Adaptive Multi-round Smoothing Based on the Savitzky-Golay Filter

József Dombi[1(✉)] and Adrienn Dineva[2,3]

[1] Institute of Informatics, University of Szeged, 6720 Szeged Árpád tér 2, Szeged, Hungary
dombi@inf.u-szeged.hu

[2] Doctoral School of Applied Informatics and Applied Mathematics, Óbuda University, 1034 Budapest Bécsi út 96/b, Budapest, Hungary
dineva.adrienn@kvk.uni-obuda.hu

[3] Doctoral School of Computer Science, Department of Information Technologies, Universitá degli Studi di Milano, Crema Campus, 65 Via Bramante, 26013 Crema (CR), Italy

Abstract. Noise cancellation is the primary issue of the theory and practice of signal processing. The Savitzky-Golay (SG) smoothing and differentiation filter is a well studied simple and efficient technique for noise eliminating problems. In spite of all, only few book on signal processing contain this method. The performance of the classical SG-filter depends on the appropriate setting of the windowlength and the polynomial degree. Thus, the main limitations of the performance of this filter are the most conspicious in processing of signals with high rate of change. In order to evade these deficiencies in this paper we present a new adaptive design to smooth signals based on the Savitzky-Golay algorithm. The here provided method ensures high precision noise removal by iterative multi-round smoothing. The signal approximated by linear regression lines and corrections are made in each step. Also, in each round the parameters are dynamically change due to the results of the previous smoothing. The applicability of this strategy has been validated by simulation results.

Keywords: Savitzky-Golay filter · Adaptive multi-round smoothing · Iterative smoothing · Denoising

1 Introduction

Many areas of signal processing require highly efficient processing methods in order to achieve the desired precision of the result. In a particular class of tasks, for e.g. chemical spectroscopy, smoothing and differentiation is very significant. An ample number of studies have been revealed the details of the smoothing filters. In 1964 a great effort has been devoted to the study of Savitzky and Golay, in which they introduced a particular type of low-pass filter, the so-called digital smoothing polynomial filter (DISPO) or Savitzky-Golay (SG) filter [17].

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The main advantage of the SG-filter in contrast to the classical filters - that require the characterzation and model of the noise process-, is that both the smoothed signal and the derivatives can be obtained by a simple calculation. Critical reviews and modifications of the original method can be read, for instance in [15,21]. The core of this algorithm is fitting a low degree polynomial in least squares sense on the samples within a sliding window. The new smoothed value of the centerpoint obtained from convolution. There is a rapidly growing literature discussing the properties and improvements of SG filters [1,3,6,7,12,16,24,25]. Also the importance and applicability of a digital smoothing polynomial filter in chemometric algorithms are well established [8,11,22]. The frequency domain properties of SG-filters are addressed in [2,10,18,19]. In [14], the properties of the SG digital differentiator filters and also the issue of the choice of filter length are discussed. Paper [13] concerns the calculation of the filter coefficients for even-numbered data. Also the fractional-order SG differentiators have been investigated, for e.g., by using the Riemann-Lioueville fractional order definition in the SG-filter. For instance, the fractional order derivative can be calculated of corrupted signals as published in [4]. There are several sources and types of noise that may distort the signal, for e.g., eletronic noise, electromagnetic and electrostatic noise, etc. [23]. In the theory of signal processing it is commonly assumed that the noise is an additive white Gaussian noise (AWGN) process. However, in engineering practice often nonstationary, impulsive type disturbances, etc., can degrade the performance of the processing system. Since, for the noise removal issue of signals with a large spectral dynamic or with a high rate of change, the classical SG filtering is an unefficient method. Additionally, the performance depends on the appropriate selection of the polynomial order and the window length. The arbitrary selection of these parameters is difficulty for the users. Usually the Savitzky-Golay filters perform well by using a low order polynomial with long window length or low degree with short window. This latter case needs the repetition of the smoothing. It has also been declared that the performance decreases by applying low order polynomial on higher frequencies. Nonetheless, it is possible to further improve the efficiency. With this goal, in this work we introduce an adaptive smoothing approach based on the SG filtering technique that ensures acceptable performance independent of the type of noise process.

2 Mathematical Background of the Savitzky-Golay Filtering Technique

In this section we briefly outline the premise behind the SavitzkyGolay filtering according to [9]. Let us consider equally spaced input data of n$\{x_j; y_j\}$, $j = 1, \ldots, n$. The smoothed values derives from convolution, given by

$$g_i = \sum_{i=-m}^{m} c_i y_{k+i}, \tag{1}$$

where the window length $M = 2m+1$, $i = -m, \ldots, \lambda, \ldots, m$, and λ denotes the index of the centerpoint. The k^{th} order polynomial P can be written as

$$P = a_0 + a_1(x - x_\lambda) + a_2(x - x_\lambda)^2 + \ldots + a_k(x - x_\lambda)^k \tag{2}$$

The aim is to calculate the coefficients of Eq. (2) by minimizing the fitting error in the least squares sense. The Jacobian matrix is as follows

$$J = \frac{\partial P}{\partial a} \tag{3}$$

The polynomial at $x = x_\lambda$ is a_0, hence in order to evaluate the polynomial in the window we have to solve a system of M equations which can be written in matrix form

$$J \cdot a = y \tag{4}$$

$$\begin{pmatrix} 1 & (x_{\lambda-m} - x_\lambda) & \cdots & (x_{\lambda-m} - x_\lambda)^k \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (x_{\lambda+m} - x_\lambda) & \cdots & (x_{\lambda+m} - x_\lambda)^k \end{pmatrix} \times \begin{pmatrix} a_0 \\ \vdots \\ \vdots \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} y_{\lambda-m} \\ \vdots \\ \vdots \\ \vdots \\ y_{\lambda+m} \end{pmatrix}$$

The coefficients are found from the normal equation in the following writing

$$J^T(Ja) = (J^T J)a \tag{5}$$

so

$$a = (J^T J)^{-1}(J^T y). \tag{6}$$

Since

$$P(x_\lambda) = a_0 = (J^T J)^{-1}(J^T y), \tag{7}$$

by replacing y with a unit vector in Eq. (6) the c_0 coefficient can be calculated as

$$c_j = \sum_{i=1}^{k+1} |(J^T J)^{-1}|_{0i} J_{ij}. \tag{8}$$

With a size of $(2m+1)$x$(k+1)$ the G matrix of the convolution coefficients

$$G = J(J^T J) = [g_0, g_1, \ldots, g_j]. \tag{9}$$

Figure 1 demonstrates the performance of the classical SG-filter. It can be observed that the smoothing is not precise. To overcome this problem, the following section will present an adaptive strategy (Table 1).

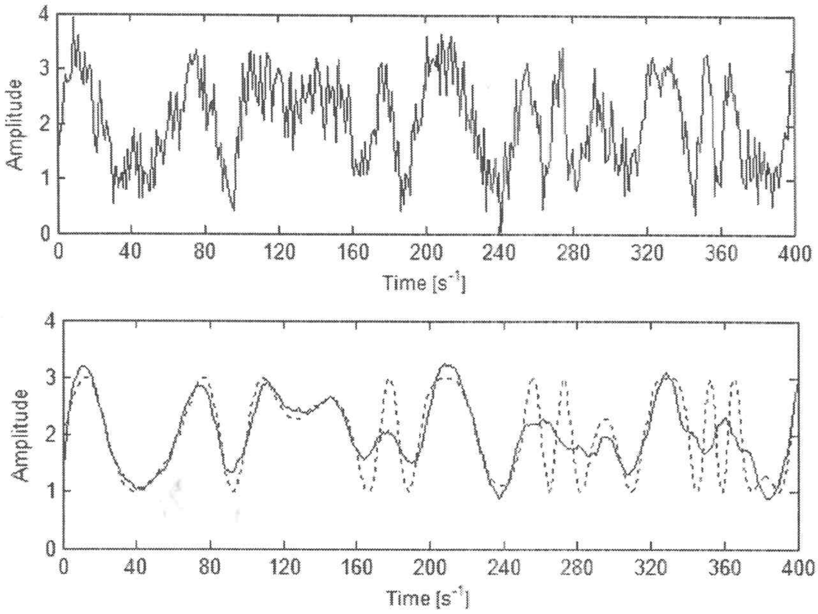

Fig. 1. Performance of the SG filter. Upper chart: signal with contaminating noise. Lower chart: dotted line - original signal, solid line - smoothed signal, $k = 3$, $M = 35$

Table 1. Some SG coefficients. $M = 2m + 1$ is the window length and k denotes the polynomial degree

Savitzky-Golay coefficients		
M	k	**Coefficients**
2*9	2	-0.0909 0.0606 0.1688 0.2338 0.2554 0.2338 0.1688 0.0606 -0.0909
	4	0.0350 -0.1282 0.0699 0.3147 0.4172 0.3147 0.0699 -0.1282 0.0350
2*11	3	-0.0839 0.0210 0.1026 0.1608 0.1958 0.2075 0.1958 0.1608 0.1026 0.0210 -0.0839
	5	0.0420 -0.1049 -0.0233 0.1399 0.2797 0.3333 0.2797 0.1399 -0.0233 -0.1049 0.0420

3 Adaptive Multi-round Smoothing Based-On the SG Filtering Technique

3.1 Multi-round Smoothing and Correction

This new adaptive strategy aims setting automatically the suitable polynomial order and window length at the different frequency components of the signal. Hence, it is possible to avoid the undershoots and preserve the peaks that could be important from different data analysis aspects. Since we perform in the time domain, this method provides efficient results independent of the type of contaminating noise. At first, the classical Savitzky-Golay filtering is performed. Assuming that only the corrupted signal is available, this step serves for revealing the peaks, hence the window length and degree of the polynomial may be arbitrary. After the first smoothing, the coordinates of the local minimum and maximum points can be obtained. From now on, we can also define the d distace vector which contains the number of samples between two neighboring points of local minima and maxima. Then, the next step is the separation of the high- and low frequency components using the bordering points and setting the proper parameters for the smoothing. The window should match the scale of the signal and the polynomial degree should vary by depending on the framesize and frequency. Since the next fuzzy relation can be defined between the section length;

$$F(d_{max} >> \bar{d_R}) = \frac{1}{1 + e^{-(\delta_{max} - \bar{d_R})}} \in [0, 1] \tag{10}$$

where $\bar{\delta_R}$ stands for the average length of the sections in the current R parts of the signal, while $\delta_{max} = max(d)$ in the observed signal. If $g(d_{max}, \bar{d_R}) = 1$, the current part of the signal contains high freqency componenst. Hence, the following rules are applied:

$$\begin{aligned} &\text{if } 1 > g(d_{max}, \bar{d_R}) > 0.9 \text{ then } k = 5, M = nint(0.3\bar{\delta_R}) \\ &\text{if } 0.89 > g(d_{max}, \bar{d_R}) > 0.75 \text{ then } k = 4, M = nint(0.5\bar{\delta_R}) \\ &\text{if } 0.75 > g(d_{max}, \bar{d_R}) > 0.45 \text{ then } k = 3, M = nint(\bar{\delta_R}) \\ &\text{if } 0.44 > g(d_{max}, \bar{d_R}) > 0.2 \text{ then } k = 2, M = nint(0.5R_n) \\ &\text{else } k = 1, M = nint(0.8R_n), \end{aligned} \tag{11}$$

where R_n is the total number of samples of the R part, and we can assign the k and M values to each R part of the signal. The values for the bounds have been determined according to the forlmula 2^k modified by experimental results.

3.2 Approximation Using Modified Shephard Method for Corretion

The correction carried out with taking the linear approximation of the obtained signal. Then, it is extracted from the smoothed one. This step reveals the higher deviation, thus the next smoothing procedure can be modified accordint to its result. As we have the coordinates of the local minimum and maximum points and the vector d, we can easily fit a regression line on the points between two local extrema. In this case, the ending and starting points of two consecutive lines do not necessary follow so as to match at the same value. In order to ensure the continuous joining of the lines we can perform this step by appling the Lagrange-multiplicator method given by

$$\sum_{x_i \in [x_1, x_2]} (m_1 x^{(i)} + b_1 - y^{(i)})^2 + \sum_{x_i \in [x_1, x_2]} (m_2 x^{(i)} + b_2 - y^{(i)})^2 \Rightarrow \min, \tag{12}$$

with the following constraints:

$$m_1 x_2 + b_1 - m_2 x_2 + b_2. \tag{13}$$

However, in some cases the peaks can contain the information of interest. Therefore, the form of the peak or valley should be processed with special care. To addres this issue, a modified Shepard - method can be applied. Let us consider

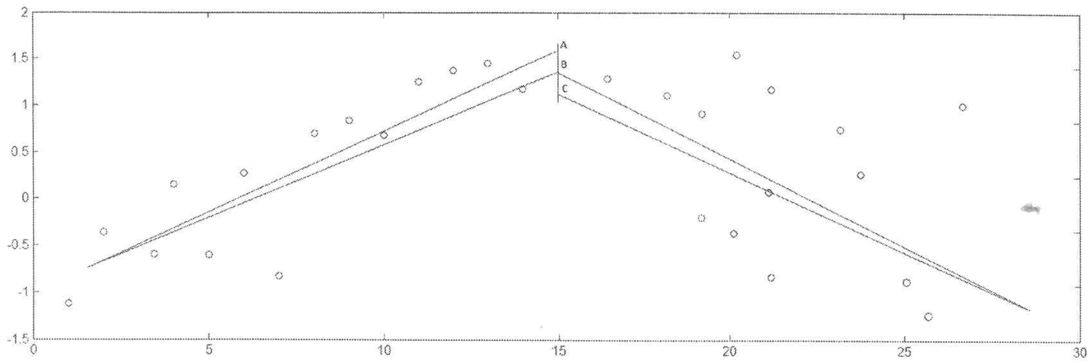

Fig. 2. Illustration of the problem of joining of the regression lines

the points around the local extrema in radius r. The new values are calculated by weighting according to the neighboring points distance. There are several variations of the Shepard method [20], now let us consider the *GMS* (*Groundwater Modeling System*) form below

$$w_i = \frac{\left(\frac{d-d_i}{dd_i}\right)^2}{\sum_{i=1}^{n}\left(\frac{d-d_i}{dd_i}\right)^2}. \tag{14}$$

Equation 14 can be trasformed into

$$w_i = \frac{\left(\frac{1-u_i}{u_i}\right)^2}{\sum_{j=1}^{n}\left(\frac{u-u_j}{u_j}\right)^2}, \tag{15}$$

in which $u_i(x) = \frac{d^i(x)}{d(x)}$. Now, using the similarity between the form of Eq. 15 and the *Dombi* operator [5] we can define the following new parametric weighting function:

$$w_i = \frac{1}{1 + \left(\frac{u_i}{1-u_i}\right)^2 \sum_{j=1}\left(\frac{1-u_j}{u_j}\right)^\lambda} \tag{16}$$

in which the setting of λ and radius r (where it performs) have the effect on the smoothness of the result (Fig. 2).

4 Simulation Results

The performance of the proposed method have been tested on a noisy signal (see, Fig. 3). The simulation has been carried out by using Matlab 8. Figure 4 shows the approximated signal after the first round. In Fig. 5 the resulted and the original signal can be seen after two rounds. It can be observed that the applied technique can efficiently recover the signal.

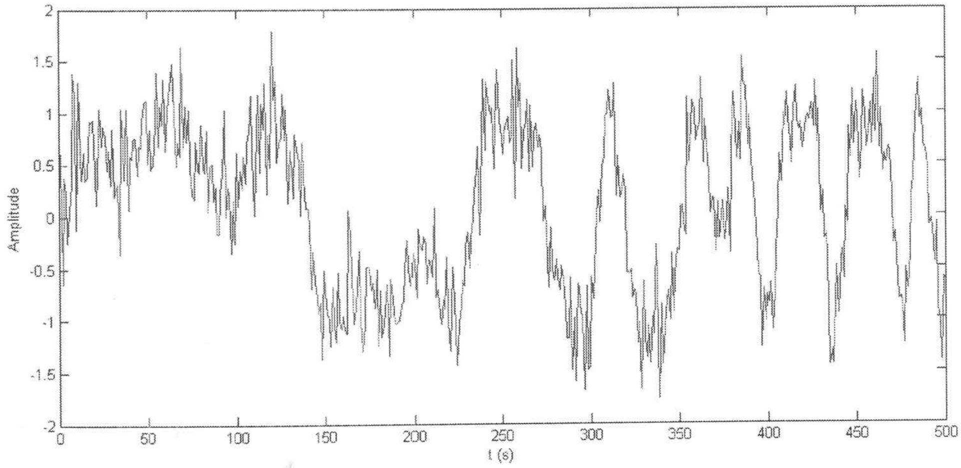

Fig. 3. The corrupted signal

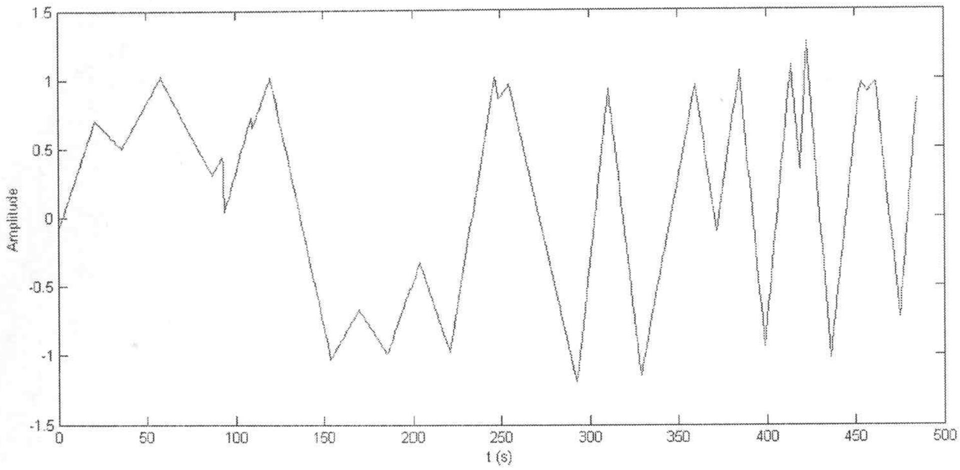

Fig. 4. The approximation of the signal after the first round.

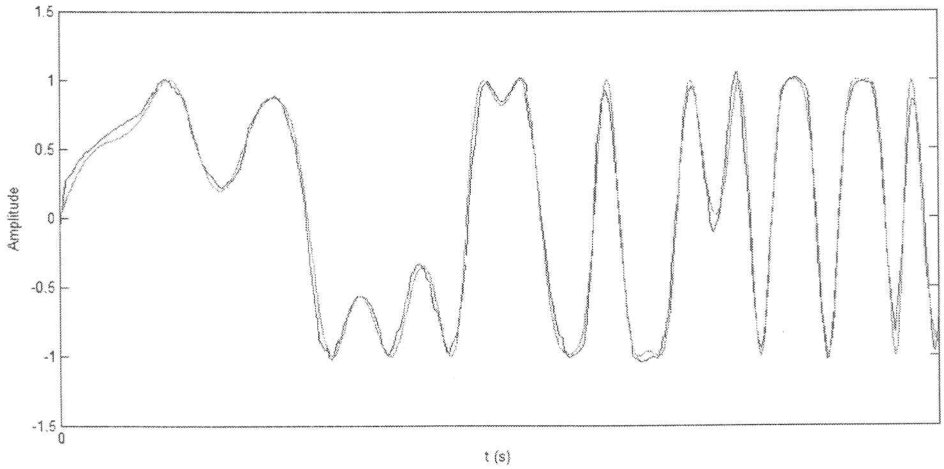

Fig. 5. The recovered (blue) and the original (magenta) signal.

5 Conclusions

In this paper a new adaptive smooting strategy has been introduced based on the Savitzky-Golay filtering technique. The proposed method allows to evade the main difficulties of the original SG filter by automatically setting the smoothing parameters. Furthermore, for the precise reconstruction of the signal a multi round correction has been applied using the linear approximation of the signal. For the reconstruction of the peaks and valleys that may contain the important information, a new weighting function has been introduced with the combination of the GMS method and the *Dombi* operator. The applicability and efficiency of this new strategy have been validated by simulation results.

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