig. 4. Hence, $6n$ layers are sufficient if and only if the variable gadgets represent a truth assignment that satisfies all the clauses of Φ . The mixed interval graph G_Φ has polynomial size and can be constructed in polynomial time.

A *proper interval graph* is an interval graph that admits an interval representation of the underlying graph in which none of the intervals properly contains another interval. We can slightly adjust the reduction presented in the proof of Theorem 2 to make G_Φ a *mixed proper interval graph*.

Corollary 2 (\star). *Given a mixed proper interval graph G and a number k , it is NP-complete to decide whether G admits a proper coloring with at most k colors.*

4 Recognizing Directional Interval Graphs

In this section we present a recognition algorithm for directional interval graphs. Given a mixed graph G , our algorithm decides whether G is a directional interval graph, and additionally if the answer is yes, it constructs a set of intervals representing G . The algorithm works in two phases. The first phase carefully selects a rotation of the PQ-tree of $U(G)$. This fixes the order of maximal cliques in the interval representation of $U(G)$. In the second phase, the endpoints of the intervals are perturbed so that the edges and arcs in G are represented correctly. This is achieved by checking that an auxiliary poset is two-dimensional.

PQ-trees of interval graphs [12] and realizers of two-dimensional posets [13] can be constructed in linear time. Our algorithm runs in quadratic time, but we suspect that a more involved implementation can achieve linear running time.

For a set of pairwise intersecting intervals on the real line, let the *clique point* be the leftmost point on the real line that lies in all the intervals. Given an interval representation of an interval graph G , we get a linear order of the maximal cliques of G by their clique points from left to right. Booth and Lueker [12] showed that a graph G is an interval graph if and only if the maximal cliques of G admit a *consecutive arrangement*, i.e., a linear order such that, for each vertex v , all the maximal cliques containing v occur consecutively in the order. They have also introduced a data structure called PQ-tree that encodes all possible consecutive arrangements of G . We present our algorithm in terms of modified PQ-trees (MPQ-trees, for short) as described by Korte and Möhring [10,11]. We briefly describe MPQ-trees in the next few paragraphs; see [11] for a proper introduction.

An *MPQ-tree* T of an interval graph G is a rooted, ordered tree with two types of nodes: P-nodes and Q-nodes, joined by links. Each node can have any number of children and a set of consecutive links joining a Q-node x with children is called a *segment* of x . Further, each vertex v in G is assigned either to one of the P-nodes, or to a segment of some Q-node. Based on this assignment, we *store* v in the links of T . If v is assigned to a P-node x , we store v in the link

just above x in T (adding a dummy link above the root of T). If v is assigned to a segment of a Q-node x , we store v in each link of the segment. For a link $\{x, y\}$, let S_{xy} denote the set of vertices stored in $\{x, y\}$. We say that v is *above* (*below*, resp.) a node x if v is stored in any of the links on the upward path (in any of the links on some downward path, resp.) from x in T . We write A_x^T (B_x^T , resp.) for the set of all vertices in G that are above (below, resp.) node x .

The *frontier* of T is the sequence of the sets A_x^T , where x goes through all leaves in T in the order of T . Given an MPQ-tree T , one can obtain another MPQ-tree, which is called a *rotation* of T , by arbitrarily permuting the order of the children of P-nodes and by reversing the orders of the children of some Q-nodes. The defining property of the MPQ-tree T of a graph G is that each leaf x of T corresponds to a maximal clique A_x^T of G and the frontiers of rotations of T correspond bijectively to the consecutive arrangements of G . Observe that any two vertices adjacent in G are stored in links that are connected by an upward path in T . We say that T *agrees* with an interval representation \mathcal{I} of G if the order of the maximal cliques of G given by their clique points in \mathcal{I} from left to right is the same as in the frontier of T . We assume the following properties of the MPQ-tree (see [11], Lemma 2.2):

- For a P-node x with children y_1, \dots, y_k , for every $i = 1, \dots, k$, there is at least one vertex stored in link $\{x, y_i\}$ or below y_i , i.e., $S_{xy_i} \cup B_{y_i}^T \neq \emptyset$.
- For a Q-node x with children y_1, \dots, y_k , we have $k \geq 3$. Further, for $S_i = S_{xy_i}$, we have:
 - $S_1 \cap S_k = \emptyset$, $B_{y_1}^T \neq \emptyset$, $B_{y_k}^T \neq \emptyset$, $S_1 \subsetneq S_2$, $S_k \subsetneq S_{k-1}$,
 - $(S_i \cap S_{i+1}) \setminus S_1 \neq \emptyset$, $(S_{i-1} \cap S_i) \setminus S_k \neq \emptyset$, for $i = 2, \dots, k-1$.

A *partially ordered set*, or a *poset* for short, is a transitive directed acyclic graph. A poset P is *total* if, for every pair of vertices u and v , there is either an arc (u, v) or an arc (v, u) in P . We can conveniently represent a total poset P by a linear order of its vertices $v_1 < v_2 < \dots < v_n$ meaning that there is an arc (v_i, v_j) for each $1 \leq i < j \leq n$. A poset P is *two-dimensional* if the arc set of P is the intersection of the arc sets of two total posets on the same set of vertices as P . McConnell and Spinrad [13] gave a linear-time algorithm that, given a directed graph D as input, decides whether D is a two-dimensional poset. If the answer is yes, the algorithm also constructs a *realizer*, that is, (in this case) two linear orders (R_1, R_2) on the vertex set of D such that

$$\text{arc } (u, v) \text{ is in } D \iff [(u < v \text{ in } R_1) \wedge (u < v \text{ in } R_2)].$$

The main result of this section is the following theorem.

Theorem 3 (\star). *There is an algorithm that, given a mixed graph G , decides whether G is a directional interval graph. The algorithm runs in $O(|V(G)|^2)$ time and produces a directional representation of G if G admits one.*

The algorithm runs in two phases that we introduce in separate lemmas.

Lemma 2 (Rotating PQ-trees). *There is an algorithm that, given a directional interval graph G , constructs an MPQ-tree T that agrees with some directional representation of G .*

Proof. Given a mixed graph G , if G is a directional interval graph, then clearly $U(G)$ is an interval graph and we can construct an MPQ-tree T of $U(G)$ in linear time using the algorithm by Korte and Möhring [11]. We call a rotation of T *directional* if it agrees with some directional representation of G . As we assume G to be a directional interval graph, there is at least one directional rotation \tilde{T} of T , and our goal is to find some directional rotation of T . Our algorithm decides the rotation of each node in T independently.

Rotating Q-nodes. Let y_1, \dots, y_k be the children of a Q-node x in T . We are to decide whether to reverse the order of the children of x . Let $S_i = S_{xy_i}$, let $\ell = \max \{i : S_1 \cap S_i \neq \emptyset\}$, and let $u \in S_1 \cap S_\ell$. We have $\ell < k$, and there is some vertex $v \in (S_\ell \cap S_{\ell+1}) \setminus S_1$. This implies that u and v are assigned to overlapping segments of x . Thus, the intervals representing u and v overlap in every interval representation of $U(G)$. Hence, u and v are connected by an arc in G , and the direction of this arc determines the only possible rotation of x in any directional rotation of T , e.g., if (u, v) is an arc in G and the segment of u is to the right of the segment of v , then reverse the order of the children of x .

Rotating P-nodes. Let y_1, \dots, y_k be the children of a P-node x in T . For each $i = 1, \dots, k$, let $B_i = S_{xy_i} \cup B_{y_i}^T$, and let $B = \bigcup_{i=1}^k B_i$. The properties of the MPQ-tree give us that (i) every vertex in A_x^T is adjacent in $U(G)$ to every vertex in B , (ii) none of the B_i is empty, and (iii) for any two vertices $b_i \in B_i, b_j \in B_j$ with $i \neq j$, we have that b_i and b_j are independent in G .

Assume that there is an arc (b_i, a) directed from some $b_i \in B_i$ to some $a \in A_x^T$. We claim that any rotation T' of T that does not put y_i as the first child of x is not directional. Assume the contrary. Let $y_j, j \neq i$ be the first child of x in T' , let \mathcal{I} be a directional representation that agrees with T' , and let b_j be some vertex in B_j . The left endpoint of $\mathcal{I}(a)$ is to the right of the left endpoint of $\mathcal{I}(b_i)$ as (b_i, a) is an arc. The right endpoint of $\mathcal{I}(b_j)$ is to the left of the left endpoint of $\mathcal{I}(b_i)$ as T' puts y_j before y_i . Thus, $\mathcal{I}(b_j)$ and $\mathcal{I}(a)$ are disjoint, a contradiction.

Similarly, there are directed arcs from A_x^T to at most one set of type B_i . If there are any, the corresponding child y_i is in the last position in every directional rotation of T . Our algorithm rotates the child y_i (y_j) with an arc from B_i to A_x^T (from A_x^T to B_j) to the first (last) position, should such children exist, and leaves the other children as they are in T . It remains to show that the resulting rotation of T is directional; see App. C.

Lemma 3 (Perturbing Endpoints). *There is an algorithm that, given an MPQ-tree T that agrees with some directional representation of a graph G , constructs a directional representation \mathcal{I} of G such that T agrees with \mathcal{I} .*

Proof. The frontier of T yields a fixed order of maximal cliques C_1, \dots, C_k of G . Given this order, we construct the following auxiliary poset D . First, we add two independent chains of length $k+1$ each: vertices a_1, \dots, a_{k+1} with arcs (a_i, a_j) for $1 \leq i < j \leq k+1$, and vertices b_1, \dots, b_{k+1} with arcs (b_i, b_j) for $1 \leq i < j \leq k+1$.

Then, for each vertex v in G , let $\text{lc}(v)$ and $\text{rc}(v)$ denote the indices of the leftmost and of the rightmost clique in which v is present, respectively. Now we add to D vertex v plus, for $1 \leq i \leq \text{lc}(v)$, the arc (a_i, v) and, for $1 \leq i \leq \text{rc}(v)$, the arc (b_i, v) . Further, for each arc (u, v) in G , we add (u, v) to D . Lastly, for any two vertices u and v that are independent in G and that fulfill $\text{rc}(u) < \text{lc}(v)$, we add an arc (u, v) to D . We claim that G is a directional interval graph if and only if D is a two-dimensional poset.

First assume that G is a directional interval graph and fix a directional interval representation of G whose intervals all have distinct endpoints. For $i = 1, \dots, k$, let L_i be the sequence of all the vertices v in G for which $\text{lc}(v) = i$, in the order of their left endpoints. Similarly, let R_i be the sequence of all the vertices v in G for which $\text{rc}(v) = i$, in the order of their right endpoints. The following two linear orders L and R of the vertices of D yield a realizer of D :

$$\begin{aligned} L &= b_1 < b_2 < \dots < b_k < a_1 < L_1 < a_2 < L_2 < \dots < a_k < L_k < a_{k+1}, \\ R &= a_1 < a_2 < \dots < a_k < b_1 < R_1 < b_2 < L_2 < \dots < b_k < R_k < b_{k+1}. \end{aligned}$$

Now, for the other direction, assume that we have a two-dimensional realizer of D . As b_{k+1} and a_1 are independent in D , we have that $b_{k+1} < a_1$ in exactly one of the orders in the realizer. We call this order L , and the other one R . As a_{k+1} and b_1 are independent in D and $b_1 < b_{k+1} < a_1 < a_{k+1}$ in L , we have that $a_{k+1} < b_1$ in R . For each $i = 1, \dots, k$, define L_i as the sequence of vertices in G appearing between a_i and a_{i+1} in the order L . Similarly, let R_i be the sequence of vertices in G appearing between b_i and b_{i+1} in the order R . Observe that, for every vertex v , we have that $a_{\text{lc}(v)} < v$ in D and that $a_{\text{lc}(v)+1}$ and v are independent in D . As $a_{\text{lc}(v)+1} \leq a_{k+1} < b_1 \leq b_{\text{rc}(v)} < v$ in R , we have $v < a_{\text{lc}(v)+1}$ in L . Thus, v is in $L_{\text{lc}(v)}$ and, by a similar argument, v is in $R_{\text{rc}(v)}$.

Now we are ready to construct a directional interval representation \mathcal{I} of G . For each $i = 1, \dots, k$, we select $|L_i|$ different real points in $(i - \frac{1}{2}, i)$ and $|R_i|$ different real points in $(i, i + \frac{1}{2})$. For a vertex v that appears on the i -th position in $L_{\text{lc}(v)}$ and on the j -th position in $R_{\text{rc}(v)}$, we choose the i -th point in $(\text{lc}(v) - \frac{1}{2}, \text{lc}(v))$ as the left endpoint, and the j -th point in $(\text{rc}(v), \text{rc}(v) + \frac{1}{2})$ as the right endpoint. Such a set of intervals is a directional interval representation of G . First, observe that any two intervals intersect if and only if they have a common clique. Next, if there is an arc (u, v) in G , then the arc (u, v) is also in D , $u < v$ holds both in L and in R , the corresponding intervals overlap, and $\mathcal{I}(u)$ starts and ends to the left of $\mathcal{I}(v)$. Last, if there is an edge $\{u, v\}$ in G , then u and v are independent in D , $u < v$ in one of the orders in the realizer, and $v < u$ in the other. Thus, one of the intervals $\mathcal{I}(u)$ and $\mathcal{I}(v)$ must contain the other.

Theorem 3 follows easily from Lemmas 2 and 3. See App. D for details.

5 Open Problems

Can we recognize directional interval graphs in linear time? Can we recognize bidirectional interval graphs in polynomial time? Can we color bidirectional interval graphs optimally, or at least find α -approximate solutions with $\alpha < 2$?

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A Speeding Up the Greedy Coloring Algorithm

Lemma 1 (\star). *The greedy algorithm can be implemented to color n intervals in $O(n \log n)$ time, which is optimal assuming the comparison-based model.*

Proof. We describe a sweep-line algorithm sweeping from left to right. In a first step, we show how to achieve a running time of $O(m + n \log n)$, where m is the number of edges of the directional interval graph G induced by the given set V of n intervals. Then we use an additional data structure in order to avoid the $O(m)$ term in the running time. Note that m can be quadratic in n . For the faster implementation, we do not assume knowledge of G .

Build a balanced binary search tree \mathcal{T} to keep track of the currently available colors. Initially, \mathcal{T} contains the colors 1 to n . Fill a list \mathcal{L} with the $2n$ endpoints of the intervals in V (which we can assume to be pairwise different). Sort \mathcal{L} . Then traverse \mathcal{L} in this order, which corresponds to a left-to-right sweep. There are two types of events.

LEFT: If the current endpoint is the left endpoint of an interval v , let x be the largest color over all intervals that have an arc to v , that is, $x = \max\{c(v) : (u, v) \in A(G)\} \cup \{0\}$. Then search in \mathcal{T} for the smallest color y greater than x , delete y from \mathcal{T} , and set $c(v) = y$.

RIGHT: If the current endpoint is the right endpoint of an interval v , we insert $c(v)$ into \mathcal{T} because $c(v)$ is available again.

Clearly, this implementation runs in $O(m + n \log n)$ time. To avoid the $O(m)$ term, we use a second binary search tree \mathcal{T}' that maintains the currently active intervals, sorted according to color. We augment \mathcal{T}' by storing, in every node ν , the leftmost right endpoint r^ν in its subtree. Any interval that contains the current endpoint in \mathcal{L} is *active*.

At a LEFT event, this allows us to determine, in $O(\log n)$ time, the interval u with the largest color x among all active intervals that overlap the new interval v (that is, $r_u < r_v$), as follows. We find u by descending \mathcal{T}' from its root. From the current node, we go to its right child ρ whenever $r^\rho < r_v$. If such an interval does not exist, we set $x = 0$. Then we continue as above, querying \mathcal{T} for the smallest available color $y > x$. Finally, we set $c(v) = y$ and add v to \mathcal{T}' .

At a RIGHT event, we update \mathcal{T} as above. Additionally, we need to update \mathcal{T}' . We do this by deleting the interval v that is about to end.

We now argue that, for outputting the greedy solution of our coloring problem, the running time of $O(n \log n)$ is worst-case optimal assuming the comparison-based model of computation. Suppose that a coloring algorithm would run in $o(n \log n)$ time. Then, we could use it to sort any set $\{a_1, \dots, a_n\}$ of n numbers in $o(n \log n)$ time by coloring the set $\{[a_1 - M, a_1 + M], \dots, [a_n - M, a_n + M]\}$ of intervals, where $M = \max\{a_1, \dots, a_n\} - \min\{a_1, \dots, a_n\}$. Namely, the corresponding directional interval graph is a tournament graph and for each $i \in \{1, \dots, n\}$, the color of the interval $[a_i - M, a_i + M]$ in an optimal coloring corresponds to the rank of a_i in a sorted version of $\{a_1, \dots, a_n\}$.

Corollary 1 (\star). *There is an $O(n \log n)$ -time algorithm that, given a bidirectional interval representation, computes a 2-approximation of an optimal proper coloring of the corresponding bidirectional interval graph.*

Proof. Let \mathcal{I} be the set of intervals of G . We split \mathcal{I} into a set of left-going intervals \mathcal{I}_1 and into a set of right-going intervals \mathcal{I}_2 . These sets induce the directional graphs G_1 and G_2 , respectively. Now we color G_1 and G_2 independently with our greedy coloring algorithm and we re-combine them by using different sets of colors for G_1 and G_2 . This is a proper coloring of G with $\chi = \chi(G_1) + \chi(G_2)$ colors since between any interval in \mathcal{I}_1 and any interval in \mathcal{I}_2 , there may be an edge but no arc. Clearly, $\chi \leq 2 \max\{\chi(G_1), \chi(G_2)\} \leq 2\chi(G)$.

B Coloring Mixed Proper Interval Graphs

Corollary 2 (\star). *Given a mixed proper interval graph G and a number k , it is NP-complete to decide whether G admits a proper coloring with at most k colors.*

Proof. The general idea is as follows. We start the construction with the same set of intervals as in the proof of Theorem 2. Then, we set $x_{\text{left}} = 0$, and x_{right} to the very right of all intervals, i.e., $x_{\text{right}} = 4(m + 1)$. Next, we describe a procedure that extends every interval so that it has the left endpoint in x_{left} , or has the right endpoint in x_{right} . The procedure adds some new intervals and merges some groups of intervals into one interval. The total height of the interval representation increases to $4n + 2nm$ during the procedure. Finally, we extend every interval at x_{left} (x_{right}) to the left (right) by a length inverse to its current total length. This trick guarantees that in the end, no interval contains another interval. In the remainder of the proof, we describe the procedure of extending, adding and merging intervals.

The intervals of the frame and all v_i^{true} and v_i^{false} with $i = 1, \dots, n$ already end at x_{left} or x_{right} . Currently, we have that in any drawing of G_Φ with $6n$ layers and a fixed $i \in \{1, \dots, n\}$, all the intervals b_i^j , and σ_i^j with $j = 1, \dots, m$ are drawn in the layers of f_i^3 and f_i^6 . Additionally, each dummy interval and each s_j is drawn in one of these layers. We divide these layers into m copies each so that each pair of b_i^j and σ_i^j has its own two layers.

First we divide each f_i^3 and f_i^6 into m copies each. Accordingly, we adjust the height of the drawing to be $4n + 2nm$. Then, we make m copies of each dummy interval and virtually assign each copy to a distinct layer of the drawing. For each b_i^j we virtually assign it to the layer of the j -th copy of f_i^3 and extend it to the left up to x_{left} . In this process, we merge b_i^j with every dummy interval on the left and with the j -th copy of f_i^3 while keeping all involved arcs. We call the merged interval $f_i^{3,j}$. If there is no b_i^j , we obtain $f_i^{3,j}$ by extending the j -th copy of f_i^3 up to x_{right} and merging it with all dummy intervals virtually assigned to its layer.

Symmetric to b_i^j , we extend each o_i^j to the right up to x_{right} and merge o_i^j with all dummy intervals virtually assigned to the layer of $f_i^{3,j}$, but here we drop the arcs of the dummy intervals. We call the merged interval o_i^j . Similarly, for every clause c_j with variables v_i, v_k, v_ℓ , we merge all dummy intervals virtually assigned to the layer of the j -th copy of f_z^6 , for $z = i, k, \ell$ that are to the right of s_j and drop all the arcs as in the previous case. We obtain three copies d_j^1, d_j^2 , and d_j^3 of the same interval and we merge one of these copies, say d_j^3 , with s_j . We denote that new interval by s'_j . We drop all arcs of d_j^1, d_j^2 , and s'_j to preserve the freedom we had for placing s_j in our original construction. The only unmerged dummy intervals are in the layer of the j -th copy of f_i^6 to the left of s_j or in the layer of the j -th copy of f_i^6 if there is no occurrence of the variable v_i in the clause c_j . In each of these layers, we merge the dummy intervals together with the corresponding copy of f_i^6 and obtain intervals ending at x_{left} . For $j = 1, \dots, m$, we call the merged interval $f_i^{6,j}$.

For $i = 1, \dots, n$ and $j = 1, \dots, m-1$, we add the arcs $(f_i^2, f_i^{3,1}), (f_i^{3,j}, f_i^{3,j+1}), (f_i^{3,m}, f_i^4), (f_i^5, f_i^{6,1}), (f_i^{6,j}, f_i^{6,j+1}),$ and $(f_i^{6,m}, f_{i+1}^1)$ to have a frame as in the original hardness construction. Observe that this new frame now has exactly $4n+2nm$ intervals with their left endpoint at x_{left} and, in the whole construction, there are $2n+6m$ other intervals with their right endpoint at x_{right} , i.e., the $2n$ intervals v_i^{true} and v_i^{false} for $i = 1, \dots, n$ and the $6m$ intervals o_i^j, d_j^1, d_j^2 , and s'_j for $j = 1, \dots, m$.

Next, we argue that the functionality described in the proof of Theorem 2 is retained. Intervals of the (new) frame either block a complete layer from x_{left} to x_{right} or they end at position 1 (each f_i^2 and f_i^4) or within the clause gadget of a variable c_j if a variable v_i occurs in c_j (each $f_i^{3,j}$ and $f_i^{6,j}$). Any other interval starting in a clause gadget of a clause c_j needs to be matched with a frame interval that ends in the clause gadget of c_j . Therefore, to have a construction with a total height of at most $4n+2nm$, we need to combine $f_i^{3,j}$ and $f_i^{6,j}$ with o_i^j and some of $\{d_j^1, d_j^2, s'_j\}$, while $f_i^{3,j}$ and s'_j are not combinable. This ensures that the correctness argument from the proof of Theorem 2 remains valid.

C Rotation Is Directional

Claim. The rotation of MPQ-tree T of a directional interval graph G constructed in the proof of Lemma 2 is a directional rotation.

Let T' denote the tree T after applying the rotations described in the proof of Lemma 2. We claim that T' is directional. Let \tilde{T} be an arbitrary directional rotation of T . By construction, T' and \tilde{T} differ only in the ordering of children of P-nodes x that do not have arcs from/to vertices in A_x^T . To prove that T' is directional, it suffices to show that the rotation of \tilde{T} obtained by swapping two children of a P-node x that have no arcs from/to vertices in A_x^T is directional.

Consider a directional interval representation \mathcal{I} whose clique ordering corresponds to the rotation \tilde{T} and let y_k, y_ℓ be two children of some P-node x such that

neither B_k nor B_l contains a vertex with an arc from/to a vertex in A_x^T . Let I_k be the smallest interval that contains $\mathcal{I}(v)$ for all $v \in B_k$ and let I_l be defined analogously for B_l . Note that I_k and I_l are disjoint and that each of them is properly contained in $\bigcap_{v \in A_x^T} \mathcal{I}(v)$ as otherwise it would have incoming or outgoing arcs. After suitably stretching the real line, we may assume that I_k and I_l have the same length. Let x_k, x_l denote the left endpoints of I_k and I_l , respectively. We obtain a directional representation whose clique ordering corresponds to T' simply by exchanging the positions of the representations of the subgraphs induced by B_l and by B_k . More formally, for each $v \in B_k$ set $\mathcal{I}'(v) = \mathcal{I}(v) - x_k + x_l$ and for each $v \in B_l$ set $\mathcal{I}'(v) = \mathcal{I}(v) - x_l + x_k$. For all other vertices $v \in V \setminus (B_k \cup B_l)$ set $\mathcal{I}'(v) = \mathcal{I}(v)$. It follows that each of the subgraphs induced by B_k and B_l is still represented correctly. Moreover, by construction the vertices in B_l and B_k still have edges (and not arcs) to all vertices in A_x^T .

D Recognition Algorithm

Theorem 3 (\star). *There is an algorithm that, given a mixed graph G , decides whether G is a directional interval graph. The algorithm runs in $O(|V(G)|^2)$ time and produces a directional representation of G if G admits one.*

Proof. Our algorithm, given a directional interval graph G , applies the algorithm from Lemma 2 to obtain a directional MPQ-tree of G . Then, using Lemma 3, it constructs a directional representation of G . If any of the phases fails, then we know that G is not a directional interval graph, and we can reject the input. Otherwise, our algorithm accepts the input and produces the directional representation of G . It is easy to see that both algorithms from Lemmas 2 and 3 can be implemented to run in $O(|V(G)|^2)$ time. For Lemma 2 it is enough to notice that:

- the MPQ-tree of an interval graph $U(G)$ is of size $O(|V(G)|)$ and can be constructed in time $O(|V(G)| + |E(G) + A(G)|)$ [11],
- when deciding the rotation of a Q-node x , the pair of vertices that decide the rotation of x can be found in $O(|V(G)|)$ time,
- when deciding the rotation of a P-node x , the first, and the last child of x can be found in $O(|V(G)|)$ time.

For Lemma 3 it is enough to notice that:

- there are $O(|V(G)|)$ maximal cliques in an interval graph,
- there are $O(|V(G)|)$ vertices and $O(|V(G)|^2)$ arcs in the auxiliary poset D ,
- two-dimensional realizer of the auxiliary poset D can be constructed in time $O(|V(D) + A(D)|)$ [13].

It is also quite easy to speed up the implementation of the rotation algorithm in Lemma 2 to linear time. Sadly, the auxiliary poset D that we construct in Lemma 3 has quadratic size and is thus the main obstacle for obtaining a linear-time algorithm. We suspect that an explicit construction of D can be avoided.