Biomedical, Industrial & Human Factors
Engineering Faculty Publications

Biomedical, Industrial & Human Factors
Engineering

2017

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ABSTRACT. Probabilistic methods are applied to the study of fatigue wear of sliding surfaces. A variance of time to failure (to occurrence of maximum allowable wear depth) is evaluated as a function of a mean wear rate of normal wear and a size of wear particles. A method of estimating probability of failure-free work during a certain time interval (reliability) is presented. An effect of the bedding-in phase of wear on the reliability is taken into account. Experimental data for Ultra High Molecular Weight Polyethylene (UHMWPE) cups of artificial hip implants is used to make numerical calculations.

Introduction. Every year more than a million patients worldwide have a joint prosthesis implanted, the majority of which are hips and knees. The wear of artificial joints poses a particular challenge to engineers, medical scientists and clinicians, and this subject requires further development. This paper is devoted to estimation of probabilistic reliability of cups of artificial hip implants, made of Ultra High Molecular Weight Polyethylene (UHMWPE) with the use of experimental data available to the authors.

During the sliding contact of surfaces, in near-surface material layers, prone to the friction damage, the stresses are distributed non-uniformly, because of discreteness of the surface contact. The actual contact area A_a is of the order $10 \div 10^3$. Therefore, the average actual pressure p_a at contact spots (defined as the ratio of the total contact force F to the actual contact area, $p_a = F/A_a$) is $10 \div 10^3$ times higher than the nominal pressure $p_n = F/A_n$. Experimental and theoretical research shows that the average actual contact pressure p_a does not change much upon the change of the total contact force F, but depends mainly on roughness parameters and mechanical properties of interacting surfaces [1]. This fact indicates the presence of plastic deformation in the near-surface layers of the interacting bodies. The plastic deformation causes displacement of the contact spots during the sliding contact, leading to the cyclic change of stress at points of the contacting surfaces. The cyclic variation of stress components and their high amplitude in the near surface layers (the average actual pressure p_a is usually larger than the fatigue limit) causes cyclic fatigue in the near-surface layers. The fatigue damage and the resulting separation of particles of contacting surfaces occurs because of interaction of their ridges, the size and shape of which have random character. Besides, material properties in the near-surface layers can vary randomly too. Therefore the stress components in the near-surface layers and the wear depth are random functions of time. This leads to the need of using probabilistic methods to the study of wear and to the need of evaluating reliability of contacting surfaces, i.e. probability of failure-free work during a certain time interval (with failure understood as occurrence of the maximum allowable wear depth). It is established experimentally that the size of the wear particles in the fatigue wear is comparable with diameters of the contact spots, which vary from 10^{-6} m to 10^{-5} m [7]. More accurate experimental data on the average size of the wear particles is required for each particular contact pair.

Some obvious formulas, needed for the further exposition, are

$$W(t) = \int_0^t \dot{W}(\tau)\, d\tau \tag{1}$$

$$m_{\dot{w}} \equiv E[\dot{w}(t)] \equiv \int_{-\infty}^{\infty} \dot{W}(\tau)(\dot{W},t)d\dot{W} \tag{2}$$

$$\sigma_{\dot{w}}^2(t) \equiv D[\dot{W}(t)] = E[\dot{W}^2] - \left(E[\dot{W}]\right)^2 = \int_{-\infty}^{\infty} \dot{W}^2 f(\dot{W},t)d\dot{W} - \left(\int_{-\infty}^{\infty} \dot{W} f(\dot{W},t)d\dot{W}\right)^2 \tag{3}$$

$$R_{\dot{w}}(\tau) \equiv E[\dot{W}(t)\dot{W}(t+\tau)] \tag{4}$$

$$r_{\dot{w}}(\tau) = \frac{1}{R_{\dot{w}}(0)} E[\{\dot{W}(t) - m_{\dot{w}}\}\{\dot{W}(t+\tau) - m_{\dot{w}}\}] \tag{5}$$

After the bedding -in phase of the wear, the amount of wear can be small as compared to the maximum allowable wear depth W_m, or the bedding-in phase of the wear can be performed by a manufacturer of the implant, before its use. In this case, it can be considered that the non-linear bedding-in phase of the wear process is not present on the graph of the wear depth versus time (Figure 2), and then the reliability calculations can be done by the method, presented below.
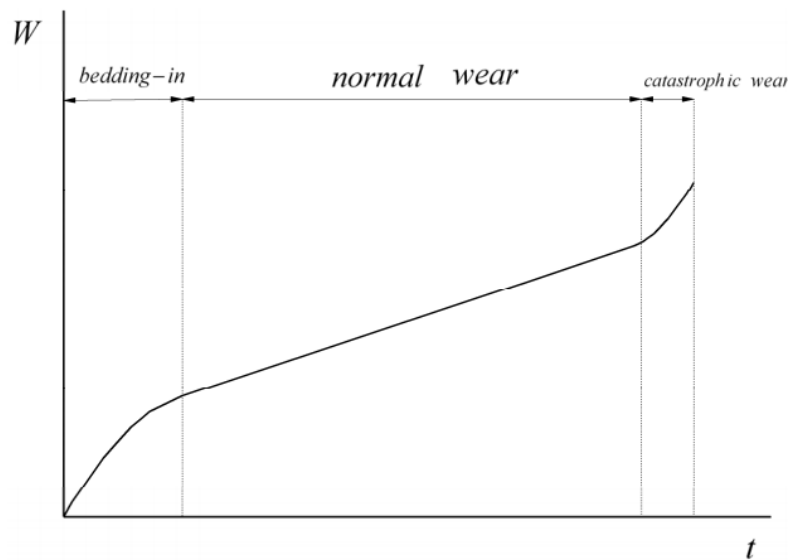

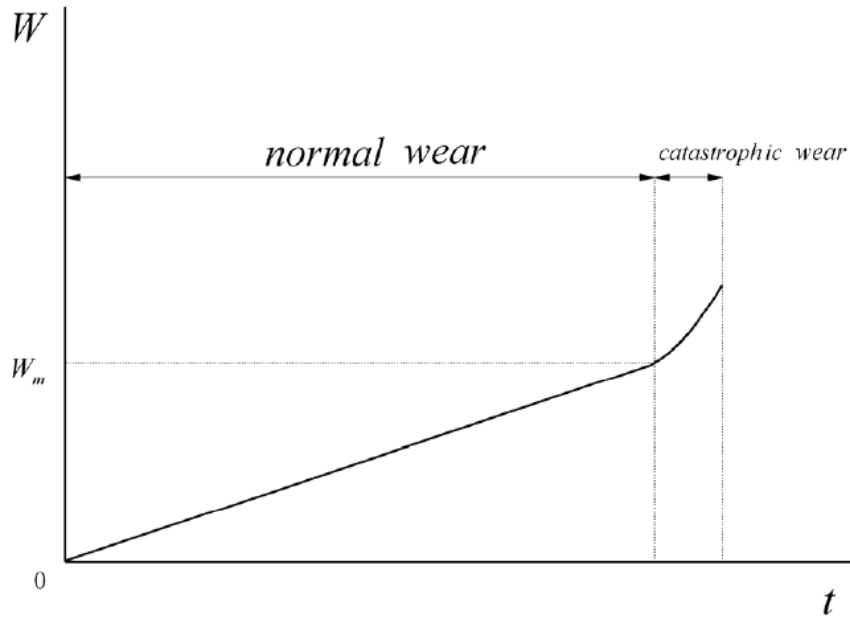*Fig. 1. Typical phases of wear depth growth with time.*

Fig. 2. Wear depth growth with time if the bedding -in phase is absent.

For the steady-state (normal) phase of wear, the wear rate $\dot{W}(t)$ can be treated as a stationary, ergodic random function of time, therefore the mean values (mathematical expectations) can be substituted with time-averaged quantities, leading to the formulas

$$m_{\dot{w}} \equiv E[\dot{w}(t)] = \frac{1}{T_n} \int_0^{T_n} \dot{W}(t)\, dt = const = a \qquad (6)$$

$$\sigma_{\dot{w}}^2(t) \equiv D[\dot{W}(t)] = \frac{1}{T_n} \int_0^{T_n} \dot{W}^2(t)\, dt - \left(\frac{1}{T_n} \int_0^{T_n} \dot{W}(t)\, dt \right)^2 = const \qquad (7)$$

$$R_{\dot{w}}(\tau) = \frac{1}{T_n} \int_0^{T_n} \dot{W}(t)\dot{W}(t+\tau)\, dt \qquad (8)$$

$$r_{\dot{w}}(\tau) = \frac{1}{R_{\dot{w}}(0)} \frac{1}{T_n} \int_0^{T_n} [\dot{W}(t) - a][\dot{W}(t+\tau) - a]\, dt \qquad (9)$$

For discrete experimental data, the autocorrelation function can be approximated by the autocorrelation sequence [6].

$$R(l) = \sum_{n=i}^{N-|k|-1} x(n)\, x(n-l) \qquad (10a)$$

Where

$$i = l, k = 0, for\ l \geq 0$$
$$i = 0, k = l, for\ l < 0$$

and the mean wear rate can be calculated as

$$a = \frac{1}{N} \sum_{i=1}^{N} \dot{W}_i \qquad (10b)$$

If for any choice of time instants $t_0 < t_1 < ... < t_n$, the random variables $W(t_0)$, $W(t_1)-W(t_0),..., W(t_n)-W(t_{n-1})$ are mutually independent, then the process W (t) is called the process with independent increments [8]. A process $W(t)$ with independent increments is said to have stationary independent increments, if $W(0) = 0$, and the distribution of $W(t + h)W(t)$ is independent of t for all positive h. For this process the mean value m w and the variance σ_w^2 are proportional to t [8]. If, in addition to being a stationary random process, the wear rate $\dot{W}(t)$ is a highly random process, then the wear depth $W(t)$ is a random process with stationary independent increments. In this case

$$m_w \equiv E[W(t)] = at \qquad (11)$$

$$\sigma_{\dot{w}}^2(t) \equiv D[W(t)] = bt \qquad (12)$$

where b – is a constant.

To verify that $\dot{W}(t)$ is a highly random function of time, one needs to verify that the normalized autocorrelation function $r_{\dot{W}}(\tau)$ has a sharp spike at $\tau = 0$ that drops off rapidly to zero as τ moves away from zero.

The graph of the wear rate versus time [4] is presented in Figure 4. The graph of the normalized autocorrelation for the total wear rate (including bedding-in and steady-state phases of wear) is presented in Figure 5. It can be seen from this graph that at small values of time since the beginning of the wear process, the values of the autocorrelation are positive, and at large values of time ñ negative. The negativeness of the autocorrelation means that the initial increase of the wear rate leads to decrease of the wear rate upon the wear progression. Such behaviour of the wear rate is caused by presence of the bedding-in phase. By removing the first 28 values of the wear rate (corresponding to the first 3.5 years) from the data, used to plot the graph in Figure 4, one can remove the bedding-in phase and plot the normalized autocorrelation of the steady-state wear rate (Figure 6). One can see from the Figure 6 that for the normal wear, the autocorrelation of the wear rate indeed behaves in a manner that is characteristic for a highly random process: it has a sharp spike initially, and then drops o§ rapidly and oscillates near the zero value subsequently. From this follows that for the normal wear, the random process $W(t)$ (wear depth as a function of time) is a process with stationary independent increments, for which the formulas (11) and (12) are true. Obviously, the formulas (11) and (12) cannot be applied for the wear process with the bedding-in phase present, because in this case the random function $W(t)$ is not stationary. For the normal phase of wear, which has stationary independent increments of the wear depth $W(t)$, the time interval to a wear depth W, $\theta(W)$, is also a random function of W with stationary independent increments, therefore its mean value m_θ is proportional to W:
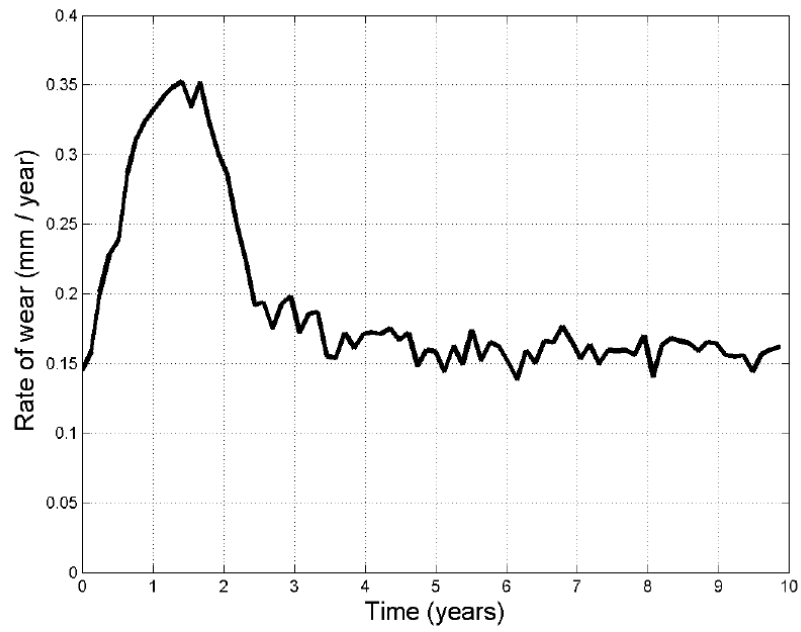
$$m_\theta = \frac{W}{a} \qquad (13)$$
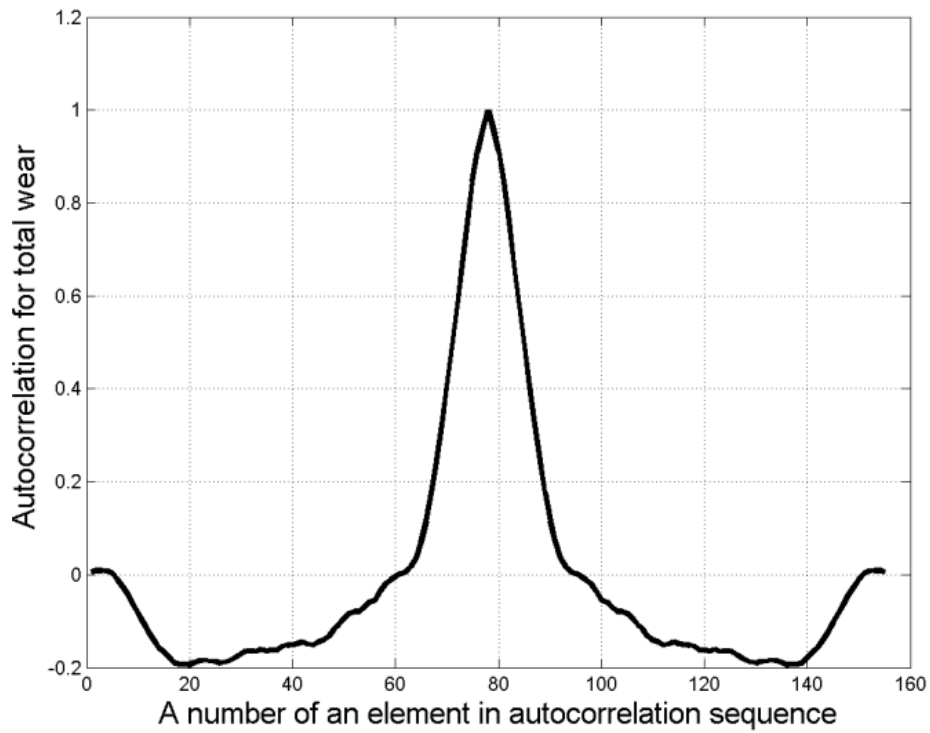
Fig. 4. Wear rate versus implantation time.

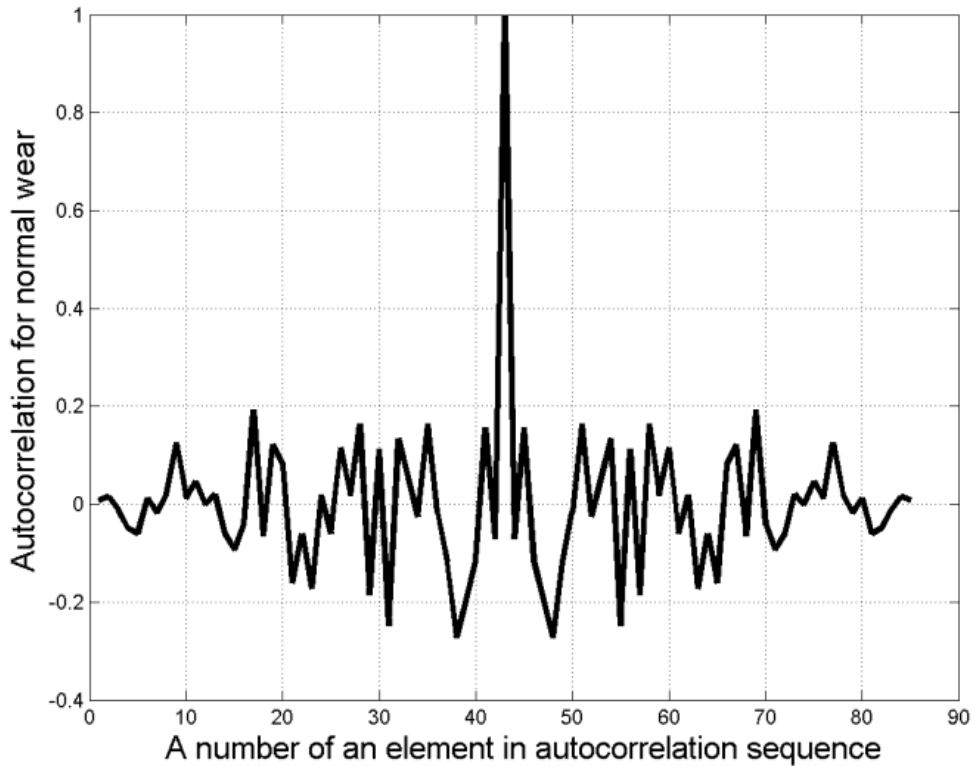*Fig. 5. Autocorrelation sequence for the total wear.*

Fig. 6. Autocorrelation sequence for the normal wear.

Let us consider a time interval Δt (between instants t and $t+\Delta t$), during which one of the following events occurs: either a particle of size h is separated from the surface with probability γ, or the particle is not separated form the surface (with probability $1-\gamma$, obviously) is assumed that γ is proportional to Δt:

$$\gamma = \lambda \Delta t \tag{14}$$

The wear increment for the time interval Δt is

$$\Delta W = W(t + \Delta t) - W(t) \tag{15}$$

and its mean value is

$$E(\Delta W) = \gamma h + (1 - \gamma) \cdot 0 = \gamma h = \lambda \, \Delta t \, h \tag{16}$$

From the last equation, we have

$$E\left[\frac{\Delta W}{\Delta t}\right] = \lambda h \tag{17}$$

or, if $\Delta t \rightarrow 0$,

$$E\left[\frac{dW}{dt}\right] = \lambda h \tag{18}$$

The left side of eq. (18) is the mean rate of wear. According to this equation, the mean rate of wear is constant, and this is a consequence of the assumption in eq. (14). Therefore, the assumption in eq. (14) is valid for the normal wear. The mean value of di§erence of random variables is equal to the difference of their mean values, therefore

$$E\left[\frac{dW}{dt}\right] = \frac{dE[W(t)]}{dt} \tag{19}$$

If the wear process is modelled as separation of discrete particles, then the function $\theta(W)$ can be treated as a random function of the wear depth with the gamma-distribution [2]:

$$\phi(\theta, W) = \begin{cases} \frac{1}{\Gamma(n)}\lambda^n\theta^{n-1}\exp(-\lambda\theta) \ for \ \theta \geq 0 \\ 0 \ for \ \theta < 0 \end{cases} \tag{20}$$

where n − is a number of separated particles necessary for the wear depth to become equal to W. Obviously,

$$n = \frac{W}{h} \tag{21}$$

In eq. (20),

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx \tag{22}$$

The mean m_θ and the variance σ_θ^2 of the function $\theta(W)$, having the gamma-distribution, is [5]

$$m_\theta \equiv E[\theta(W)] = \int_0^\infty \theta \ \phi(\theta, W)d\theta = \int_0^\infty \theta \frac{1}{\Gamma(n)}\lambda^n\theta^{n-1}\exp(-\lambda\theta)\,d\theta = \frac{n}{\lambda} \tag{23}$$

$$\sigma_\theta^2 \equiv D[\theta(W)] = \int_0^\infty \theta^2 \ \phi(\theta, W)d\theta - m_\theta^2 = \int_0^\infty \theta^2 \frac{1}{\Gamma(n)}\lambda^n\theta^{n-1}\exp(-\lambda\theta)\,d\theta - \left(\frac{n}{\lambda}\right)^2 = \frac{n}{\lambda^2} \tag{24}$$

If the size h of the particles, separated from the surface, is very small, then the number n of the separated particles for a given wear depth W is very large. With a large number n in the gamma-distribution (20), the distribution becomes symmetrical and tends to the form [2]

$$\phi(\theta, W) = \frac{1}{\sqrt{2\pi}\sqrt{n/\lambda^2}}exp\left[-\frac{(\theta - \frac{n}{\lambda})^2}{2n/\lambda^2}\right] \tag{25}$$

i.e. the distribution becomes normal with the mean value

$$m_\theta \equiv E[\theta(W)] = \frac{n}{\lambda} \tag{26}$$

and the variance

$$\sigma_\theta^2 \equiv D[\theta(W)] = \frac{n}{\lambda^2} = \frac{m_\theta}{\lambda} \tag{27}$$

where, according to eq. (18)

$$\lambda = \frac{E[\dot{w}(t)]}{h} = \frac{m_{\dot{w}}}{h} = \frac{a}{h} \tag{28}$$

So,

$$\sigma_\theta^2 = \frac{m_\theta}{\lambda} = \frac{m_\theta}{a/h} \tag{29}$$

where, according to eq. (13),

$$m_\theta = \frac{W}{a}$$

Substituting eq. (13) into eq. (29), we find

$$\sigma_\theta^2 = \frac{h}{a^2} \tag{30}$$

where h is a mean size of a particle, separated from the surface, and a is the mean rate of the steady-state (normal) wear. Introducing notations

$$\mu = \frac{1}{a}, \eta = \frac{h}{a^2} \tag{31}$$

we will write eqs. (13) and (30) as

$$m_\theta = \mu W, \sigma_\theta^2 = \eta W \tag{32}$$

According to eq. (25), for the normal phase of wear with stationary independent increments of $W(t)$, the probability density of $\theta(W)$ can be taken as normal.

$$\phi(\theta, W) = \frac{1}{\sigma^2 \sqrt{2\pi}} exp\left(-\frac{(\theta - m_\tau)^2}{2\sigma_\theta^2}\right) = \frac{1}{\sqrt{2\pi}\sqrt{\eta W}} exp\left(-\frac{(\theta - \mu W)^2}{2\eta W}\right) \qquad (33)$$

Therefore, the probability that the time to some specified wear depth W is less than some specified time interval T is

$$P\{\theta(W) < T\} = \frac{1}{\sqrt{2\pi}\sqrt{\eta W}} = \int_{-\infty}^{T} exp\left(-\frac{(\theta - \mu W)^2}{2\eta W}\right) d\theta \qquad (34)$$

Then, the probability that the time to a maximum allowable wear depth W_m is less than some specifed time T (probability of failure during the time interval $[0; T]$) is

$$P\{\theta(W_m) < T\} = \frac{1}{\sqrt{2\pi}\sqrt{\eta W_m}} = \int_{-\infty}^{T} exp\left(-\frac{(\theta - \mu W_m)^2}{2\eta W_m}\right) d\theta \qquad (35)$$

Performing the change of the variable in the last integral

$$u = \frac{\theta - \mu W_m}{2\eta W_m} \qquad (36)$$

we obtain

$$P\{\theta(W_m) < T\} = \frac{1}{\sqrt{2\pi}} = \int_{-\infty}^{\frac{T - \mu W_m}{\sqrt{\eta W_m}}} exp\left(-\frac{u^2}{2}\right) du = \Phi\left(\frac{T - \mu W_m}{\sqrt{\eta W_m}}\right) \qquad (38)$$

Where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp\left(-\frac{u^2}{2}\right) du \qquad (39)$$

Then, the probability that the time to a maximum allowable wear depth W_m is larger than some specified time T (probability of failure-free work during the time interval $[0; T]$) is

$$P\{\theta(W_m) > T\} = 1 - P\{\theta(W_m) < T\} = 1 - \Phi\left(\frac{T - \mu W_m}{\sqrt{\eta W_m}}\right) \qquad (40)$$

Now let us consider a situation in which the effect of the bedding-in phase on probability of failure is not negligible. In Figure 7, the end of the bedding-in phase coincides with the time instant t = 0, for convenience. The wear depth at the end of the bedding-in phase will be denoted as

$$W_0 \equiv W(0) \tag{41}$$

and it will be treated as a random quantity. Then

$$W^*(t) = W(t) - W_0 \tag{42}$$

is the wear depth during the normal wear. The maximum allowable wear depth, measured from the beginning of the wear process (from $t = t_0 < 0$) is denoted as W_m. The maximum allowable wear depth, measured from the beginning of the normal wear process (from $t = 0$) is denoted as W^*_m. Then
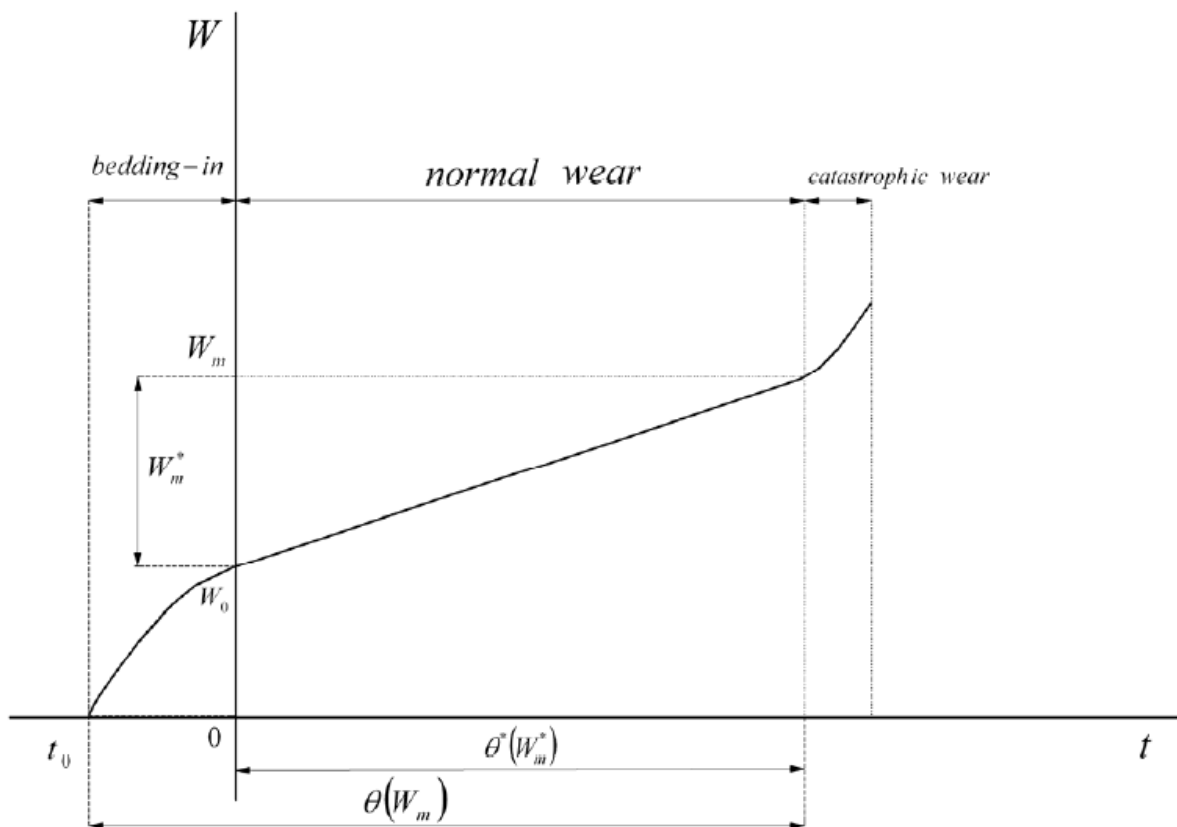
$$W^*_m = W_m - W_0 \tag{43}$$

*Fig. 3. Illustration to reliability calculation with account of the bedding -in phase. W0 and * Wm are random quantities.*

The quantity W_m is not random, and the quantity W_0 is random, so the quantity W^*_m is random. The time interval, measured from $t = t_0$, to a predetermined wear depth W (measured from $t = t_0$), will be denoted as $\theta(W)$. The function $\theta(W)$ is random. The time interval, measured from $t = 0$, to a wear depth W^* (measured from $t = 0$), will be denoted as $\theta^*(W^*)$. The function $\theta^*(W^*)$ is random. Obviously

$$\theta(W) = |t_0| + \theta^*(W^*) \tag{44}$$

Therefore

$$\theta(W_m) = |t_0| + \theta^*(W_m^*) \tag{45}$$

The length of the bedding-in time interval is usually much less than the length of the time interval of normal wear:

$$|t_0| \ll \theta(W_m) \tag{46}$$

So,

$$\theta(W_m) \approx \theta^*(W_m^*) \tag{47}$$

Then, the probability that the time to a maximum allowable wear depth W_m is less than some specified large time T is

$$P\{\theta(W_m) < T\} \approx P\{\theta^*(W_m^*) < T\} \tag{48}$$

So, with account of the bedding-in phase, i.e. considering that the maximum allowable wear depth during the normal wear, W^*_m, is a random quantity, the formula (38) can be substituted with the formula [8]

$$P\{\theta(W_m) < T\} \approx P\{\theta^*(W_m^*) < T\} = \Phi\left(\frac{T - \mu E[W_m^*]}{\sqrt{\mu D[W_m^*] + \eta E[W_m^*]}}\right) = \Phi\left(\frac{T - \mu(W_m - E[W_0])}{\sqrt{\mu D[W_0] + \eta(W_m - E[W_0])}}\right) \tag{49}$$

Where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp\left(-\frac{u^2}{2}\right) du$$

If the bedding-in phase of the wear is absent, then $W_0 = 0$, and the formula (49) reduces to the formula (38). The probability that the time to the maximum allowable wear depth W_m is larger than some specified time T (probability of failure-free work during the time interval $[0; T]$) is

$$P\{\theta(W_m) > T\} = 1 - P\{\theta(W_m) < T\} = 1 - \Phi\left(\frac{T - \mu(W_m - E[W_0])}{\sqrt{\mu D[W_0] + \eta(W_m - E[W_0])}}\right) \tag{50}$$

The mean value and variance of the wear depth at the beginning of the normal wear phase, $E[W_0]$ and $D[W_0]$, should be known from experimental data. The maximum allowable wear depth W_m is the wear depth at transition from the normal to catastrophic phase of wear, and it should be known from

experimental data also. So, the formula (50) can be used for evaluating probability of failure-free work during a time interval $[0; T]$.

For the data on wear rate of UHMWPE cups of artifficial hip joints, presented in the reference Kurtz, 2004, the mean wear rate during the normal wear is $a = 0.159 \frac{mm}{year}$; the mean value of the wear depth at the beginning of the normal wear is $E [W_0] = 0,35$ mm ; the variance of the wear depth at the beginning of the normal wear is $D [W_0] = 10^{-4}$ mm^2 ; the maximum allowable wear depth is $W_m = 1,4$ mm. Taking an average size of particles, separated from the surface, as $h = 10^{-3}$ mm, we find the following dependence of the probability of failure-free work of the hip joint during a time period $[0; T]$ on the value of T (Table 1).

Table 1. Value of T.

T (years)	$P\left\{\theta\left(W_m\right) > T\right\}$
5.5	1
6	0.998
6.5	0.693
7	0.027
7.5	0.001
8	0

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