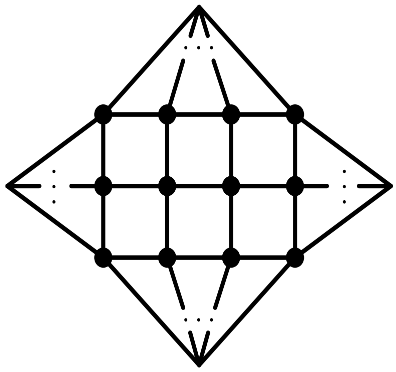
Conformal Feynman Integrals and Correlation Functions in Fishnet Theory



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Abstract

In this thesis we study various aspects of correlation functions in the (dynamical) fishnet theory. This theory can be derived as a strong-twist limit of $\mathcal{N} = 4$ super Yang–Mills theory, and is conjectured to retain its integrability in the planar limit.

We begin with the study of one of the simplest correlators of the fishnet theory, represented by the conformal box integral, in Minkowski space. While this integral is conformally invariant in Euclidean space, this symmetry is subtly broken in Minkowski space. We demonstrate the mechanism behind this symmetry breaking, and explicitly quantify the extent to which conformal symmetry is broken by analysing the functional form of the box in each kinematic region. We find that there are up to four values of the box integral on any conformal trajectory. We propose a new method to calculate the box integral directly in Minkowski space, by introducing a family of configurations with two points at infinity. These configurations help to expose the geometry behind the breaking of conformal symmetry. Furthermore, we investigate the extent to which the box integral is constrained by Yangian symmetry. We constrain the functional form of the box integral in all kinematic regions up to twelve undetermined constants, which we fix by three separate analytic continuations from the Euclidean region.

Next, we study the Basso–Dixon graphs, which are four-point conformal integrals and represent higher-loop versions of the box integral. We derive and study Yangian Ward identities for this class of integrals. These symmetry equations follow from interpreting the respective Feynman integrals as correlation functions in the bi-scalar fishnet theory. The Ward identities take the form of inhomogeneous extensions of the partial differential equations defining the Appell hypergeometric functions. We employ a manifestly conformal tensor reduction in order to express these inhomogeneities in compact form, which are given by linear combinations of Basso–Dixon integrals with shifted dimensions and propagator powers. The Ward identities naturally generalise to a one-parameter family of D-dimensional integrals representing correlators in the generalised fishnet theory of Kazakov and Olivucci. When specified to two spacetime dimensions, the Yangian Ward identities decouple. Using separation of variables, we explicitly bootstrap the solution for the conformal two-dimensional box integral. The result is a single-valued linear combination of products of Legendre functions, which reduce to elliptic K integrals for an isotropic choice of propagator powers.

Finally, we study the dilatation operator in a particular three scalar sector of the dynamical fishnet theory, which has been dubbed the eclectic model. The dilatation operator in undeformed $\mathcal{N} = 4$ super Yang–Mills is one of the hallmarks of its planar integrability, and can be mapped to integrable spin chain models. In the strongly-twisted models various subtleties emerge: the dilatation operator is rendered non-diagonalisable in various operator sectors, in particular in the three scalar sector we consider. This leads to logarithmic spacetime dependence in the corresponding two-point functions. Although the model is integrable in the Yang–Baxter sense, approaches to solve it based on the Bethe ansatz have been shown to fail. Using combinatorial arguments, we introduce a generating function which fully characterises the Jordan block spectrum of a related model: the hypereclectic spin chain. This function is found by purely combinatorial means and can be expressed in terms of the q-binomial coefficient. We provide further evidence for the universality hypothesis, which is the claim that the Jordan block spectra of both models coincide under certain filling conditions.

Zusammenfassung

Wir untersuchen unterschiedliche Aspekte im Zusammenhang mit Korrelationsfunktionen in der Fischnetz-Theorie. Dieses Model lässt sich aus dem Grenzwert einer Deformation aus der sogenannten $\mathcal{N} = 4$ supersymmetrischen Yang–Mills–Theorie ableiten und es wird vermutet, dass sie deren Integrabilität im planaren Grenzfall erbt.

Zunächst betrachten wir einen der einfachsten Korrelatoren der Fischnetz Theorie, das konforme Box-Integral, in Minkowski Signatur. Während dieses Integral in Euklidischer Signatur eine konforme Symmetrie aufweist, wird diese Symmetrie in Minkowski–Raumzeit subtil gebrochen. Wir demonstrieren den Mechanismus, der hinter dieser Symmetriebrechung steckt und beschreiben die Brechung der konformen Symmetrie quantitativ, indem wir die funktionale Form des Box-Integrals in allen kinematischen Regionen untersuchen. Auf jeder konformen Trajektorie nimmt das Box-Integral bis zu vier unterschiedliche Werte an. Wir entwickeln eine neue Methode zur direkten Berechnung des Integrals im Minkowski-Raum, die auf der Einführung von kinematischen Konfigurationen beruht, bei der zwei externe Punkte im Unendlichen liegen. Außerdem untersuchen wir das Ausmaß zu dem das Box integral durch seine Yangian–Symmetrie festgelegt ist. Wir fixieren die funktionale Form des Integrals in allen kinematischen Regionen bis auf zwölf unbestimmte Konstanten, die wir dann durch drei unterschiedliche analytische Forsetzungen bestimmen.

Als nächstes widmen wir uns den Basso–Dixon–Graphen, die ebenfalls konforme Vier–Punkt–Integrale sind und Verallgemeinerungen des Box-Integrals zu höheren Schleifenordnungen darstellen. Wir leiten die Yangian–Ward–Identitäten ab, die diese Klasse von Integralen erfüllen. Die Ward–Identitäten sind einhomogene Erweiterungen der partiellen Differentialgleichungen, die im homogenen Fall durch Appell-Hypergeometrische Funktionen gelöst werden. Wir bringen die Inhomogenitäten in eine kompakte Form, explizit sind sie durch Linearkombinationen von Basso–Dixon–Integralen mit veränderten Dimensionen und Propagatorgewichten gegeben. Die Ward–Identitäten können natürlicherweise auf eine Ein–Parameter–Familie von D–dimensionalen Integralen erweitert werden, die Korrelatoren in der verallgemeinerten Fischnetz–Theorie von Kazakov und Olivucci darstellen. In zwei Raumzeit-Dimensionen, entkoppeln die Yangian–Ward–Identitäten. Mit Hilfe der Methode der Trennung der Variablen konstruieren wir die Lösung der Identität für das konforme zwei-dimensionale Box-Integral mittels eines Bootstrap–Verfahrens.

Schließlich untersuchen wir den Dilatationsoperator in einem Drei–Skalar–Sektor der dynamischen Fischnetztheorie, der auch als Eklektisches Modell bezeichnet wird. In diesem stark deformierten Modellen gibt es zahlreiche Subtilitäten, im Vergleich zur undeformieren $\mathcal{N} = 4$ SYM Theorie: der Dilatationsoperator nimmt in unterschiedlichen Operator–Sektoren nicht–diagonalisierbare Form an. Das führt dazu, dass die Zwei–Punkt–Korrelationsfunktionen eine logarithmische Abhängigkeit von der Raumzeitseparierung der Operatoren annimmt. Obwohl das Modell im Yang–Baxter–Sinn integrabel ist, konnte es bisher nicht mit Bethe–Ansatz–Methoden gelöst werden. Unter Zuhilfenahme von kombinatorischen Argumenten führen wir eine generierende Funktion ein, die das Jordan–Block–Spektrum eines verwandten Modells, der hypereklektischen Spinkette, vollständig charakterisiert. Diese generierende Funktion lässt sich durch q–binomische Koeffizienten ausdrücken. Weiterhin liefern wir Indizien für die Universalitäts–Hypothese, die besagt, dass die Jordan–Block–Spektren für beide Modelle unter bestimmten Füllungsbedingungen übereinstimmen.

List of Publications

This thesis is based on the peer-reviewed publications [1–4] of the author, listed chronologically by date of publication:

[1] L. Corcoran and M. Staudacher, *"The dual conformal box integral in Minkowski space"*,
Nucl. Phys. B 964, 115310 (2021), `arxiv:2006.11292`

[2] L. Corcoran, F. Loebbert, J. Miczajka and M. Staudacher, *"Minkowski box from Yangian bootstrap"*,
JHEP 2104, 160 (2021), `arxiv:2012.07852`

[3] C. Ahn, L. Corcoran and M. Staudacher, *"Combinatorial solution of the eclectic spin chain"*,
JHEP 2203, 028 (2022), `arxiv:2112.04506`

[4] L. Corcoran, F. Loebbert and J. Miczajka, *"Yangian Ward identities for fishnet four-point integrals"*,
JHEP 2204, 131 (2022), `arxiv:2112.06928`

For the purpose of this thesis, the text of these publications was rewritten and expanded on. However, some parts of the technical discussions were adapted with minor changes only. Specifically, sections 4.2 and 4.3 are based on [1], section 4.4 is based on [2], section 6.3 is based on [3], and section 5 is based on [4].

Declaration of Independent Work

I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 42/2018 on 11/07/2018.

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I'm so grateful to have a family who loves and supports me so much. Thanks to Mum and Dad who make everything so easy for me. Thanks to my twin sister Rebecca who is my best friend and somebody with whom I share an inseparable bond. Thanks to my brothers Simon and Philip with whom I share so much and who I look up to a lot.

Joining the SAGEX network was one of the reasons I decided to do my PhD in Berlin. I got to experience many things different from pure research, including making a video project, online exhibition, and even spending 3 months in Champaign, IL working at Wolfram Research (where indeed most of this thesis was written)! Thanks to all my fellow SAGEX students: Anne, Stefano, Manuel, Lorenzo, Nikolai, Canxin, Riccardo, Gabriele, Sebastian, Ingrid, Davide, Kays, Andrea, and Marco. We made so many fun memories together at the various SAGEX organised events. In particular I fondly remember living with Nikolai and Lorenzo in Champaign, where *Nutellaman* was born and we almost burned down the Airbnb on five separate occasions.

And of course, for the last time I have to say: this project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 764850 "SAGEX".

Contents

Chapter 1

Introduction

Feynman integrals lie at the heart of perturbative calculations in quantum field theory. A prominent example is the Standard Model of Particle Physics, which is a gauge theory based on the gauge group $G = SU(3) \times SU(2) \times U(1)$. This theory unifies the strong, weak, and electromagnetic forces, and is to this day the most well-tested theory in all of theoretical physics. To test a theory, one must perform experiments to measure *observables*. To compute observables in a theory, for example scattering amplitudes[1] (or more generally correlation functions), one is usually forced to work perturbatively in the coupling constant. In this case, the problem of computing the observable in question to a given loop order is reduced to the problem of writing down all the Feynman diagrams associated to that physical process and calculating and further summing up the corresponding Feynman integrals as functions of the external data. For scattering amplitudes of scalar particles the external data is the on-shell momenta $p_i^2 = m_i^2$ of the external particles. For position-space correlation functions the external data is simply the positions x_i of the operators.

The Feynman diagram approach to calculating observables is not without issues, however. As we consider physical processes with higher numbers of contributing particles, at a higher number of loops, the number of contributing Feynman integrals increases factorially. Furthermore, the individual integrals can be difficult to calculate. For one, they are often plagued by UV/IR divergences, and must be appropriately regulated to even be well-defined. When they are well-defined, they can still be very complicated objects to calculate in terms of the appropriate function class, if it can be identified. For decades it has been understood that many Feynman integrals can be expressed in terms of a class of iterated integrals known as polylogarithms. More recently, Feynman integrals have been identified which evaluate to more exotic functions, for example harmonic polylogarithms and elliptic polylogarithms. The study of the mathematical properties of Feynman integrals and efficient ways to evaluate them is at the forefront of modern research in QFT. However, easier methods with which we can calculate and understand observables are still very welcome.

When there is a simple way to calculate an observable which bypasses a brute-force Feynman diagram calculation, it typically relies on a *symmetry* of the underlying theory. There are several types of symmetry relevant to the action of a QFT: for example gauge symmetry, spacetime symmetry, and internal symmetry. Therefore, if

[1] Technically the observable is the cross section $\mathcal{M}$, related to the scattering amplitude $\mathcal{A}$ via $\mathcal{M} \sim |\mathcal{A}|^2$.

one can identify a theory which has many of these symmetries, there is raised hope to get a good grasp on the observables of the theory. A case in point is $\mathcal{N}=4$ super Yang–Mills (SYM) theory in four spacetime dimensions, which is a gauge theory based on the gauge group $SU(N)$. This theory possesses a maximal amount of supersymmetry in four dimensions. Furthermore, its spacetime symmetry is enhanced beyond the usual Poincaré symmetry to a conformal symmetry, which combines with supersymmetry into a superconformal symmetry. Conformal symmetry vastly constrains the kinematic dependence of observables. Perturbatively these observables can typically be expressed in terms of *conformal* Feynman integrals. These integrals represent a special point in the space of Feynman integrals: they are more tractable than their non-conformal counterparts and possess a number of interesting analytic properties.

Notably, the conformal group on four-dimensional Minkowski space $\mathrm{Conf}(\mathbb{R}^{1,3})$ is incompatible with quantum field theory. This is because it does not respect causality: timelike separated points can be mapped to spacelike separated points via a global conformal transformation, and vice versa. However, it is still possible for a Minkowskian QFT to be *locally* conformally invariant, such that observables are constrained in each kinematic region by conformal symmetry. A notable kinematic region is the so-called *Euclidean* region, where the values of the observables agree with those computed in the corresponding Euclidean QFT. In Euclidean space there are no issues with global conformal transformations, and a Euclidean QFT can be honestly conformally invariant. Mapping between Minkowskian and Euclidean QFTs is a tricky business: often when faced with a Feynman integral representing an observable in Minkowskian QFT, one argues that it can be 'Wick rotated' in the Euclidean region to a Euclidean Feynman integral, which is easier to calculate. On the other hand, there is a prescription for analytically continuing Euclidean correlation functions to Minkowskian correlation functions via the Osterwalder-Schrader theorem. One way to avoid these subtleties is to do computations *directly* in Minkowski space, which is of course the realm in which physics takes place. However, computations in Minkowski space are usually tricky in their own right, due to the $i\epsilon$ prescription which regulates propagators and implements causality.

Putting the subtleties with Minkowski space aside, $\mathcal{N}=4$ SYM has received great attention due to its connection to *holography* and *integrability*. The AdS/CFT correspondence remains the most successful realisation of the holographic principle. This correspondence provides a duality between planar $\mathcal{N}=4$ SYM and free strings on an $AdS_5 \times S^5$ background, and a potential window into the mysteries of quantum gravity. In this thesis we are mainly concerned with integrability: this appears when a physical system is so constrained by symmetry that the relevant dynamical variables can be solved for analytically. A typical feature of integrable systems is the existence of a large number of conserved quantities. There is by now a generally accepted definition for integrability of classical mechanical systems via the Arnold-Liouville theorem. A generally accepted definition of integrability for quantum mechanical models is still missing. However, there are a wide range of quantum models, for example the $\mathfrak{su}(2)$ Heisenberg spin chain, which are generally accepted to be quantum integrable. This is because they are related to an algebraic structure known as an *R-matrix*, which satisfies the *Yang–Baxter equation*. This equation leads to a web of non-trivial algebraic relations, and using these the spectrum can often be solved for exactly using the *algebraic Bethe ansatz*. There are various other Bethe ansätze that are used to solve integrable models, for example the coordinate Bethe ansatz and the thermodynamic Bethe ansatz. The Yang–Baxter equation is related to an intricate algebraic structure known as the *Yangian* algebra, which is believed to play a central role in quantum integrability.

There is a special limit in which $\mathcal{N} = 4$ SYM simplifies dramatically, called the *planar* limit [5]. This is a double-scaling limit[2]

$$g_{\mathrm{YM}} \to 0, \qquad N \to \infty, \qquad g := g_{\mathrm{YM}}^2 N \text{ fixed}, \tag{1.0.1}$$

where g_{YM} is the Yang–Mills coupling and N is the rank of the gauge group. $\mathcal{N} = 4$ SYM is believed to integrable in the planar limit. This is because structures from quantum integrability appear time and time again in the calculation of observables in this theory. The most well-known is the calculation of the dilatation operator of the theory, which encodes quantum corrections to the two-point correlation functions known as *anomalous dimensions*. This operator can be calculated tediously by a Feynman diagram approach. However, miraculously, this dilatation operator can be identified in perturbation theory with an integrable (super) spin chain based on the superconformal algebra $\mathfrak{psu}(2,2|4)$. Restricting to a particular sector of operators one recovers the $\mathfrak{su}(2)$ Heisenberg spin chain at one-loop order. Yangian symmetry has also been detected in various guises, for example in the amplitudes of the theory, the dilatation operator, and even directly at the level of the action. There are many other places where integrability can be used for calculations in this theory, for example the *quantum spectral curve* for computing anomalous dimensions and the *hexagon* approach to calculating three-point functions.

Although integrability is ubiquitous in $\mathcal{N} = 4$ SYM, its origin is still shrouded in mystery. It seems plausible that the large amount of symmetry of the theory is the main mechanism behind this integrability. But which symmetry is responsible? Is the combination of symmetries really required? An interesting simplification of $\mathcal{N} = 4$ SYM which makes it possible to probe these questions was proposed recently in [7]. It begins with an integrable deformation of the $\mathcal{N} = 4$ SYM Lagrangian by three complex parameters γ_j, which breaks supersymmetry [8]. This is followed by a double scaling limit

$$g \to 0, \qquad \gamma_j \to i\infty, \qquad \xi_j := g e^{-\frac{i}{2}\gamma_j} \text{ fixed}. \tag{1.0.2}$$

After this limit the gauge field decouples and one recovers the so-called *dynamical fishnet* theory. Setting $\xi_1 = \xi_2 = 0$ one recovers the remarkably simple bi-scalar fishnet theory

$$\mathcal{L}_{\mathrm{FN}} = N \operatorname{tr}(\partial_\mu X \partial^\mu \bar{X} + \partial_\mu Z \partial^\mu \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}). \tag{1.0.3}$$

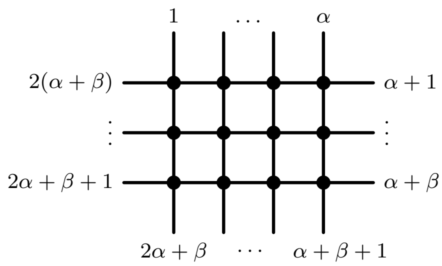
Although supersymmetry and local gauge symmetry is absent from (1.0.3), a global $SU(N)$ symmetry still allows for the notion of a planar limit. Furthermore, the theory (1.0.3) has conformal symmetry .[3] Despite this vast reduction in symmetry compared to undeformed $\mathcal{N} = 4$ SYM, the fishnet theory appears to be integrable in the planar limit. In many cases, the integrability manifests itself in very simple ways. This is a direct consequence of the simplicity of the planar Feynman diagrammatics of the theory

[2] There are several ways to take this double-scaling limit, which lead to two independent models [6].

[3] At least up to double trace terms which are introduced by renormalisation. There is a fixed point where the beta functions corresponding to these couplings vanish, discussed in section 3.3.2.

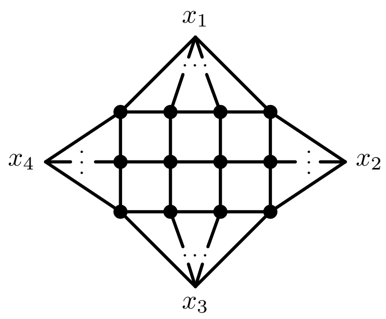
(1.0.3). There are very explicit connections to integrable non-compact conformal spin chains, and using this various correlation functions and scattering amplitudes of the theory can be obtained analytically. Most of the integrability constructions of undeformed $\mathcal{N} = 4$ SYM carry over, and they are typically more tractable mathematically. There is a price to pay for this simplicity, however. The theory (1.0.3) is *non-unitary*, since the interaction $\mathrm{tr}(XZ\bar{X}\bar{Z})$ has no Hermitian conjugate partner. Although non-unitary models do appear in certain contexts in physics, this certainly renders the theory unphysical in the conventional sense. This non-unitarity has very direct implications for the correlation functions of the fishnet theory. In certain operator sectors, the form of the two-point functions of the theory differs from the conventional power-like two-point functions of conformal field theory by the introduction of logarithms. Therefore such non-unitary conformal field theories are referred to as *logarithmic*. This non-unitarity and logarithmicity poses curious challenges for some aspects of integrability in the fishnet theory, most notably for the spectral problem of the theory. In particular the dilatation operator is non-diagonalisable in logarithmic sectors of operators [9]. The sizes and multiplicities of the corresponding Jordan blocks determine the powers to which logarithms can appear in the two-point functions in this sector.

Yangian symmetry has been detected in the fishnet theory in a very explicit setting. The so-called fishnet Feynman graphs

$\tilde{I}_{\alpha\beta} =$ 

(1.0.4)

represent the single contribution to a class of planar correlation functions in the fishnet theory (1.0.3). In Euclidean signature they are annihilated by the conformal Yangian generators $\widehat{\mathrm{J}}^A \in Y[\mathfrak{so}(1,5)]$, which take the form of second order differential operators in the external coordinates $x_1, x_2, \ldots, x_{2(\alpha+\beta)}$. This is a manifestation of the integrability of the fishnet theory directly at the level of its Feynman graphs. Since integrability is best used to *calculate* quantities, it was proposed in [10] to constrain these and similar Yangian invariant integrals from this symmetry, in an approach dubbed the *Yangian bootstrap*. In a four-point limit the fishnet Feynman graphs (1.0.4) reduce to the Basso–Dixon graphs

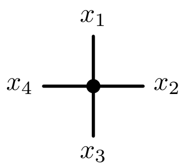
$I_{\alpha\beta} =$ 

. (1.0.5)

The integrals (1.0.5) can be expressed in terms of polylogarithms via the elegant Basso–Dixon formula: $I_{\alpha\beta}$ can be expressed as a determinant of a matrix, whose entries are proportional to the ladder integrals $I_{\alpha 1}$. Although this formula has been proven via matrix model techniques, its simplicity still begs for a derivation based purely on integrability. Notably, the explicit function form of the fishnet integrals (1.0.4) is known only for $\alpha = \beta = 1$. Although the fishnet theory is a toy model, one can hope that a deep understanding of its integrability can shed some light on the corresponding problem in the mother theory $\mathcal{N} = 4$ SYM.

In this thesis we study various aspects of correlation functions in conformal fishnet theory, and the interplay between Euclidean and Minkowski signature with respect to conformal and Yangian symmetry. We discuss three main topics:

Conformal Box Integral in Minkowski Space. One of the simplest (time-ordered) correlators in the fishnet theory is represented by the four-point box integral



$$= \int \frac{\mathrm{d}^4 x_a}{i\pi^2} \frac{1}{(x_{a1}^2 + i\epsilon)(x_{a2}^2 + i\epsilon)(x_{a3}^2 + i\epsilon)(x_{a4}^2 + i\epsilon)}. \tag{1.0.6}$$

In Euclidean space, this integral is represented essentially by the famous Bloch–Wigner function $D(z, \bar{z})$ of the conformal variables z and $\bar{z}$, which are defined via

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}). \tag{1.0.7}$$

As already mentioned, conformal symmetry is incompatible with Minkowskian QFT. In Minkowski space the integral further depends on the kinematic region, which is specified by the signs of the kinematic invariants x_{ij}^2. In this thesis we do four main things:

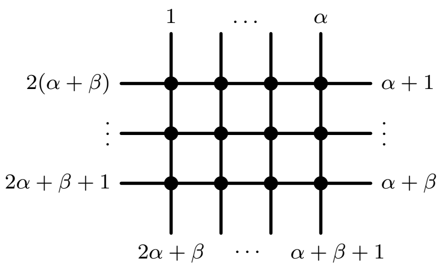
- We classify the set of conformally equivalent configurations of four points in Minkowski space. In Euclidean space any two configurations of four points with the same z and $\bar{z}$ are conformally equivalent. In Minkowski space it further depends on the kinematic region, for which there are $2^6 = 64$ possibilities. We find that the kinematic regions split up into 8 groups of equal size, K_i and $\bar{K}_i$ for $i = 1, \ldots, 4$. We prove that if two configurations of four points with the same z and $\bar{z}$ further have a kinematic region in the same set K_i (or $\bar{K}_i$), then they are conformally equivalent. We do this with the novel notion of *Minkowskian conformal planes.*

- We analyse the expression for the box integral in each kinematic region, and write it explicitly in terms of the conformal variables z and $\bar{z}$. Combining this with our classification of conformally equivalent configurations of four points, we conclude that the box integral can assume up to **four** values on any conformal trajectory. When the functional form of the box integral away from the Euclidean regions differs from the Bloch–Wigner function, it differs by a discontinuity thereof in *one* of the conformal variables, z or $\bar{z}$. Therefore we understand for this very simple example the extent to which global conformal symmetry is broken for the box integral, and how the symmetry manifests locally in each kinematic region.

- We introduce a new method to calculate the conformal box integral directly in Minkowski space, based on the introduction of *double infinity* configurations of four points. These configurations make use of the rich structure of 'infinity' in Minkowski space, where it is a three-dimensional surface rather than a single point. These are 'boundary' configurations, in that their kinematic region can change by a local conformal transformation. In these configurations the mechanism behind the breaking of global conformal symmetry is clear geometrically; the value of the box integral can change if a conformal transformation moves a point through infinity.

- We study the extent to which the box integral is constrained by Yangian symmetry, extending the Yangian bootstrap to Minkowski space. Due to the rich structure of kinematic regions the permutation symmetry of the box integral at the level of conformal invariants is reduced. We study these constraints and fix the functional form of the box integral in all kinematic regions up to twelve undetermined constants. We fix these constants by an explicit analytic continuation of the integral from the Euclidean region.

Yangian Symmetry for Basso–Dixon Correlators. While the many-point fishnet integrals (1.0.4) are Yangian invariant, there are subtleties in taking the coincidence limit to the simpler four-point Basso–Dixon graphs (1.0.5). If we take the four-point limit of the level-one invariance equation

$\widehat{\mathrm{J}}^a$ 

$= 0,$ (1.0.8)

we find that the equation develops a non-zero right hand side. In this thesis we take the first steps to understanding the implications of Yangian symmetry on the Basso–Dixon graphs. In particular:

- We carefully analyse the limit of (1.0.8) to four external points, for the particular case of the level-one momentum generator $\widehat{\mathrm{P}}^\mu$. An important part of our derivation is the interpretation of the equation (1.0.8) as a Ward identity for the corresponding correlators in the fishnet theory. The consistency of our derivation hints at a Yangian symmetry for the fishnet theory at the level of the action, although this has not been shown yet. We find that the right hand side can be written as a linear combination of correlators with a single field $\varPhi$ replaced by its conformal descendant $\mathrm{P}^\mu\varPhi$. We analyse this equation at the level of the conformal function $\phi_{\alpha\beta}(u,v)$, related to the Basso–Dixon integral (1.0.5) via $x_{13}^{2\alpha}x_{24}^{2\beta}I_{\alpha\beta} = \phi_{\alpha\beta}$. We dub the resulting equations the *Yangian Ward identities*:

$$[\mathcal{D}_{uv}^{\alpha\beta} - \mathrm{d}^+\mathcal{A}_{\alpha\beta}]\phi_{\alpha\beta} = 0, \tag{1.0.9}$$

where $\mathcal{D}_{uv}^{\alpha\beta}$ is a certain second order differential operator in the conformal variables u and v, d^+ is an operator which shifts the dimension of an integral from D to

$D+2$, and $\mathcal{A}_{\alpha\beta}$ is a linear combination of operators which modify the propagator powers of an integral. We provide explicit examples of this equation for all examples up to four loops.

- We generalise the equation (1.0.9) to an integrable two-parameter family of square fishnet models $\mathcal{L}_{\text{FN}}^{\omega D}$ [11], where D is the dimension of the model and ω is an anisotropy which is essentially the difference between the horizontal and vertical propagator powers. For the special case $D = 4, \omega = 1$ this model reduces to the ordinary bi-scalar fishnet model (1.0.3). The Ward identites for the two-parameter model take a similar form to (1.0.9):

$$[\mathcal{D}_{uv}^{\alpha\beta,\omega D} - \mathrm{d}^{+}\mathcal{A}_{\alpha\beta}]\phi_{\alpha\beta}^{\omega D} = 0, \tag{1.0.10}$$

i.e. only the differential operators $\mathcal{D}_{uv}^{\alpha\beta}$ are modified.

- We specialise the modified Ward identities (1.0.10) to two dimensions, and find the remarkable property that they separate in the variables $z, \bar{z}$. For the special case of the two-dimensional box integral, we use this separation to describe one of the simplest incarnations of the Yangian bootstrap. We derive a new result for the two-dimensional anisoptropic box integral as a single-valued combination of Legendre functions, which reduce to elliptic K integrals in the isotropic case.

Dilatation Operator of the Dynamical Fishnet Theory. As already mentioned, the dilatation operator in the dynamical fishnet theory is non-diagonalisable in certain operator sectors, although it is still integrable in the sense that it can be derived from an R-matrix which satisfies the Yang–Baxter equation. So far, there has been no Bethe ansatz which fully describes the Jordan block spectrum of any non-diagonalisable sector. In [6] this was shown very explicitly for a particular three scalar sector, where the dilatation operator has been dubbed the *eclectic spin chain*. This is a spin chain of three states: ϕ_1, ϕ_2, and ϕ_3. It was shown that the strong-twist limit of Bethe states from the γ-twisted model all reduce to a trivial *locked* state. We propose a method to fully classify the Jordan blocks of the eclectic spin chain:

- Using combinatorial arguments we introduce a generating function $\mathcal{Z}(q)$, which enumerates the spectrum of a model closely related to the eclectic model, namely the *hypereclectic* model. In the hypereclectic model the ϕ_3 field is distinguished as a non-mover. Our generating function is related to the sizes and the multiplicities of the Jordan blocks via

$$\mathcal{Z}(q) = \sum_{j=1}^{\infty} N_j [j]_q, \tag{1.0.11}$$

where N_j is the number of Jordan blocks of length j, and $[j]_q$ is a q-number defined in (6.3.41).

- We provide explicit formulas for the generating function for various subsectors of operators. For example, for operators of length L with $L - M$ ϕ_2 fields and a single ϕ_3 field we find

$$\mathcal{Z}_{L,M}(q) = L \begin{bmatrix} L-1 \\ M-1 \end{bmatrix}_q \tag{1.0.12}$$

where $\begin{bmatrix} L-1 \\ M-1 \end{bmatrix}_q$ is a *q-binomial coefficient*, which reduces to an ordinary binomial coefficient $\binom{L-1}{M-1}$ as $q \to 1$. For higher numbers of ϕ_3 fields, the generating function can be written as a sum of products of q-binomial coefficients.

- The spectrum of the hypereclectic model was conjectured to coincide with the spectrum of the eclectic model for generic couplings, provided special filling conditions on the numbers of fields are satisfied. We provide further evidence for this conjecture, and sketch a proof for the case of a single ϕ_3 field.

This thesis is structured as follows. In chapter 2 we review the foundational concepts that make up this thesis. In particular, we review Euclidean/Minkowskian correlation functions and their expansion in Feynman diagrams in section 2.1. In section 2.2 we review (logarithmic) conformal field theory, discussing constraints on correlation functions and the distinction between the conformal group on Euclidean and Minkowski space. In section 2.3 we discuss several aspects of Feynman integrals which are relevant to this thesis, and in particular discuss conformal Feynman integrals. In chapter 3 we discuss the (dynamical) fishnet theory and discuss its integrability. In section 3.1 we give an overview of classical and quantum integrability, focusing on the example of the Heisenberg spin chain in the quantum case. In section 3.2 we introduce $\mathcal{N} = 4$ SYM theory and discuss its integrability. In section 3.3 we describe in detail how to recover the dynamical fishnet Lagrangian from that of $\mathcal{N} = 4$ SYM, and discuss the conformality and integrability of the theory. In chapter 4 we discuss various aspects of the conformal box integral in Minkowski space described above. In chapter 5 we derive the Yangian Ward identities for the four-point Basso–Dixon graphs. In chapter 6 we introduce the eclectic spin chain and describe its integrability and solution. Finally, we conclude and give some outlook in chapter 7.

Chapter 2

Correlation Functions in Conformal Field Theory

We begin with a review of the core concepts that underlie this thesis. These are correlation functions, conformal field theory, and Feynman integrals.

2.1 Correlation Functions in Quantum Field Theory

First, we will discuss time-ordered correlation functions of scalar fields in Minkowskian quantum field theory:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle \coloneqq \langle \Omega | \mathbb{T} \phi(x_1) \cdots \phi(x_n) | \Omega \rangle, \tag{2.1.1}$$

where $x_1, \ldots, x_n$ are external points in Minkowski space $\mathbb{R}^{1,3}$, $|\Omega\rangle$ is the vacuum of the theory, and $\mathbb{T}$ is the time-ordering operator. Such objects are naturally defined in the path integral formulation of QFT. They are quite general objects; for example they contain all the information of the scattering amplitudes $\mathcal{A}$ of a theory, which describe interactions between asymptotic on-shell states, via the Lehmann–Symanzik–Zimmermann (LSZ) reduction formula. The structure of scattering amplitudes can be directly probed in collider experiments through the scattering cross section $\mathcal{M} \sim |\mathcal{A}|^2$, and their calculation is currently a huge field of study [12]. Correlation functions are often calculated perturbatively, as a sum over Feynman diagrams. This can be done in either momentum space or position space. In this thesis we will mainly focus on the position space picture, where correlation functions are naturally defined.

There is an important distinction between correlation functions in Euclidean space and those in Minkowski space. Euclidean correlators are more relevant for statistical mechanical systems, for example the Ising model. In Euclidean space there is no notion of time ordering, and there are no light cone singularities (except at coincident points). As such, Euclidean correlation functions are simpler in structure, and are single-valued, permutation invariant functions of the external positions. Quantum field theory is formulated in Minkowski space, however, and the analytic structure of the correlation functions is more intricate. There is a well-defined procedure to analytically continue Euclidean correlators to Minkowski space, however, via the Osterwalder–Schrader reconstruction theorem [13, 14].

In this section we will define correlation functions of scalar operators in Minkowski space, and outline how to calculate them perturbatively as a sum over Feynman diagrams. We will then discuss Wick rotation and the issue of analytically continuing Euclidean correlators to Minkowski space. Much of what we describe here is available in the standard textbooks on quantum field theory [15–18]. Therefore we will be fairly brief in our discussion and only give salient details.

2.1.1 Correlators and Feynman Rules

In the path integral formulation of quantum field theory, a theory of a single scalar field ϕ is defined by the partition function[1]

$$\mathcal{Z}[0] = \int \mathcal{D}\phi \, e^{iS(\phi)}, \tag{2.1.2}$$

where the integration measure $\mathcal{D}\phi$ is a formal sum over all classical configurations of the field $\phi(x) = \phi(t, \boldsymbol{x})$, where $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^3$. $S(\phi)$ is the classical action of the theory

$$S(\phi) = \int \mathrm{d}^4x \, \mathcal{L}, \tag{2.1.3}$$

where $\mathcal{L}$ is the Lagrangian. The sum over configurations (2.1.2) is formal because strictly speaking the sum is highly oscillatory and even divergent at large times t. To regulate the divergences there are a couple possibilities. One would be to begin with a partition function defined in Euclidean space, which is exponentially damped, and analytically continue results obtained in this theory to Minkowski space. We discuss this option in section 2.1.2. Another option is to shift all times by a small imaginary part:

$$t \to t(1 - i\epsilon). \tag{2.1.4}$$

Although (2.1.4) is a formal way to make the sum converge, it is still extremely difficult to show that the sum over configurations exists in a strict mathematical sense. Indeed, there is so far no Minkowskian partition function in four dimensions which has been shown to exist mathematically. For the remainder of the thesis we ignore this subtle question of existence and simply assume it, as all physicists do.

A key fact in the path integral formulation is that the time-ordered correlation functions (2.1.1) can be calculated as path integrals:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) \, e^{iS(\phi)}}{\mathcal{Z}[0]}. \tag{2.1.5}$$

We stress that while the fields $\phi(x_i)$ on the left hand side of (2.1.5) are quantum fields in the Heisenberg picture, the fields $\phi(x_i)$ on the right hand side are purely classical.

There is a convenient way to calculate path integrals, using an auxiliary current $J(x)$. We define the *generating functional*

$$\mathcal{Z}[J] := \int \mathcal{D}\phi \, \exp\left(iS(\phi) + i \int d^4x J(x)\phi(x) \right), \tag{2.1.6}$$

[1] Of course there are more complicated theories than (2.1.2), containing for example gauge fields or fermions. Moreover, the fields could transform in a non-trivial representation of some matrix group, and have extra index structure. In these cases the basic results of this section hold true, with some modifications to the Feynman rules.

which coincides with the partition function (2.1.2) when $J(x)$ is the zero field, which justifies the notation $\mathcal{Z}[0]$. Moreover, the correlation function (2.1.5) can be calculated using functional derivatives of the generating functional:

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{(-i)^n}{\mathcal{Z}[0]}\frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}. \tag{2.1.7}$$

The functional derivatives $\delta/\delta J(x_i)$ satisfy

$$\frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y), \tag{2.1.8}$$

where $\delta^4(x-y)$ is the four-dimensional Dirac delta distribution. From (2.1.7) we see that if we can calculate the generating functional $\mathcal{Z}[J]$ of a theory, we can access all the correlation functions just by taking functional derivatives, and then sending the currents $J(x_i)$ to zero.

Free Theory. It is possible to calculate the generating functional $\mathcal{Z}[J]$ for the free theory, $\mathcal{Z}_0[J]$, using simple (formal) Gaussian integration. The free theory is defined via the Lagrangian

$$\mathcal{L}_0 = -\frac{1}{2}\phi\left(\partial_\mu\partial^\mu + m^2\right)\phi, \tag{2.1.9}$$

where m is the mass of the scalar field. The result is

$$\begin{aligned}\mathcal{Z}_0[J] &= \mathcal{Z}_0[0]\exp\left(-\frac{i}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y J(x)\left(-\partial_\mu\partial^\mu - m^2 + i\epsilon\right)^{-1} J(y)\right) \\ &:= \mathcal{Z}_0[0]\exp\left(-\frac{i}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y J(x)\Delta_F(x-y)J(y)\right)\end{aligned} \tag{2.1.10}$$

where the $i\epsilon$ originates form the time shift (2.1.4), and we denoted the inverse functional by $\Delta_F(x-y)$. We note that the factor $\mathcal{Z}_0[0]$ contains a divergent functional determinant, which cancels when considering correlation functions (2.1.5). The inverse functional $\Delta_F(x-y)$ should satisfy

$$\left(-\partial_\mu\partial^\mu - m^2 + i\epsilon\right)\Delta_F(x-y) = \delta^4(x-y). \tag{2.1.11}$$

This equation is solved by the Feynman propagator

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}, \tag{2.1.12}$$

which is (up to a factor of i) exactly the time-ordered two-point function encountered in canonical quantisation. This propagator can be evaluated in terms of the Bessel function [19][2]

$$\begin{aligned}\Delta_F(x-y) &= -\frac{im}{4\pi^2}\frac{K_1(im\sqrt{-(x-y)^2+i\epsilon})}{\sqrt{-(x-y)^2+i\epsilon}} \\ &= \frac{1}{4\pi^2}\frac{1}{(x-y)^2-i\epsilon} + O(m\log m),\end{aligned} \tag{2.1.13}$$

[2]This equation is valid for spacelike separations $(x-y)^2 < 0$. For timelike separations $(x-y)^2 > 0$ the propagator can be evaluated in terms of Hankel functions of the first kind. These results agree in the massless limit $m \to 0$.

where we note that for massless scalar fields the sign of the infinitesimal regulator changes $i\epsilon \to -i\epsilon$, with respect to (2.1.12).

Using (2.1.7) and (2.1.10) all of the correlators in the free theory can be calculated. For example, the two-point function is

$$\langle \phi(x_1)\phi(x_2)\rangle_0 = (-i)^2 \frac{\delta}{\delta J(x_2)} \frac{\delta}{\delta J(x_1)} \exp\left(-\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y J(x) \varDelta_F(x-y)J(y)\right)\Bigg|_{J=0} \tag{2.1.14}$$

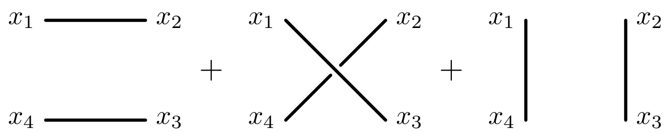
$$= i\varDelta_F(x_1 - x_2),$$

as expected. This equation can be expressed diagrammatically as

$$\langle \phi(x_1)\phi(x_2)\rangle_0 = \quad$$  , (2.1.15)

which is the simplest example of a Feynman diagram. The three-point function, or more generally any correlator with an odd number of points, is easily seen to vanish in the free theory. This is because an odd number of functional derivatives will necessarily leave at least one factor of the current J in front of the exponential, which vanishes upon sending the current to zero. When there is an even number of functional derivatives, it is possible for one functional derivative $\delta/\delta J(x_i)$ to 'pull down' a factor of $J(x_i)$ from the exponential, and another functional derivative $\delta/\delta J(x_j)$ to annihilate this factor, resulting in a propagator $\varDelta_F(x_i - x_j)$. The next non-trivial correlator in the free theory is the four-point function, given by

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_0 = \varDelta_{12}\varDelta_{34} + \varDelta_{13}\varDelta_{24} + \varDelta_{14}\varDelta_{23} \tag{2.1.16}$$

$=$  ,

where we abbreviated $\varDelta_{ij} \coloneqq i\varDelta_F(x_i - x_j)$. We see that calculating the correlation functions in the free theory is essentially a combinatorial problem; one just needs to find all the possible ways to contract the external points in pairs. The $2n$-point function is a sum of $(2n-1)!!$ products of n propagators. This can be summarised with Wick's theorem.

Interacting Theory. In an interacting theory, it is no longer possible to calculate the generating functional $\mathcal{Z}[J]$ analytically. However, it is possible to proceed perturbatively. As an example, take the scalar ϕ^4 theory, defined by the Lagrangian

$$\mathcal{L}_{\phi^4} = \mathcal{L}_0 - \frac{g}{4!}\phi^4, \tag{2.1.17}$$

where $\mathcal{L}_0$ is the free Lagrangian (2.1.9) and g is the coupling constant. For small g the generating functional can be expanded

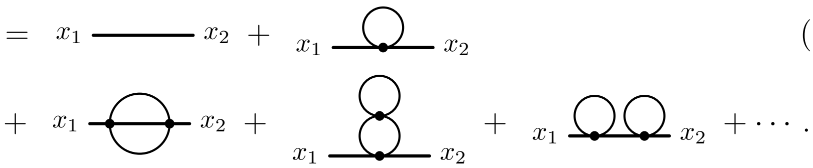
$$\mathcal{Z}[J] = \mathcal{Z}_0[J] + \frac{-ig}{4!}\int \mathrm{d}^4x(-i)^4 \frac{\delta^4 \mathcal{Z}_0[J]}{\delta^4 J(x)} + \left(\frac{-ig}{4!}\right)^2 \frac{1}{2!}\int \mathrm{d}^4x\mathrm{d}^4y(-i)^8 \frac{\delta^8 Z_0[J]}{\delta^4 J(x)\delta^4 J(y)} + \cdots \tag{2.1.18}$$

and the correlation function can be calculated order by order in g using (2.1.7). For example, at order g the correlation function is calculated as

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle_{g^1} = \frac{(-i)^n}{\mathcal{Z}[0]}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\frac{-ig}{4!}\int \mathrm{d}^4x(-i)^4\frac{\delta^4\mathcal{Z}_0[J]}{\delta^4 J(x)}\bigg|_{J=0} \tag{2.1.19}$$
$$= (-i)^{n+5}\frac{\mathcal{Z}_0[0]}{\mathcal{Z}[0]}\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\frac{g}{4!}\int \mathrm{d}^4x\frac{\delta^4}{\delta^4 J(x)}\exp\left(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4yJ(x)\Delta_{xy}J(y)\right)\bigg|_{J=0}$$

which is combinatorially much more intricate to calculate than in the free theory.

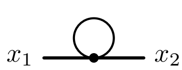
Miraculously, the whole correlation function (2.1.7) can be calculated at each order in g by adding up all connected Feynman diagrams with n external points compatible with the *Feynman rules* at this order in g, and multiplying by appropriate symmetry factors for each diagram. This fact is non-trivial and we simply state it here. The reason is that the $\mathcal{Z}[0]$ factor in the denominator of (2.1.7) can be written as the exponential of the sum of connected vacuum bubbles, i.e. Feynman diagrams with no external points, and the numerator of (2.1.7) factors into a product (exp of connected vacuum bubbles)×(sum of connected diagrams). As an example, the two-point function in ϕ^4 theory is

$\langle\phi(x_1)\phi(x_2)\rangle =$  (2.1.20)

The diagrams in (2.1.20) are associated to functions of the external points x_1, x_2 via the Feynman rules as follows:

- For each internal four-vertex at position x, include a factor $-ig\int \mathrm{d}^4x$.
- For each line joining points x and y, include a factor $D_{xy} = i\Delta_F(x-y)$.
- Include the symmetry factor of the diagram.

The symmetry factor of the diagram accounts for contractions of fields which give rise to the same Feynman diagram. We do not give details how to calculate it; it is explained for example in [15]. As an example of these rules, we have[3]

 $= \dfrac{-ig}{2}\displaystyle\int \mathrm{d}^4xD_{x_1x}D_{xx}D_{xx_2},$ (2.1.21)

where the symmetry factor of this diagram is $\frac{1}{2}$. We note that this integral and the other integrals in (2.1.20) are divergent, and the theory (2.1.17) requires renormalisation for a finite two-point function.

For a general theory, the interaction Lagrangian determines the vertices which appear in the Feynman rules. For example, a scalar interaction $\phi^n/n!$ will produce an n-valent vertex. The propagators are different for particles of different spin, but they still represent functional inverses of the kinetic part of the Lagrangian, as in (2.1.11).

[3]Note that we will typically not label internal points.

Then correlation functions for a given number of external fields can be calculated by drawing all Feynman diagrams with these external fields, compatible with appropriately modified Feynman rules. Correlation functions can also be calculated in momentum space, using the *momentum space Feynman rules*. We will mainly use the (massless) position space rules in this thesis, and so we omit them from the discussion here.

2.1.2 Wick Rotation

The structure of correlation functions in Minkowski space is quite intricate: the $i\epsilon$ prescription, which is used to regulate the path integral (2.1.4), appears in the Feynman rules as a regulation of the propagator $\Delta_F(x-y)$, which turns out to implement causality by time ordering the correlation functions. Since things are much more convenient to calculate in Euclidean space, physicists often do calculations there, with the promise that results can be 'Wick rotated' to Minkowski space. We will discuss this issue a bit; first the problem of mapping a Minkowski space calculation to a Euclidean one at the level of Feynman integrals, and then the problem of mapping a Euclidean calculation to a Minkowski one at the level of the full correlation function. One way to avoid these subtleties is to proceed with a calculation directly in Minkowski space, while keeping careful track of the $i\epsilon$ factors at each stage of the calculation. We do such a calculation explicitly in section 4.3.2.

Minkowski to Euclidean: Rotating Contours. In Minkowski space Lorentz invariant squares can be calculated[4]

$$x = (t, \boldsymbol{x}) \quad \longrightarrow \quad x^2 = t^2 - \boldsymbol{x}\cdot\boldsymbol{x}, \tag{2.1.22}$$

where $\boldsymbol{x}\cdot\boldsymbol{x}$ is the usual Euclidean dot product. By making the change of variables $t \to -i\tau$ we see that

$$x \to (-i\tau, \boldsymbol{x}) \quad \longrightarrow \quad x^2 = -\tau^2 - \boldsymbol{x}\cdot\boldsymbol{x}, \tag{2.1.23}$$

which is now a square of definite Euclidean signature. This is an analytic continuation of the time variable, which has to be justified in any given calculation, for example if x is an integration variable. If it can be justified, we are left with a calculation in Euclidean space, which is usually more convenient than the corresponding calculation in Minkowski space. In practice this is most commonly seen at the level of Feynman integrals. A common example is a one-loop integral of the form

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - \Delta + i\epsilon}, \tag{2.1.24}$$

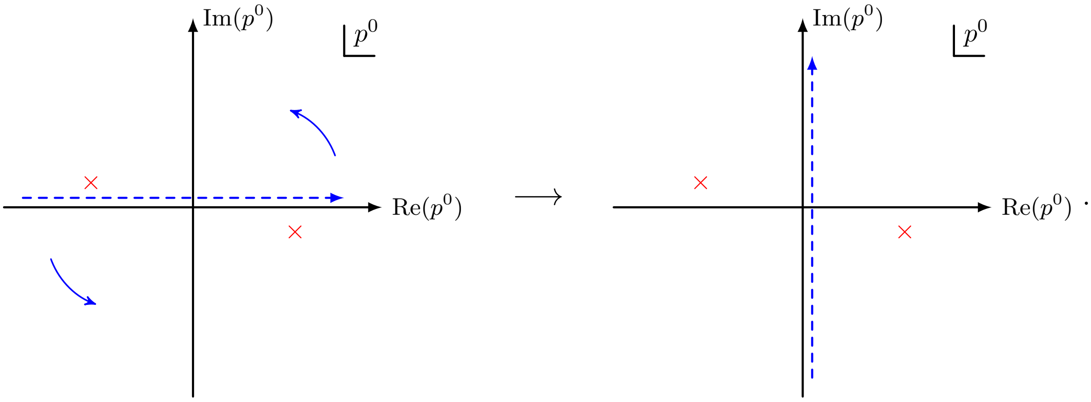
where $\Delta > 0$ is an effective mass. The denominator of (2.1.24) can be factorised

$$p^2 - \Delta + i\epsilon = (p^0 - \omega + i\epsilon')(p^0 + \omega - i\epsilon'), \tag{2.1.25}$$

where $\omega := \sqrt{\boldsymbol{p}\cdot\boldsymbol{p} + \Delta}$ and $\epsilon' = \epsilon/2\omega$ is a new infinitesimal regulator. Due to this singularity structure, one can safely rotate the integration contour *anticlockwise* onto

[4] Throughout this thesis we use the mostly minus $(+\,-\,-\,-)$ spacetime signature.

the imaginary p^0 axis, without crossing any poles of the integrand:



One can achieve this exactly by the substitution $p^0 \to ip^0$. In the integral (2.1.24) the Lorentz square becomes $p^2 \to -(p^0)^2 - \boldsymbol{p} \cdot \boldsymbol{p} := -p_E^2$ and the measure transforms $\mathrm{d}^4p \to i\mathrm{d}^4p_E$, where p_E is a Euclidean momentum. Overall we have

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{1}{p^2 - \Delta + i\epsilon} = -i \int \frac{\mathrm{d}^4p_E}{(2\pi)^4} \frac{1}{p_E^2 + \Delta}, \tag{2.1.26}$$

where the regulator ϵ' can now safely be ignored because there are no poles on the integration contour. (2.1.26) can be straightforwardly integrated using hyperspherical Euclidean coordinates. There is a tradeoff: one could also compute the integral (2.1.24) by using an appropriate semicircular contour and the residue theorem for the p^0 integral, and then integrating over the remaining Euclidean degrees of freedom in spherical coordinates. This is slightly more technical than computing the integral (2.1.26), however one must be careful to justify the Wick rotation properly. For example, one would get a wrong result for the integral if they made the substitution $p^0 \to -ip^0$, which rotates the contour anticlockwise through the poles. For higher numbers of integrations the singularity structure becomes more intricate. More complicated examples of Wick rotations are discussed in [20].

Euclidean to Minkowski: Osterwalder–Schrader Theorem. There is the natural question: if we compute some correlation function in a Euclidean quantum field theory, is it possible to analytically continue this result to the corresponding time-ordered correlator in Minkowski space? The answer is yes for sufficiently well-behaved Euclidean correlators, and it can be done using the Osterwalder–Schrader reconstruction theorem [13, 14]. A modern summary of these results is given in [21]. Loosely, it states that well-behaved Euclidean correlators can be analytically continued to Lorentzian correlators which satisfy the Wightman axioms. We will not go into detail on what 'well-behaved' means or what the Wightman axioms are, and simply sketch the results.

First, we should describe how correlation functions are calculated in a Euclidean quantum field theory. Under a formal Wick rotation $x^0 = -ix_E^0$, the action of ϕ^4 theory transforms

$$\begin{aligned} S &= \int \mathrm{d}^4x \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 \right) \\ &\longrightarrow -i \int \mathrm{d}^4x_E \left(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 \right) := iS_E. \end{aligned} \tag{2.1.27}$$

Therefore, under this Wick rotation the Minkowskian partition function transforms

$$\mathcal{Z} = \int \mathcal{D}\phi\, e^{iS} \to \int \mathcal{D}\phi\, e^{-S_E} := \mathcal{Z}_E. \tag{2.1.28}$$

We see that $\mathcal{Z}_E$ is the partition function encountered in statistical mechanics, which explains the colloquial statement that QFT and statistical mechanics are related by Wick rotation. Since $\mathcal{Z}_E$ is exponentially damped, it has a better chance of being mathematically well-defined, although it may still require some kind of regulation to be finite.

One can define a Euclidean quantum field theory beginning with the partition function $\mathcal{Z}_E$. In this case there is no notion of time ordering, and correlation functions should be permutation invariant functions of the positions x_i. For scalar field theories the correlation functions can be defined

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle := \frac{\int \mathcal{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, e^{-S_E(\phi)}}{\int \mathcal{D}\phi\, e^{-S_E(\phi)}}. \tag{2.1.29}$$

These correlation functions can be calculated in complete analogy to the Minkowski case, as a sum over connected Feynman diagrams with n external legs. The only difference is that we should of course integrate over Euclidean space in the Feynman rules, and Euclidean propagators should be used:

$$\mathit{\Delta}_{F,E}(x-y) := \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2+m^2}, \tag{2.1.30}$$

which similarly to (2.1.13) can be evaluated in terms of a Bessel K function, and in the massless limit $m \to 0$ behaves as

$$\mathit{\Delta}_{F,E}(x-y) = \frac{1}{4\pi(x-y)^2} + O(m\log m). \tag{2.1.31}$$

Consider a Euclidean correlator

$$G(x_1,\ldots,x_n) := \langle \phi(x_1)\cdots\phi(x_n)\rangle, \qquad x_i = (\tau_i, x_i^1, x_i^2, x_i^3) \in \mathbb{R}^4, \tag{2.1.32}$$

where we suggestively denote the first component of the coordinates x_i as the *Euclidean time* $\tau_i \in \mathbb{R}$. This function is analytic away from coincident points $x_i = x_j$. If we analytically continue τ_i away from the real axis, the function G has an intricate branch cut structure. To obtain a Minkowski correlator, the hope would be to safely rotate the Euclidean times to Lorentzian times $\tau_i \to it_i$, so that

$$x_i = (\tau_i, x_i^1, x_i^2, x_i^3) \to (it_i, x_i^1, x_i^2, x_i^3) := y_i \tag{2.1.33}$$

Given the branch cut structure of G, there may be ambiguities in how to perform this analytic continuation, for example one could pass through a branch cut, picking up a discontinuity, or around it. In fact, the choice of which analytic continuation to make is equivalent to the choice of time-ordering of the Minkowskian correlator [21].[5]

[5]While we have only mentioned time-ordered correlators so far, other time orderings are possible. Out-of-time-order correlators have received much interest recently, see for example [22].

This branch cut structure depends sensitively on the causal order of the points in Minkowski space, i.e. the signs of $(y_i - y_j)^2 := y_{ij}^2$. Notably, if all the Minkowskian points are spacelike separated $y_{ij}^2 < 0$ for $i \neq j$, then there are no branch cuts and all time-orderings of the Lorentzian correlator are equal to the Euclidean correlator. This is the so-called *Euclidean region* of kinematics. If some of these variables are timelike, for example $y_{12}^2 > 0$, then there is the possibility that analytically continuing the Euclidean correlator to the time-ordered Minkowskian correlator yields some extra discontinuities obtained from continuing through a branch cut. We will see examples of this explicitly in section 4.4.3.

The Osterwalder–Schrader theorem, as well as guaranteeing a safe analytic continuation of Euclidean correlators to Minkowski correlators, also gives an explicit algorithm to compute this continuation. In particular, we have

$$\langle \phi(t_1, \boldsymbol{x}_1) \cdots \phi(t_n, \boldsymbol{x}_n) \rangle_{\mathbb{R}^{1,3}} = \lim_{\epsilon_j \to 0} G(\,(it_1 + \epsilon_1, \boldsymbol{x}_1), \ldots, (it_n + \epsilon_n, \boldsymbol{x}_n)\,) \tag{2.1.34}$$

where the limit is taken with $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > 0$. Depending on the order of the t_i, this order of the ϵ_j encodes the way the τ_i are continued through the branch cuts of the function $G(x_1, \ldots, x_n)$. If $t_1 > t_2 > t_3 > t_4$ then the correlator is time-ordered.

In practice, the limit (2.1.34) may be tricky to calculate. In particular it depends sensitively on the causal order of the points in Minkowski space. It can be useful to calculate the correlator directly in Minkowski space, rather than computing it in Euclidean space and performing the analytic continuation. The direct computation in Minkowski space can also serve as a check that the analytic continuation from Euclidean space was performed correctly. In chapter 4 we will look at a particular four-point correlator, which is represented by a single Feynman diagram, from many different perspectives. This will allow us to investigate the relationship between the Euclidean correlator and the Minkowski correlator in detail.

2.2 (Logarithmic) Conformal Field Theory

An important class of quantum field theories are those which are invariant under the conformal group, which is an extension of the Poincaré group. Classical conformal symmetry is present in some of the most famous equations in physics, for example the massless Yang–Mills equations of motion in four dimensions:

$$D^\mu F^a_{\mu\nu} = 0, \tag{2.2.1}$$

where $D^\mu = \partial^\mu + g\,[A^\mu, \;]$ is a gauge covariant derivative. (2.2.1) includes as a special case Maxwell's equations in the absence of sources, which was noticed to have conformal symmetry in the early 1900s [23, 24].

A necessary condition for conformal symmetry is invariance under both Poincaré transformations $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$ and *dilatations*

$$x^\mu \to c\,x^\mu, \qquad c > 0, \tag{2.2.2}$$

which is a uniform scaling of space(time). In many cases[6] invariance of a physical theory

[6]Conformal invariance has been shown to follow from scale invariance in two dimensions under a few technical assumptions, including unitarity [25]. Under similar assumptions it is expected to be the case in four dimensions. A scale invariant theory without conformal symmetry has been proposed by Cardy and Riva [26], however this model is non-unitary.

under these two transformations implies invariance under so-called *special conformal transformations*

$$x^\mu \to \frac{x^\mu - x^2 b^\mu}{1 - 2x \cdot b + x^2 b^2}, \tag{2.2.3}$$

where b^μ is a fixed vector, with dimensions of inverse length.

Notably, invariance under dilatations requires a theory to have no length scale, and hence only involve massless particles. At the quantum level, renormalisation in general introduces an energy dependence to the coupling constants, which necessarily breaks conformal invariance. Crucially, conformal invariance can be restored at fixed points g^* where the beta functions corresponding to these running couplings vanish:

$$\beta_i(g^*) = 0. \tag{2.2.4}$$

In this case the couplings do not depend on the energy scale. Therefore it is the machinery of conformal field theory which describes QFTs at fixed points (2.2.4), which should already be more than enough motivation for their study. They also play a prominent role in the AdS/CFT correspondence, one of the most fruitful frameworks for understanding theories of quantum gravity at strong coupling [27–29].

Conformal symmetry places very stringent constraints on the correlation functions in these theories. In many cases this allows for their calculation non-perturbatively. In particular, the two-point functions of scalar operators (after an appropriate rescaling) are completely determined by the operators' scaling dimensions $\varDelta$. Similarly, the spacetime dependence of the three-point functions is fixed by these scaling dimensions, up to an overall normalisation λ. Higher-point functions can be reduced to sums of three-point functions via the operator product expansion, and thus all correlation functions can be characterised by the set of *CFT data* $\{\{\varDelta, \mathcal{R}\}, \lambda\}$. Here $\mathcal{R}$ is the representation of the Lorentz group under which the operator with dimension $\varDelta$ transforms, and λ represents the aforementioned three-point normalisations. In recent years there has been enormous progress in the *conformal bootstrap*, a method to constrain the conformal data of a CFT using the operator product expansion as well as consistency conditions coming from crossing symmetry [30]. This has been most famously applied in the case of the 3D Ising model [31].

There is an important distinction between the conformal group $\mathrm{Conf}(\mathbb{R}^D)$ in Euclidean space and the conformal group $\mathrm{Conf}(\mathbb{R}^{1,D-1})$ in Minkowski space.[7] The Euclidean conformal group is more relevant to statistical mechanics and the Minkowskian conformal group is more relevant for QFT, although locally they have a similar structure. In many cases we will consider the Euclidean conformal group, even in the case of QFT, with the understanding that the obtained results can be analytically continued to Minkowski space after an appropriate Wick rotation, see section 2.1.2. For $D = 4$ these conformal groups are 15-dimensional and we have $\mathrm{Conf}(\mathbb{R}^4) \simeq SO^+(1,5)$ and $\mathrm{Conf}(\mathbb{R}^{1,3}) \simeq SO^+(2,4)/\mathbb{Z}_2$. For $D = 2$ however we have $\mathrm{Conf}(\mathbb{R}^2) \simeq SO^+(1,3)$, whereas for Minkowski space the conformal group is infinite-dimensional $\mathrm{Conf}(\mathbb{R}^{1,1}) \simeq \mathrm{Diff}_+(\mathbb{S}^1) \times \mathrm{Diff}_+(\mathbb{S}^1)$ and is essentially a direct product of orientation preserving diffeomorphisms on the circle. In this thesis we mainly consider dimensions $D \geq 2$, with an exception in chapter 5.

[7] We will always use the symbol Conf to denote the component of the group of all conformal transformations on a given space which is connected to the identity.

One of the holy principles in quantum field theory is *unitarity*, which roughly states that sum of the probabilities of all outcomes in any experiment is unity. In conformal field theory unitarity imposes lower bounds on the scaling dimensions of operators in the theory, for example $\varDelta \geq \frac{D-2}{2}$ for scalar operators. Notably, this bound is saturated for scalar primaries.[8] There are certain conformal theories where the assumption of unitarity can be given up, which leads to many new features. Notably, the representation of the conformal group no longer needs to be decomposable[9], which can lead to a non-diagonalisable dilatation operator in certain operator sectors. In such cases the spacetime dependence of the two-point functions can contain logarithms of the spacetime separation $x_1 - x_2$. Therefore such theories are often referred to as *logarithmic conformal field theories*. Logarithmic conformal field theories play an important role in two dimensions [32]. There, due to their direct connection with two-dimensional statistical mechanics models, they are of great physical interest. Important examples include models of self-avoiding walks, polymers, and percolation. In higher dimensions, logarithmic CFTs have been much less studied, although some reviews exist [33]. The fishnet theory, one of the main subjects of this thesis, is non-unitary and contains sectors of operators which are logarithmic.

A vast literature exists on the subject of conformal field theory. [34] gives a comprehensive review of mathematical aspects of the conformal group. [35] is the standard textbook on conformal field theory, but focuses mainly on two dimensions. There are several reviews on the conformal group in higher dimensions, which also explain the exciting recent developments of the conformal bootstrap [36–39].

In this section we give a review of the basic concepts of conformal field theory which are relevant for this thesis. We define conformal transformations, both globally and locally, and describe their action on scalar fields. We discuss the constraints of global conformal transformations on correlation functions. Locally these take the form of the conformal Ward identities, which are explicit differential constraints on the correlation functions in terms of the generators of the conformal group. We discuss geometrically the constraints of conformal symmetry on four-point functions in the conformal plane picture. We describe the necessary modifications to the Ward identities for logarithmic CFTs, and the explicit form their solution takes. Finally, we describe the subtle issue of the conformal compactification of Minkowski space.

2.2.1 Conformal Symmetry

In this section we describe the basics of conformal symmetry. We consider a spacetime in D dimensions with metric $g_{\mu\nu}$ and a transformation of the coordinates $x \to x'(x)$.[10] Such a transformation is *conformal* if it acts as a Weyl transformation on the metric

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = \Lambda(x) g_{\mu\nu}(x), \tag{2.2.5}$$

where $\Lambda(x)$ is the *conformal factor*. (2.2.5) implies that the *angle* between curves crossing at any point is preserved. Indeed if a^μ and b^μ are tangent vectors to some

[8]These are scalar fields $\phi(x)$ which are annihilated by the special conformal generator K^μ at $x = 0$.

[9]A representation is decomposable if it can be brought into block diagonal form by a change of basis.

[10]We consider only Euclidean space $g = \mathrm{diag}(+ + \cdots +)$ and Minkowski space $g = \mathrm{diag}(+ - \cdots -)$.

curves λ_a and λ_b intersecting at x, then the angle ϕ between them is

$$\cos(\phi) = \frac{g_{\mu\nu}(x)a^\mu b^\nu}{\sqrt{g_{\alpha\beta}(x)a^\alpha a^\beta g_{\gamma\delta}(x)b^\gamma b^\delta}}, \tag{2.2.6}$$

which is invariant under $x \to x'(x)$ when (2.2.5) is satisfied. Conformal transformations form a group under composition, and indeed successive conformal transformations with conformal factors Λ_1, Λ_2 correspond to a conformal transformation with conformal factor $\Lambda_1\Lambda_2$. If the conformal factor $\Lambda(x) = 1$ for all x, then the conformal transformation is a Poincaré transformation, i.e. a combination of a translation and a (Lorentz) rotation.

In order to determine the set of possible conformal transformations, it is useful to first work locally. Considering infinitesimal coordinate transformations

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{2.2.7}$$

then (2.2.5) takes the form

$$g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) + O(\epsilon^2). \tag{2.2.8}$$

Therefore the vector $\epsilon^\mu(x)$ determines a local conformal transformation, provided

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = f(x)g_{\mu\nu} \tag{2.2.9}$$

for all x. In this case we have

$$g'_{\mu\nu} = (1 - f(x))g_{\mu\nu}. \tag{2.2.10}$$

Clearly (2.2.9) is satisfied from constant vectors $\epsilon^\mu(x) = a^\mu$, and $\epsilon^\mu(x) = m^{\mu\nu}x_\nu$ whenever $m^{\mu\nu} = -m^{\nu\mu}$. These cases correspond to infinitesimal translations and rotations respectively, whereby $f(x) = 0$. If $\epsilon^\mu(x) = d^{\mu\nu}x_\nu$ with $d^{\mu\nu}$ arbitrary, then it is straightforward to show that the symmetric part of $d^{\mu\nu}$ must be proportional to the metric if (2.2.9) is to be satisfied. $\epsilon^\mu(x) = \frac{1}{2}\alpha g^{\mu\nu}x_\nu = \frac{1}{2}\alpha x^\mu$ corresponds to infinitesimal dilatations (2.2.2), and in this case $f(x) = \alpha$. Finally, and perhaps least intuitively, it is possible for (2.2.9) to be satisfied when $\epsilon^\mu(x) = c^{\mu\nu\rho}x_\nu x_\rho$ is quadratic in x. One can check that

$$\epsilon^\mu(x) = c^{\mu\nu\rho}x_\nu x_\rho = 2(x\cdot b)x^\mu - x^2 b^\mu \tag{2.2.11}$$

is the most general quadratic solution to (2.2.9). This transformation is the infinitesimal version of the special conformal transformation (2.2.3), and in this case $f(x) = 4x \cdot b$. There are no further solutions to the equation (2.2.9), and in summary we have the infinitesimal conformal transformations

$$\text{Translation}: \quad x^\mu \to x^\mu + \epsilon^\mu. \tag{2.2.12}$$

$$\text{Rotation}: \quad x^\mu \to x^\mu + m^{\mu\nu}x_\nu, \qquad m^{\mu\nu} = -m^{\nu\mu}. \tag{2.2.13}$$

$$\text{Dilatation}: \quad x^\mu \to (1 + \tfrac{1}{2}\alpha)x^\mu. \tag{2.2.14}$$

$$\text{SCT}: \quad x^\mu \to x^\mu + 2(x\cdot b)x^\mu - x^2 b^\mu. \tag{2.2.15}$$

These transformations are parametrised by constant vectors ϵ^μ and b^μ, an antisymmetric matrix $m^{\mu\nu}$, and a positive real number α. Therefore the conformal group has

$D + D + D(D-1)/2 + 1 = (D+1)(D+2)/2$ parameters. The corresponding finite transformations are given by

$$\begin{aligned}
&\text{Translation :} && x^\mu \to x^\mu + a^\mu. && (2.2.16)\\
&\text{Rotation :} && x^\mu \to \Lambda^\mu{}_\nu x^\nu, \qquad \Lambda^T g \Lambda = 1. && (2.2.17)\\
&\text{Dilatation :} && x^\mu \to c\, x^\mu. && (2.2.18)\\
&\text{SCT :} && x^\mu \to \frac{x^\mu - x^2 b^\mu}{1 - 2x\cdot b + x^2 b^2}. && (2.2.19)
\end{aligned}$$

Note that (2.2.19) already appeared at the beginning of the section (2.2.3). These are just the conformal transformations which are connected to the identity. A notable transformation which is not connected to the identity is the conformal *inversion*

$$\mathcal{I} : x^\mu \to x'^\mu(x) = \frac{x^\mu}{x^2}. \tag{2.2.20}$$

Under (2.2.20) the metric transforms

$$g_{\mu\nu} \to \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = \frac{1}{x^4}\left(\delta^\alpha{}_\mu - 2\frac{x^\alpha x_\mu}{x^2}\right)\left(\delta^\beta{}_\nu - 2\frac{x^\beta x_\nu}{x^2}\right) g_{\alpha\beta} = \frac{1}{x^4} g_{\mu\nu}, \tag{2.2.21}$$

showing that the inversion is indeed a conformal transformation, with conformal factor $\Lambda_{\text{inv}}(x) = 1/x^4$. Notably, a special conformal transformation can be expressed as an inversion, followed by a translation, followed by another inversion:

$$\frac{\frac{x^\mu}{x^2} - b^\mu}{(\frac{x^\mu}{x^2} - b^\mu)^2} = \frac{x^\mu - x^2 b^\mu}{1 - 2x\cdot b + x^2 b^2}. \tag{2.2.22}$$

From this we see explicitly that the parameter b^μ has units of inverse length, since it represents a translation in the inverted space. Using (2.2.21) and (2.2.22) one can show that a special conformal transformation is conformal with $\Lambda_{\text{SCT}}(x) = 1/(1 - 2x\cdot b + x^2 b^2)$. There can be further conformal transformations not connected to the identity depending on the spacetime signature. In Minkowski space the space and time reflections $\mathcal{P}$ and $\mathcal{T}$ are examples.

2.2.2 Conformal Algebra

Since we are concerned with field theories invariant under the conformal group, we need to describe the representations under which the fields transform. The precise representation depends on the properties of the particle which the field represents. For example, a scalar field $\phi(x)$ transforms trivially under the Lorentz group

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu, \qquad \phi(x) \to \phi(\Lambda^{-1}x), \tag{2.2.23}$$

where the transformation is considered in the active sense. On the other hand, a spin-$\frac{1}{2}$ particle can be represented by a Dirac spinor $\psi^\alpha(x)$, and under the same Lorentz transformation transforms

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x), \tag{2.2.24}$$

where $S[\Lambda]^{\alpha}{}_{\beta}$ constitute the spin-$\frac{1}{2}$ representation of the Lorentz group, built from the Clifford algebra. In this thesis we will mostly consider scalar representations of the conformal group. However, scalar fields still transform with a non-trivial scaling dimension Δ.

At the local level, there is the notion of a *generator* of a transformation of fields. Suppose we have an infinitesimal transformation

$$x^\mu \to x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}, \qquad \phi(x) \to \phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}(\phi(x))}{\delta \omega_a}, \tag{2.2.25}$$

where ω_a are infinitesimal parameters and the function $\mathcal{F}$ parametrises the transformation of the field $\phi(x) \to \phi'(x') \coloneqq \mathcal{F}(\phi(x))$. The *generators* J_a corresponding to this transformation are defined by

$$\delta_\omega \phi(x) = \phi'(x) - \phi(x) \coloneqq -i\omega_a \mathrm{J}_a \phi(x), \tag{2.2.26}$$

where J_a can be expressed in terms of the variations as

$$i\mathrm{J}_a\phi = \frac{\delta x^\mu}{\delta \omega_a}\partial_\mu \phi - \frac{\delta F}{\delta \omega_a}. \tag{2.2.27}$$

For the conformal group we denote by P^μ the generators of translations, $\mathrm{L}^{\mu\nu}$ the generators of rotations, D the generator of dilatations, and K^μ the generators of special conformal transformations. A scalar field transforms under scaling as follows:

$$x \to cx, \qquad \phi(x) \to \phi'(cx) = c^{-\Delta}\phi(x), \tag{2.2.28}$$

where Δ is the *scaling dimension*, or simply dimension, of ϕ. A scalar field with dimension Δ transforms under special conformal transformations as

$$x^\mu \to x'^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2x\cdot b + x^2 b^2}, \qquad \phi(x) \to \phi'(x') = (1 - 2x\cdot b + x^2b^2)^{\Delta}\phi(x). \tag{2.2.29}$$

The transformation rule (2.2.29) ensures that the field $\phi(x)$ does not feel the effect of the variation of the scale factor $\Lambda_{\mathrm{SCT}}(x) = 1/(1 - 2x\cdot b + x^2b^2)$. Using (2.2.27), (2.2.28), and (2.2.29), the generators of the conformal group for scalar fields with dimension Δ can be calculated. For example, for an infinitesimal dilatation we have $x^\mu \to (1+\omega)x^\mu + O(\omega^2), \phi(x) \to (1+\omega)^{-\delta}\phi(x) = (1 - \Delta\omega)\phi(x) + O(\omega^2)$, so

$$\frac{\delta x^\mu}{\delta \omega} = x^\mu, \qquad \frac{\delta \mathcal{F}}{\delta \omega} = -\Delta\phi(x). \tag{2.2.30}$$

Using (2.2.27) we calculate the dilatation generator to be $\mathrm{D} = -i(x\cdot\partial + \Delta)$. Similarly, the rest of the generators of the conformal group in this representation can be calculated, and constitute a set of first order differential operators in x:

$$\mathrm{P}^\mu = -i\partial^\mu, \tag{2.2.31}$$
$$\mathrm{L}^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu), \tag{2.2.32}$$
$$\mathrm{D} = -i(x\cdot\partial + \Delta), \tag{2.2.33}$$
$$\mathrm{K}^\mu = -i(2x^\mu(x\cdot\partial + \Delta) - x^2\partial^\mu). \tag{2.2.34}$$

(2.2.31)-(2.2.34) generate the infinitesimal transformations (2.2.12)-(2.2.15) acting on scalar fields $\phi(x)$. These generators are closed under commutation and make up the *conformal algebra* with scaling dimension $\mathit{\Delta}$. For example, we can calculate $[\mathrm{D}, \mathrm{P}^\mu]\phi$ as

$$[\mathrm{D}, \mathrm{P}^\mu]\phi = i^2(x_\nu\partial^\nu + \mathit{\Delta})\partial^\mu\phi - i^2\partial^\mu(x_\nu\partial^\nu\phi + \mathit{\Delta}\phi) = \partial^\mu\phi = i\mathrm{P}^\mu\phi, \tag{2.2.35}$$

and so we see $[\mathrm{D}, \mathrm{P}^\mu] = i\mathrm{P}^\mu$. The full algebra reads

$$[\mathrm{D}, \mathrm{P}^\mu] = i\mathrm{P}^\mu, \tag{2.2.36}$$

$$[\mathrm{D}, \mathrm{K}^\mu] = -i\mathrm{K}^\mu, \tag{2.2.37}$$

$$[\mathrm{K}^\mu, \mathrm{P}^\nu] = 2i(g^{\mu\nu}\mathrm{D} - \mathrm{L}^{\mu\nu}), \tag{2.2.38}$$

$$[\mathrm{K}^\rho, \mathrm{L}^{\mu\nu}] = i(g^{\rho\mu}\mathrm{K}^\nu - g^{\rho\nu}\mathrm{K}^\mu), \tag{2.2.39}$$

$$[\mathrm{P}^\rho, \mathrm{L}^{\mu\nu}] = i(g^{\rho\mu}\mathrm{P}^\nu - g^{\rho\nu}\mathrm{P}^\mu), \tag{2.2.40}$$

$$[\mathrm{L}^{\mu\nu}, \mathrm{L}^{\rho\sigma}] = i(g^{\nu\rho}\mathrm{L}^{\mu\sigma} + g^{\mu\sigma}\mathrm{L}^{\nu\rho} - g^{\mu\rho}\mathrm{L}_{\nu\sigma} - g^{\nu\sigma}\mathrm{L}^{\mu\rho}). \tag{2.2.41}$$

This algebra can be recast into a more recognisable form via the definitions

$$\mathrm{L}^{-1,\mu} := \frac{1}{2}(\mathrm{P}^\mu - \mathrm{K}^\mu), \qquad \mathrm{L}^{0,\mu} := \frac{1}{2}(\mathrm{P}^\mu + \mathrm{K}^\mu), \qquad \mathrm{L}^{-1,0} := \mathrm{D}. \tag{2.2.42}$$

Then the commutation relations (2.2.36)$-$(2.2.41) can be summarised

$$[\mathrm{L}^{ab}, \mathrm{L}^{cd}] = i(\eta^{bc}\mathrm{L}^{ad} + \eta^{ad}\mathrm{L}^{bc} - \eta^{ac}\mathrm{L}^{bd} - \eta^{bd}\mathrm{L}^{ac}) \text{ for } a, b, c, d = -1, 0, 1, \ldots, D, \tag{2.2.43}$$

where $\eta_{ab} = \mathrm{diag}(+ \mp - - \cdots -)$ for Euclidean and Minkowski space respectively. (2.2.43) is recognised to be the defining relations for the special orthogonal algebra $\mathfrak{so}(1, D+1)$ ($\mathfrak{so}(2, D)$) in Euclidean (Minkowski) space. Therefore conformal transformations, which act rather non-trivially on our D-dimensional spacetime, act simply as rotations on a larger $(D+2)$-dimensional space. This observation forms the basis of the *embedding space formalism* [36,40], which is a very neat arena to do computations in a CFT.

2.2.3 Constraints on Correlation Functions

As already mentioned, correlation functions are vastly constrained by conformal symmetry. In this section we first discuss these constraints for *Euclidean* CFTs, and describe the functions which solve them. The Minkowski case is more subtle, and indeed time-ordered correlation functions are not invariant under global conformal transformations, although they are invariant locally. We discuss this subtlety at the end of this section.

We first consider a Euclidean quantum field theory of some scalar fields ϕ_i, defined by an action $S[\phi_i]$, which is invariant under conformal transformations $x \to x'$, $\phi_i(x) \to \mathcal{F}_i(\phi_i(x))$. Recall that the precise form of $\mathcal{F}_i$ for a given transformation depends on the scaling dimension $\mathit{\Delta}_i$ of the field ϕ_i. Moreover, we assume that the path integral measure $\prod_i \mathcal{D}\phi_i$ is invariant under these transformations. The n-point correlation function of operators ϕ_i can be expressed in terms of the path integral as

$$\langle\phi_{j_1}(x_1)\cdots\phi_{j_n}(x_n)\rangle = \frac{1}{\mathcal{Z}}\int\left(\prod_i \mathcal{D}\phi_i\right)\ \phi_{j_1}(x_1)\cdots\phi_{j_n}(x_n)e^{-S[\phi_i]}, \tag{2.2.44}$$

analogously to (2.1.29). $\mathcal{Z}$ is the partition function of the theory:

$$\mathcal{Z} = \int \left(\prod_i \mathcal{D}\phi_i \right) e^{-S[\phi_i]}. \tag{2.2.45}$$

Consider a conformal transformation of the external points $x_i \to x_i'$. Using conformal symmetry, one can show that the transformed correlation function is related to the original:

$$\langle \phi_{j_1}(x_1') \dots \phi_{j_n}(x_n') \rangle = \langle \mathcal{F}_{j_1}(\phi_{j_1}(x_1)) \dots \mathcal{F}_{j_n}(\phi_{j_n}(x_n)) \rangle. \tag{2.2.46}$$

As an illustration, we consider the constraints that (2.2.46) imposes on the two-point function $\langle \phi_1(x_1)\phi_2(x_2) \rangle$. Translation and Lorentz invariance imply that

$$\langle \phi_1(x_1 + a)\phi_2(x_2 + a) \rangle = \langle \phi_1(x_1)\phi_2(x_2) \rangle, \qquad \forall a \in \mathbb{R}^D, \tag{2.2.47}$$

$$\langle \phi_1(\Lambda^\mu{}_\nu x_1^\nu)\phi_2(\Lambda^\mu{}_\nu x_2^\nu) \rangle = \langle \phi_1(x_1)\phi_2(x_2) \rangle, \qquad \forall \Lambda \in SO(D). \tag{2.2.48}$$

(2.2.47) and (2.2.48) imply that $\langle \phi_1(x_1)\phi_2(x_2) \rangle$ can only be a function of the separation

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = f(x_{12}^2), \tag{2.2.49}$$

where $x_{12}^\mu \coloneqq x_1^\mu - x_2^\mu$. Scale invariance leads to the constraint

$$c^{\Delta_1+\Delta_2} f(c^2 x_{12}^2) = f(c^2 x_{12}^2), \tag{2.2.50}$$

for all $c > 0$, where Δ_i is the scaling dimension of ϕ_i. This implies that

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \frac{d_{12}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}}, \tag{2.2.51}$$

where d_{12} is a constant. We now compute the constraints imposed by invariance under special conformal transformations. Under an SCT the squared distance between points x_{12}^2 transforms as

$$x_{12}^2 \to \frac{x_{12}^2}{b(x_1)b(x_2)}, \tag{2.2.52}$$

where $b(x) = 1/\Lambda_{\text{SCT}}(x) = 1 - 2x \cdot b + x^2 b^2$. (2.2.52) can be proven by proving the analogous relation for inversions $x^\mu \to x^\mu/x^2$ with $b_{\text{inv}}(x) = 1/\Lambda_{\text{inv}} = x^2$

$$(x_1' - x_2')^2 = \left(\frac{x_1^\mu}{x_1^2} - \frac{x_2^\mu}{x_2^2} \right)^2 = \frac{(x_1 - x_2)^2}{x_1^2 x_2^2} = \frac{(x_1 - x_2)^2}{b_{\text{inv}}(x_1) b_{\text{inv}}(x_2)}, \tag{2.2.53}$$

and then using the fact that SCT = inversion $\circ$ translation $\circ$ inversion. Thus, using (2.2.29), (2.2.46) for SCT invariance reads

$$(b(x_1)b(x_2))^{\frac{\Delta_1+\Delta_2}{2}} \frac{d_{12}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}} = b(x_1)^{\Delta_1} b(x_2)^{\Delta_2} \frac{d_{12}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}} \tag{2.2.54}$$

for all x_1, x_2, which is only satisfied for $\Delta_1 = \Delta_2$, provided $d_{12} \neq 0$. For general scalars ϕ_i, ϕ_j we can normalise our fields so that $d_{ij} = \delta_{ij}$, and the two-point function is diagonal

$$\langle \phi_i(x_1)\phi_j(x_2) \rangle = \frac{\delta_{ij}}{x_{12}^{2\Delta_i}}. \tag{2.2.55}$$

A similar procedure can be followed for three-point functions. The final result is

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_1+\Delta_3-\Delta_2}}, \tag{2.2.56}$$

Importantly, the coefficients λ are undetermined after the normalisation of the two-point functions has been fixed. For higher points, conformal symmetry is no longer sufficient to constrain the spacetime dependence of the correlation functions. Indeed, with four points it is possible to form two independent conformally invariant *cross ratios*

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{2.2.57}$$

These are clearly invariant under translations and rotations, since they only depend on the squared differences x_{ij}^2. Under dilatations $x_i \to c\, x_i$ we have

$$u \to \frac{c^2 x_{12}^2 c^2 x_{34}^2}{c^2 x_{13}^2 c^2 x_{24}^2} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u, \tag{2.2.58}$$

and similarly $v \to v$. Under special conformal transformations we use (2.2.52) to see that

$$u \to \frac{\frac{x_{12}^2 x_{34}^2}{b(x_1)b(x_2)b(x_3)b(x_4)}}{\frac{x_{13}^2 x_{24}^2}{b(x_1)b(x_3)b(x_2)b(x_4)}} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u, \tag{2.2.59}$$

and similarly $v \to v$. Therefore four-point functions in a conformal quantum field theory can a priori have an arbitrary functional dependence on the cross ratios u and v. We can express this as

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \left(\frac{x_{34}^2}{x_{14}^2}\right)^{\frac{\Delta_1-\Delta_3}{2}} \left(\frac{x_{14}^2}{x_{12}^2}\right)^{\frac{\Delta_2-\Delta_4}{2}} \frac{g(u,v)}{x_{13}^{\Delta_1+\Delta_2}x_{24}^{\Delta_3+\Delta_4}}. \tag{2.2.60}$$

In this thesis we will often consider primary scalars in four dimensions, whereby (2.2.60) reduces to

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{g(u,v)}{x_{13}^2 x_{24}^2}. \tag{2.2.61}$$

We stress that all expressions in this chapter apply to conformal correlation functions of scalar operators. For spinning operators there can be different tensor structures which contribute to the correlation function, see for example [36]. We will often consider another pair of conformal variables $z, \bar{z}$, which are functions of the cross ratios (2.2.57). They are defined by the equations

$$u = z\bar{z} \qquad v = (1-z)(1-\bar{z}). \tag{2.2.62}$$

Many correlators take a simpler form in terms of the variables $z, \bar{z}$. In Euclidean space z is a complex number, and $\bar{z}$ is its complex conjugate. This fact is non-trivial and is proven in section 2.2.4. For $N \geq 4$ it is possible to form $N(N-3)/2$ independent cross ratios. For example, at five points there are five cross ratios $u_1, u_2, \ldots, u_5$. This is dependent on the spatial dimension being large enough, namely $D \geq N-2$. If $D < N-2$ then there are $DN - (D+1)(D+2)/2$ independent cross ratios, due to extra non-trivial algebraic relations between the cross ratios. For example, if $D = 1$ then there are always $N-3$ independent cross ratios for $N \geq 4$.

Conformal Ward Identities. Above we have discussed the constraints global conformal symmetry imposes on correlation functions, summarised in equation (2.2.46). Often it is useful to consider this equation for infinitesimal transformations (2.2.25), which implies constraints on the correlation functions in terms of the generators of the conformal algebra $\mathrm{J}^a = \{\mathrm{P}^\mu, \mathrm{L}^{\mu\nu}, \mathrm{D}, \mathrm{K}^\mu\}$. To see this, one should insert (2.2.25) into (2.2.46), expand to first order in the infinitesimal parameters ω_a, and then use the definition of the generators (2.2.27). This leads to the *conformal Ward identities*[11][12]

$$\langle[\mathrm{J}_1^a\phi(x_1)]\phi(x_2)\dots\phi(x_n)\rangle + \langle\phi(x_1)[\mathrm{J}_2^a\phi(x_2)]\dots\rangle + \dots + \langle\phi(x_1)\phi(x_2)\dots[\mathrm{J}_1^a\phi(x_n)]\rangle = 0, \tag{2.2.63}$$

where J_i^a denotes the action of the conformal generator on the i^{th} coordinate, and is henceforth referred to as a *generator density.* In (2.2.31)−(2.2.34) we saw that these generators densities are simply first order differential operators in these coordinates. The total conformal generator can be defined as a sum over generator densities

$$\mathrm{J}^a := \sum_{i=1}^{n} \mathrm{J}_i^a. \tag{2.2.64}$$

Then the conformal Ward identities can be summarised

$$\mathrm{J}^a\langle\phi(x_1)\dots\phi(x_n)\rangle = 0, \qquad \mathrm{J}^a \in \{\mathrm{P}^\mu, \mathrm{L}^{\mu\nu}, \mathrm{D}, \mathrm{K}^\mu\}, \tag{2.2.65}$$

where we have further assumed that the action of the generators commutes with the path integral. For example, invariance of the two-point function under infinitesimal translations leads to the Ward identity

$$\mathrm{P}_\mu\langle\phi(x_1)\phi(x_2)\rangle = -i\left(\frac{\partial}{\partial x_1^\mu} + \frac{\partial}{\partial x_2^\mu}\right)\langle\phi(x_1)\phi(x_2)\rangle = 0, \tag{2.2.66}$$

which implies that the two-point function can only depend on the difference x_{12}^μ. Indeed, further invariance of the two-point function under $\mathrm{L}^{\mu\nu}, \mathrm{D}, \mathrm{K}^\mu$ fixes in a new way the form (2.2.55) for suitably normalised scalars ϕ_i, ϕ_j.

Conformal Correlators in Minkowski Space. In Minkowski space time-ordered correlators are defined by the formula (2.1.5), where the imaginary time shift (2.1.4) is required if $\mathcal{Z}[0]$ is to have a chance of converging. The $i\epsilon$ time shift further imposes causality by time-ordering the fields, and appears at the level of Feynman diagrams in the Feynman propagator (2.1.12).

Notably, the conformal group on Minkowski space is known to be incompatible with causality. In particular, finite special conformal transformations can change the signs of the kinematic invariants x_{ij}^2. We recall (2.2.52), which implies that under SCTs the kinematics transform as

$$x_{ij}^2 \to \frac{x_{ij}^2}{b(x_i)b(x_j)}, \tag{2.2.67}$$

[11]For notational convenience we state the identities for identical fields ϕ. The story is the for fields with differing scaling dimensions, one should just be careful to use the correct representation of the conformal group on each field.

[12]This result requires the assumption that a certain surface integral at infinity vanishes, see [35].

where $1 - 2x \cdot b + x^2 b^2$. In Minkowski space $b(x_i)$ can be positive, negative, or zero. In particular, if $b(x_i)$ and $b(x_j)$ have different signs then x_{ij}^2 changes sign under this SCT. Geometrically this corresponds to a point 'crossing infinity', as we will describe in section 2.2.6, and see an explicit example of in section 4.3.2.

The question of formulating a globally conformally invariant QFT in Minkowski space was considered already in the seventies [41, 42], and is reviewed nicely in [43]. We state the result for $D = 4$. If the correlators in a Euclidean QFT are invariant under the Euclidean conformal group $\mathrm{Conf}(\mathbb{R}^4)$, then the time-ordered correlation functions (more generally the Wightman functions) of the corresponding theory in Minkowski space can be analytically continued to an infinite sheeted covering of the conformal compactification of Minkowski space, $\tilde{\mathbb{R}}_c^{1,3}$, in a way that is consistent with causality. These analytically continued correlation functions are invariant under the universal covering group of the conformal group on (compactified) Minkowski space $\mathrm{Conf}^*(\mathbb{R}^{1,3})$. It is only possible to define globally conformally invariant Minkowskian QFTs in this more abstract setting, due to the issues with causality in regular Minkowski space.

However, if a Euclidean theory is conformally invariant, the corresponding theory in Minkowski space will still be *locally* conformally invariant. This is because local special conformal transformations are still consistent with causality, and cannot change the signs of the kinematic invariants. This implies that in each *kinematic region*, defined by an assignment of signs to the x_{ij}^2, the correlation function depends only on the cross ratios u_i, and moreover satisfies the conformal Ward identities (2.2.65). For example, a four-point correlation function in a locally conformally invariant QFT depends only on the conformal variables z and $\bar{z}$, defined in (2.2.62), in each kinematic region. In chapter 4 we will see this explicitly in the case of the box integral, which represents a four-point correlation function in the fishnet theory. In different kinematic regions the correlation function differs by discontinuities of the correlation function in Euclidean space. This reflects the fact that the Minkowski correlation function can be obtained from the Euclidean one by analytic continuation, cf. (2.1.34).

2.2.4 Euclidean Conformal Plane Configuration

Here we introduce the notion of the *conformal plane configuration* for conformally invariant four-point functions in Euclidean space. We argued above that the only non-trivial spacetime dependence of such four-point functions is through the conformal cross ratios u, v, or equivalently the conformal variables $z, \bar{z}$. Here we will see this fact from a geometric perspective. This discussion will also set the stage for the Minkowskian conformal plane configurations, a new result of this thesis, discussed in section 4.2.2.

Consider a four-point function $f(x_1, x_2, x_3, x_4)$ which is invariant under conformal transformations:

$$f(x_1, x_2, x_3, x_4) = f(Ax_1, Ax_2, Ax_3, Ax_4), \tag{2.2.68}$$

for any conformal transformation $A : \mathbb{R}^D \to \mathbb{R}^D$. Because of this invariance, we can make arbitrary conformal transformations to the points $x_1, \ldots, x_4$, and not change the value of f. We can use this fact to put the points into a canonical position. Firstly, we translate all points by $-x_4$ and then make a conformal inversion $\mathcal{I}$ (see (2.2.20)).

This maps the point x_4 to ∞[13], and the other points to $y_i \coloneqq \mathcal{I}(x_i - x_4), i = 1, 2, 3.$[14] We then make a translation by $-y_1$, so that the configuration becomes

$$\{x_1, x_2, x_3, x_4\} \to \{0, y_2 - y_1, y_3 - y_1, \infty\}. \tag{2.2.69}$$

The points 0 and ∞ are invariant under $SO(D)$ rotations. The point $y_3 - y_1$ has some magnitude $|y_3 - y_1| = \rho > 0$. Considering an orthogonal coordinate system of $\mathbb{R}^D$, it is always possible to find a rotation $R \in SO(D)$ such that $R(y_3 - y_1) = (\rho, 0, \ldots, 0)$. We can then make a dilatation by $1/\rho$ to map this point to $(1, 0, \ldots, 0) \coloneqq e_1$. Under these transformations $y_2 - y_1$ will be mapped to some arbitrary point w_2. We note that $0, \infty$, and e_1 are all invariant under the $SO(D-1)$ subgroup of $SO(D)$, which acts on the space spanned by the unit vectors $e_2, \ldots, e_D$. We can thus find a rotation $R' \in SO(D-1)$ which maps w_2 into the two-dimensional plane spanned by e_1 and e_2:

$$w_2 \to R'w_2 = (w_2^1, w_2^2, 0, \ldots, 0) \coloneqq (r\cos\phi, r\sin\phi, 0, \ldots, 0). \tag{2.2.70}$$

Thus conformal transformations can be used to map any four points to a two-dimensional subspace of $\mathbb{R}^D$

$$\{0, (r\cos\phi, r\sin\phi, 0, \ldots, 0), e_1, \infty\}, \tag{2.2.71}$$

where r and ϕ represent the only remaining degrees of freedom of the configuration. Since the conformal cross ratios are invariant under conformal transformations, we can just as well compute them for the configuration (2.2.71):

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = r^2, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = r^2 - 2r\cos\phi + 1. \tag{2.2.72}$$

Computing the conformal invariants z and $\bar{z}$, we see the interesting fact

$$z = re^{i\phi}, \qquad \bar{z} = re^{-i\phi}, \tag{2.2.73}$$

from which we see that z is generically not real[15], and that $\bar{z}$ is the complex conjugate of z. We also notice that the non-zero coordinates of the second point in (2.2.71) are $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. We can therefore alternatively represent this point as a complex number in the $e_1 + ie_2$ plane. Note that since z and $\bar{z}$ are calculated as the roots of a quadratic equation depending on u, v, we could alternatively take $z = re^{-i\phi}, \bar{z} = re^{i\phi}$. We can use this freedom to place z in the closed upper half-plane $\bar{\mathbb{H}}$, i.e. $\mathrm{Im}(z) \geq 0$.

We will call the complex $e_1 + ie_2$ plane the *Euclidean conformal plane*, and the configuration (2.2.71) the *Euclidean conformal plane configuration*, visualised in figure 2.1. Any four points with conformal invariant $z \in \bar{\mathbb{H}} \setminus \{0, 1\}$ can be mapped to the configuration (2.2.71). This shows that any two configurations of four points with the same conformal invariant z can be mapped into each other by conformal transformations. Indeed, if A_1 and A_2 are the respective conformal transformations which map these configurations to the conformal plane configuration (2.2.71), then the transformation

[13] Strictly speaking ∞ only makes sense in the conformal compactification, we discuss this in section 2.2.6.

[14] We assume the points x_i are initially distinct, so that y_i are distinct.

[15] If it so happens that $\phi = 0$, then $z = \bar{z} = r \in \mathbb{R}$. This is not, however, a generic situation.

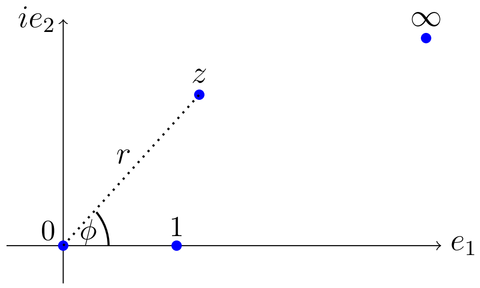


Figure 2.1: Euclidean conformal plane configuration.

$A_2^{-1}A_1$ maps configuration 1 into configuration 2. Mathematically this fact can be formulated as follows. Define the set of four-point configurations

$$V_z = \left\{ \{x_1, x_2, x_3, x_4\} \;\middle|\; x_i \in \mathbb{R}^D, \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = zz^*, \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-z^*) \right\}, \quad (2.2.74)$$

together with the action of the conformal group on V_z

$$\{x_1, x_2, x_3, x_4\} \to \{Ax_1, Ax_2, Ax_3, Ax_4\}, \qquad A \in \mathrm{Conf}(\mathbb{R}^D). \quad (2.2.75)$$

Then the above argument with the conformal planes implies that $\mathrm{Conf}(\mathbb{R}^D)$ acts transitively[16] on the set V_z. In section 4.2.2 we will investigate whether $\mathrm{Conf}(\mathbb{R}^{1,D-1})$ acts transitively on the analogously defined sets in Minkowski space. This will be used to describe the invariance properties of the conformal box integral under conformal transformations in Minkowski space.

2.2.5 Logarithmic CFT

In the previous sections we silently assumed unitary representations of the conformal group, where dilatations act diagonally on our operators

$$\phi(x) \to c^{-\Delta}\phi(x). \quad (2.2.76)$$

For non-unitary theories it is possible for the dilatation operator to be non-diagonalisable in certain operator sectors, leading to *logarithmic multiplets* of fields. In a logarithmic multiplet $\{\phi_a\}_{a=1,\ldots,r}$ of rank r, dilatations act as

$$\phi_a(x) \to c^{-\Delta_{ab}}\phi_b(x), \qquad a = 1, \ldots, r, \quad (2.2.77)$$

where Δ_{ab} is a non-diagonalisable matrix, with all eigenvalues equal to the scaling dimension Δ of the multiplet. In general we can change the operator basis such that Δ_{ab} is in *Jordan normal form*

$$\Delta_{ab} = \begin{pmatrix} \Delta & 1 & & & 0 \\ & \Delta & 1 & & \\ & & \Delta & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \Delta \end{pmatrix}. \quad (2.2.78)$$

[16] Recall that a group G acts transitively on a set X if for all $x, y \in X$, there exists $g \in G$ such that $gx = y$.

In these logarithmic sectors, the dilatation and conformal generators D and K^μ now mix different fields in the multiplet:

$$\mathrm{D}\phi_a(x) = -i(\Delta_a{}^b + \delta_a^b(x\cdot\partial))\phi_b(x), \tag{2.2.79}$$

$$\mathrm{K}^\mu\phi_a(x) = -i(2x^\mu(\delta_a^b x\cdot\partial + \Delta_a{}^b)\phi_b(x) - x^2\partial^\mu\phi_a(x)). \tag{2.2.80}$$

The deviation of the generators (2.2.79) and (2.2.80) in the logarithmic multiplet from the original generators (2.2.33) and (2.2.34) has striking implications on the solution of the conformal Ward identities. We consider the two-point function of operators in the same logarithmic multiplet of rank two

$$\langle\phi_a(x_1)\phi_b(x_2)\rangle \coloneqq f_{ab}(x_{12}^2), \qquad a,b=1,2. \tag{2.2.81}$$

For this multiplet the dilatation generator acts explicitly as

$$i\mathrm{D}\phi_1(x) = [\Delta + (x\cdot\partial)]\phi_1(x) + \phi_2(x), \tag{2.2.82}$$

$$i\mathrm{D}\phi_2(x) = [\Delta + (x\cdot\partial)]\phi_2(x). \tag{2.2.83}$$

The conformal Ward identities corresponding to translations and rotations are unchanged, and ensure that the correlator (2.2.81) depends on the squared separation x_{12}^2. The Ward identities corresponding to dilatations and special conformal transformations lead to the four constraints

$$0 = (x_1\cdot\partial_1 + x_2\cdot\partial_2 + 2\Delta)f_{21} + f_{22}, \tag{2.2.84}$$

$$0 = (x_1\cdot\partial_1 + x_2\cdot\partial_2 + 2\Delta)f_{11} + f_{12} + f_{21}, \tag{2.2.85}$$

$$0 = (2(x_1^\mu + x_2^\mu)\Delta + 2x_1^\mu(x_1\cdot\partial_1) + 2x_2^\mu(x_2\cdot\partial_2) - x_1^2\partial_1 - x_2^2\partial_2)f_{21} + 2x_2^\mu f_{22}, \tag{2.2.86}$$

$$0 = (2(x_1^\mu + x_2^\mu)\Delta + 2x_1^\mu(x_1\cdot\partial_1) + 2x_2^\mu(x_2\cdot\partial_2) - x_1^2\partial_1 - x_2^2\partial_2)f_{11} + 2x_2^\mu f_{12} + 2x_2^\mu f_{21}. \tag{2.2.87}$$

The solution of the system (2.2.84)-(2.2.87) is fairly technical, and is described for general logarithmic multiplets in [33], see also [44]. The result for the two-point function of the multiplet is

$$f_{ab} = \frac{c}{x_{12}^{2\Delta}}\begin{pmatrix} -\log(\mu^2 x_{12}^2) & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.2.88}$$

Notably, two point functions in a logarithmic multiplet can contain logarithmic dependance on the spacetime separation, with a scale μ.[17] For higher rank logaritmic multiplets, the two point functions can be brought into a canonical form. For example, the corresponding rank three multiplet can be written

$$f_{ab}^{(3)} = \frac{c}{x_{12}^{2\Delta}}\begin{pmatrix} \frac{1}{2}\log^2(\mu^2 x_{12}^2) & -\log(\mu^2 x_{12}^2) & 1 \\ -\log(\mu^2 x_{12}^2) & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2.2.89}$$

where we notice that for higher rank multiplets, higher powers of logarithms can appear. In [33], many more aspects of logarithmic CFTs are described, including the constraints on higher-point functions, the logarithmic OPE, and conformal blocks. However in this thesis we will only be concerned with two-point functions, in particular the dilatation operator in logarithmic sectors of the dynamical fishnet theory.

[17] The appearance of such a scale in a CFT seems troubling at first, but it can be removed by a further change of operator basis, see [33].

2.2.6 Conformal Compactification: Euclidean vs. Minkowski

There is a slight subtlety in defining the action of the conformal group on Euclidean space $\mathbb{R}^4$. Since the inversion operation $\mathcal{I} : x^\mu \to x^\mu/x^2$ is not well-defined on the origin $x = 0$, we should add a point '∞' to $\mathbb{R}^4$ which corresponds to the image of the origin under $\mathcal{I}$. The conformal group is well-defined and acts transitively on the *conformal compactification* $\mathbb{R}^4_c := \mathbb{R}^4 \cup \{\infty\}$.

The situation is more subtle in Minkowski space $\mathbb{R}^{1,3}$. Defining the light cone of the origin

$$L_0 := \{x \in \mathbb{R}^{1,3} \mid x^2 = 0\} \tag{2.2.90}$$

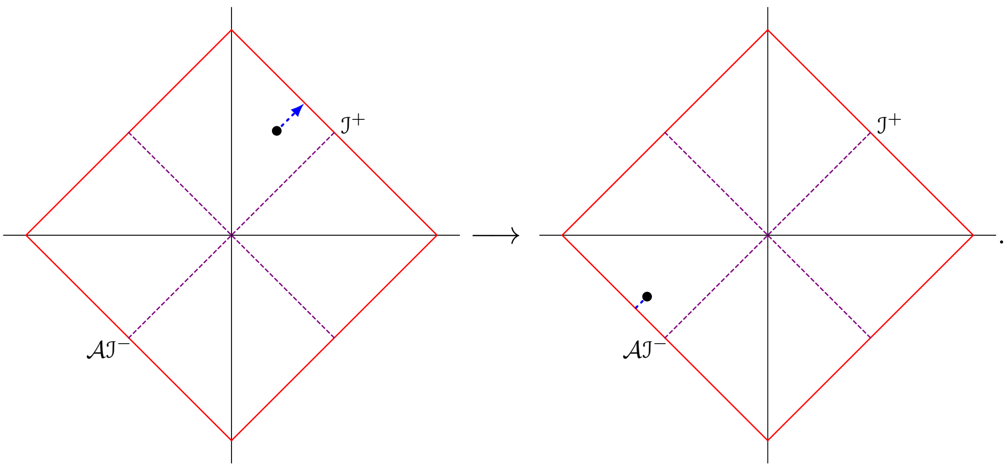
we note in this case that $\mathcal{I}$ not well-defined on the entire space L_0. We thus are interested in a compactification $\mathbb{R}^{1,3} \to \mathbb{R}^{1,3}_c$ such that the conformal group is well-defined and acts transitively and continuously on $\mathbb{R}^{1,3}_c$. The points 'at infinity' will be identifiable[18] with the image $\mathcal{I}(L_0)$, and so will constitute a 3−dimensional surface.

The most common way of compactifying Minkowski space is the *Penrose compactification* [45] and replaces $\mathbb{R}^{1,3} \to \mathbb{R}^{1,3}_p := \mathbb{R}^{1,3} \cup \partial\mathbb{R}^{1,3}_p$, where $\partial\mathbb{R}^{1,3}_p := \iota^+ \cup \iota^- \cup \iota^0 \cup \mathcal{J}^+ \cup \mathcal{J}^-$. ι^0 is called *spatial* infinity, and in spherical coordinates (t, r, θ, ϕ) corresponds to $r \to \infty$. $\iota^\pm$ are called *future* and *past* infinity, and correspond to $t \to \pm\infty$. Both ι^0 and $\iota^\pm$ correspond to individual points on $\partial\mathbb{R}^{1,3}_p$. $\mathcal{J}^\pm$ are called *future null* and *past null* infinity and correspond to $t \pm r \to \infty$ with $t \mp r$ constant. $\mathcal{J}^\pm$ are 3-dimensional submanifolds of $\partial\mathbb{R}^{1,3}_p$, parametrised by the coordinates $(t \mp r, \theta, \phi)$. Useful coordinates for describing $\mathbb{R}^{1,3}_p$ are

$$\chi_+ := \arctan(t + r), \qquad \chi_- := \arctan(t - r), \tag{2.2.91}$$

and $\tau := \chi_- + \chi_+$, $\rho := \chi_+ - \chi_-$. The range of the τ, ρ coordinates in $\mathbb{R}^{1,3}_p$ is $\tau \in [-\pi, \pi]$, $\rho \in [0, \pi]$. We visualise $\mathbb{R}^{1,3}_p$ in (τ, ρ) coordinates using an extended Penrose diagram (figure 2.2).

We want to consider conformal transformations on the Penrose compactification $\mathbb{R}^{1,3}_p$. There are a few issues at $\partial\mathbb{R}^{1,3}_p$. Using infinitesimal special conformal transformations it is possible to move between $\iota^\pm$ and ι^0, and also between $\mathcal{J}^\pm(\chi_\mp, \theta, \phi)$ and $\mathcal{J}^\mp(\chi_\pm, \mathcal{A}(\theta, \phi))$, where $\mathcal{A}$ is the antipodal map on S^2. An example of such a conformal trajectory is



(2.2.92)

[18] Up to an extra 'null sphere at infinity' S^*, see (2.2.106) and figure 2.3.

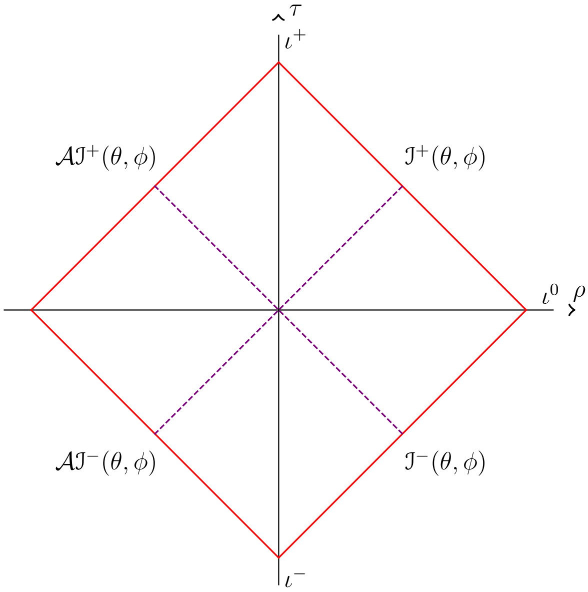


Figure 2.2: Extended Penrose diagram of Minkowski space. Every two-sphere $S^2(\tau, \rho > 0)$ is represented by two points, one on the left and one on the right, which are exchanged by the antipodal map $\mathcal{A}$. The dotted lines represent the light cone of the origin [46].

(2.2.92) is not a continuous operation, and indeed it represents a point crossing infinity. The fact that it is possible is the basic reason the conformal group does not preserve causality, as discussed in section 2.2.3. In order for the conformal group to act continuously on compactified Minkowski space, we must make the identifications[19]

$$\iota^+ = \iota^- = \iota^0 := \iota, \qquad \mathcal{I}^+ = \mathcal{A}\mathcal{I}^- := \mathcal{I}. \tag{2.2.93}$$

We define $\partial\mathbb{R}^{1,3}_c := \iota \cup \mathcal{I}$ as $\partial\mathbb{R}^{1,3}_p$ subject to the identifications (2.2.93). $\partial\mathbb{R}^{1,3}_c$ can be visualised as a 'pinched torus' or croissant (figure 2.3), and can be parametrised in terms of $\chi \in (-\pi/2, \pi/2]$ and a $2-$sphere angle (θ, ϕ) for $\chi \neq 0$ [47]. We define the conformal compactification $\mathbb{R}^{1,3}_c := \mathbb{R}^{1,3} \cup \partial\mathbb{R}^{1,3}_c$.

$\mathbb{R}^{1,3}_c$ can be identified with the group $U(2)$, on which the conformal group acts naturally by $SU(2,2)/\mathbb{Z}_4$. To see this, we first note the bijection $H : \mathbb{R}^{1,3} \to H_{2\times 2}$ between Minkowski space and the space of 2×2 Hermitian matrices[20]

$$H : x = (x^0, x^1, x^2, x^3) \longrightarrow H(x) := \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{2.2.94}$$

In what follows we will label the Hermitian matrices corresponding to points x, y, z in Minkowski space by X, Y, Z. The inverse map is given by

$$H^{-1} : X \to x^\mu = \frac{1}{2}\mathrm{tr}(X\sigma^\mu), \tag{2.2.95}$$

[19]The identification $\mathcal{I}^+ = \mathcal{A}\mathcal{I}^-$ resembles boundary conditions on fields in asymptotic symmetries [46].

[20]This is the same identification used in the spinor helicity representation of on-shell momenta in scattering amplitudes. However, here we do not consider light-like points in general, and so the matrices are not rank 1.

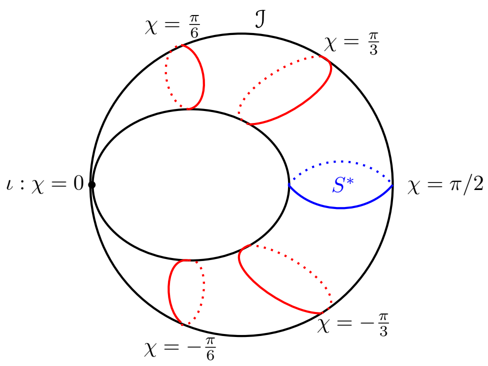


Figure 2.3: Conformal infinity in terms of ι and $\mathcal{I}$. Surfaces of fixed χ are 2−spheres. The parameter $\chi \in (-\pi/2, \pi/2]$ is chosen as $\chi = \pi/2 - \chi_+ \mod \pi$, so $\chi = 0$ corresponds to ι.

where $\sigma^\mu = (\mathbb{I}_2, \sigma^1, \sigma^2, \sigma^3)$ is the covariant vector of Pauli matrices. In spherical coordinates $x = (t, r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, the Hermitian matrix corresponding to x is

$$X = \begin{pmatrix} t + r\cos\theta & re^{-i\phi}\sin\theta \\ re^{i\phi}\sin\theta & t - r\cos\theta \end{pmatrix} = \Omega(\theta,\phi)^\dagger M(t,r)\Omega(\theta,\phi), \tag{2.2.96}$$

where

$$\Omega(\theta,\phi) := \begin{pmatrix} \cos(\theta/2) & e^{-i\phi}\sin(\theta/2) \\ -e^{i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \in SU(2), \qquad M(t,r) := \begin{pmatrix} t+r & 0 \\ 0 & t-r \end{pmatrix}. \tag{2.2.97}$$

$M(t,r)$ represents the event in Minkowski space at a time t, and spatially on the north pole ($\theta = 0$) of a sphere of radius r. The adjoint action of $\Omega(\theta,\phi)$ rotates this point from the north pole to a general angle (θ,ϕ) on the sphere. In terms of Hermitian matrices we can calculate spacetime intervals via determinants

$$(x-y)^2 = \det(X - Y). \tag{2.2.98}$$

To compactify, we use the injective Cayley map[21] $J : H_{2\times 2} \to U(2)$ defined by

$$J : X \longrightarrow J(X) := \frac{i\mathbb{I}_2 - X}{i\mathbb{I}_2 + X}, \tag{2.2.99}$$

which is well-defined because $\det(i\mathbb{I}_2 + X) \neq 0$ and $[i\mathbb{I}_2 - X, (i\mathbb{I}_2 + X)^{-1}] = 0$ for all $X \in H_{2\times 2}$. The map (2.2.99) is not surjective. The inverse is given

$$J^{-1} : U \to i\frac{\mathbb{I}_2 - U}{\mathbb{I}_2 + U}, \tag{2.2.100}$$

which is not well-defined for those $U \in U(2)$ which satisfy $\det(\mathbb{I}_2 + U) = 0$. $\mathrm{Im}(J)$ can be identified with Minkowski space $\mathbb{R}^{1,3}$ and $U_\infty := U(2)\backslash\mathrm{Im}(J) = \{U \in U(2) \,|\, \det(\mathbb{I}_2 + U) = 0\}$ can be identified with the conformal boundary $\partial\mathbb{R}^{1,3}_c$ defined above. If x_1 and

[21]This is analogous to the one-dimensional Cayley map $\mathbb{R} \to U(1)$, $x \to \frac{i-x}{i+x}$.

x_2 are points in compactified Minkowski space represented by unitary matrices U_1 and U_2, their spacetime interval can be calculated

$$(x_1 - x_2)^2 = -4\frac{\det(U_1 - U_2)}{\det(\mathbb{I}_2 + U_1)\det(\mathbb{I}_2 + U_2)}. \tag{2.2.101}$$

We can define the spacetime interval between arbitrary unitary matrices U_1, U_2 using the formula (2.2.101), noting that the expression is of course infinite if either U_1 or U_2 are on U_∞ so that the interval is not well-defined. To define a finite interval between all unitary matrices an unphysical metric $\langle U_1, U_2\rangle := \det(U_1 - U_2)$ must be used.

Conformal transformations can be implemented on $\mathbb{R}^{1,3}_c \simeq U(2)$ by $SU(2,2)$ fractional linear transformations [47, 48]:

$$U \longrightarrow \mathfrak{C}_G U := (AU + B)(CU + D)^{-1}, \qquad G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,2), \tag{2.2.102}$$

$$SU(2,2) = \{G \in SL(4,\mathbb{C}) \mid G^\dagger K G = K\}, \qquad K = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}. \tag{2.2.103}$$

$\mathfrak{C}$ is the fourfold covering homomorphism $\mathfrak{C} : SU(2,2) \to \mathrm{Conf}(\mathbb{R}^{1,3})$. $\mathfrak{C}$ acts trivially on the centre $Z(SU(2,2)) = \{\pm\mathbb{I}_4, \pm i\mathbb{I}_4\} \simeq \mathbb{Z}_4$ and $\mathrm{Conf}(\mathbb{R}^{1,3}) \simeq SU(2,2)/\mathbb{Z}_4$. For example, translations $x \to x + b$ can be implemented by the $SU(2,2)$ matrices

$$T(b) = \begin{pmatrix} -\mathbb{I}_2 & i\mathbb{I}_2 \\ \mathbb{I}_2 & i\mathbb{I}_2 \end{pmatrix} \begin{pmatrix} \mathbb{I}_2 & B \\ 0 & \mathbb{I}_2 \end{pmatrix} \begin{pmatrix} -\mathbb{I}_2 & i\mathbb{I}_2 \\ \mathbb{I}_2 & i\mathbb{I}_2 \end{pmatrix}^{-1}, \tag{2.2.104}$$

where B is the Hermitian matrix constructed from $b \in \mathbb{R}^{1,3}$ via (2.2.94).

The set U_∞ can be parametrised explicitly

$$U_\infty = \left\{ \Omega(\theta,\phi)^\dagger \begin{pmatrix} -1 & 0 \\ 0 & -e^{2i\chi} \end{pmatrix} \Omega(\theta,\phi) \in U(2) \;\middle|\; \chi \in (-\pi/2, \pi/2] \right\}, \tag{2.2.105}$$

where $\Omega(\theta,\phi)$ is as defined in (2.2.97). With the $U(2)$ formalism it can be checked that $\partial\mathbb{R}^{1,3}_c = U_\infty$ and $U_\infty = \mathcal{I}(L_0) \cup S^*$. S^* is the *null sphere at infinity* and can be parametrised

$$S^* = \left\{ \Omega(\theta,\phi)^\dagger \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Omega(\theta,\phi) \in U(2) \right\}. \tag{2.2.106}$$

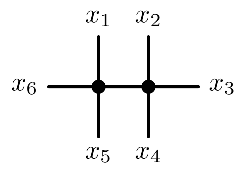
S^* is stable under inversions $\mathcal{I}S^* = S^*$. It corresponds to the largest sphere ($\chi = \pi/2$) on the pinched torus (figure 2.3) and can be reached from the bulk of Minkowski space by taking limits to infinity along the light cone L_0. ι is the image of the origin under $\mathcal{I}$ and corresponds to the $U(2)$ matrix $U = -\mathbb{I}_2$.

This picture of the conformal compactification of Minkowski space is useful when considering the invariance properties of the conformal box integral, as discussed in chapter 4. It is also used explicitly in the double infinity configurations which we introduce in section 4.3.

2.3 Conformal Feynman Integrals

As described in section 2.1, calculations of correlation functions (and scattering amplitudes) can be done perturbatively at weak coupling, by summing over Feynman diagrams. The calculation can be expressed as a power series in the coupling, with the dominant effects occurring at tree-level and the sub-leading effects being representable by integrals of certain functions over multiple copies of space(time).[22] The number of such copies is the *loop order* ℓ of the integral.[23] Such integrals are referred to as *Feynman integrals*. These integrals can be put in one-to-one correspondence with Feynman diagrams, with the Feynman rules providing a dictionary between the two. A typical example of a position space Euclidean Feynman integral in a massless theory and its corresponding Feynman diagram is

$$I(x_1, x_2, x_3, x_4, x_5, x_6) = \int \frac{\mathrm{d}^4 x_a}{\pi^2} \frac{\mathrm{d}^4 x_b}{\pi^2} \frac{1}{x_{a1}^2 x_{a5}^2 x_{a6}^2 x_{ab}^2 x_{b2}^2 x_{b3}^2 x_{b4}^2} =$$



(2.3.1)

In general the *propagators* $1/x_{ij}^2$ can have masses $1/x_{ij}^2 \to 1/(x_{ij}^2 + m_{ij}^2)$.[24] However, in this thesis we only consider diagrams with massless lines $m_{ij}^2 = 0$.

One issue with Feynman integrals is their complexity, both in how they appear in QFT calculations and in their own right as integrals to be calculated. In non-abelian gauge theories the Feynman rules can be quite complex, and the number of Feynman diagrams contributing to a given process proliferates very quickly with the loop order and particle number. For example, the tree-level 4-gluon amplitude in QCD has four diagrams, whereas the 6-gluon amplitude has 220 diagrams at tree-level. Moreover, the representation of scattering processes in terms of Feynman integrals spoils gauge invariance in each individual Feynman integral, and is only restored thanks to (possibly complicated) cancellations in the sum. As such, there have been attempts in many different contexts to mitigate the use of Feynman integrals to calculate physical quantities, and rather use symmetries of the model and other consistency conditions. These can lead to more efficient methods to constructing integrands which should then be integrated, or in some cases bypass integration entirely. A prototypical example of this spinor-helicity formulation of QFT, which is on-shell by design. This formalism led to a huge progress in calculations in $\mathcal{N} = 4$ super Yang–Mills and related theories, thanks to methods like on-shell recursion relations and generalised unitarity, see [12] for a recent review. Other examples of using symmetry techniques to constrain observables include the conformal bootstrap, referred to in the previous section, and the Steinmann cluster bootstrap for amplitudes in $\mathcal{N} = 4$ SYM [49].

As mathematical objects, Feynman integrals are quite complicated in general. For one, they can suffer from UV- and IR-divergences which require regularisation and/or

[22] In this section we mostly restrict to Euclidean space.

[23] In momentum space there are as many integrations as there are loops in the corresponding Feynman diagram. This is not true in position space, where internal points are integrated over, although we will still conveniently refer to the number of integrations as the loop order.

[24] This is the form of massive propagators in *dual momentum space*, see section 2.3.1. In position space massive propagators are more complicated, cf. (2.1.13), however as $m \to 0$ they are proportional to $1/x_{ij}^2$.

renormalisation. UV-divergences occur when the integrand is singular at large values of the loop momentum, and IR-divergences arise from singularities arising at small values of the loop momentum. In both cases, a regularisation is required to make the integral well-defined. There are many methods one can use: For UV-divergences there is *cutoff* regularisation, where an integral over all momentum is replaced by an integral over a large momentum sphere $|k| < \Lambda$. IR-divergences can be cured by giving a finite mass to propagators which diverge in the low energy limit. A very common approach to regularising both UV- and IR-divergent Feynman integrals is the use of *dimensional regularisation*. This is the replacement of the dimension d in which the integral is divergent, with a general dimension $D = d - 2\epsilon$.[25] The divergence of the integral can then be studied as $\epsilon \to 0$. When a physical quantity in a QFT is represented by a sum of Feynman integrals which is divergent, a renormalisation of the theory is required. In this case the behaviour of the Feynman integrals as a function of the regulator is necessary to calculate the beta functions of the couplings.

The evaluation of Feynman integrals as functions of the kinematic data is a field of study in its own right. There is no method to calculate a Feynman integral in general, and the complexity increases with both the loop order and kinematical complexity. However, many strategies have emerged over the years. The first thing to try would be a direct integration of the so-called Feynman parametric representation of the integral, which replaces an integral over ℓ copies of space(time) with a projective integral over N Feynman parameters $\alpha_1, \alpha_2, \ldots, \alpha_N$, where N is the number of lines in the corresponding Feynman diagram. Sometimes (possibly after some sophisticated changes of variables in the Feynman parameter space) the integral can be integrated into polylogarithms. This can be a difficult task, however, and indeed sometimes there are explicit algebraic obstructions in the Feynman parameter space to obtaining polylogs. In these cases the integral may be representable in terms of elliptic polylogarithms [50, 51], or even more complicated functions associated to higher-genus surfaces [52]. Another option is to pass to a Mellin-Barnes representation of the integral, which trades integrals over spacetime with integrals over infinite imaginary lines in Mellin space. The general hope is then to compute the Mellin integrals as a sum over poles of the appearing gamma functions. We will use the Mellin–Barnes representation occasionally in this thesis. Finally, one of the most powerful techniques for calculating Feynman integrals in dimensional regularisation is the *differential equations* method. This involves considering a large class of integrals in the same topology, with generic values of propagator powers and dimension. Using so-called integration by parts (IBP) relations these integrals can be efficiently related to each other, and a set of *master integrals* identified. The vector of master integrals can be related to each other via a set of differential equations in the kinematic space. If these differential equations can be brought to a certain canonical form, then they can be directly integrated into polylogs [53]. By now there are several computational packages available for performing this IBP reduction, for example FIRE [54] and LiteRed [55]. A detailed summary of each of these methods is contained in [56,57]. While these are the most common techniques for the evaluation of Feynman integrals, there are other approaches which can work for isolated classes of diagrams.

[25] Using Poincaré invariance it is possible to express a Feynman integral in terms of the kinematic invariants x_{ij}^2 and define them in non-integer dimensions.

In this thesis we focus on a special class of Feynman integrals, namely *conformal* Feynman integrals. These are Euclidean Feynman integrals in D dimensions which are covariant under simultaneous conformal transformations of the external points $x_i \to Ax_i$. As such, up to a conformal weight, these integrals have a reduced dependence on the kinematics x_{ij}^2, and are expressible in terms of cross ratios of these kinematics. These integrals tend to be finite, i.e. they do not suffer from UV- or IR- divergences. In Minkowski space the corresponding Feynman integrals depend locally on the cross ratios, although global conformal transformations can change the value of the integral. This situation is discussed in detail in section 4.2 in the case of the box integral, and so we focus here on the Euclidean case.

This section is organised as follows. We first define the notion of a massless Feynman integral in position space, and explain the dual space picture. We define the Feynman/Schwinger parametrisations, and explain how these can be read off from the topology of the Feynman graph. We explain the tensor decomposition of Feynman integrals with vector insertions of the loop momenta, which will be relevant in chapter 5. We then define conformal Feynman integrals and explain a method to recover the conformal Feynman parametrisation. We give several examples of conformal Feynman integrals and discuss their relevance.

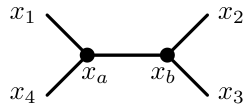
2.3.1 Feynman Integrals and Parametric Representation

We begin by defining Feynman integrals in position space with massless internal lines. Such integrals are defined by a connected graph G with n external vertices $x_1, \ldots, x_n$, ℓ internal vertices $x_{a_1}, \ldots, x_{a_\ell}$, and N lines. The Feynman integral in D dimensions with propagator powers $\nu_1, \ldots, \nu_N$ corresponding to G is defined by

$$I_{\nu,D}^G(x_1, \ldots, x_n) := \int \prod_{i=1}^{\ell} \frac{\mathrm{d}^D x_{a_i}}{\pi^{D/2}} \prod_{\langle jk \rangle \in G} \frac{1}{x_{jk}^{2\nu_{jk}}}, \tag{2.3.2}$$

where $\langle jk \rangle$ is the line joining points x_j and x_k in G. In other words, given a graph G, we insert a factor of $1/x_{ij}^2$ for each line connecting points x_i and x_j, and integrate over the internal points $x_{a_1}, \ldots, x_{a_\ell}$, including a factor of $\pi^{-D/2}$ per integration.[26] For example, if the graph is[27]

$G =$



, (2.3.3)

then

$$I_{\nu_j=1,D=4}^G(x_1, x_2, x_3, x_4) = \int \frac{\mathrm{d}^4 x_a}{\pi^2} \frac{\mathrm{d}^4 x_b}{\pi^2} \frac{1}{x_{a1}^2 x_{a4}^2 x_{ab}^2 x_{b2}^2 x_{b3}^2}. \tag{2.3.4}$$

Integrals of the form (2.3.2) are invariant under translations $x_i \to x_i + a$ and rotations $x_i^\mu \to M^\mu{}_\nu x_i^\nu$ of the external points, and therefore depend only on the *kinematics* x_{ij}^2.

We can alternatively represent the Feynman integrals (2.3.2) in *dual position space*. This space is labelled by coordinates $p_1, \ldots, p_n$, which are related to the position space

[26]This factor is inserted to cancel a corresponding factor which appears in the Feynman parametrisation.

[27]Note that we typically will not mark the external vertices by dots.

coordinates via

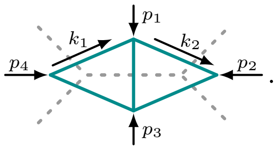
$$p_i = x_i - x_{i+1}, \tag{2.3.5}$$

where we take summation in the indices mod n, i.e. $x_{n+1} \equiv x_1$. Note that the p coordinates automatically satisfy $p_1 + p_2 + \cdots + p_n = 0$. We could alternatively start from a momentum space Feynman diagram depending on momenta $p_1, \ldots, p_n$ and define dual momentum space coordinates $x_1, \ldots, x_n$ via (2.3.5). In this case there is a redundancy in defining the x's due to momentum conservation, which can be fixed, for example by taking $x_1 = 0$. The change of coordinates (2.3.5) can often reveal hidden symmetries in physical quantities, for example the scattering amplitude/Wilson loop duality in planar $\mathcal{N} = 4$ SYM [58].

Letting $k_1 \coloneqq x_{a1}, k_2 \coloneqq x_{b2}$, the integral (2.3.4) in dual position space reads

$$\int \frac{\mathrm{d}^4 k_1}{\pi^2} \frac{\mathrm{d}^4 k_2}{\pi^2} \frac{1}{k_1^2 (k_1 + p_{123})^2 (k_1 - k_2 + p_1)^2 k_2^2 (k_2 + p_2)^2}, \tag{2.3.6}$$

where $p_{123} \coloneqq p_1 + p_2 + p_3 = -p_4$. The integral (2.3.6) can be associated to the dual graph $\tilde{G}$ of G, and is extracted from $\tilde{G}$ using *momentum space* Feynman rules:

$\tilde{G} =$  . (2.3.7)

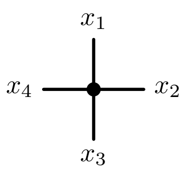
Specifically, $\tilde{G}$ is extracted from G by placing a point in each region determined by G, and then connecting these points. Furthermore, points in the region between x_i and x_{i+1} are assigned an incoming momentum p_i. A line in each loop of the resulting graph is assigned a loop momentum k_i, and the remaining lines can be assigned momenta via momentum conservation. While they are reminiscent, we stress that dual position space is not momentum space, i.e. it is not related to x space via Fourier transformation.

Feynman Parametrisation. Any integral of the form (2.3.2) can be written in its *Feynman parametrisation*, also called its *parametric form*. This representation exposes the dependence of the integral on the kinematics x_{ij}^2, and replaces the ℓ-fold integration over D-dimensional space with a projective integral over Feynman parameters $\alpha_1, \ldots, \alpha_N$. The key formula for deriving the Feynman parametrisation is

$$\frac{1}{A_1^{\nu_1} \ldots A_k^{\nu_k}} = \frac{\Gamma_\nu}{\Gamma_{\nu_1} \ldots \Gamma_{\nu_k}} \int_0^\infty \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_k \frac{\alpha_1^{\nu_1 - 1} \ldots \alpha_k^{\nu_k - 1} \delta(h(\alpha) - 1)}{(\alpha_1 A_1 + \cdots + \alpha_k A_k)^\nu}, \tag{2.3.8}$$

where we abbreviate $\Gamma_x \coloneqq \Gamma(x)$ and $\nu \coloneqq \nu_1 + \cdots + \nu_k$. $h(\alpha)$ is any polynomial in the Feynman parameters which is homogenous of degree one, i.e. rescaling $\alpha_i \to \lambda \alpha_i$ maps $h(\alpha) \to \lambda h(\alpha)$. Common choices are $h(\alpha) = \alpha_1 + \cdots + \alpha_k$ and $h(\alpha) = \alpha_i$ for some $i \in \{1, 2, \ldots, k\}$. The identity (2.3.8) can be applied to a product of propagators corresponding to a single integration variable x_{a_i} in (2.3.2), and that integral can then be performed straightforwardly. This procedure can be carried out loop by loop.

As a simple example, consider the four-dimensional cross integral

$I =$  $= \int \frac{\mathrm{d}^4 x_a}{\pi^2} \frac{1}{x_{a1}^2 x_{a2}^2 x_{a3}^2 x_{a4}^2}$. (2.3.9)

Using (2.3.8) we can rewrite the product of propagators as an integral over Feynman parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$:

$$I = \int \frac{\mathrm{d}^4 x_a}{\pi^2} \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \frac{\delta(h(\alpha)-1)}{(\alpha_1 x_{a1}^2 + \alpha_2 x_{a2}^2 + \alpha_3 x_{a3}^2 + \alpha_4 x_{a4}^2)^4}. \tag{2.3.10}$$

By completing the square the denominator of (2.3.10) can be written as $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^4 (l^2 + \Delta)^4$, where

$$l^\mu = x_a^\mu - \frac{\sum_{i=1}^4 \alpha_i x_i^\mu}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}, \qquad \Delta = \frac{\sum_{i<j} \alpha_i \alpha_j x_{ij}^2}{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^2}. \tag{2.3.11}$$

Exchanging the order of integration and introducing hyperspherical coordinates, the integral can thus be written

$$I = \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \int \frac{\pi^2 l^2 \mathrm{d}l^2}{\pi^2} \frac{\delta(h(\alpha)-1)}{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^4 (l^2 + \Delta)^4}. \tag{2.3.12}$$

Performing the integral over l^2 leads to the Feynman parametrisation

$$I = \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \frac{\delta(h(\alpha)-1)}{(\sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^2} := \int_0^\infty [\mathrm{d}^3\alpha] \frac{1}{(\sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^2}, \tag{2.3.13}$$

where we defined the projective integral $\int_0^\infty [\mathrm{d}^3\alpha] := \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \delta(h(\alpha)-1)$.

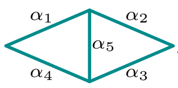
Remarkably, it is possible to extract the Feynman parametrisation for a Feynman integral $I_{\nu,D}^G$ just from the topology of the dual graph $\tilde{G}$ [59, 60].[28] To do this, we associate to each internal line in $\tilde{G}$ with propagator power ν_l a Feynman parameter α_l, $l = 1, \ldots, N$. (2.3.2) can be rewritten

$$I_{\nu,D}^G(x_{ij}^2) = \frac{\Gamma_{\nu - \ell D/2}}{\prod_{i=1}^N \Gamma_{\nu_i}} \int_0^\infty [\mathrm{d}^{N-1}\alpha] (\prod_{k=1}^N \alpha_k^{\nu_k - 1}) \frac{\mathcal{U}^{\nu - (\ell+1)D/2}}{\mathcal{F}(x_{ij}^2)^{\nu - \ell D/2}}, \tag{2.3.14}$$

where here $\nu = \nu_1 + \cdots + \nu_N$. $\mathcal{U}$ and $\mathcal{F}(x_{ij}^2)$ are the *Symanzik polynomials* of the dual graph $\tilde{G}$, and are polynomials in the Feynman parameters. To define them we need some notions from graph theory. A *tree* of $\tilde{G}$ is any connected subdiagram of $\tilde{G}$ which contains all vertices of $\tilde{G}$ and is free from loops. A *chord* of a tree is any line that is in $\tilde{G}$, but not in the tree. Clearly, any tree has ℓ chords, where ℓ is the number of loops in $\tilde{G}$. A *two-forest* of $\tilde{G}$ is a subdiagram which contains all vertices of $\tilde{G}$, is free from loops, and has two connected components. A two-forest can be obtained from a tree by deleting an appropriate edge, and so necessarily contains $\ell + 1$ chords. The polynomials $\mathcal{U}$ and $\mathcal{F}$ can then be defined as

$$\mathcal{U} = \sum_{\text{trees}} \prod_{\text{chords } i} \alpha_i, \quad \mathcal{F} = \sum_{\text{two-forests } j} q_j^2 \prod_{\text{chords } i} \alpha_i, \tag{2.3.15}$$

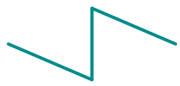
where α_i is the Feynman parameter corresponding to chord i and q_j is the momentum flowing into each of the trees in the two-forest j. As an example we consider the graph (2.3.7), to which we assign Feynman parameters as follows:

$\tilde{G} =$ 

. (2.3.16)

[28] It is not clear to the author whether there is an efficient way to extract the Feynman parametrisation directly from the position space graph G.

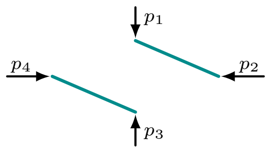
There are eight trees associated to this graph, for example one obtained by removing the legs corresponding to Feynman parameters α_1 and α_3:


, (2.3.17)

which gives a contribution $\alpha_1\alpha_3$ to $\mathcal{U}$. Overall the $\mathcal{U}$ polynomial is

$$\mathcal{U} = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3) + \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4). \tag{2.3.18}$$

There are ten two-forests, for example one obtained by removing the legs corresponding to α_1, α_3, and α_5:



. (2.3.19)

In this case there is momentum $(p_1 + p_2)^\mu = (p_3 + p_4)^\mu = x_{13}^\mu$ flowing into each tree, and so this two-forest gives a contribution $x_{13}^2\alpha_1\alpha_3\alpha_5$ to $\mathcal{F}$. Overall we have

$$\begin{aligned}\mathcal{F}(x_{ij}^2) = {} & x_{12}^2\alpha_1\alpha_2\alpha_5 + x_{13}^2\alpha_1\alpha_3\alpha_5 + x_{14}^2(\alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_4\alpha_5) \\ & + x_{23}^2(\alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_5) + x_{24}^2\alpha_2\alpha_4\alpha_5 + x_{34}^2\alpha_3\alpha_4\alpha_5.\end{aligned} \tag{2.3.20}$$

Therefore using (2.3.14) the integral (2.3.4) can be rewritten

$$I^G_{\nu_j=1,D=4}(x_{ij}^2) = \int [\mathrm{d}^4\alpha] \frac{((\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3) + \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4))^{-1}}{\mathcal{F}(x_{ij}^2)}, \tag{2.3.21}$$

where $\mathcal{F}(x_{ij}^2)$ is defined in (2.3.20).

Schwinger Parametrisation. Another way to express the integral (2.3.2) in terms of its $\mathcal{U}$ and $\mathcal{F}$ polynomials is via its *Schwinger parametrisation*:

$$I^G_{\nu,D}(x_{ij}^2) = \left(\prod_{j=1}^{N} \int_0^\infty \frac{\mathrm{d}\alpha_j \alpha_j^{\nu_j - 1}}{\Gamma_{\nu_j}}\right) \frac{\exp(-\mathcal{F}(x_{ij}^2)/\mathcal{U})}{\mathcal{U}^{D/2}}. \tag{2.3.22}$$

(2.3.14) can be obtained from (2.3.22) via some integration tricks, see [61], or can alternatively be derived started from the formula analogous to (2.3.8):

$$\frac{1}{A^\nu} = \frac{1}{\Gamma_\nu} \int_0^\infty \mathrm{d}x\, x^{\nu-1} e^{-xA}. \tag{2.3.23}$$

The formula (2.3.23) allows one to convert products of propagators into an exponential of sums of propagators. The loop integrations can then be performed as Gaussian integrals.

2.3.2 Tensor Decomposition

A natural generalisation of the family of integrals (2.3.2) are those with tensor insertions of integration variables in the numerators. For example, one could consider the integral

$$\int \frac{\mathrm{d}^4 x_a}{\pi^2} \frac{\mathrm{d}^4 x_b}{\pi^2} \frac{x_{b1}^\mu}{x_{a1}^2 x_{a3}^2 x_{a4}^2 x_{ab}^2 x_{b1}^4 x_{b2}^2 x_{b3}^2}. \tag{2.3.24}$$

Such integrals occur in gauge/higher-spin theories where the Feynman rules have a tensor structure. They can also occur in correlation functions of conformal descendants of scalar fields, as we will see in chapter 5.

There are several approaches in the literature for decomposing such integrals into scalar integrals. The most well-known is the *Passarino-Veltman* reduction, which decomposes one-loop integrals with arbitrary tensor insertions of loop momenta in the numerator into a sum of equal and lower-point scalar integrals [62]. Some more general approaches have appeared; in particular Tarasov proposed a procedure to decompose ℓ-loop integrals with arbitrary insertions of the loop momenta into a sum of higher-dimensional scalar integrals, which we will use in this thesis [63, 64]. In chapter 5 we will mainly focus on the case with a single vector insertion:

$$I^G_{\nu,D}(x^\mu_{a_m l}) = \int \prod_{i=1}^{\ell} \frac{\mathrm{d}^D x_{a_i}}{\pi^{D/2}} x^\mu_{a_m l} \prod_{\langle jk\rangle \in G} \frac{1}{x_{jk}^{2\nu_{jk}}}, \tag{2.3.25}$$

for some $m = 1, \ldots, \ell$ and $l = 1, \ldots, n$. By considering its properties under translations and rotations, (2.3.25) can be decomposed as

$$I^G_{\nu,D}(x^\mu_{a_m l}) = -\sum_{k \neq l} x^\mu_{lk} f_{k,l,m}(x^2_{ij}), \tag{2.3.26}$$

and the result of Tarasov is that $f_{k,l,m}(x^2_{ij})$ can be expressed as a linear combination of $(D+2)$-dimensional scalar integrals with the same topology G as (2.3.25), but with modified propagator powers.

To compute the functions $f_{k,l,m}$ from the graph G, where each edge is labelled by a Feynman parameter $\alpha_1, \ldots, \alpha_N$, one should proceed as follows. We first form an $\ell \times \ell$ symmetric matrix B and an ℓ-vector c. B_{ij} and c_i are indexed by the integration points $a_1, \ldots, a_\ell$. B_{ij} is minus the sum of the Feynman parameters corresponding to the legs connecting internal point i and internal point j for $i \neq j$, and it is the sum of the Feynman parameters corresponding to all legs connected to internal point i for $i = j$. c_i is the sum of Feynman parameters connecting internal point i to some external point x^μ_k, multiplied by that external point. Using these one should compute the quantity

$$(B^{-1} \cdot c)_{a_m} - x^\mu_l = -\frac{1}{\mathcal{U}} \sum_{k \neq l} x^\mu_{lk} \mathcal{P}_{k,l,m}(\alpha), \tag{2.3.27}$$

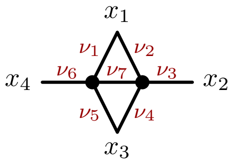
where $\mathcal{P}_{k,l,m}(\alpha)$ are polynomials in the Feynman parameters. Then at the level of the Schwinger parametrisation we have

$$I^G_{\nu,D}(x^\mu_{a_m l}) = -\sum_{k \neq l} x^\mu_{lk} \left(\prod_{j=1}^{N} \int_0^\infty \frac{\mathrm{d}\alpha_j \alpha_j^{\nu_j - 1}}{\Gamma_{\nu_j}} \right) \frac{\mathcal{P}_{k,l,m}(\alpha) \exp(-\mathcal{F}(x^2_{ij})/\mathcal{U})}{\mathcal{U}^{(D+2)/2}}. \tag{2.3.28}$$

Comparing (2.3.28) and (2.3.26) with (2.3.22) we see the vector coefficients $f_{k,l,m}(x_{ij}^2)$ are linear combinations of scalar integrals of dimension $D+2$, with propagator powers modified appropriately by the polynomials $\mathcal{P}_{k,l,m}$.

As an example, consider the integral (2.3.24), which has double ladder topology

$$G_{\text{double ladder}} = \quad ,\tag{2.3.29}$$



where we have indicated how we assign Feynman parameters to the legs. From the dual graph we can use (2.3.15) to extract the $\mathcal{U}$ and $\mathcal{F}$ polynomials as

$$\begin{aligned}\mathcal{U} =&\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_5+\alpha_3\alpha_5+\alpha_4\alpha_5+\alpha_2\alpha_6+\alpha_3\alpha_6\\&+\alpha_4\alpha_6+\alpha_1\alpha_7+\alpha_2\alpha_7+\alpha_3\alpha_7+\alpha_4\alpha_7+\alpha_5\alpha_7+\alpha_6\alpha_7,\end{aligned}\tag{2.3.30}$$

$$\begin{aligned}\mathcal{F} =&x_{12}^2(\alpha_1\alpha_2\alpha_3+\alpha_2\alpha_3\alpha_5+\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_3\alpha_7+\alpha_2\alpha_3\alpha_7)+x_{13}^2(\alpha_1\alpha_2\alpha_4\\&+\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_5+\alpha_1\alpha_4\alpha_5+\alpha_2\alpha_4\alpha_5+\alpha_2\alpha_4\alpha_6+\alpha_1\alpha_4\alpha_7\\&+\alpha_2\alpha_4\alpha_7+\alpha_1\alpha_5\alpha_7+\alpha_2\alpha_5\alpha_7)+x_{14}^2(\alpha_1\alpha_2\alpha_6+\alpha_1\alpha_3\alpha_6+\alpha_1\alpha_4\alpha_6\\&+\alpha_1\alpha_6\alpha_7+\alpha_2\alpha_6\alpha_7)+x_{23}^2(\alpha_1\alpha_3\alpha_4+\alpha_3\alpha_4\alpha_5+\alpha_3\alpha_4\alpha_6+\alpha_3\alpha_4\alpha_7\\&+\alpha_3\alpha_5\alpha_7)+x_{34}^2(\alpha_5\alpha_6\alpha_7+\alpha_2\alpha_5\alpha_6+\alpha_3\alpha_5\alpha_6+\alpha_4\alpha_5\alpha_6+\alpha_4\alpha_6\alpha_7)\\&+x_{24}^2\alpha_3\alpha_6\alpha_7.\end{aligned}\tag{2.3.31}$$

For this topology we have

$$B_{ij}=\begin{pmatrix}\alpha_1+\alpha_5+\alpha_6+\alpha_7 & -\alpha_7\\ -\alpha_7 & \alpha_2+\alpha_3+\alpha_4+\alpha_7\end{pmatrix},\qquad c_i=\begin{pmatrix}\alpha_1x_1^\mu+\alpha_5x_3^\mu+\alpha_6x_4^\mu\\ \alpha_2x_1^\mu+\alpha_3x_2^\mu+\alpha_4x_3^\mu\end{pmatrix},\tag{2.3.32}$$

and using these we compute (2.3.27) for the integral (2.3.24) to be

$$\begin{aligned}\mathcal{U}[(B^{-1}\cdot c)_b-x_1^\mu]=&-x_{12}^\mu(\alpha_1\alpha_3+\alpha_3\alpha_5+\alpha_3\alpha_6+\alpha_3\alpha_7)\\&-x_{13}^\mu(\alpha_1\alpha_4+\alpha_4\alpha_5+\alpha_4\alpha_6+\alpha_4\alpha_7+\alpha_5\alpha_7)-x_{14}^\mu\alpha_6\alpha_7.\end{aligned}\tag{2.3.33}$$

Using this we can expand (2.3.24) as

$$\int\frac{\mathrm{d}^4x_a}{\pi^2}\frac{\mathrm{d}^4x_b}{\pi^2}\frac{x_{b1}^\mu}{x_{a1}^2x_{a3}^2x_{a4}^2x_{ab}^2x_{b1}^4x_{b2}^2x_{b3}^2}=-x_{12}^\mu f_2-x_{13}^\mu f_3-x_{14}^\mu f_4,\tag{2.3.34}$$

where f_2, f_3, f_4 can be written as a sum of six-dimensional double ladder integrals with modified propagator powers:

$$f_2=I^6_{2,2,2,1,1,1,1}+I^6_{1,2,2,1,2,1,1}+I^6_{1,2,2,1,1,2,1}+I^6_{1,2,2,1,1,1,2},\tag{2.3.35}$$
$$f_3=I^6_{1,2,1,1,2,1,2}+I^6_{2,2,1,2,1,1,1}+I^6_{1,2,1,2,2,1,1}+I^6_{1,2,1,2,1,2,1}+I^6_{1,2,1,2,1,1,2},\tag{2.3.36}$$
$$f_4=I^6_{1,2,1,1,1,2,2},\tag{2.3.37}$$

where we adopt the notation

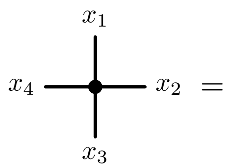
$$I^6_{\nu_1,\ldots,\nu_7}:=I^{G_{\text{double ladder}}}_{\nu,D=6}.\tag{2.3.38}$$

2.3.3 Conformal Invariance

A Feynman integral $I^G_{\nu,D}(x^2_{ij})$, defined in (2.3.2), is a *conformal* Feynman integral if after multiplication by some appropriate function of the kinematics $w(x^2_{ij})$, it is invariant under conformal transformations of the external points $x_i \to A x_i$, for $i = 1, \ldots, n$. We will call the function $w(x^2_{ij})$ a *conformal weight* of the integral.[29] Note that while only massless Feynman integrals have a chance of being conformally invariant, an extension of conformal symmetry which applies to some classes of massive diagrams was proposed in [65].

Since it is a central part of this thesis, we will show that the box integral[30] is conformally invariant, with a conformal weight $x^2_{13}x^2_{24}$. After multiplication by this weight, this integral is

$$\phi := x^2_{13}x^2_{24}$$



$$= \int \frac{\mathrm{d}^4 x_a}{\pi^2} \frac{x^2_{13}x^2_{24}}{x^2_{a1}x^2_{a2}x^2_{a3}x^2_{a4}}. \tag{2.3.39}$$

The strategy to show that this function is invariant under conformal transformations is to map all the points $x_i \to A x_i$, where A is a translation, rotation, dilatation, or SCT, and integrate over a new variable

$$x_b = A^{-1} x_a. \tag{2.3.40}$$

This ensures that $x_{ai} \to A x_{bi}$ under this conformal transformation, i.e. the external points and integration point are on the same footing. One should just be careful to include the Jacobian of the transformation (2.3.40) in the integration measure.

Let's see how this procedure works with translations. If we translate all points $x_i \to x_i + a$, and further define a translated loop variable $x_b = x_a - a$, we see that under this translation

$$\phi \to \int \frac{\mathrm{d}^4(x_b + a)}{\pi^2} \frac{x^2_{13}x^2_{24}}{x^2_{b1}x^2_{b2}x^2_{b3}x^2_{b4}} = \int \frac{\mathrm{d}^4 x_b}{\pi^2} \frac{x^2_{13}x^2_{24}}{x^2_{b1}x^2_{b2}x^2_{b3}x^2_{b4}} = \phi, \tag{2.3.41}$$

because $x_{ij} \to (x_i + a - (x_j + a)) = x_{ij}$ and $x_{ai} \to (x_b + a - (x_i + a)) = x_{bi}$. Similarly, ϕ is invariant under rotations of the external points $x^\mu_i \to M^\mu{}_\nu x^\nu_i$, which is clear after introducing the rotated loop variable $x^\mu_b = (M^{-1})^\mu{}_\nu x^\nu_a$ and using the fact that $\mathrm{d}^4 x_a = \mathrm{d}^4 x_b$ since $\det M = 1$. Then each of the x^2_{ij} are clearly invariant under rotations and $x^2_{ai} \to x^2_{bi}$. Invariance of (2.3.39) under translations and rotations is not a surprise, and indeed any Feynman integral of the form (2.3.2) is invariant as such.

As general Feynman integral is *not* invariant under dilatations however, and here the conformal weight $x^2_{13}x^2_{24}$ is necessary for invariance. Under dilatations $x_i \to c\, x_i$, we integrate over a scaled variable $x_b = \frac{1}{c} x_a$. Then we have

$$\phi \to \int \frac{c^4 \mathrm{d}^4 x_b}{\pi^2} \frac{c^2 x^2_{13} c^2 x^2_{24}}{c^2 x^2_{b1} c^2 x^2_{b2} c^2 x^2_{b3} c^2 x^2_{b4}} = \int \frac{\mathrm{d}^4 x_b}{\pi^2} \frac{x^2_{13}x^2_{24}}{x^2_{b1}x^2_{b2}x^2_{b3}x^2_{b4}} = \phi. \tag{2.3.42}$$

[29] The conformal weight $w(x^2_{ij})$ is not unique. Indeed, given a conformal weight $w(x^2_{ij})$ one can multiply it by any rational function of the cross ratios to get a new conformal weight.

[30] We will mainly refer to this integral as the box integral, since the dual graph resembles a box. It is also referred to as the cross integral.

Finally, we should show that ϕ is invariant under special conformal transformations (2.2.3). Since we saw in (2.2.22) that SCT = inversion $\circ$ translation $\circ$ inversion, it suffices to demonstrate the invariance of (2.3.39) under inversions $x_i^\mu \to x_i^\mu/x^2$. We integrate over an inverted variable $x_b^\mu = x_a^\mu/x_a^2$, so that $x_a^\mu = x_b^\mu/x_b^2$. The integration measure transforms

$$\mathrm{d}^4x_a = \det\left(\frac{\partial x_a^\mu}{\partial x_b^\nu}\right)\mathrm{d}^4x_b = \det\left(\delta^\mu{}_\nu - 2\frac{x_b^\mu x_{b,\nu}}{x_b^2}\right)\mathrm{d}^4x_b = \frac{\mathrm{d}^4x_b}{x_b^8}, \tag{2.3.43}$$

where the determinant was evaluated using

$$\det\left(\frac{\partial x_a^\mu}{\partial x_b^\nu}\right) = \sqrt{\det\left(\frac{\partial x_a^\mu}{\partial x_b^\alpha}\frac{\partial x_a^\alpha}{\partial x_b^\nu}\right)} = \sqrt{\det\left(\frac{\delta^\mu{}_\nu}{x_b^4}\right)} = \frac{1}{x_b^8}, \tag{2.3.44}$$

which is essentially the relation $\det M = \sqrt{\det MM^T}$. We also recall the equation (2.2.53), which implies that

$$\left(\frac{x_b^\mu}{x_b^2} - \frac{x_i^\mu}{x_i^2}\right)^2 = \frac{x_{bi}^2}{x_b^2x_i^2}, \quad i = 1,2,3,4. \tag{2.3.45}$$

Combining these facts, we conclude that under inversions ϕ transforms

$$\phi \to \int \frac{\mathrm{d}^4x_b}{\pi^2 x_b^8}\frac{\frac{x_{13}^2}{x_1^2x_3^2}\frac{x_{24}^2}{x_2^2x_4^2}}{\frac{x_{b1}^2}{x_b^2x_1^2}\frac{x_{b2}^2}{x_b^2x_2^2}\frac{x_{b3}^2}{x_b^2x_3^2}\frac{x_{b4}^2}{x_b^2x_4^2}} = \int\frac{\mathrm{d}^4x_b}{\pi^2}\frac{x_{13}^2x_{24}^2}{x_{b1}^2x_{b2}^2x_{b3}^2x_{b4}^2} = \phi. \tag{2.3.46}$$

Since ϕ is invariant under conformal transformations, it depends only on the conformal cross ratios u and v defined in (2.2.57), or equivalently the conformal variables $z, \bar{z}$ defined in (2.2.62). It has been evaluated in terms of a combination of logarithms and dilogarithms known as the Bloch–Wigner function. This function has many interesting properties, and is discussed in detail in section 4.1.2.

In general, an integral (2.3.2) is conformal if given any integration point x_a, the sum of the propagator powers for the propagators attached to x_a is equal to the spatial dimension D. In this case, the part of the integral containing x_a transforms

$$\mathrm{d}^Dx_a\prod_{l=1}^{v_a}\frac{1}{x_{al}^{2\nu_l}} \to \frac{\mathrm{d}^Dx_a}{x_a^{2D}}x_a^{2(\nu_1+\cdots+\nu_{v_a})}\prod_{l=1}^{v_a}\frac{x_l^{2\nu_l}}{x_{al}^{2\nu_l}} = \mathrm{d}^Dx_a\prod_{l=1}^{v_a}\frac{x_l^{2\nu_l}}{x_{al}^{2\nu_l}}, \tag{2.3.47}$$

if $\nu_1 + \cdots + \nu_{v_a} = D$. The factors $\prod_l x_l^{2\nu_l}$ which appear for each integration variable can be cancelled by choosing an appropriate conformal weight.[31] If we only consider unit propagator powers, then a D-dimensional Feynman integral (2.3.2) can only be conformal if the graph G is built from vertices which are D-valent. In particular, in four dimensions the vertices must be four-valent and any conformal Feynman integral with unit propagator powers has a fishnet topology. An n-point conformal Feynman integral depends on $n(n-3)/2$ cross-ratios.[32]

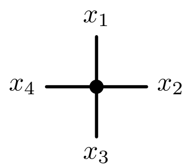
[31]Note that x_l could be another integration variable x_b. In this case it will simply be cancelled by the Jacobian of d^Dx_b if the conformal condition is also satisfied at this vertex.

[32]Provided the spatial dimension is large enough, see the discussion below (2.2.62).

Conformal Feynman Parametrisation. The Feynman parametrisation (2.3.14) exposes the dependence of a Feynman integral on the kinematics x_{ij}^2. It is natural to ask whether there is a corresponding representation for conformal Feynman integrals; one which exposes their dependence on the conformal cross ratios. In most cases it is possible to derive such a representation. A few examples were discussed in [66], although the basic mathematical tricks in deriving it were already seen in [67]. There does not appear to systematic way for deriving this representation, however, and there is certainly no graph-theoretic representation analogous to (2.3.14). The general spirit is to introduce Feynman parameters, integrate over one of the Feynman parameters for each loop in the integral, and rescale the remaining Feynman parameters by an appropriate kinematic factor. We will show how the derivation goes for the box integral. We state the conformal Feynman parametrisations relevant for this thesis in appendix A.

We start from the Feynman parametrisation for the conformal function of the box, taken from (2.3.13):

$$\phi = x_{13}^2 x_{24}^2$$



$$= \int_0^\infty [\mathrm{d}^3\alpha] \frac{x_{13}^2 x_{24}^2}{(\sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^2} \tag{2.3.48}$$

$$= \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \frac{x_{13}^2 x_{24}^2 \delta(\alpha_2 - 1)}{(\sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^2}, \tag{2.3.49}$$

where we picked a specific de-projection $\alpha_2 = 1$. Localising α_2 and performing the α_4 integration, we are left with

$$\phi = \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_3 \frac{x_{13}^2 x_{24}^2}{(\alpha_1 x_{12}^2 + \alpha_1 \alpha_3 x_{13}^2 + \alpha_3 x_{23}^2)(\alpha_1 x_{14}^2 + x_{24}^2 + \alpha_3 x_{34}^2)}. \tag{2.3.50}$$

This integral can be made manifestly conformal invariant by a judicious rescaling of Feynman parameters.[33] Indeed, letting $\alpha_1 = \frac{x_{23}^2}{x_{13}^2}\beta_1$ and $\alpha_3 = \frac{x_{12}^2}{x_{13}^2}\beta_2$, we have

$$\phi(u,v) = \int_0^\infty \mathrm{d}\beta_1 \mathrm{d}\beta_3 \frac{1}{(\beta_1 + \beta_1\beta_3 + \beta_3)(\beta_1 v + 1 + \beta_3 u)}, \tag{2.3.51}$$

which exposes the dependence of ϕ on u and v.

Conformal Feynman parametrisations are quite robust to work with numerically, especially at lower loops, and the Mathematica function NIntegrate is sufficient for their numerical integration. The precision of the agreement between analytic results and numerical integration of conformal Feynman parametrisations depends on the loop order of the appearing integrals. We find that at one, two, three, and four-loop orders there is a typical relative precision of $10^{-9}, 10^{-6}, 10^{-3}$, and 10^{-1}, respectively. When there is no analytic form available, numerical integration of the conformal Feynman parametrisation is valuable, since it is a convenient way to extract numerical values for the conformal integral.

In the following sections we provide a few examples of families of conformal integrals, and discuss some of their properties. The box integral (2.3.39) is a member of all of these families.

[33] Picking the right rescaling requires some experience. At higher loops when there are many Feynman parameters it can be tricky.

2.3.4 Conformal n-gons and the Star-Triangle Relation

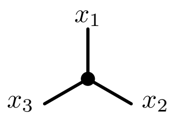
In n dimensions the canonical conformal vertex is n-valent. We can therefore consider the one-loop integral built from one of these vertices:

$$I_n = \int \frac{\mathrm{d}^n x_a}{\pi^{n/2}} \frac{1}{x_{a1}^2 x_{a2}^2 \cdots x_{an}^2} = \mathit{\Gamma}_{n/2} \int_0^\infty [\mathrm{d}^{n-1}\alpha] \frac{1}{(\sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^{n/2}}, \tag{2.3.52}$$

where we also wrote the corresponding Feynman parametrisation. In dual position space the graphs for this family of integrals are regular polygons, and so they are also referred to as the conformal n-gons, which up to a conformal weight depend on $n(n-3)/2$ cross ratios. Mathematically, the n-gons (2.3.52) (and massive generalisations) are known to compute volumes of hyperbolic manifolds [68, 69]. The conformal invariance of these integrals can be enhanced to conformal Yangian symmetry, described more in section 3.3.3.

For $n = 2$ the integral is divergent and requires regularisation. For $n = 4$ this is the box integral already discussed. For $n = 3$, we recover the *conformal star integral*. After multiplication by the conformal weight $\sqrt{x_{12}^2 x_{13}^2 x_{23}^2}$ this integral is completely fixed by conformal symmetry to be a constant:

$$\phi_3 \coloneqq \sqrt{x_{12}^2 x_{13}^2 x_{23}^2}$$



$$= \int \frac{\mathrm{d}^3 x_a}{\pi^{3/2}} \frac{\sqrt{x_{12}^2 x_{13}^2 x_{23}^2}}{x_{a1}^2 x_{a2}^2 x_{a3}^2} \tag{2.3.53}$$

$$= \mathit{\Gamma}_{3/2} \int_0^\infty [\mathrm{d}^2\alpha] \frac{\sqrt{x_{12}^2 x_{13}^2 x_{23}^2}}{(\alpha_1\alpha_2 x_{12}^2 + \alpha_1\alpha_3 x_{13}^2 + \alpha_2\alpha_3 x_{23}^2)^{3/2}}.$$

By performing the α_3 integral and making an appropriate rescaling of Feynman parameters, the kinematic dependence of the integral disappears and we are left with a constant:

$$\phi_3 = 2\mathit{\Gamma}_{3/2} \int_0^\infty [\mathrm{d}\alpha] \frac{1}{(\alpha_1 + \alpha_2)\sqrt{\alpha_1\alpha_2}} = 4\mathit{\Gamma}_{3/2} \tan^{-1}(\infty) = \pi^{3/2}, \tag{2.3.54}$$

where the final integral was easily performed using the de-projection $\alpha_2 = 1$. The fact that (2.3.53) is a constant reflects the fact that one cannot form a conformal cross ratio with $n = 3$ external points.

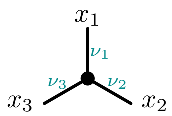
The conformal star in D dimensions is built from propagators with propagator powers ν_1, ν_2, ν_3 which sum to the dimension D. This integral is also a constant, up to a conformal weight:

$$\int \frac{\mathrm{d}^D x_a}{\pi^{D/2}} \frac{1}{x_{a1}^{2\nu_1} x_{a2}^{2\nu_2} x_{a3}^{2\nu_3}} = \frac{X_{123}}{x_{12}^{2\bar{\nu}_3} x_{13}^{2\bar{\nu}_2} x_{23}^{2\bar{\nu}_1}}, \qquad \nu_1 + \nu_2 + \nu_3 = D, \tag{2.3.55}$$

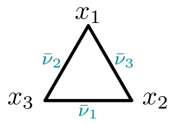
where we defined $\bar{\nu}_i \coloneqq D/2 - \nu_i$ and

$$X_{123} \coloneqq \frac{\mathit{\Gamma}_{\bar{\nu}_1} \mathit{\Gamma}_{\bar{\nu}_2} \mathit{\Gamma}_{\bar{\nu}_3}}{\mathit{\Gamma}_{\nu_1} \mathit{\Gamma}_{\nu_2} \mathit{\Gamma}_{\nu_3}}. \tag{2.3.56}$$

Graphically, the identity (2.3.55) can be represented



$$= X_{123}$$

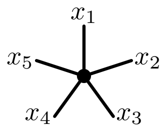


$$, \tag{2.3.57}$$

and is thus referred to as the *star-triangle* relation. This relation is useful for the computation of many integrals which contain conformal three-point vertices, and is central to various integrability constructions involving conformal integrals [70–73].

For $n = 5$ we have the conformal pentagon

$$I_5 =$$



$$= \int \frac{\mathrm{d}^5 x_a}{\pi^{5/2}} \frac{1}{x_{a1}^2 x_{a2}^2 x_{a3}^2 x_{a4}^2 x_{a5}^2} = \frac{\phi_5(u_1, u_2, u_3, u_4, u_5)}{\sqrt{x_{13}^2 x_{14}^2 x_{24}^2 x_{25}^2 x_{35}^2}}. \tag{2.3.58}$$

(2.3.58) is a rare example of a conformal integral computed at higher than four points. In [74] the authors derived a recursion relation for the symbols of the conformal n-gons, and in [75] the symbol of the pentagon was integrated to recover the expression[34]

$$\phi_5 = \frac{\pi^{3/2}}{2\sqrt{-\Delta^{(5)}}} \sum_{i<j} \log R_{ij}. \tag{2.3.59}$$

The conformal variables $\sqrt{-\Delta^{(5)}}$ and R_{ij} are defined from the gram matrix Q_5, whose entries are simply the external kinematics:

$$(Q_5)_{ij} := x_{ij}^2, \qquad i, j = 1, \ldots, 5. \tag{2.3.60}$$

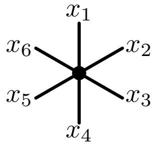
In particular we have

$$\Delta^{(5)} := \frac{1}{2} \frac{\det Q_5}{x_{13}^2 x_{14}^2 x_{24}^2 x_{25}^2 x_{35}^2}, \qquad R_{ij} := \frac{(Q_5^{-1})_{ij} - \sqrt{(Q_5^{-1})_{ij}^2 - (Q_5^{-1})_{ii}(Q_5^{-1})_{jj}}}{(Q_5^{-1})_{ij} + \sqrt{(Q_5^{-1})_{ij}^2 - (Q_5^{-1})_{ii}(Q_5^{-1})_{jj}}}. \tag{2.3.61}$$

From the recursion relation for their symbols it is clear that the conformal n-gons can be expressed as harmonic polylogarithms of the conformal varibles, times a *leading singularity*, which is the value of the integral when all propagators are sent on-shell [76]. These leading singularities can be expressed as $1/\sqrt{-\Delta^{(n)}}$, where $\Delta^{(n)}$ is calculated from the n-point gram matrix Q_n analogously to (2.3.61). At four points this leading singularity is simply $1/(z-\bar{z})$, where $z, \bar{z}$ are the conformal variables defined in (2.2.62).

The explicit polylogarithmic representation of the conformal n-gons for $n > 5$ is not known. The conformal hexagon

$$I_6 =$$



$$= \int \frac{\mathrm{d}^6 x_a}{\pi^3} \frac{1}{x_{a1}^2 x_{a2}^2 x_{a3}^2 x_{a4}^2 x_{a5}^2 x_{a6}^2} = \frac{\phi_6(u_1, \ldots, u_9)}{x_{14}^2 x_{25}^2 x_{36}^2}. \tag{2.3.62}$$

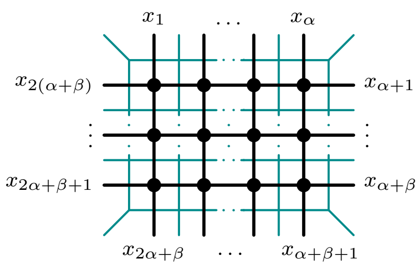
[34] This expression is only valid provided the correct branch of the logarithm is chosen. The author is not aware of a systematic procedure to determine this branch for a given configuration of points $x_1, \ldots, x_5$.

evaluates to a currently-unknown combination of weight three polylogs times a leading singularity, although a formal series representation has been identified [77]. This integral would be very interesting to calculate: it is related to the conformal six-point elliptic double box, described in the next section, by a simple differential equation. Six-dimensional hexagons are also known to contribute to tensor integrals appearing in amplitudes in $\mathcal{N}=4$ SYM [78]. Several on-shell limits of (2.3.62) have been calculated [79–81], although the fully off-shell case remains an open problem.

2.3.5 Fishnets and Basso–Dixon

Another interesting class of conformal Feynman integrals are the fishnet integrals in four dimensions, which are built from conformal four-point vertices. These are known to contribute to amplitudes in $\mathcal{N}=4$ SYM, and like the conformal n-gons they have been shown to have an enhanced conformal Yangian symmetry, discussed in section 3.3.3. The fishnet Feynman integrals are defined by an $\alpha \times \beta$ lattice of conformal vertices:

$$\tilde{I}_{\alpha\beta} =$$

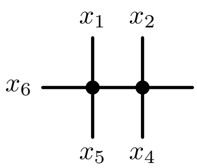


, (2.3.63)

where we also indicated the graph in dual position space. $\tilde{I}_{\alpha\beta}$ is a conformal integral depending on $n = 2(\alpha+\beta)$ external points. For $\alpha=\beta=1$ we recover the conformal cross/box integral, which we have already mentioned. It can be expressed in terms of weight two polylogs of the conformal variables, and is discussed more in chapter 4. This is, however, the only integral in the family (2.3.63) which is known analytically.

For $\alpha = 2, \beta = 1$, we recover the six-point conformal double box:

$$\tilde{I}_{21} =$$



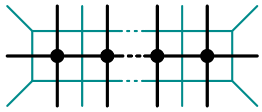
$$= \int \frac{\mathrm{d}^4 x_a}{\pi^2}\frac{\mathrm{d}^4 x_b}{\pi^2}\frac{1}{x_{a1}^2 x_{a5}^2 x_{a6}^2 x_{ab}^2 x_{b2}^2 x_{b3}^2 x_{b4}^2} = \frac{\psi_{21}(u_1,\ldots,u_9)}{x_{14}^2 x_{25}^2 x_{36}^2}. \tag{2.3.64}$$

This integral has been evaluated in terms of elliptic polylogarithms [82, 51] in a seven cross-ratio limit, where the integral contributes to a ten-point amplitude in $\mathcal{N}=4$ SYM [83]. The result for the fully off-shell integral has not been expressed as such, although a formal power series representation was derived in [77]. It is also related to the six-dimensional hexagon integral discussed in the previous section:

$$\partial_{u_8}\psi_{21} = -\pi\phi_6, \tag{2.3.65}$$

where ϕ_6 is defined in (2.3.62), and the cross ratios have been defined appropriately, see for example [10].

The sub-family of fishnet Feynman integrals (2.3.63) with one-dimensional lattice structure, which generalises the above box and double box integral to $2(n+1)$ external points, has been dubbed *train tracks* due to the shape of their graphs in dual position space:

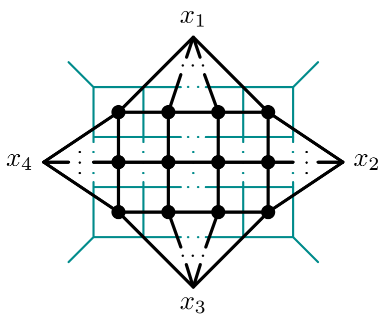


$$. \tag{2.3.66}$$

For on-shell kinematics, these integrals were conjectured to be associated with Calabi–Yau geometries [84]. For off-shell kinematics, the integrals (2.3.63) (except for the box and perhaps the double box) appear to be beyond the reach of modern methods of Feynman integral calculation.

An interesting simplification of the integral family (2.3.63) is the four-point limit:

$$I_{\alpha\beta} =$$



$$= \frac{\phi_{\alpha\beta}(u,v)}{x_{13}^{2\alpha} x_{24}^{2\beta}}. \tag{2.3.67}$$

For general α and $\beta = 1$ we recover the ladder integrals, named after their form in dual position space. These integrals have been known analytically since the work of Usyukina and Davydychev [85]. They can be expressed as

$$\phi_{\alpha 1}(z,\bar{z}) = -\frac{1}{z-\bar{z}} L_\alpha\left(\tfrac{z}{z-1}, \tfrac{\bar{z}}{\bar{z}-1}\right), \tag{2.3.68}$$

where L_α denotes the ladder functions, which are single-valued polylogarithms of z and $\bar{z}$, see [86]:

$$L_\alpha(z,\bar{z}) = \sum_{r=0}^{\alpha} \frac{(-1)^r (2\alpha - r)!}{r!(\alpha - r)!\alpha!} \log(z\bar{z})^r \left(\mathrm{Li}_{2\alpha-r}(z) - \mathrm{Li}_{2\alpha-r}(\bar{z})\right). \tag{2.3.69}$$

Remarkably, for general α, β, the functions $\phi_{\alpha\beta}$ can be expressed as a determinant of these ladder integrals, which was conjectured first in [87] by Basso and Dixon and proven recently in [88]. The result is

$$\phi_{\alpha\beta} = \det M^{\alpha\beta}, \qquad M_{ij}^{\alpha\beta} = c_{ij}^{\alpha\beta} \phi_{\beta-\alpha-1+i+j,1}, \tag{2.3.70}$$

where the coefficients $c_{ij}^{\alpha\beta}$ are defined according to

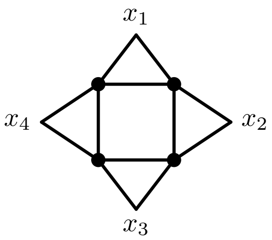
$$c_{ij}^{\alpha\beta} = \begin{cases} \prod_{k=j+1}^{i} p_{jk}(p_{jk}-1), & \text{for } i > j, \\ 1, & \text{for } i = j, \\ \prod_{k=i+1}^{j} [p_{jk}(p_{jk}-1)]^{-1}, & \text{for } i < j, \end{cases} \tag{2.3.71}$$

with

$$p_{jk} = \beta - \alpha - 1 + j + k. \tag{2.3.72}$$

The formula (2.3.70) is colloquially known as the *Basso–Dixon* formula, after its discoverers. As an example, if $\alpha = \beta = 2$ then we have

$$\phi_{22}(z,\bar{z}) = x_{13}^4 x_{24}^4$$



$$= \begin{vmatrix} \phi_{11} & 2\phi_{21} \\ \frac{1}{6}\phi_{21} & \phi_{31} \end{vmatrix} = \phi_{11}\phi_{31} - \frac{1}{3}(\phi_{21})^2. \tag{2.3.73}$$

In fact, (2.3.73) was already calculated via a differential equations approach in [89], although its expression in terms of ladder integrals was probably not known at this point.

Both integral families (2.3.63) and (2.3.67) represent exact correlation functions in the fishnet theory, as will be described in section 3.3. Due to their simple analytic structure the integrals (2.3.67), especially the ladders, have received a lot of attention in many different contexts, even very recently [90, 91]. The first few ladder integrals have famously been related to other four-point conformal integrals via so-called 'magic identities' [92]. Fascinatingly, the sum of all ladder diagrams been computed at strong coupling via a Bethe-Salpeter approach [93].

We conclude our discussion of correlation functions in conformal field theory, and proceed to discuss the conformal fishnet theory and its integrability.

Chapter 3

Fishnet Theory and Integrability

In this chapter we introduce the main QFT we study in this thesis, namely the *dynamical fishnet theory*. This theory can be obtained via a specific double-scaling limit of $\mathcal{N} = 4$ super Yang–Mills theory. In the most degenerate limit we recover the bi-scalar fishnet theory, often referred to as simply the fishnet theory. The fishnet theory is believed to be an example of an integrable logarithmic conformal field theory. We have already discussed logarithmic CFT, so we begin this section with a discussion of integrability in the context of quantum mechanics. We then discuss the $\mathcal{N} = 4$ SYM theory, and explain why it is believed to be an integrable conformal field theory. Finally, we explain how the fishnet theory is obtained from $\mathcal{N} = 4$ SYM, and describe how integrability manifests itself in this theory.

3.1 Integrability

Physicists have long been fascinated by models which are *exactly solvable*. These are models which are so constrained by symmetry that the dynamical variables thereof can be solved for analytically. Such models have been very fruitful for mathematics, as the powerful tools developed for their solution have proven very interesting from a theoretical standpoint. Usually such models are idealised versions of more physical models, however they can often be perturbed/deformed to represent more realistic models. In these cases the exactly solvable model represents a zeroth order approximation to the more realistic one, and the corrections to this model may be calculated perturbatively.

In many cases the reason for the exact solvability of a system can be explained by *integrability*. What it means precisely for a model to be integrable depends on what type of model it is. We will first discuss the case of classical mechanical models, for which there is a generally accepted definition of integrability, via the Arnold–Liouville theorem [94]. The main requirement for a model to be integrable in this case is the existence of sufficiently many conserved quantities along the flow of the Hamiltonian. This constrains the motion sufficiently such that the equations of motion can be integrated *by quadratures.* [1]

For quantum mechanical models, there is no analogous criterion for the integrability of a system. However, there are numerous common features which appear in the

[1]This is a historic term, which roughly means that the solutions can be obtained by solving a finite number of algebraic equations and computing a finite number of integrals.

solution of quantum models which are called integrable. We will discuss these features in the case of the most famous quantum integrable model, the Heisenberg XXX spin chain. This model has an $\mathfrak{su}(2)$ algebra symmetry, and it can be diagonalised using a so-called Bethe ansatz, which encodes the eigensystem of the Hamiltonian in a set of Bethe equations. The two most well-known forms of the Bethe ansatz are the coordinate Bethe ansatz and the algebraic Bethe ansatz. The algebraic Bethe ansatz is usually more powerful, and it is based on the fact that the Hamiltonian can be expressed in terms of a so-called R-matrix which satisfies the Yang–Baxter equation. This equation encodes a Yangian algebra, which is a structure believed to be at the heart of quantum integrability. Fascinatingly, these features have also appeared in the calculation of observables in certain quantum field theories. In these cases the field theory can be argued to be integrable in its own right. We will discuss this more in section 3.2 and section 3.3.

There are numerous resources which discuss integrability from different perspectives. The book [94] gives a mathematical formulation of classical mechanics and formulates the notion of Arnold-Liouville integrability. [95] provides an overview of classical integrability and the relevant mathematical concepts, with several examples. [96] is the canonical introduction to techniques in quantum integrability, where the Bethe ansatz approach to solving various integrable spin chains is described. More recent reviews on various aspects of integrability have appeared [97–101], as well as the book [102].

3.1.1 Classical Integrability: Arnold–Liouville and Lax Pairs

A classical mechanical model of n particles has a $2n$-dimensional phase space M, with generalised coordinates $q_1, \dots, q_n$ and corresponding canonically conjugate momenta $p_1, \dots, p_n$. The model is defined by a Hamiltonian function $H : M \to \mathbb{R}$. Together with a set of initial conditions $q_i(0), p_i(0)$, the Hamiltonian generates a time flow on M via *Hamilton's equations*[2]

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2, \dots, n. \tag{3.1.1}$$

Typical examples of such models are the harmonic oscillator ($n = 1$)

$$H(p, q) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}, \tag{3.1.2}$$

and the Kepler model ($n = 3$)

$$H(p_i, q_i) = \frac{p_1^2 + p_2^2 + p_3^2}{2m} + \frac{\beta}{\sqrt{q_1^2 + q_2^2 + q_3^2}}. \tag{3.1.3}$$

A classical mechanical model is exactly solvable if the equations of motion (3.1.1) can be integrated by quadratures. It is for this reason that exactly solvable models are also referred to as *integrable* models. There is a well-defined criterion of when this is possible in the case of classical mechanics, given by the Arnold–Liouville theorem [94]. It relies

[2] This is most elegantly formulated in the language of symplectic geometry [94]. Here we state it for the usual symplectic structure on $\mathbb{R}^{2n}$.

on the existence of enough *conserved quantities*. A dynamical variable $F : M \to \mathbb{R}$ is conserved on the flow generated by the Hamiltonian if the Poisson bracket of F and H is zero:

$$\frac{dF}{dt} = \{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} \right) = 0. \tag{3.1.4}$$

More generally, F_i is invariant under the flow generated by F_j (and vice versa) if $\{F_i, F_j\} = 0$.

One of the conditions of the Arnold–Liouville theorem is that there are n independent mutually Poisson-commuting functions $F_i : M \to \mathbb{R}$, of which the Hamiltonian is one $F_1 = H$. Then all of these functions are constant $F_i(p_j, q_j) = f_i$ under the Hamiltonian's flow. If the functions F_i are independent, then the $2n$-dimensional phase space M reduces to an n-dimensional level manifold M_f, defined by

$$M_f \coloneqq M|_{F_i = f_i}. \tag{3.1.5}$$

For example, in the case of the harmonic oscillator (3.1.2), the phase space is $M = \mathbb{R}^2$, spanned by the coordinates p and q. The Hamiltonian is conserved under its own flow

$$\frac{p(t)^2}{2m} + \frac{m\omega^2 q(t)^2}{2} = E, \tag{3.1.6}$$

for all times t. (3.1.6) determines a one-dimensional submanifold of $\mathbb{R}^2$, diffeomorphic to the circle $\mathbb{S}^1$.

The Arnold–Liouville theorem states that if a system has n independent mutually commuting quantities F_i, and the level manifold M_f is compact and connected, then

- M_f is invariant under the flow generated by the Hamiltonian $F_1 = H$, and the flows generated by $F_2, \dots, F_n$.
- M_f is diffeomorphic to the n-torus $\mathbb{T}^n \simeq \mathbb{S}^1(\varphi_1) \times \cdots \times \mathbb{S}^1(\varphi_n)$.
- There exists a canonical transformation to *action-angle* variables $(p_i, q_i) \to (I_i, \varphi_i)$. The equations of motion in these variables take the form

$$\dot{\varphi}_i = \omega_i, \qquad \dot{I}_i = 0, \tag{3.1.7}$$

  and so can be integrated trivially.

The angle variables φ_i represent coordinates on the torus, and the action variables are functions of the conserved quantities F_i, chosen appropriately so that the transformation $(p_i, q_i) \to (I_i, \varphi_i)$ is canonical.[3] We do not go through the proof here, although we mention that the action variables can be constructed as

$$I_j = \frac{1}{2\pi} \int_{\gamma_j} (p_1 dq_1 + \cdots + p_n dq_n), \tag{3.1.8}$$

where γ_j is the j^{th} cycle on the torus, parametrised by the angle φ_j.

[3] A change of coordinates on M is canonical if it preserves the symplectic structure. This means that the form of the Poisson brackets and Hamilton's equations are unchanged.

In the Hamilton–Jacobi formulation of classical mechanics, canonical transformations are typically defined through a *generating function* S. This can also be done for the canonical transformation $(p_i, q_i) \to (I_i, \varphi_i)$, where the generating function is defined

$$S(q_i, I_i) = \int_{C(q_0,q)} (p_1 dq_1 + \cdots + p_n dq_n), \tag{3.1.9}$$

where $C(q_0, q)$ is an open, not self-intersecting path on M_f, which begins at $q_0 \in \mathbb{R}^n$ and ends at $q \in \mathbb{R}^n$. Then we have

$$\frac{\partial S}{\partial I_i} = \varphi_i, \qquad \frac{\partial S}{\partial q_i} = p_i. \tag{3.1.10}$$

Example: Kepler Model. The Kepler model (3.1.3) is a Liouville integrable model. In spherical coordinates (r, θ, ϕ) the Hamiltonian reads

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + \frac{\beta}{r}. \tag{3.1.11}$$

For this model H is conserved, as well as the components of the angular momentum J_i. These are not independent quantities, as they satisfy $\{J_i, J_j\} = \epsilon_{ijk} J_k$. The independent conserved quantities are typically chosen to be

$$H, \qquad J^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}, \qquad J_3 = p_\phi. \tag{3.1.12}$$

The generating function S defined in (3.1.9) can be found by writing the momenta p_r, p_θ, p_ϕ in terms of the conserved quantities using (3.1.12):

$$S = \int \mathrm{d}r \sqrt{2\left(H - \frac{\beta}{r}\right) + \frac{J^2}{r^2}} + \int \mathrm{d}\theta \sqrt{J^2 - \frac{J_3^2}{\sin^2\theta}} + \int \mathrm{d}\phi\, J_3. \tag{3.1.13}$$

Then the angular coordinates can be recovered using (3.1.10):

$$\varphi_H = \frac{\partial S}{\partial H}, \qquad \varphi_{J^2} = \frac{\partial S}{\partial J^2}, \qquad \varphi_{J_3} = \frac{\partial S}{\partial J_3}. \tag{3.1.14}$$

Fascinatingly, there are a further two independent conserved quantities for the Kepler model, which reflects a hidden enhancement of the symmetry group $SO(3)$ to $SO(4)$. The Hamiltonian (3.1.11) commutes with the components of the *Laplace-Runge-Lenz* vector $\vec{A}$, defined as

$$\vec{A} = \vec{p} \times \vec{L} - m\beta \frac{\vec{r}}{r^2}. \tag{3.1.15}$$

The components of this vector satisfy $\{A_i, A_j\} = \epsilon_{ijk} A_k$. Systems with more than n conserved quantities are called *super-integrable.* The maximum number of independent conserved quantities a system can have is $2n - 1$, and such systems are called *maximally super-integrable.* The Kepler model is a example of a maximally super-integrable classical model, with independent conserved quantities H, J^2, J_3, A^2, A_3.

Classical integrable systems are few and far between in physics, and as such they are treasured when found. The harmonic oscillator (3.1.2) is the simplest example of a classical integrable system with $n = 1$. For this model this transition to action angle variables is very natural, since the phase space is already an ellipse. There is also the Neumann model, which describes n particles moving on the surface of a sphere, constrained by harmonic forces of various frequencies [103]. The existence of sufficiently many conserved quantities was exhibited by Uhlenbeck [104]. There are also several examples of integrable spinning top models, for example the Euler top, summarised in [95]. Moving towards systems with an infinite number of degrees of freedom, the classical closed Toda lattice with N particles is an integrable model:

$$H = \sum_{n=1}^{N} \left(\frac{p_i^2}{2m} + \exp(q_n - q_{n+1}) \right) , \tag{3.1.16}$$

where we identify $q_{N+1} \equiv q_N$. In the limit $N \to \infty$ we can obtain an integrable two-dimensional field theory. The conserved quantities of (3.1.16) are best described in the *Lax pair* formalism.

Lax Pairs. The existence of sufficiently many conserved quantities is one of the keys for a system to be integrable. One way to find conserved quantities for a system is to construct a *Lax pair.* These are a pair of matrices L, M, whose entries depend on the phase space variables q_i, p_i, such that the equation

$$\frac{dL}{dt} - [L, M] = 0 \tag{3.1.17}$$

is equivalent to the equations of motion (3.1.1). For example, if

$$L = \frac{1}{\sqrt{m}} \begin{pmatrix} p & m\omega q \\ m\omega q & -p \end{pmatrix} , \qquad M = \frac{\omega}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{3.1.18}$$

then the combination (3.1.17) evaluates to

$$\frac{dL}{dt} - [L, M] = \frac{1}{\sqrt{m}} \begin{pmatrix} m\omega^2 q + \dot{p} & \omega(-p + m\dot{q}) \\ \omega(-p + m\dot{q}) & -m\omega^2 q - \dot{p} \end{pmatrix} . \tag{3.1.19}$$

The vanishing of (3.1.19) is equivalent to the equations of motion of the harmonic oscillator (3.1.2), $\dot{q} = p/m$ and $\dot{p} = -m\omega^2 q$.

Given a Lax pair, one can easily generate conserved quantities. Indeed, for any $k \in \mathbb{N}$ the quantity

$$O_k \coloneqq \operatorname{tr} L^k \tag{3.1.20}$$

is conserved under the flow of the Hamiltonian. This is because

$$\frac{d}{dt} \operatorname{tr} L^k = \sum_{i=1}^{k} \operatorname{tr} \left(L^{i-1} \frac{dL}{dt} L^{k-i} \right) = \sum_{i=1}^{k} \operatorname{tr}(L^{i-1}[L, M]L^{k-i}) = 0 \tag{3.1.21}$$

by the cyclicity of the trace. A priori, it might seem that this generates an infinite number of of conserved quantities. However, the operators O_k are of course not independent. For example, for the Lax pair of the harmonic oscillator (3.1.18), we have

$$O_{2k} = 2^{k+1} H^k , \qquad O_{2k+1} = 0 , \qquad k = 0, 1, 2, \ldots , \tag{3.1.22}$$

where H is the Hamiltonian (3.1.1). This is not surprising, as a one-dimensional system can only have one conserved quantity.

Lax pairs form the algebraic foundation for classical integrable models. The construction of such a pair is a non-trivial task and must be done on a case by case basis. Notice that the equation (3.1.17) makes no reference to the Poisson bracket structure underlying the corresponding Hamiltonian system. This can be incorporated by introducing an algebraic structure known as the *r-matrix*. Specifically, the eigenvalues of the matrix L will poisson commute if and only if the classical Yang–Baxter equation is satisfied [95]. We will discuss the quantum version of this equation in section 3.1.2.

Lax pairs are also central to integrability constructions for classical field theories. Although we do not discuss integrable classical field theories here, they constitute a large number of the integrable models which are known. They are typically defined in $(1+1)$ spacetime dimensions, for a field $\phi(t,x)$. The model can be defined by a Lagrangian, and it is integrable if the field equations can be written in terms of an operatorial Lax pair, analogously to (3.1.17). Often it is the field equations themselves that are called integrable. Famous examples are the sine-Gordon equation and the Korteweg–de Vries equation. That these equations admit solitonic solutions is typical for integrable field theories.

3.1.2 Quantum Integrability: Heisenberg Spin Chain

While the notion of Arnold–Liouville integrability is the generally accepted definition for a system for be classically integrable, there is no corresponding theorem for quantum mechanical models [105]. One reason for this is provided in [106]: in general a quantum mechanical system is defined on a (possibly infinite-dimensional) Hilbert space $\mathcal{H}$, and it is not easy to find a consistent definition for 'degree of freedom'. Therefore it is difficult to know how many conserved quantities one should expect for a system to be integrable. Moreover, a theorem by Von Neumann states that given commuting operators A and B on a Hilbert space $\mathcal{H}$, there necessarily exists an operator C such that both A and B are functions of C. Therefore, it is less clear how to define independence of operators.

Nevertheless, many quantum models have been discovered which deserve to be called integrable. This is because there exist powerful algebraic methods to extract the spectrum of these models, i.e. the eigenvectors and eigenvalues of the Hamiltonian H.[4] This is best described in the case of the famous quantum spin chain models. A possible working definition for quantum integrability is the following: a model is quantum integrable if the Hamiltonian can be derived from a R-matrix which satisfies the (quantum) Yang–Baxter equation.[5] In this case a transfer matrix $t(u) : \mathcal{H} \to \mathcal{H}$ can be constructed from the R-matrix, where $u \in \mathbb{C}$ is the *spectral parameter*. These form a one-parameter family of commuting operators $[t(u), t(u')] = 0$. The Hamiltonian can be identified from this transfer matrix, typically as a logarithmic derivative thereof. The transfer matrix can be diagonalised using a Bethe ansatz, named after Hans Bethe's solution of the spin-$\frac{1}{2}$ XXX (also known as Heisenberg) spin chain [109]. The eigensystem is encoded is a set of rational equations known as Bethe

[4] In this section we will discuss operators acting on Hilbert spaces, however we will not use the usual 'hat' notation for operators commonly seen in quantum mechanics.

[5] In fact, this definition of integrability was used to bootstrap new integrable models [107, 108].

equations. This simultaneously diagonalises the Hamiltonian as well as an infinite tower of operators which appear in the expansion of $\log t(u)$. There is a weaker form of the Bethe ansatz known as the coordinate Bethe ansatz. A priori this diagonalises only the Hamiltonian, however it is a much more direct approach to deriving the Bethe equations of the integrable model.

In this section we focus on models which are diagonalisable. In chapter 6 we will discuss non-diagonalisable models, derived from broken algebra symmetries. The discussion in this section will be useful to compare to later, as we discuss the integrability of these non-diagonalisable models.

Heisenberg Spin Chain. We will describe all of the basic concepts of quantum integrability for the case of the spin-$\frac{1}{2}$ Heisenberg spin chain of length L. This model is defined on the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes L}$, where we take a local basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ at each site. A general state is then defined as linear combination of lists of up and down arrows. As an example, for $L = 2$ we could have

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \in \mathcal{H}, \tag{3.1.23}$$

where $|\uparrow\downarrow\rangle$ is shorthand for $|\uparrow\rangle \otimes |\downarrow\rangle$. The Hamiltonian for the Heisenberg spin chain is

$$H = \sum_{i=1}^{L}(1 - 4(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z)), \tag{3.1.24}$$

where $S_i^{x,y,z}$ are spin operators acting non-trivially only on site i of the spin chain, and we identify $S_{L+1}^a \equiv S_1^a$. They are related to the Pauli matrices via $S^a = \frac{1}{2}\sigma^a$, where

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1.25}$$

These spin matrices furnish a local representation of $\mathfrak{su}(2)$ at each site:

$$[S_i^a, S_j^b] = i\delta_{ij}\epsilon^{abc}S_j^c. \tag{3.1.26}$$

The Hamiltonian (3.1.24) is also conveniently written in terms of raising and lowering operators $S_i^\pm \coloneqq S_i^x \pm iS_i^y$. Then the operators $\{S^+, S^-, S^z\}$ act on local states as

$$\begin{aligned} &S^+|\uparrow\rangle = 0, \qquad S^-|\uparrow\rangle = |\downarrow\rangle, \qquad S^z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle, \\ &S^+|\downarrow\rangle = |\uparrow\rangle, \qquad S^-|\downarrow\rangle = 0, \qquad S^z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle. \end{aligned} \tag{3.1.27}$$

In terms of these operators the Hamiltonian (3.1.24) reads

$$H = \sum_{i=1}^{L}(1 - 2(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) - 4S_i^z S_{i+1}^z). \tag{3.1.28}$$

From (3.1.28) one can see that Heisenberg spin chain, as defined, is a rudimentary quantum model of ferromagnetism, since the spins will tend to align. Although not as physical, probably the most transparent representation of the Hamiltonian (3.1.24) is in terms of the *permutation operator* $\mathbb{P} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$, which acts as

$$\mathbb{P}\, x \otimes y = y \otimes x. \tag{3.1.29}$$

Similarly, $\mathbb{P}$ acts on any tensor product space $V \otimes V$ by swapping the order of terms. In terms of $\mathbb{P}$ the Hamiltonian (3.1.24) is simply

$$H = 2\sum_{i=1}^{L}(1 - \mathbb{P}_{i,i+1}). \tag{3.1.30}$$

(3.1.30) is very simple to work with computationally, although some of the symmetries of the Hamiltonian are obscured in this representation. For example, all of the operators

$$S^a := \sum_{i=1}^{L} S_i^a, \qquad a = x, y, z \tag{3.1.31}$$

commute with the Hamiltonian:

$$[H, S^a] = 0, \qquad a = x, y, z, \tag{3.1.32}$$

which is easily proven using the algebra (3.1.26). S^z is the operator which counts (half) the difference between the number of up spins and down spins in a state. Therefore, when diagonalising H one can consider states with a fixed number of down spins. More concretely, we can grade the Hilbert space

$$\mathcal{H} = \bigoplus_{M=0}^{L} \mathcal{H}_M, \tag{3.1.33}$$

where $\mathcal{H}_M$ is the vector subspace of $\mathcal{H}$ spanned by states with M flipped spins, or *magnons*. By simple combinatorics the dimension of $\mathcal{H}_M$ is $\binom{L}{M}$. $\mathcal{H}_0$ is spanned by a single state, the *ferromagnetic vacuum* $|0\rangle := |\uparrow\uparrow\cdots\uparrow\rangle$. This is an eigenstate of the Hamiltonian with eigenvalue 0

$$H|0\rangle = 0. \tag{3.1.34}$$

Because of (3.1.32) H further commutes with the total spin lowering operator $[H, S^-] = 0$. Therefore if $\psi \in \mathcal{H}_M$ is an eigenstate of H, so is $S^-\psi \in \mathcal{H}_{M-1}$.[6] This causes the eigenstates of H to arrange in $\mathfrak{su}(2)$ multiplets. For example, for $L = 2$ the eigenstates arrange into a spin-1 representation and a spin-0 representation

$$|\uparrow\uparrow\rangle \longrightarrow |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \longrightarrow |\downarrow\downarrow\rangle, \tag{3.1.35}$$
$$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle, \tag{3.1.36}$$

respectively. We proceed to describe two methods for the diagonalisation of the Hamiltonian (3.1.24), the coordinate and algebraic Bethe ansätze.

[6]Note that $S^-\psi$ might be the zero vector.

Coordinate Bethe Ansatz. The coordinate Bethe ansatz makes use of the fact that rather than diagonalising H on the whole Hilbert space $\mathcal{H}$, we can diagonalise it on the restriction to each $\mathcal{H}_M, M = 0, 1, \ldots, L$. It searches for eigenstates of H in the form of discretised plane waves, motivated by the interpretation of magnons as particles moving along the spin chain, with a definite momentum p.

The first non-trivial case is $M = 1$, where we look for eigenstates using the plane wave ansatz

$$\psi_p = \sum_{n=1}^{L} e^{ipn} S_n^- |0\rangle \in \mathcal{H}_1. \tag{3.1.37}$$

Acting with H on this state, we see that it is an eigenstate with eigenvalue

$$E(p) = 4 \sin^2 \frac{p}{2}, \tag{3.1.38}$$

provided p satisfies

$$e^{ipL} = 1. \tag{3.1.39}$$

(3.1.39) is imposed by the periodicity along the spin chain, and is the simplest example of a Bethe equation. Its L solutions in terms of the L^{th} roots of unity exhaust the space $\mathcal{H}_1$ of eigenvectors. Note that if $p = 0$ we recover the $\mathfrak{su}(2)$ descendant of the vacuum

$$\psi_{p=0} = |{\downarrow}{\uparrow} \cdots {\uparrow}\rangle + |{\uparrow}{\downarrow} \cdots {\uparrow}\rangle + \cdots + |{\uparrow}{\uparrow} \cdots {\downarrow}\rangle = S^-|0\rangle. \tag{3.1.40}$$

The structure becomes much more intricate for the case of two magnons $M = 2$. In this case we make the modified plane wave ansatz

$$\psi_{p_1 p_2} = \sum_{n_1 < n_2} (\mathcal{A}_{12}(p_1, p_2) e^{ip_1 n_1 + ip_2 n_2} + \mathcal{A}_{21}(p_1, p_2) e^{ip_1 n_2 + ip_2 n_1}) S_{n_1}^- S_{n_2}^- |0\rangle. \tag{3.1.41}$$

The summand of (3.1.41) behaves as two independent $M = 1$ eigenstates under H if $n_2 > n_1 + 1$, and thus has eigenvalue $E(p_1) + E(p_2)$. For this to remain true even in the case $n_2 = n_1 + 1$, a quick calculation shows that we must have

$$\mathcal{A} := \frac{\mathcal{A}_{21}}{\mathcal{A}_{12}} = \frac{\cot \frac{p_1}{2} - \cot \frac{p_2}{2} - 2i}{\cot \frac{p_1}{2} - \cot \frac{p_2}{2} + 2i}. \tag{3.1.42}$$

$\mathcal{A}$ is interpreted as a *scattering matrix*, and is the factor the wavefunction picks up when a magnon of momentum p_1 moves past a magnon of momentum p_2. Periodicity along the chain implies the Bethe equations

$$e^{ip_1 L} = \mathcal{A}(p_2, p_1), \qquad e^{ip_2 L} = \mathcal{A}(p_1, p_2). \tag{3.1.43}$$

In summary, the state $\psi_{p_1 p_2} \in \mathcal{H}_2$ is an eigenstate of H with eigenvalue $E(p_1) + E(p_2)$, provided the Bethe equations (3.1.43) are satisfied. It is less clear that these states exhaust the $\binom{L}{2} = \frac{L(L-1)}{2}$ states in $\mathcal{H}_2$. This is the question of *completeness* of the Bethe equations (3.1.43) [110].

Fascinatingly, the eigenstates in $\mathcal{H}_M$ for general M can be established using only the equations written down so far. The general coordinate Bethe ansatz is

$$\psi_{p_1 \ldots p_M} = \sum_{n_1 < n_2 < \cdots < n_M} \sum_{\sigma \in S_M} \mathcal{A}_\sigma(p_1, \ldots, p_M) e^{i(p_{\sigma_1} n_1 + \cdots + p_{\sigma_M} n_M)}, \tag{3.1.44}$$

where S_M is the symmetric group on M letters. The coefficients $\mathcal{A}_\sigma$ can be factorised into two-particle scattering processes:

$$\mathcal{A}_\sigma = \prod_{i<j} \mathcal{A}_{\sigma_i \sigma_j}. \tag{3.1.45}$$

Then $\psi_{p_1 \dots p_M}$ is an eigenstate of H with eigenvalue $E(p_1) + \cdots + E(p_M)$, provided the Bethe equations are satisfied

$$e^{ip_i L} = \prod_{j \neq i} \mathcal{A}(p_j, p_i). \tag{3.1.46}$$

The factorisation of a many-body scattering process into two-body processes is indicative of integrability.

Algebraic Bethe Ansatz. The main objects for the algebraic Bethe ansatz are the Lax operator and the R-matrix.[7] We denote the quantum spaces of the spin chain as $V_n = \mathbb{C}^2, n = 1, \dots, L$ and introduce an auxiliary space $V_a = \mathbb{C}^2$. The Lax operator $\mathcal{L}_{na} : V_n \otimes V_a \to V_n \otimes V_a$ intertwines the phyiscal and auxiliary space, and is defined as

$$\mathcal{L}_{na}(u) = u \mathbb{I}_n \otimes \mathbb{I}_a + i \sum_{c=x,y,z} S_n^c \otimes \sigma_a^c = \begin{pmatrix} u + iS_n^z & iS_n^- \\ iS_n^+ & u - iS_n^z \end{pmatrix} = \left(u - \tfrac{i}{2}\right) \mathbb{I} + i \mathbb{P}_{na}, \tag{3.1.47}$$

where $u \in \mathbb{C}$ is the *spectral parameter* and $\mathbb{P}$ is the permutation operator defined in (3.1.29). Depending on the calculation, one of the representations of the Lax operator given in (3.1.47) may be more convenient than the others. The R-matrix $R_{ab} : V_a \otimes V_b \to V_a \otimes V_b$ acts on two copies of the auxiliary space, and is defined

$$R_{ab}(u) = u\mathbb{I} + i\mathbb{P}_{ab} = \mathcal{L}_{ab}\left(u + \tfrac{i}{2}\right). \tag{3.1.48}$$

As a matrix on $V_a \otimes V_b$, R_{ab} can be written as

$$R_{ab} = \begin{pmatrix} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{pmatrix}. \tag{3.1.49}$$

From the Lax operator (3.1.47), one may build a monodromy along the spin chain $T_a(u) : \mathcal{H} \otimes V_a \to \mathcal{H} \otimes V_a$

$$T_a(u) \coloneqq \mathcal{L}_{La} \mathcal{L}_{L-1,a} \cdots \mathcal{L}_{1a} = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{3.1.50}$$

which is interpreted as a matrix in the auxiliary space V_a, whose entries act on the whole spin chain $A, B, C, D : \mathcal{H} \to \mathcal{H}$. From (3.1.47) we notice that each of these

[7] In this case the operators are simply related by a shift of the spectral parameter. However, for more complicated spin chains the auxiliary space does not coincide with the physical spaces of the spin chain, and these operators act on different spaces.

operators are polynomials of degree L in the spectral parameter u. Taking the trace over the auxiliary space, we can recover the *transfer matrix* $t(u) : \mathcal{H} \to \mathcal{H}$

$$t(u) \coloneqq \mathrm{tr}_a T_a(u) = A(u) + D(u) = \sum_{j=0}^{L} u^j t_j(u). \tag{3.1.51}$$

The operators A, B, C, D appearing in (3.1.50) obey a number of intricate algebraic relations, which stem from the fact that the R-matrix satisfies the *quantum Yang–Baxter equation*:

$$R_{ab}(u_1 - u_2) R_{ac}(u_1) R_{bc}(u_2) = R_{bc}(u_2) R_{ac}(u_1) R_{ab}(u_1 - u_2). \tag{3.1.52}$$

(3.1.52) can be verified explicitly by expressing each operator as a matrix in $V_a \otimes V_b \otimes V_c$ and performing the matrix multiplication, for example with Mathematica. For example, we have

$$R_{ab}(u) \otimes \mathbb{I}_c = \begin{pmatrix} u+i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u+i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & u & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & u & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u+i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u+i \end{pmatrix}.$$

R_{ac} can be obtained with the help of the permutation operator $\mathbb{P}$, more details on this are given in section 6.2.4. The Yang–Baxter equation lies at the heart of all algebraic relations which appear in the algebraic Bethe ansatz, and is crucial to the integrability of a quantum model. For one, it implies that the monodromy matrix (3.1.50) satisfies

$$R_{ab}(u_1 - u_2) T_a(u_1) T_b(u_2) = T_b(u_2) T_a(u_1) R_{ab}(u_1 - u_2), \tag{3.1.53}$$

which can be established by expressing the monodromy matrix (3.1.50) in terms of R-matrices using (3.1.48), and repeatedly applying (3.1.52). Taking the trace over both auxiliary spaces, one can then establish that

$$[t(u_1), t(u_2)] = 0, \qquad u_1, u_2 \in \mathbb{C}, \tag{3.1.54}$$

i.e. the transfer matrix $t(u)$ constitutes a one-parameter family of commuting operators on $\mathcal{H}$. Remarkably, the Heisenberg spin chain Hamiltonian (3.1.24) is a member of this family. It can be obtained from the logarithmic derivative of the transfer matrix:

$$i \frac{d}{du} \log t(u) \Big|_{u=i/2} \coloneqq i t^{-1}(i/2) \frac{d}{du} t(u) \Big|_{u=i/2} = L\mathbb{I} - \frac{H}{2}. \tag{3.1.55}$$

Specifically, the logarithmic derivative of $t(u)$ at $u = i/2$ is proportional to the interacting part of the Hamiltonian (3.1.30):

$$\frac{d}{du} \log t(u) \Big|_{u=i/2} = -i \sum_n \mathbb{P}_{n,n+1}. \tag{3.1.56}$$

Other logarithmic derivatives of $t(u)$ at $u = i/2$ give rise to interesting operators on the spin chain. The transfer matrix at $u = i/2$ is proportional to the shift operator

$$t(i/2) = i^L U, \tag{3.1.57}$$

where U maps any state $|i_1 i_2 \cdots i_L\rangle \to |i_L i_1 \cdots i_{L-1}\rangle$. In general the logarithm of the transfer matrix can be expanded

$$\log t(u) = \sum_{j=1}^{\infty} (u - i/2)^{j-1} Q_j, \tag{3.1.58}$$

where $Q_j : \mathcal{H} \to \mathcal{H}$ are commuting operators on the spin chain. For $j \leq L$, Q_j is an operator of range j. For example we have

$$Q_3 = \sum_{i=1}^{L} [\mathbb{P}_{i,i+1}, \mathbb{P}_{i+1,i+2}]. \tag{3.1.59}$$

If one can diagonalise the transfer matrix $t(u)$ at all values of the spectral parameter u, then the operators Q_j are automatically diagonalised. All of the above equations can be derived by expanding the expression (3.1.50) for the monodromy matrix and using simple properties of permutation opertors $\mathbb{P}$. Notice that the Lax operators simplify substantially at $u = i/2$; we simply have $\mathcal{L}_{i,a}(i/2) = i\mathbb{P}_{i,a}$. We give more details for similar calculations appearing in section 6.2.4.

The ground state $|0\rangle$ is an eigenstate of the transfer matrix:

$$t(u)|0\rangle = [A(u) + D(u)]|0\rangle = \left[\left(u + \tfrac{i}{2}\right)^L + \left(u - \tfrac{i}{2}\right)^L\right] |0\rangle. \tag{3.1.60}$$

The algebraic Bethe ansatz is an ansatz for eigenstates of $t(u)$ with M flipped spins with respect to the vacuum $|0\rangle$, i.e. eigenstates in $\mathcal{H}_M$. It is heavily based on the algebra of the operators A, B, C, D appearing in the monodromy (3.1.50), which follows from the Yang–Baxter equation (3.1.52). To motivate the ansatz, we first notice that the state $B(u_1)|0\rangle$ is a linear combination of states with exactly one flipped spin, i.e. $B(u_1)|0\rangle \in \mathcal{H}_1$. This is because when expanding the $(1, 2)$ entry of the monodromy matrix (3.1.50), if there is more than one occurrence of S^-, there is necessarily an occurrence of S^+, which annihilates $|0\rangle$. For example, for $L = 3$ we have

$$\tfrac{1}{i} B(u)|0\rangle = \left[(u - \tfrac{i}{2})^2 S_1^- + (u - \tfrac{i}{2})(u + \tfrac{i}{2}) S_2^- + (u + \tfrac{i}{2})^2 S_3^-\right] |{\uparrow\uparrow\uparrow}\rangle \in \mathcal{H}_1. \tag{3.1.61}$$

This motivates the idea to search for eigenstates of $t(u)$ of the form $|u_1\rangle := B(u_1)|0\rangle$. We investigate the action of $t(u)$ on this state:

$$t(u)|u_1\rangle = [A(u) + D(u)]B(u_1)|0\rangle. \tag{3.1.62}$$

We would like to 'commute' $A(u)$ and $D(u)$ past $B(u_1)$, since these operators have a very simple action on the vacuum:

$$A(u)|0\rangle = (u + \tfrac{i}{2})^L |0\rangle, \qquad D(u)|0\rangle = (u - \tfrac{i}{2})^L |0\rangle. \tag{3.1.63}$$

To do this, we use the following algebra of the entries of the monodromy matrix:

$$B(u)B(v) = B(v)B(u), \tag{3.1.64}$$
$$A(u)B(v) = \frac{u-v-i}{u-v}B(v)A(u) + \frac{i}{u-v}B(u)A(v), \tag{3.1.65}$$
$$D(u)B(v) = \frac{u-v+i}{u-v}B(v)D(u) - \frac{i}{u-v}B(u)D(v), \tag{3.1.66}$$

which can be derived with the help of the Yang–Baxter equation (3.1.52). Using (3.1.63), (3.1.65), and (3.1.66), we can compute (3.1.62) to be

$$\begin{aligned} t(u)|u_1\rangle = &\left[(u+\tfrac{i}{2})^L\frac{u-u_1-i}{u-u_1} + (u-\tfrac{i}{2})^L\frac{u-u_1+i}{u-u_1}\right]|u_1\rangle \\ &+ \left[(u+\tfrac{i}{2})^L - (u-\tfrac{i}{2})^L\right]\frac{i}{u-u_1}B(u)|0\rangle. \end{aligned} \tag{3.1.67}$$

Therefore $|u_1\rangle$ is an eigenstate of $t(u)$ if

$$\left(\frac{u_1+\frac{i}{2}}{u_1-\frac{i}{2}}\right)^L = e^{ip_1L} = 1, \tag{3.1.68}$$

where we denoted

$$u_i \coloneqq \tfrac{1}{2}\cot\tfrac{p_i}{2}. \tag{3.1.69}$$

The equation (3.1.68) is solved by the L^{th} roots of unity $p_1 = \exp(2\pi ik/L)$, $k = 0,1,\dots,L-1$, which exhausts the L eigenstates in $\mathcal{H}_1$. Note that $p_1 = 0$ corresponds to $u_1 = \infty$.[8]

The story proceeds analogously for higher numbers of excitations. For eigenstates in $\mathcal{H}_M$, we make the ansatz

$$|u_1,\dots,u_M\rangle \coloneqq B(u_1)\cdots B(u_M)|0\rangle. \tag{3.1.70}$$

By using the algebra (3.1.64)-(3.1.66), we can similarly show that

$$t(u)|u_1,\dots,u_M\rangle = \left[(u+\tfrac{i}{2})^L\prod_{j=1}^{M}\frac{u-u_j-i}{u-u_j} + (u-\tfrac{i}{2})^L\prod_{j=1}^{M}\frac{u-u_j+i}{u-u_j}\right]|u_1,\dots,u_M\rangle, \tag{3.1.71}$$

provided $u_1,\dots,u_M$ satisfy the *Bethe equations*

$$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^L = \prod_{k\neq j}\frac{u_j-u_k+i}{u_j-u_k-i}. \tag{3.1.72}$$

This set of equations coincides with the Bethe equations obtained from the coordinate Bethe ansatz (3.1.46), which can be seen by using the change of variables (3.1.69). Notice that the operator $D(u)$ did not appear in this construction. This operator would be used if we had chosen the vacuum $|0'\rangle \coloneqq |{\downarrow\downarrow}\cdots{\downarrow}\rangle$. The eigenvalue of the

[8]Bethe states with finite Bethe roots are highest weight states, satisfying $S^+|u\rangle = 0$. In the case $u_1 = \infty$ the state is an $\mathfrak{su}(2)$ descendant $|u_1\rangle = S^-|0\rangle$.

XXX Hamiltonian can be derived by taking the logarithmic derivative of the eigenvalue in (3.1.71):

$$E = \sum_{i=1}^{M} \frac{1}{u_i^2 + 1/4} = E(p_1) + \cdots + E(p_M), \tag{3.1.73}$$

where $E(p)$ is defined in (3.1.38). Therefore the eigenvalues derived from both Bethe ansätze coincide.[9] The algebraic Bethe ansatz can be modified to solve more complex spin chains. There are various ways to construct different spin chains, the simplest being to mimic the definition of the Heisenberg spin chain (3.1.24), but use a spin-s representation of $\mathfrak{su}(2)$ and take the physical spaces to be $\mathbb{C}^{2s+1}$. For these models, the form of the Hamiltonian derived from the transfer matrix differs slightly from the spin-$\frac{1}{2}$ case (3.1.24). For example, for $s = 1$ we have [96]

$$H_{s=1} \simeq \sum_{i=1}^{L} \sum_{a=x,y,z} \left[S_i^a S_{i+1}^a - (S_i^a S_{i+1}^a)^2 \right]. \tag{3.1.74}$$

The XXX spin chain can be deformed in many ways, for example by adding an inhomogeneity in one of the spin axes. This leads to the XXZ spin chain, and the algebra symmetry $\mathfrak{su}(2)$ is replaced by a quantum group symmetry $U_q(\mathfrak{su}(2))$. In this case the Lax operators contain trigonometric functions of the spectral parameter. One can also consider spin chains with symmetries corresponding to higher rank gauge groups, for example $\mathfrak{su}(n)$ with $n > 2$. In this case the algebraic Bethe ansatz can be modified to a *nested* Bethe ansatz, which is based on solving a nested system of $n - 1$ $\mathfrak{su}(2)$ Bethe equations [100,111]. Not all integrable spin chains admit a known algebraic description. For example, the Inozemtsev model is a long-range elliptic spin chain which admits a coordinate Bethe ansatz, although there is no known algebraic Bethe ansatz [112].

3.1.3 Yangian Algebra

Yangian symmetry is an extension of the common Lie algebra symmetry of physical models. It is intimately tied to the Yang–Baxter equation, already discussed in section 3.1.2, which was first discovered in a one-dimensional scattering problem by Yang [113], and independently discovered by Baxter in the context of the eight vertex model [114]. Later, Drinfel'd did work on the algebraic foundations of the Yangian algebra as a Hopf algebra, in an effort to better understand solutions of the Yang–Baxter equation [115–117]. He introduced three *realisations* of the Yangian algebra, which are equivalent descriptions of the algebra, each useful in different contexts. Yangian symmetry appears in many quantum integrable models, commonly in spin chains and two-dimensional quantum field theory. A prime example is the two-dimensional chiral Gross–Neveu model [118]. More recently Yangian symmetry has been exhibited in four-dimensional planar $\mathcal{N} = 4$ SYM, both at the level of observables [119] and the action [120].

In this section, we will first review some concepts from Lie algebras, and then describe the first of the realisations of the Yangian algebra proposed by Drinfel'd. This realisation is in terms of abstract generators. The second realisation is based

[9] The eigenstates differ by a normalisation which depends on the Bethe roots u_i [101].

on the Chevally–Serre basis of the underlying Lie algebra, and the third is based on the so-called RTT relation, an example of which we already saw in (3.1.53). We will not discuss the second and third realisations in this thesis. We will then explain the Hopf algebra structure of the Yangian, and its evaluation representations. We finally describe how the Yangian algebra appears in the context of spin chains, focusing on the example of the Heisenberg spin chain. Later, in sections 3.2.2 and 3.3.3 we will describe how Yangian symmetry manifests itself in integrable quantum field theory. We will be brief in our discussion, since in this thesis we are mainly focused on applying the Yangian for a very specific algebra (the conformal algebra) in specific *evaluation* representations. A modern review of Yangian symmetry is given in [97].

Lie Algebras. We first briefly introduce some concepts of Lie algebras, mainly to fix notation. A nice review of Lie algebras in the context of particle physics is given in [121]. There are many textbooks which offer a more mathematical treatment, for example [122].

A Lie algebra $\mathfrak{g}$ is a vector space over some field (typically $\mathbb{R}$ or $\mathbb{C}$) spanned by generators J^A, $A = 1, 2, \dots, \dim(\mathfrak{g})$, together with a bilinear bracket operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is antisymmetric and satisfies the Jacobi identity:

$$[\mathrm{J}^A, \mathrm{J}^B] = -[\mathrm{J}^B, \mathrm{J}^A], \qquad \mathrm{J}^A, \mathrm{J}^B \in \mathfrak{g}, \tag{3.1.75}$$

$$[\mathrm{J}^A, [\mathrm{J}^B, \mathrm{J}^C]] + [\mathrm{J}^C, [\mathrm{J}^A, \mathrm{J}^C]] + [\mathrm{J}^B, [\mathrm{J}^C, \mathrm{J}^A]] = 0, \qquad \mathrm{J}^A, \mathrm{J}^B, \mathrm{J}^C \in \mathfrak{g}. \tag{3.1.76}$$

The bracket can be defined by a set of *structure constrants* $f^{AB}{}_C$, which take values in the field $\mathbb{K}$ underlying the Lie algebra. The Lie bracket then takes the form

$$[\mathrm{J}^A, \mathrm{J}^B] = f^{AB}{}_C \mathrm{J}^C. \tag{3.1.77}$$

A representation ρ of $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a vector space of dimension $n \in \mathbb{N}$ (for finite-dimensional representations), which is possibly the vector space underlying $\mathfrak{g}$. $\mathfrak{gl}(V)$ is the general linear algebra over V, which is isomorphic to $\mathrm{Mat}_{n\times n}$ over the corresponding field. Specifically, ρ associates to each $\mathrm{J}^A \in \mathfrak{g}$ a linear map $\rho(\mathrm{J}^A) : V \to V$, in a way compatible with the Lie bracket:

$$\rho([\mathrm{J}^A, \mathrm{J}^B]) = [\rho(\mathrm{J}^A), \rho(\mathrm{J}^B)], \qquad \mathrm{J}^A, \mathrm{J}^B \in \mathfrak{g}, \tag{3.1.78}$$

where the bracket on $\mathfrak{gl}(V)$ is taken as the usual matrix commutator

$$[A, B] = AB - BA, \qquad A, B \in \mathfrak{gl}(V). \tag{3.1.79}$$

A representation is *reducible* if it has no non-trivial invariant subspaces. [10]

One of the simplest examples of a Lie algebra over $\mathbb{R}$ is $\mathfrak{su}(2)$. This is generated by three elements $\mathrm{J}^1, \mathrm{J}^2, \mathrm{J}^3$ which satisfy

$$[\mathrm{J}^i, \mathrm{J}^j] = \epsilon^{ijk} \mathrm{J}^k. \tag{3.1.80}$$

This algebra can be represented by the Pauli matrices (3.1.25) via

$$\mathrm{J}^i = -\tfrac{i}{2}\sigma^i. \tag{3.1.81}$$

[10] The trivial invariant subspaces are the zero vector space and V itself.

General representations of this algebra are best expressed in terms of its *complexification* $\mathfrak{sl}_2 \simeq \mathfrak{su}(2) + i\mathfrak{su}(2)$. The algebra can be rewritten in terms of generators $e^{\pm}$ and h, where the commutators are defined as

$$[h, e^{\pm}] = \pm e^{\pm}, \qquad [e^+, e^-] = 2h. \tag{3.1.82}$$

For this algebra there is a single finite-dimensional irreducible representation for each dimension $n \in \mathbb{N}$, dubbed the spin-$\frac{n-1}{2}$ representation, which can be constructed as highest weight representations. For $n = 2$ we have the spin-$\frac{1}{2}$ representation, which can be realised in terms of Pauli matrices:

$$e^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y), \qquad h = \frac{1}{2}\sigma^z, \tag{3.1.83}$$

c.f. (3.1.25) and the discussion below. A typical representation of a Lie algebra is a *fundamental* representation. These are highest weight representations possessing a highest weight vector with a single non-zero component. Every Lie algebra acts on itself via the *adjoint* representation, with $V = \mathfrak{g}$. Here the representation maps are defined via $\rho(\mathrm{J}^A) = \mathrm{ad}_{\mathrm{J}^A} := [\mathrm{J}^A, \ ]$.

Note that while we can multiply representation matrices $\rho(\mathrm{J}^A)\rho(\mathrm{J}^B)$, there is no notion of multiplication in $\mathfrak{g}$. One way to allow for this is to embed $\mathfrak{g}$ in the *universal enveloping algebra* $U(\mathfrak{g})$. This is defined to be the space of formal polynomials of elements in $\mathfrak{g}$, modulo the identifications $\mathrm{J}^A\mathrm{J}^B - \mathrm{J}^B\mathrm{J}^A \sim f^{AB}_{\ \ C}\mathrm{J}^C$. Formally this can be expressed as the tensor algebra over $\mathfrak{g}$, modulo the Lie bracket:

$$U(\mathfrak{g}) = \left(\bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}\right) / (\mathrm{J}^A \otimes \mathrm{J}^B - \mathrm{J}^B \otimes \mathrm{J}^A - [\mathrm{J}^A, \mathrm{J}^B]), \tag{3.1.84}$$

where $\mathfrak{g}^{\otimes 0} := \mathbb{K}$, the underlying field. For example, we have the following identification of elements in $U(\mathfrak{sl}_2)$:

$$e^+e^-h \sim e^-e^+h + 2h^2, \tag{3.1.85}$$

where we used the algebra (3.1.82).

A Lie algebra is *semisimple* if it is a direct sum of simple Lie algebras, i.e non-abelian Lie algebras with no non-trivial ideals. We mention that semisimple Lie algebras over $\mathbb{C}$ admit a *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_{\mathfrak{g}}} \mathfrak{g}_\alpha, \tag{3.1.86}$$

where $\mathfrak{h}$ is a *Cartan subalgebra* of $\mathfrak{g}$, i.e. a maximal commuting Lie subalgebra of $\mathfrak{g}$. The dimension of $\mathfrak{h}$ is called the *rank* of $\mathfrak{g}$. $R_{\mathfrak{g}}$ is the *root system* of $\mathfrak{g}$, which is a geometrical object underlying the Lie algebra. The decomposition (3.1.86) can be made explicit in the Cartan–Weyl basis of $\mathfrak{g}$. This can be refined to a Chevally–Serre basis of $\mathfrak{g}$, where the generators are expressed in terms of *simple* roots $\alpha^{(i)}, i = 1, \ldots, \mathrm{rank}(\mathfrak{g})$. The whole algebra is exhausted by a set of extra relations known as the Serre relations.

First Realisation of the Yangian and Hopf Algebras. In this section we describe Drinfel'd's first realisation of the Yangian algebra $Y[\mathfrak{g}]$. This is an extension of the enveloping algebra $U(\mathfrak{g})$ by *level-one* generators $\widehat{\mathrm{J}}^A$, where the original generators J^A are referred to as *level-zero* generators. $Y[\mathfrak{g}]$ is the algebra generated by $\mathrm{J}^A, \widehat{\mathrm{J}}^A$, with the relations

$$[\mathrm{J}^A, \mathrm{J}^B] = f^{AB}{}_C \mathrm{J}^C, \qquad [\mathrm{J}^A, \widehat{\mathrm{J}}^B] = f^{AB}{}_C \widehat{\mathrm{J}}^C. \tag{3.1.87}$$

In particular, J^A generate $\mathfrak{g}$, and $\widehat{\mathrm{J}}^A$ transform in the adjoint representation of $\mathfrak{g}$. The generators J^A and $\widehat{\mathrm{J}}^A$ are subject to additional *Serre relations*

$$[\widehat{\mathrm{J}}^A, [\widehat{\mathrm{J}}^B, \mathrm{J}^C]] - [\mathrm{J}^A, [\widehat{\mathrm{J}}^B, \widehat{\mathrm{J}}^C]] = \frac{1}{4!} f^{ADI} f^{BEJ} f^{CFK} f_{IJK} \mathrm{J}^{(D} \mathrm{J}^E \mathrm{J}^{F)}. \tag{3.1.88}$$

Since the algebraic structure is rather involved, checking whether a given algebraic structure is isomorphic to a Yangian algebra is a non-trivial task.

Note that the commutation relations (3.1.87) do not specify the commutator of two level-one generators in terms of existing generators. Therefore from the level-one generators we can form independent *level-two* generators $\mathrm{J}^A_{(2)} \simeq f^A{}_{BC}[\widehat{\mathrm{J}}^B, \widehat{\mathrm{J}}^C]$, and similarly for higher generators $\mathrm{J}^A_{(n)}$. In this way one can obtain an infinite set of generators, such that the Yangian algebra is infinite-dimensional.

The Yangian, as defined above, can be upgraded to a so-called *Hopf algebra*. Hopf algebras appear in many contexts in mathematics and physics, and have an elegant representation theory [123]. A Hopf algebra $\mathcal{A}$ has the structure of both an algebra and a coalgebra. Because of its algebra structure, it admits an associative product $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. There is a corresponding unit map $\iota : \mathbb{C} \to \mathcal{A}$ which maps $c \to c\mathbb{I}$, where the identity element $\mathbb{I} \in \mathcal{A}$ satisfies

$$\mathbb{I}x = x\mathbb{I} \qquad \forall x \in \mathcal{A}. \tag{3.1.89}$$

Due to the coalgebra structure, $\mathcal{A}$ admits a coassociative *coproduct* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Coassociativity implies that

$$(\Delta \otimes \mathbb{I})\Delta(x) = (\mathbb{I} \otimes \Delta)\Delta(x) \qquad \forall x \in \mathcal{A}. \tag{3.1.90}$$

There is a corresponding counit $\varepsilon : \mathcal{A} \to \mathbb{C}$, which satisfies

$$(\varepsilon \otimes \mathbb{I})\Delta(x) = \mathbb{I} = (\mathbb{I} \otimes \varepsilon)\Delta(x) \qquad \forall x \in \mathcal{A}. \tag{3.1.91}$$

The coproduct and counit should be compatible with the multiplication in the algebra:

$$\Delta(xy) = \Delta(x)\Delta(y), \qquad x, y \in \mathcal{A}, \tag{3.1.92}$$

$$\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}, \tag{3.1.93}$$

$$\epsilon(xy) = \epsilon(x)\epsilon(y), \qquad x, y \in \mathcal{A}. \tag{3.1.94}$$

So far we have discussed the condition for $\mathcal{A}$ to be a *bialgebra*. A Hopf algebra is a bialgebra together with an antipode map $S : \mathcal{A} \to \mathcal{A}$, which satisfies the consistency conditions

$$m \circ (S \otimes \mathbb{I}) \circ \Delta(x) = m \circ (\mathbb{I} \otimes S) \circ \Delta(x) = i \circ \epsilon(x), \qquad \forall x \in \mathcal{A}. \tag{3.1.95}$$

More concretely, if $\Delta(x) = c_{ij} x^i \otimes x^j$, then (3.1.95) implies

$$c_{ij} S(x^i) x^j = c_{ij} x^i S(x^j) = \epsilon(x)\mathbb{I}. \tag{3.1.96}$$

The coproduct which extends the Yangian algebra (3.1.87) to a Hopf algebra is

$$\Delta(\mathrm{J}^A) = \mathbb{I} \otimes \mathrm{J}^A + \mathrm{J}^A \otimes \mathbb{I}, \qquad \Delta(\widehat{\mathrm{J}}^A) = \mathbb{I} \otimes \widehat{\mathrm{J}}^A + \widehat{\mathrm{J}}^A \otimes \mathbb{I} + \frac{1}{2} f^A{}_{BC} \mathrm{J}^B \otimes \mathrm{J}^C, \tag{3.1.97}$$

and the antipode is defined to act on J^A and $\widehat{\mathrm{J}}^A$ via

$$S(\mathrm{J}^A) = -\mathrm{J}^A, \qquad S(\widehat{\mathrm{J}}^A) = -\widehat{\mathrm{J}}^A - \frac{1}{2} f^A{}_{BC} \mathrm{J}^B \mathrm{J}^C, \tag{3.1.98}$$

and we furthermore note $S(\mathbb{I}) = \mathbb{I}$. The counit acts trivially as

$$\varepsilon(\mathrm{J}^A) = \varepsilon(\widehat{\mathrm{J}}^A) = 0, \tag{3.1.99}$$

with $\varepsilon(\mathbb{I}) = 1$. By using (3.1.97), (3.1.98), and (3.1.99) it is easy to verify that the consistency conditions (3.1.95) are satisfied for $x = \mathrm{J}^A$ and $x = \widehat{\mathrm{J}}^A$.

In the context of Hopf algebras there is the notion of a *universal R-matrix* $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$, which relates the coproduct Δ of the Hopf algebra with the 'opposite' coproduct $\Delta^{\mathrm{op}} := \mathbb{P}\Delta\mathbb{P}$:

$$\mathcal{R}\Delta = \Delta^{\mathrm{op}}\mathcal{R}. \tag{3.1.100}$$

$\mathcal{R}$ should further satisfy a quasitriangularity property, which implies that it satisfies a generalised Yang–Baxter equation in $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ [124]

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{3.1.101}$$

After specifying the Hopf algebra to a particular representation, one can obtain an R-matrix for a specific physical model, for example (3.1.48). The power of the universal R-matrix is that it doesn't refer to any representation; it is a more general object. If a Hopf algebra possesses such a universal R-matrix then it is called a *quasitriangular Hopf algebra*. Drinfel'd showed, by introducing a so-called *boost automorphism*, that given a Yangian algebra, one can construct such a (pseudo-) universal R-matrix $\mathcal{R}(u)$ which solves the Yang–Baxter equation (3.1.101) [115]. This construction underlines the connection of the Yangian algebra to the Yang–Baxter equation.

Evaluation Representations. Due to its complicated algebraic structure it is non-trivial to construct representations of the Yangian. We discuss a simple example, namely *evaluation* representations. These can be constructed from representations of the underlying Lie algebra $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Evaluation representations are a one-parameter family of representations $\rho_s : Y[\mathfrak{g}] \to \mathfrak{gl}(V)$, defined via

$$\rho_s(\mathrm{J}^A) = \rho(\mathrm{J}^A), \qquad \rho_s(\widehat{\mathrm{J}}^A) = s\,\rho(\mathrm{J}^A), \tag{3.1.102}$$

where $s \in \mathbb{R}$ is the *evaluation parameter*. It is easy to see that if ρ is a representation of $\mathfrak{g}$, then $\rho(\mathrm{J}^A)$ and $\rho_s(\widehat{\mathrm{J}}^A)$ satisfy the Yangian algebra (3.1.87).

We will be interested in representations of the Yangian on tensor product spaces $V^{\otimes L}$. These are relevant in the context of spin chains and multi-leg Feynman diagrams. Given evaluation representations ρ_{s_i} of the Yangian, $i = 1, 2, \dots, L$, we can construct a representation on $V^{\otimes L}$ by an iterated coproduct

$$\Delta^{L-1}(\mathrm{J}^A) = \sum_{k=1}^{L} \mathrm{J}_k^A, \quad \Delta^{L-1}(\widehat{\mathrm{J}}^A) = \sum_{k=1}^{L} \widehat{\mathrm{J}}^A + \frac{1}{2} f^A{}_{BC} \sum_{k<l} \mathrm{J}_k^B \mathrm{J}_l^C, \tag{3.1.103}$$

where J_k^A denotes the action of J^A on the k^{th} copy of V. The corresponding L-site representation then reads

$$\rho_L(\mathrm{J}^A) = \sum_{k=1}^{L} \rho(\mathrm{J}_k^A), \qquad \rho_L(\widehat{\mathrm{J}}^A) = \frac{1}{2} f^A{}_{BC} \sum_{k<l} \rho(\mathrm{J}_k^B)\rho(\mathrm{J}_l^C) + \sum_{i=1}^{L} s_i \rho(\mathrm{J}_i^B), \qquad (3.1.104)$$

where s_i are a set of L evaluation parameters which characterise the representation.

Heisenberg Spin Chain. Here we briefly show how the Yangian algebra appears in the context of integrable quantum spin chains, for the example of the Heisenberg spin chain [125]. The Yangian is often only an exact symmetry of the Hamiltonian for spin chains of infinite length, and thus eigenstates do not typically arrange into representations of the Yangian.[11]

Consider the expansion of the monodromy matrix (3.1.50) of the Heisenberg spin chain in powers of the spectral parameter u:

$$T(u) = u^L \mathbb{I} + u^{L-1} i \sum_{n=1}^{L} S_n^c \otimes \sigma_a^c + u^{L-2} i^2 \sum_{n<m} S_m^c S_n^d \otimes (i\epsilon_{cde}\sigma_a^e + \delta_{cd}\mathbb{I}_a) + O(u^{L-3}), \quad (3.1.105)$$

where we used the identity for Pauli matrices $\sigma^c \sigma^d = i\epsilon^{cde}\sigma^e + \delta^{cd}\mathbb{I}$, and the explicit representation of the Lax operators (3.1.47). We consider an evaluation representation (3.1.102) with $s = 0$ of $Y[\mathfrak{su}(2)]$ at each site of the spin chain, where the underlying representation ρ of $\mathfrak{su}(2)$ is taken as the spin-$\frac{1}{2}$ representation. Then we have

$$\mathrm{J}^a = \sum_{n=1}^{L} S_n^a, \qquad \widehat{\mathrm{J}}^a = \frac{1}{2} f^a{}_{bc} \sum_{n<m} S_n^b S_m^c, \qquad (3.1.106)$$

where here $f^a{}_{bc} = i\epsilon^a{}_{bc}$. Therefore, the expansion of the monodromy matrix (3.1.105) can be rewritten

$$\begin{aligned} T(u) &= u^L \mathbb{I} + (u^{L-1} i \mathrm{J}^c + 2u^{L-2} i^2 \widehat{\mathrm{J}}^c) \otimes \sigma_a^c \\ &\quad + u^{L-2} i^2 \sum_{n<m} S_m^c S_n^d \otimes \delta_{cd}\mathbb{I} + O(u^{L-3}). \end{aligned} \qquad (3.1.107)$$

We see that the level-zero generators J^a appear at the order u^{L-1} in the expansion of $T(u)$ about $u = \infty$, and the level-one generators appear at order u^{L-2}. Similarly, the higher generators $\mathrm{J}_{(n)}^a$ will appear at lower orders in u. The second line of (3.1.107) is less interesting algebraically, see [97].

While the level-zero generators commute with the Heisenberg Hamiltonian (3.1.24), we have

$$[H, \widehat{\mathrm{J}}^a] = 2(\mathrm{J}_L^a - \mathrm{J}_1^a). \qquad (3.1.108)$$

The terms on the right hand side of (3.1.108) can be regarded as 'boundary' terms. These are boundary terms dictated by this particular representation of the Yangian, and are not distinguished by the periodic boundary conditions of the spin chain. For a spin chain model to have an exact Yangian symmetry, we thus typically need to take the length of the chain $L \to \infty$. However, such models are less physical than finite-length models.

[11] There are still examples of periodic spin chains of finite length with an exact Yangian symmetry, for example the Haldane–Shastry model [126].

3.2 $\mathcal{N} = 4$ SYM as an Integrable Field Theory

$\mathcal{N} = 4$ super Yang–Mills (SYM) theory is one of the most widely studied quantum field theories in theoretical physics. It possesses a maximal amount of supersymmetry and is furthermore a conformal field theory, due to a vanishing β function for its coupling $g_{\rm YM}$ [127,128]. As such $\mathcal{N} = 4$ SYM is a *finite* theory, free from UV and IR divergences. This finiteness was one of the original motivations for the interest in the theory.

More recently, $\mathcal{N} = 4$ SYM has received a surge of interest in the context of the gauge-gravity duality, or AdS/CFT correspondence [27–29]. This correspondence is arguably the most inspiring and most cited conjectures of contemporary mathematical physics. Its structural deepness can be seen in the quantum integrability of free AdS-strings on the string theory side, and in planar $\mathcal{N} = 4$ SYM on the field theory side [129]. Although conjectural from a rigorous, non-perturbative point of view, integrability has allowed for the computation of numerous quantities on both the string and the gauge theory side. These include string spectrum/scaling dimensions, Wilson loops and defect lines, various correlation functions, scattering amplitudes, and much more. One of the crowning achievements in $\mathcal{N} = 4$ SYM integrability is the quantum spectral curve [130], which presumably encodes the anomalous dimensions of all local single-trace operators of the theory. This has since been generalised to other integrable field theories, such as the ABJM model [131].

In this section we briefly review the particle content of $\mathcal{N} = 4$ SYM and its symmetries. Although an exact definition of an integrable quantum field theory is missing, we explain how integrability manifests itself in this theory in two ways. We first describe the spectral problem of the theory, focusing on the case of the $\mathfrak{su}(2)$ sector. We also describe the Yangian invariance of the theory, which is visible both at the level of the action and at the level of specific observables. This will be useful to compare to later, when we consider the fishnet theory.

3.2.1 Action and Supersymmetry

$\mathcal{N} = 4$ super Yang–Mills is a four-dimensional gauge theory based on the gauge group $SU(N)$. The particle content is six massless real scalars ϕ_i, a gauge field A_μ, and four chiral fermions ψ^a_α and their conjugates $\bar\psi^{\dot\alpha}_a$, all transforming in the adjoint representation of the gauge group. As such, each field is an $N \times N$ traceless matrix. The index $i = 1, \ldots, 6$ labels the scalar fields, and the index $a = 1, \ldots, 4$ labels the fermions. The indices $\alpha, \dot\alpha = 1, 2$ are the left- and right- handed spinor indices of the Lorentz group $SO(1, 3) \simeq SU(2) \times SU(2)$. There is a global $SU(4) \simeq SO(6)$ R-symmetry, under which the scalar fields ϕ_i transform in the $\mathbf{6}$ representation, the fermions ψ^a_α transform in the fundamental $\mathbf{4}$ representation, and their conjugates transform in the anti-fundamental $\bar{\mathbf{4}}$.

The action of the theory is

$$S = \frac{2}{g^2_{\rm YM}} \int \mathrm{d}^4x \, \mathrm{tr} \Big[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} D_\mu\phi_i D^\mu\phi_i + \frac{i}{2}\bar\psi_a \Gamma^\mu D_\mu\psi^a \qquad (3.2.1)$$
$$\frac{1}{2}\bar\psi_a\Gamma^i[\phi_i, \psi^a] + \frac{1}{4}[\phi_i, \phi_j][\phi_i, \phi_j] \Big].$$

This action can be obtained by dimensionally reducing an $\mathcal{N} = 1$ supersymmetric theory in ten dimensions [132]. The Γ^i are ten-dimensional gamma matrices which describe the coupling of the ten-dimensional fermion to the bosons. $F_{\mu\nu}$ is the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g_{\mathrm{YM}}[A_\mu, A_\nu], \tag{3.2.2}$$

and D_μ is the gauge covariant derivative

$$D_\mu = \partial_\mu - ig_{\mathrm{YM}}[A_\mu, \;]. \tag{3.2.3}$$

The action (3.2.1) enjoys an enhancement of conformal symmetry by supersymmetry, which is known as *superconformal* symmetry. The conformal algebra $\{\mathrm{P}_\mu, \mathrm{L}_{\mu\nu}, \mathrm{D}, \mathrm{K}_\mu\}$ described in section 2.2.2 can be extended by supersymmetry generators $\mathcal{Q}^a_\alpha$ and $\tilde{\mathcal{Q}}^{\dot{\alpha}}_a$. A dual set of charges $\tilde{S}^a_\alpha, S^{\dot{\alpha}}_a$ can be obtained by commuting $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ with the special conformal generator K^μ. The algebra of (anti)commutation relations of $\{\mathrm{P}_\mu, \mathrm{L}_{\mu\nu}, \mathrm{D}, \mathrm{K}_\mu, \mathcal{Q}^a_\alpha, \tilde{\mathcal{Q}}^{\dot{\alpha}}_a, \tilde{S}^a_\alpha, S^{\dot{\alpha}}_a\}$ is closed after the addition of the R-symmetry generators R^{ij}, $i, j = 1, \ldots, 6$. The full algebra is known as the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2, 2|4)$, and it is written down for example in [133].

The fields of $\mathcal{N} = 4$ SYM arrange into representations of $\mathfrak{psu}(2, 2|4)$. As such, operators can be characterised by a finite set of labels associated to the representation. There is the classical scaling dimension $\varDelta_0$, the Lorentz spins s_1, s_2, and the R-symmetry labels q_1, q_2, q_3 of $\mathfrak{su}(4)$.

Due to the large number of fields in the theory, the Feynman diagrammatics can get out of hand very quickly. There is an interesting simplification of the theory (3.2.1), namely its *planar limit*, first proposed in [5] in the context of pure Yang–Mills theory. This is the limit

$$g_{\mathrm{YM}} \to 0, \qquad N \to \infty, \qquad g \coloneqq g^2_{\mathrm{YM}} N \text{ fixed.} \tag{3.2.4}$$

The combination $g = g^2_{\mathrm{YM}}N$ is known as the 't Hooft coupling. In this limit only the planar Feynman diagrams contributing to a given observable contribute, and the non-planar diagrams are suppressed by powers of $1/N$. This limit simplifies the Feynman diagrammatics somewhat, although it can still be quite involved. However, the large amount of symmetry holding the theory together renders many of the observables computable to a large number of loops, or even non-perturbatively. One example of this is the planar six-gluon amplitude, which is known by now to seven loops [134]. The corresponding calculation in for example the standard model would be unthinkably complex.

3.2.2 Spectral Problem

Techniques of quantum integrability have time and time again proved crucial in the computation of numerous quantities in planar $\mathcal{N} = 4$ SYM, and correspondingly in type IIB string theory on $AdS_5 \times S^5$ via the AdS/CFT correspondence. As such, it has been conjectured to be an integrable field theory. In the next sections we describe two manifestations of this integrability. Firstly we describe the spectral problem, which is the problem of calculating quantum corrections to the dilatation operator D. Fascinatingly, this operator at one loop can be put in correspondence with a nearest neighbour integrable spin chain [135, 136]. For the case of a very particular sector of operators, this spin chain is precisely the Heisenberg spin chain discussed in section 3.1.2. In the next section, we mention a few ways in which the Yangian algebra has been realised on the theory. This algebra has been realised on observables such as the dilatation operator [137], scattering amplitudes [119], and Wilson loops [138]. Furthermore, the theory has been shown to be Yangian invariant in the planar limit at the level of the action (3.2.1) [120].

As discussed in section 2.2.3, two point functions in a conformal field theory take the form

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{c(g_{\mathrm{YM}})}{(x_1 - x_2)^{2\Delta(g_{\mathrm{YM}})}}, \tag{3.2.5}$$

where Δ is the scaling dimension of the operator $\mathcal{O}$. Here we explicitly included the dependence on the coupling constant g_{YM}, because in general Δ will receive quantum corrections. The tree-level scaling dimension $\Delta(0)$ is simply the classical dimension of the operator $\mathcal{O}$, and the difference $\Delta(g_{\mathrm{YM}}) - \Delta(0)$ is the *anomalous dimension*.

An important class of local, gauge invariant operators for the theory (3.2.1) are single-trace operators

$$\mathcal{O}(x) = \mathrm{tr}(F_1 F_2 \cdots F_L(x)), \qquad F_i \in \{\phi_i, \psi^a, \bar{\psi}_a, \mathcal{F}_{\mu\nu}\}. \tag{3.2.6}$$

The calculation of the anomalous dimensions simplifies upon restriction to a specific *sector* of operators. For example, we could consider the so-called $\mathfrak{su}(2)$ sector, where we consider single-trace operators built from two complex scalars $X = \phi_1 + i\phi_4, Y = \phi_2 + i\phi_5$:

$$\mathcal{O} = \mathrm{tr}(X^{l_1} Y^{l_2} X^{l_3} Y^{l_4} \cdots), \tag{3.2.7}$$

where lists of X and Y fields related by a cyclic permutation are identified because of the trace, for example $\mathrm{tr}(XYX) = \mathrm{tr}(XXY)$. Since X and Y are scalar primaries in four dimensions, they have a unit classical dimension $\Delta(0) = 1$. Restricting to operators in the $\mathfrak{su}(2)$ sector with length L (and hence classical dimension $\Delta(0) = L$), a general operator takes the form

$$\mathcal{O}(x) = \Psi^{\{i_1 i_2 \cdots i_L\}}\, \mathrm{tr}(F_{i_1} F_{i_2} \cdots F_{i_L}(x)), \qquad F_i \in \{X, Y\}, \tag{3.2.8}$$

where $\Psi^I := \Psi^{\{i_1 i_2 \cdots i_L\}}$ are complex coefficients which specify the operator $\mathcal{O}$. We have $\Psi^{\{i_1 i_2 \cdots i_L\}} = \Psi^{\{i_L i_1 \cdots i_{L-1}\}}$ by cyclicity of the trace. The two-point function[12]

$$\langle \mathcal{O}_1(x_1)\bar{\mathcal{O}}_2(x_2)\rangle \tag{3.2.9}$$

can be derived via an explicit Feynman diagrammatic calculation [139]. There are a couple important features. Firstly, the correlation functions are not finite and a regularisation is required. One approach is to use dimensional regularisation

$$S = \frac{2}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4 x\, \mathcal{L} \to S_\epsilon = \frac{2}{(\mu^\epsilon g_{\mathrm{YM}})^2} \int \mathrm{d}^{4-2\epsilon} x\, \mathcal{L}, \tag{3.2.10}$$

where μ is a mass scale. Secondly, there is the phenomenon of operator mixing: at the quantum level the two-point function between single-trace operators in no longer proportional to the delta function, as in (2.2.55). Therefore a different basis of operators should be used to render the two-point function diagonal. This is the problem of

[12] $\bar{\mathcal{O}}$ is the Hermitian conjugate of $\mathcal{O}$.

diagonalising the dilatation operator. Carrying out the computation at one-loop, one finds

$$\langle \mathcal{O}_1(x_1)\bar{\mathcal{O}}_2(x_2)\rangle = L4^L\tilde{g}^{2L}\left(\frac{\pi^\epsilon\mu^{2\epsilon}\Gamma(1-\epsilon)}{x_{12}^{2-2\epsilon}}\right) \tag{3.2.11}$$
$$\times\langle\Psi_2|\mathbb{I}-\tilde{g}^2\left(\frac{1}{\epsilon}+1+\gamma_E+\log(\pi(x-y)^2)\right)H|\Psi_1\rangle+\tilde{g}^2O(\epsilon)+O(\tilde{g}^4),$$

where $\tilde{g}^2 = g/16\pi^2$ is related to the 't Hooft coupling g. This computation is nicely reviewed in [101]. Remarkably, H is precisely the Heisenberg Hamiltonian (3.1.30):

$$H = 2\sum_{i=1}^{L}(1-\mathbb{P}_{i,i+1}). \tag{3.2.12}$$

In this case, H acts on the indices of $\Psi^{\{i_1\cdots i_L\}}$, where we identify the field $X\equiv\uparrow$ and the field $Y\equiv\downarrow$. $\langle\Psi_2|\Psi_1\rangle = (\Psi_2^{I_2})^\dagger\Psi_1^{I_1}$ is the standard inner product on $\mathbb{C}^{2^L}$. For example, if $\mathcal{O}_1 = \mathrm{tr}(XYXY)$ and $\mathcal{O}_2 = \mathrm{tr}(XXYY)$, we have

$$\begin{aligned}\langle\Psi_2|H|\Psi_1\rangle &= 2(\Psi^{\{\uparrow\uparrow\downarrow\downarrow\}})^\dagger\sum_{i=1}^{4}(1-\mathbb{P}_{i,i+1})\Psi^{\{\uparrow\downarrow\uparrow\downarrow\}}\\ &= -2(\Psi^{\{\uparrow\uparrow\downarrow\downarrow\}})^\dagger 4\Psi^{\{\uparrow\uparrow\downarrow\downarrow\}} = -8.\end{aligned} \tag{3.2.13}$$

The divergence in (3.2.11) can be removed purely by a wavefunction renormalisation; no counterterms need to be added to the Lagrangian. We can change the operator basis

$$\mathcal{O}_I \to \mathcal{Z}_{IJ}(g_{\mathrm{YM}},\epsilon)\mathcal{O}_J, \tag{3.2.14}$$

and tune the coefficients $\mathcal{Z}_{IJ}$ to remove the divergent terms in ϵ. It is convenient to use an $\overline{\mathrm{MS}}$ scheme, which removes the $(1/\epsilon+1+\gamma_E+\log\pi)$ factor from (3.2.11).

After dealing with these subtleties, diagonalising the dilatation operator and finding the spectrum of anomalous dimensions for this sector is equivalent to finding the spectrum of the Heisenberg spin chain. As described in section 3.1.2, this can be done using various techniques in quantum integrability, for example the coordinate or algebraic Bethe ansätze. This rather unexpected application of techniques of quantum integrability to the calculation of planar observables in $\mathcal{N}=4$ SYM is why the theory is conjectured to be quantum integrable in the planar limit.

For more complicated sectors of operators, the one-loop dilatation operator can be put in correspondence with integrable spin chains based on higher-rank (super)algebras. For example, the sector of single-trace operators built from all six of the scalar fields $\phi_1,\dots,\phi_6$ can be put in correspondence with an integrable $\mathfrak{so}(6)\simeq\mathfrak{su}(4)$ spin chain [136]. For sectors of operators including fermions, we find integrable spin chains based on Lie superalgebras. The full one-loop dilatation operator of $\mathcal{N}=4$ SYM has famously been identified with a $\mathfrak{psu}(2,2|4)$ spin chain [140, 141]. At higher loops longer range integrable spin chains can be identified [142, 143], and even dynamical ones which do not preserve the number of spin sites [144]. Other integrable field theories have since been identified, with one-loop dilatation operators identifiable with integrable super spin chains. One example is the ABJM theory, whose one-loop dilatation operator takes the form of an $\mathfrak{osp}(2,2|6)$ spin chain [145].

3.2.3 Yangian Symmetry.

Yangian symmetry, introduced in section 3.1.3, appears in planar $\mathcal{N} = 4$ SYM in multiple instances. The relevant Yangian algebra in this case is that of the $\mathcal{N} = 4$ superconformal algebra $Y[\mathfrak{psu}(2,2|4)]$. We briefly describe how this symmetry manifests at the level of the action, scattering amplitudes, and the dilatation operator.

Action. The action (3.2.1) has been shown to be exactly Yangian invariant in the planar limit [120]:

$$\mathrm{J}^a S = 0, \quad \widehat{\mathrm{J}}^a S = 0, \qquad \mathrm{J}^a, \widehat{\mathrm{J}}^a \in Y[\mathfrak{psu}(2,2|4)]. \tag{3.2.15}$$

Care must be taken to define the representation of $\mathrm{J}^a, \widehat{\mathrm{J}}^a$ on the fields appropriately, since the action is a sum of terms with different numbers of fields:

$$S = S_2 + S_3 + S_4, \tag{3.2.16}$$

and the superconformal generators contain terms which either preserve the number of fields or increase it by one

$$\mathrm{J}^a = \mathrm{J}^a_0 + \mathrm{J}^a_1. \tag{3.2.17}$$

The full representation of the superconformal generators on the fields is given in [120]. There, they show that the level-one generator on the action S is proportional to the dual Coxeter number $\mathfrak{c}$ of the underlying Lie algebra $\mathfrak{g}$, defined by $f^A{}_{BC} f^{BC}{}_D = \mathfrak{c}\, \delta^A_D$. For $\mathfrak{psu}(2,2|4)$ this is zero, and so the action is invariant. This is also true for the algebra $\mathfrak{osp}(2,2|6)$, and so the ABJM theory action is also Yangian invariant. Yangian invariance at the level of the action is interesting because it affords the oppurtunity to derive corresponding Ward identities for this symmetry.

Scattering Amplitudes. Furthermore, certain planar amplitudes of $\mathcal{N} = 4$ SYM have been shown to be Yangian invariant. Amplitudes in gauge theories depend on a set of helicities h_i, colours a_i, and on-shell massless momenta $p_i^2 = 0$. This on-shell condition can be resolved conveniently in terms of the spinor helicity variables [146]

$$p_i^{\alpha\dot{\alpha}} = p_i^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \lambda_i^\alpha \bar{\lambda}_i^{\dot{\alpha}}. \tag{3.2.18}$$

The kinematical invariants can be expressed in terms of the spinors $\lambda, \bar{\lambda}$ as

$$\langle ij\rangle = \epsilon_{\alpha\beta}\lambda_i^\alpha \lambda_j^\beta, \quad [ij] = \epsilon_{\alpha\beta}\bar{\lambda}_i^{\dot{\alpha}} \bar{\lambda}_j^{\dot{\beta}}, \tag{3.2.19}$$

where we have $(p_i + p_j)^2 = 2p_i \cdot p_j = \langle ij\rangle[ji]$. The tree-level n-particle scattering amplitudes A_n decompose into colour-ordered components $\mathcal{A}_n$ [147]:

$$A_n(\{\lambda_i, h_i, a_i\}) = \sum_{S_n/\mathbb{Z}_n} \mathcal{A}_n(\{\lambda_{\sigma(1)}, h_{\sigma(1)}\}, \ldots, \{\lambda_{\sigma(n)}, h_{\sigma(n)}\}) \operatorname{tr}(T^{a_{\sigma}(1)} \cdots T^{a_{\sigma}(n)}), \tag{3.2.20}$$

where T^a are the generators of the gauge algebra $\mathfrak{su}(N)$ and S_n is the symmetric group on n letters.

All of the amplitudes in $\mathcal{N}=4$ SYM are conveniently described by packaging the particles of differing helicities into a *chiral on-shell superfield*

$$\begin{aligned}\Phi(\lambda,\bar\lambda,\eta) =& G^{+}(\lambda,\bar\lambda)+\eta^{A}\Gamma_{A}(\lambda,\bar\lambda)+\frac{1}{2}\eta^{A}\eta^{B}S_{AB}(\lambda,\bar\lambda)\\ &+\frac{1}{6}\epsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\bar\Gamma^{D}(\lambda,\bar\lambda)+\frac{1}{24}\epsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\eta^{D}G^{-}(\lambda,\bar\lambda),\end{aligned} \tag{3.2.21}$$

where the R-symmetry indices A,B,C,D take the values $1,\ldots,4$. The fields $G^{\pm}$ describe the gauge boson with helicity $h=\pm1$. $\Gamma_A,\bar\Gamma^A$ describe the fermions of helicity $h=+1/2$ and $h=-1/2$ respectively, and $S_{AB}=-S_{BA}$ describes the six scalars with helicity $h=0$. η^A are auxiliary Grassmann variables. A given on-shell state can be accessed from (3.2.21) by acting with appropriate η derivatives. It is then useful to consider the colour-ordered *superamplitude*

$$\mathcal{A}_n(\Phi_1,\ldots,\Phi_n)=\sum_{k=0}^{n}\mathcal{A}_{n,k}, \tag{3.2.22}$$

which has been expanded so that $\mathcal{A}_{n,k}$ is a polynomial of degree $4k$ in the η^A.[13] The component amplitudes can be extracted from the superamplitude by taking derivatives in η. The superconformal Ward identities imply that $\mathcal{A}_{n,0}=\mathcal{A}_{n,1}=\mathcal{A}_{n,n-1}=\mathcal{A}_{n,n-2}=0$, and $\mathcal{A}_{n,2+K}$ is known as the N^KMHV (maximal helicity violating) amplitude. The $K=0$ MHV amplitude is given by the remarkable Parke–Taylor formula [148, 149]

$$A_{n,2}=\frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle},\quad P^{\alpha\dot\beta}=\sum_i\lambda_i^{\alpha}\bar\lambda_i^{\dot\beta},\quad Q^{\alpha A}=\sum_i\lambda_i^{\alpha}\eta_i^{A}, \tag{3.2.23}$$

where the delta functions $\delta^4(P)$ and $\delta^8(Q)$ impose conservation of momentum and supermomentum respectively, and the spinor brackets were defined in (3.2.19).

The amplitude (3.2.23) can be shown to be exactly invariant under the superconformal Yangian $Y[\mathfrak{psu}(2,2|4)]$. The representation of the level-zero algebra J^A was given in [150]. The local algebra can written as polynomials in the spinor variables $\lambda^\alpha,\bar\lambda^{\dot\alpha}$, the Grassmann variables η^A, and their derivatives. For example, the Dilatation generator D reads

$$\mathrm{D}=\frac{1}{2}\partial_\gamma\lambda^\gamma+\frac{1}{2}\bar\lambda^{\dot\gamma}\partial_{\dot\gamma} \tag{3.2.24}$$

and the superconformal generators $\tilde S^A_\alpha, S^{\dot\alpha}_A$ read

$$\tilde S^A_\alpha=\eta^A\partial_\alpha,\qquad S^{\dot\alpha}_A=\partial^{\dot\alpha}\partial_A. \tag{3.2.25}$$

In (3.2.24) and (3.2.25) we used the abbreviations $\partial_\alpha=\partial/\partial\lambda_\alpha,\partial_{\dot\alpha}=\partial/\partial\bar\lambda_{\dot\alpha}$, and $\partial_A=\partial/\partial\eta_A$. The n-site representation of the level-zero algebra under which the amplitude $A_{n,2}$ is invariant is given simply by

$$\mathrm{J}^a=\sum_{i=1}^{n}\mathrm{J}^a_i,\qquad \mathrm{J}^a\in\{\mathrm{P}_\mu,\mathrm{L}_{\mu\nu},\mathrm{D},\mathrm{K}_\mu,\mathcal{Q}^a_\alpha,\mathcal{Q}^{\dot\alpha}_a,\tilde S^a_\alpha,S^{\dot\alpha}_a,R^{ij}\}. \tag{3.2.26}$$

[13]The order of the polynomial in η must be a multiple of four, so that the amplitude is an R-symmetry singlet.

The level-one n-site representation is simply the iterated coproduct of the evaluation representation (3.1.102) with $s = 0$, c.f. (3.1.104):

$$\widehat{\mathrm{J}}^a = f^a{}_{bc} \sum_{j<k} \mathrm{J}^b_j \mathrm{J}^c_k. \tag{3.2.27}$$

In [119] it was shown that the full tree-level superamplitude (3.2.22) is invariant under $Y[\mathfrak{psu}(2,2|4)]$:

$$\mathrm{J}^a A_n = \widehat{\mathrm{J}}^a A_n = 0. \tag{3.2.28}$$

Interestingly, $\widehat{\mathrm{J}}^a A_n$ was found to be proportional to the dual Coxeter number $\mathfrak{c}$ of the algebra, which vanishes in the case of $\mathfrak{psu}(2,2|4)$. This level-one Yangian symmetry was first understood as a so-called *dual* superconformal symmetry [151]. This is $\mathcal{N} = 4$ superconformal symmetry in the dual space (x, θ), defined via

$$x_i - x_{i+1} = p_i, \qquad \theta_i - \theta_{i+1} = \lambda_i \eta_i, \tag{3.2.29}$$

where we suppressed (super)spatial indices. Note that the bosonic part of this is simply dual conformal symmetry, which we already discussed at the level of Feynman integrals in section 2.3.1. That the superconformal and dual superconformal algebras close into a Yangian was a surprising realisation of [119]. At one-loop the Yangian symmetry is anomalous, although it can be restored via a deformation of the symmetry generators [152].

Dilatation Operator. The one-loop dilatation operator of planar $\mathcal{N} = 4$ SYM in the $\mathfrak{su}(2)$ operator sector is precisely the Heisenberg spin chain (3.1.30). As described in section 3.1.3, the level-one generators in $Y[\mathfrak{su}(2)]$ commute with this Hamiltonian up to boundary terms

$$[H, \widehat{\mathrm{J}}^a] = 2(\mathrm{J}^a_L - \mathrm{J}^a_1). \tag{3.2.30}$$

In the spin chain framework the Yangian generators appear as subleading powers in the expansion of the mondromy matrix $T(u)$ at $u = \infty$, c.f. (3.1.107). The full one-loop dilatation operator is a $\mathfrak{psu}(2,2|4)$ spin chain. In [137] it was shown that the level-one superconformal Yangian $Y[\mathfrak{psu}(2,2|4)]$ commutes with this spin chain up to boundary terms, analogously as (3.2.30). This is actually consistent with the exact Yangian symmetry of the S-matrix (3.2.28), as shown in [153].

3.3 (Dynamical) Fishnet Theory from $\mathcal{N} = 4$ SYM

Although integrability is firmly associated to planar $\mathcal{N} = 4$ SYM, there is no formal proof of it at finite coupling, and no clear indication of what its origin may be. One approach to understanding this origin is to consider an integrable deformation of the theory, that is modify the theory by some parameters $\mathcal{L}_{\text{SYM}} \to \mathcal{L}_{\text{SYM}}(q_i)$, such that the modified theory can still be argued to be integrable, and coincides with the original theory $\mathcal{L}_{\text{SYM}}$ at specific values of the parameters q_i. By tuning the parameters to values which considerably simplify the theory, one can hope to gain some more insight into the mechanisms underlying the integrability of the full theory.

One such integrable deformation of $\mathcal{N} = 4$ SYM is the so-called γ-deformation. This is a deformation of $\mathcal{L}_{\text{SYM}}$ by three complex parameters $\gamma_1, \gamma_2, \gamma_3$ which appears preserve the integrability of the theory. This deformation was first proposed in [8], demonstrating a non-supersymmetric incarnation of the AdS/CFT correspondence. On the string theory side, this corresponds to strings on $AdS_5 \times S^5$, where three separate TsT transformations with angular parameters γ_i have been applied to the S^5 factor of the background [154]. When all of these parameters coincide $\gamma_1 = \gamma_2 = \gamma_3 = -\pi\beta$, one recovers the so-called β deformation, which preserves $\mathcal{N} = 1$ supersymmetry. This was historically the first deformation that was introduced [155,156]. Despite its lack of supersymmetry, the γ-deformed theory appears to preserve integrability, see [157] for a review. This fact refutes the plausible conjecture that supersymmetry is a prerequisite for the integrability of a quantum field theory. More precisely, various integrability constructions that go through for the undeformed theory also go through for the γ-deformed theory. This includes the Bethe ansatz approach to computing anomalous dimensions [158], as well as the quantum spectral curve [159]. These constructions naturally tend to be more involved than those for the undeformed theory.

However, there is a special point in the γ_i parameter space where many interesting simplifications occur. This is a double scaling limit, where each $\gamma_i \to i\infty$ and the 't Hooft coupling $g = N g_{\mathrm{YM}}^2 \to 0$, while the double-scaled couplings $\xi_i := g e^{-\frac{1}{2}\gamma_i}$ are held fixed.[14] In this limit the gauge field decouples and we recover the *dynamical fishnet theory*, which is a theory of three scalars ϕ_i and three fermions ψ_i, all $N \times N$ matrices in $\mathfrak{su}(N)$[15] [7]. The interactions which survive are chiral Yuakawa-type interactions between the fermions and the scalars $\sim \psi\phi\psi$, and chiral quartic interactions between the scalars $\sim \phi^4$. Explicitly, the interaction Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{\mathrm{DFN}}^{\mathrm{int}} &= N_c \mathrm{tr}\left(\xi_1^2 \phi_2^\dagger \phi_3^\dagger \phi^2 \phi^3 + \xi_2^2 \phi_3^\dagger \phi_1^\dagger \phi^3 \phi^1 + \xi_3^2 \phi_1^\dagger \phi_2^\dagger \phi^1 \phi^2\right) \\ &\quad + N_c \mathrm{tr}\left(i\sqrt{\xi_2 \xi_3}(\psi^3 \phi^1 \psi^2 + \bar{\psi}_3 \phi_1^\dagger \bar{\psi}_2) + \text{cyclic}\right), \end{aligned} \tag{3.3.1}$$

In the case where two of the couplings ξ_1, ξ_2 are set to 0, one recovers the bi-scalar *fishnet* theory. This is a theory of two complex scalar fields $\phi_1 \equiv X$ and $\phi_2 \equiv Z$, which interact via a chiral quartic interaction:

$$\mathcal{L}_{\mathrm{FN}} = N \,\mathrm{tr}(\partial_\mu X \partial^\mu \bar{X} + \partial_\mu Z \partial^\mu \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}), \tag{3.3.2}$$

where $\bar{X} := X^\dagger$ and $\bar{Z} := Z^\dagger$.

The chiral nature of the interaction vertex $\mathrm{tr}(XZ\bar{X}\bar{Z})$ leads to vast simplifications in the Feynman diagrammatics. Therefore many observables can be more easily understood in perturbation theory than in full $\mathcal{N} = 4$ SYM, especially in the planar limit. Sometimes there is an iterative structure in the Feynman graphs that represent the observable at each loop order, and the corresponding *graph-building* operator can be directly connected to integrability via a description in terms of non-compact conformal spin chains [160, 161]. In very special situations quantities can be represented in perturbation theory by a single Feynman graph. In these cases integrability can be understood at the level of singular Feynman integrals. A prominent example are the fishnet correlators which are represented by the many point fishnet graphs $\tilde{I}_{\alpha\beta}$

[14] Although this is the most common, there are other ways to take this limit, which leads to one other independent theory, see [6].

[15] Since the gauge field decouples in the limit, there is no notion of a gauge transformation or gauge symmetry.

discussed in section 2.3.5. These have been shown to possess a conformal Yangian symmetry [162]. Recently, it has been proposed to leverage this symmetry to compute the Feynman integrals from scratch, in an approach called the *Yangian bootstrap* [163,10,2].

The fishnet theory has also been argued to possess at strong coupling a holographic dual known at the *fishchain* [164–166]. This potentially provides a way the understand the mechanism behind the AdS/CFT correspondence. The current status for the integrability of the full dynamical fishnet theory is less clear, however various exact results for four-point correlators have been found [167].

There is a price to pay for these simplifications. We note from inspection of the Lagrangian (3.3.2) contains the quartic interaction $\mathrm{tr}(XZ\bar{X}\bar{Z})$, but not its Hermitian conjugate $\mathrm{tr}(ZX\bar{Z}\bar{X})$. Consequently, the fishnet theory is a non-unitary field theory. This leads to the theory being a logarithmic CFT, c.f. section 2.2.5, and the dilatation operator becomes non-diagonalisable in certain operator sectors. In fact, the strict conformality of the fishnet theory has been under debate; the renormalisation of (3.3.2) and (3.3.1) leads to double trace counter terms which break conformality [168]. Indeed, these are already present in the γ-twisted theory [169]. However, in the consideration of longer length operators in the planar limit these double trace interactions can often be discarded. Furthermore, a fixed point has been identified, apparently up to seven loops, where the beta functions corresponding to these double trace couplings vanish [170].

In this section we give the γ-deformed Lagrangian of $\mathcal{N} = 4$ SYM, and show how the fishnet Lagrangian emerges in the double scaling limit. We discuss various features of the fishnet theory, namely how its Feynman diagrammatics simplify and its renormalisation. We finally discuss the various incarnations of integrability in this theory, focusing on the case of Yangian symmetry, which is relevant to the later part of this thesis.

3.3.1 γ-Deformation and Double Scaling Limit

The Lagrangian of γ-deformed $\mathcal{N} = 4$ SYM can be obtained by replacing all products of fields in the action (3.2.1) with *star products* of these fields:

$$AB \to A * B := e^{\frac{i}{2} q_A \wedge q_B} AB, \tag{3.3.3}$$

where $q_A \in \mathbb{C}^3$ is a $U(1) \times U(1) \times U(1)$ charge vector associated to the field A, and the wedge product is defined [169]

$$q_A \wedge q_B := (q_A)^T C q_B, \qquad C_{ij} = -\epsilon_{ijk}\gamma_k. \tag{3.3.4}$$

We see that the γ dependence in the deformation comes through the matrix C. In (3.2.1) we wrote the action explicitly in terms of six real scalars $\phi_1, \ldots, \phi_6$. It is convenient here to write the action in terms of three complex scalars

$$\tilde{\phi}_1 := \phi_1 + i\phi_4, \quad \tilde{\phi}_2 := \phi_2 + i\phi_5, \quad \tilde{\phi}_3 := \phi_3 + i\phi_6. \tag{3.3.5}$$

Henceforth we will drop the tildes and refer to the complex scalar fields as ϕ_1, ϕ_2, ϕ_3. The charge vectors for the fields $\{\psi_1^\alpha, \psi_2^\alpha, \psi_3^\alpha, \psi_4^\alpha, A_\mu, \phi_1, \phi_2, \phi_3\}$ can be summarised in the following table:[16]

$$
\begin{array}{c|c|c|c|c|c|c|c|c}
\Phi & \psi_1^\alpha & \psi_2^\alpha & \psi_3^\alpha & \psi_4^\alpha & A_\mu & \phi_1 & \phi_2 & \phi_3 \\
\hline
q_\Phi^1 & +\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} & 0 & 1 & 0 & 0 \\
\hline
q_\Phi^2 & -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} & 0 & 0 & 1 & 0 \\
\hline
q_\Phi^3 & -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} & 0 & 0 & 0 & 1
\end{array}
\tag{3.3.6}
$$

For example, the product $\phi_1\psi_1^\alpha$ in the SYM action gets deformed

$$
\phi_1\psi_1^\alpha \to e^{\frac{i}{4}(\gamma_3-\gamma_2)}\phi_1\psi_1^\alpha := e^{\frac{i}{2}\gamma_1^-}\,\phi_1\psi_1^\alpha. \tag{3.3.7}
$$

In the conventions of [7], the full γ-deformed Lagrangian is

$$
\mathcal{L}_\gamma = N\,\mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D^\mu\phi_i^\dagger D_\mu\phi^i + i\bar{\psi}_A^{\dot\alpha}D_{\dot\alpha}^\alpha\psi_\alpha^A\right) + \mathcal{L}_{\mathrm{int}}, \tag{3.3.8}
$$

with

$$
\begin{aligned}
\mathcal{L}_{\mathrm{int}} =& Ng\,\mathrm{tr}\Big(\frac{g}{4}\{\phi_i^\dagger,\phi^i\}\{\phi_j^\dagger,\phi^j\} - ge^{-i\epsilon_{ijk}\gamma_k}\phi_i^\dagger\phi_j^\dagger\phi^i\phi^j \\
&+ e^{-\frac{i}{2}\gamma_j^-}\,\bar\psi_j\phi^j\bar\psi_4 + e^{+\frac{i}{2}\gamma_j^-}\,\bar\psi_4\phi^j\bar\psi_j + i\epsilon_{ijk}e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+}\psi^k\phi^i\psi^j \\
&+ e^{+\frac{i}{2}\gamma_j^-}\,\bar\psi_4\phi_j^\dagger\bar\psi_j + e^{-\frac{i}{2}\gamma_j^-}\,\bar\psi_j\phi_j^\dagger\bar\psi_4 + i\epsilon_{ijk}e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+}\bar\psi^k\phi_i^\dagger\bar\psi^j\Big),
\end{aligned} \tag{3.3.9}
$$

where we abbreviated

$$
\gamma_1^\pm := -\frac{\gamma_2 \pm \gamma_3}{2}, \quad \gamma_2^\pm := -\frac{\gamma_3 \pm \gamma_1}{2}, \quad \gamma_3^\pm := -\frac{\gamma_1 \pm \gamma_2}{2}. \tag{3.3.10}
$$

The Lagrangian (3.3.8) coincides with the Lagrangian of undeformed $\mathcal{N} = 4$ SYM when $\gamma_1 = \gamma_2 = \gamma_3 = 0$.

Remarkably, many of the integrability constructions for $\mathcal{N} = 4$ SYM also go through for the γ-deformed theory (3.3.8). However, these constructions are naturally more intricate since they depend on the deformation parameters $\gamma_1, \gamma_2, \gamma_3$. In [7] the following double scaling limit was proposed, which eliminates the γ-dependence of (3.3.8) while preserving its integrability:

$$
g \to 0, \qquad \gamma_1, \gamma_2, \gamma_3 \to i\infty, \qquad \xi_j := ge^{-\frac{i}{2}\gamma_j} \text{ fixed}. \tag{3.3.11}
$$

In this limit, only some of the interactions in (3.3.9) survive. The first term in (3.3.9) clearly goes to zero

$$
\frac{g^2}{4}\,\mathrm{tr}(\{\phi_i^\dagger,\phi^i\}\{\phi_j^\dagger,\phi^j\}) \to 0 \tag{3.3.12}
$$

because of the $g \to 0$ limit. Half of the second term survives:

$$
g^2\,\mathrm{tr}(e^{-i\epsilon_{ijk}\gamma_k}\phi_i^\dagger\phi_j^\dagger\phi^i\phi^j) \sim \xi_1^2\,\mathrm{tr}(\phi_2^\dagger\phi_3^\dagger\phi^2\phi^3) + \xi_2^2\,\mathrm{tr}(\phi_3^\dagger\phi_1^\dagger\phi^3\phi^1) + \xi_3^2\,\mathrm{tr}(\phi_1^\dagger\phi_2^\dagger\phi^1\phi^2). \tag{3.3.13}
$$

Notably, the 'anti-chiral' analogues of the surviving terms in (3.3.13) vanish in the limit (3.3.11). For example

$$
g^2\,\mathrm{tr}(e^{-i\epsilon_{213}\gamma_3}\phi_2^\dagger\phi_1^\dagger\phi^2\phi^1) = g^2\,\mathrm{tr}(e^{i\gamma_3}\phi_2^\dagger\phi_1^\dagger\phi^2\phi^1) \to 0. \tag{3.3.14}
$$

[16] For the conjugate fields the charge vector is reversed, i.e. $q_{\bar\Phi} = -q_\Phi$.

Therefore the scalar fields in the strongly-twisted theory have to appear in a certain chirality. All terms involving ψ_4 or $\bar{\psi}_4$ go to zero in the limit (3.3.11), because they all contain a $\gamma^- \sim \gamma_i - \gamma_j$ factor. For example

$$g\,\mathrm{tr}(e^{-\frac{i}{2}\gamma_1^-}\bar{\psi}_1\phi^1\bar{\psi}_4) = ge^{\frac{i}{4}(\gamma_2-\gamma_3)}\,\mathrm{tr}(\bar{\psi}_1\phi^1\bar{\psi}_4) \sim \sqrt{\xi_3}e^{\frac{i}{4}\gamma_2}\,\mathrm{tr}(\bar{\psi}_1\phi^1\bar{\psi}_4) \to 0. \qquad (3.3.15)$$

However, some of the terms involving $\gamma^+ \sim \gamma_i + \gamma_j$ do survive. For example, we have

$$ige^{\frac{i}{2}\gamma_1^+}\,\mathrm{tr}(\psi^3\phi^1\psi^2) = ige^{-\frac{i}{4}(\gamma_2+\gamma_3)}\,\mathrm{tr}(\psi^3\phi^1\psi^2) \sim i\sqrt{\xi_2\xi_3}\,\mathrm{tr}(\psi^3\phi^1\psi^2). \qquad (3.3.16)$$

Similarly to (3.3.14), the 'anti-chiral' analogues of (3.3.16) vanish in the limit. For example, we have

$$-ige^{-\frac{i}{2}\gamma_2^+}\,\mathrm{tr}(\psi^3\phi^2\psi^1) = -ige^{\frac{i}{4}(\gamma_1+\gamma_3)} \to 0. \qquad (3.3.17)$$

Therefore the surviving Yukawa-like interactions are also chiral. Finally, in the limit (3.3.11), all interactions involving the gauge field A_μ vanish:

$$D_\mu = \partial_\mu + ig_{\mathrm{YM}}[A_\mu,\ ] \sim \partial_\mu. \qquad (3.3.18)$$

In this sense the gauge field 'decouples' from the strongly-twisted theory, and can be disregarded. Overall, the strongly-twisted theory reads [17]

$$\mathcal{L}_{\mathrm{DFN}} = N\,\mathrm{tr}\left(-\frac{1}{2}\partial^\mu\phi_i^\dagger\partial_\mu\phi^i + i\bar{\psi}_A^{\dot{\alpha}}(\sigma^\mu)_{\dot{\alpha}}^{\alpha}\partial_\mu\psi_\alpha^A\right) + \mathcal{L}_{\mathrm{DFN}}^{\mathrm{int}}, \qquad (3.3.19)$$

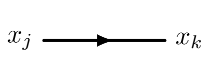
where $\mathcal{L}_{\mathrm{DFN}}^{\mathrm{int}}$ was defined in (3.3.1). In the original paper, the authors argued that the model (3.3.19) with $\xi_1 = \xi_2 = 0, \xi_3 \equiv \xi$ is integrable. For this choice of parameters all of the fermions and one of the scalars decouple, and we recover the bi-scalar fishnet theory given in (3.3.2). In the next sections we discuss some features of the fishnet model; namely the chiral Feynman graphs it generates, as well as its integrability. The dynamical fishnet theory is currently not as well understood from the perspective of integrability. However in chapter 6 we discuss one aspect of its integrability: namely its dilatation operator for a particular sector of operators.

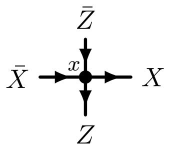
3.3.2 Chiral Graphs and Renormalisation

We describe a few features of the fishnet theory, whose Lagrangian we repeat here:

$$\mathcal{L}_{\mathrm{FN}} = N\,\mathrm{tr}(\partial_\mu X\partial^\mu\bar{X} + \partial_\mu Z\partial^\mu\bar{Z} + \xi^2 XZ\bar{X}\bar{Z}). \qquad (3.3.20)$$

Feynman Rules. The Lagrangian (3.3.20) generates a very simple set of Feynman rules. In single-line notation[18] and omitting factors of N, the position space rules are



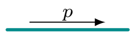
$= \frac{1}{x_{jk}^2}$,


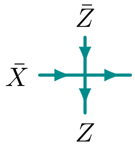
$= \xi^2 \int \mathrm{d}^4x, \qquad (3.3.21)$

[17]Note that the σ^μ here are defined in a non-standard way, see [7].

[18]One can also make the matrix indices of the fields explicit, for example $X = X^a{}_b T^b{}_a$ where $T^b{}_a$ are the generators of $\mathfrak{su}(N)$, and use a double line notation to express the propagators $\langle X^a{}_b(x_1)X^c{}_d(x_2)\rangle \sim \frac{\delta^{ac}\delta_{bd}}{x_{12}^2}$.

where as usual $x_{jk}^2 = (x_j - x_k)^2$. The corresponding momentum space rules are



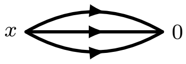
$$= \frac{1}{p^2},$$


$$= \xi^2. \tag{3.3.22}$$

The chirality of the interaction vertex vastly reduces the number of possible Feynman diagrams, especially in the planar limit. To illustrate this, we consider the two-point function

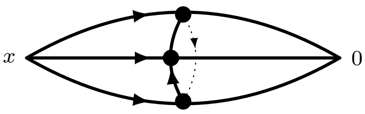
$$D(x) = \langle \mathrm{tr}(\bar{Z}^3(x))\, \mathrm{tr}(Z^3(0)) \rangle \tag{3.3.23}$$

in the planar limit. At tree-level, this correlator is represented by a single graph



$$= \frac{1}{x^6}. \tag{3.3.24}$$

Due to the chirality of the vertex, the next non-zero contribution to $D(x)$ occurs at three loops



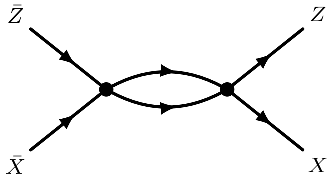
$$= \xi^6 \int \mathrm{d}^4 x_a \mathrm{d}^4 x_b \mathrm{d}^4 x_c \frac{(x-x_a)^{-2}(x-x_b)^{-2}(x-x_c)^{-2}}{x_{ab}^2 x_{bc}^2 x_{ca}^2 x_a^2 x_b^2 x_c^2}. \tag{3.3.25}$$

(3.3.25) is obtained from (3.3.24) by adding a X-field 'ring' whose orientation is consistent with the chirality of the vertex in (3.3.21). In general $D(x)$ takes the form

$$D(x) = \sum_{k=0}^{\infty} \xi^{6k} I_{3k}(x), \tag{3.3.26}$$

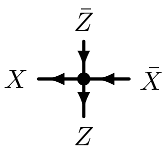
where $I_{3k}(x)$ is the $3k$-loop integral formed adding k X-field rings to the tree-level graph (3.3.24). A more general form of the two-point function (3.3.23) is discussed in [160], and various anomalous dimensions are calculated using integrability. Explicit examples of Feynman graph computations for amplitudes in fishnet theory can be found in [171].

Renormalisation. The restricted nature of the Feynman graphs in the fishnet theory has interesting consequences for the renormalisation of the theory in the planar limit. The first interesting fact is that there is no renormalisation for the coupling ξ^2 or the masses of the scalars X and Z. This is because there are simply no planar graphs which can generate quantum corrections. For example, the first planar graph which corrects the quartic coupling would be



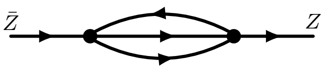
$$. \tag{3.3.27}$$

However, this diagram requires the existence of the vertex

$$\mathrm{tr}(ZX\bar{Z}\bar{X}) \sim$$


$$, \tag{3.3.28}$$

which is absent from the Lagrangian (3.3.20). Furthermore, the first planar graph which corrects the mass of the scalar Z would be



, (3.3.29)

which is similarly absent since it requires the vertex (3.3.28). The same considerations apply to the scalar X and higher-loop diagrams, and therefore the coupling ξ^2 and the masses m_Z, m_X remain uncorrected to all loops. This is encouraging if the fishnet theory is to retain the conformal invariance of undeformed $\mathcal{N} = 4$ SYM.

However, as pointed out in [169], in the γ-twisted theory (3.3.9) double trace couplings are introduced by renormalisation, which persist for the strongly twisted fishnet theories [168] and spoil conformality. These double trace terms take the form

$$\mathcal{L}_{\text{dt}} = \alpha_1^2(\operatorname{tr}(X^2)\operatorname{tr}(\bar{X}^2) + \operatorname{tr}(Z^2)\operatorname{tr}(\bar{Z}^2)) - \alpha_2^2(\operatorname{tr}(XZ)\operatorname{tr}(\bar{X}\bar{Z}) + \operatorname{tr}(X\bar{Z})\operatorname{tr}(\bar{X}Z)). \tag{3.3.30}$$

Despite the introduction of extra energy scales via α_1^2 and α_2^2, the beta functions corresponding to these couplings can be made to vanish by appropriately tuning α_i^2 as a function of ξ^2 [170]. α_2^2 is one-loop renormalisable, and indeed the fixed point where the beta function vanishes is simply

$$\alpha_2^2 = \xi^2. \tag{3.3.31}$$

The Feynman diagrams contributing to the renormalisation of the coupling α_1^2 are slightly more intricate, which leads to more complicated fixed points. Currently it is known to order ξ^{12}, where there are two possible options for α_1^2:

$$\alpha_1^2 = \alpha_\pm^2, \qquad \alpha_\pm^2 = \pm\frac{i\xi^2}{2} - \frac{\xi^4}{2} \mp \frac{3i\xi^6}{4} + \xi^8 \pm \frac{65i\xi^{10}}{48} - \frac{19\xi^{12}}{10} + O(\xi^{14}). \tag{3.3.32}$$

At the fixed points $\alpha_1^2 = \alpha_\pm^2, \alpha_2^2 = \xi^2$ the fishnet theory becomes a true logarithmic conformal field theory.

3.3.3 Yangian Symmetry

The fishnet theory is believed to be integrable in the planar limit, at the fixed points discussed in the previous section. This integrability appears to be intimately tied to the conformal group. For example, the correlation function (3.3.26) can be expressed in terms of a *graph-building operator* $\mathcal{B}$ [160]:

$$D(x) \sim \sum_{k=0}^{\infty} \xi^{6k}\mathcal{B}^k I_0, \tag{3.3.33}$$

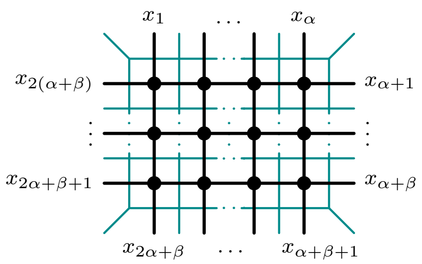
where $\mathcal{B}$ adds a ring to the tree-level graph (3.3.24). Fascinatingly, $\mathcal{B}$ can be explicitly realised as the Hamiltonian of an integrable non-compact conformal spin chain. Its R-matrix is a special case of the ones already considered in [70], and the corresponding Yang–Baxter equation is related to the star-triangle relation.

As discussed in section 3.1.3 and section 3.2.2, the Yangian algebra is central to many aspects of integrability. In the fishnet theory it appears as the conformal Yangian

$Y[\mathfrak{so}(1,5)]$.[19] Unlike the case of full $\mathcal{N}=4$ SYM, this Yangian symmetry has not yet been realised at the level of the action of the fishnet theory. However, it has been realised for a large class of fishnet Feynman graphs representing amplitudes and correlators in this theory [172, 162], a subset of which we discussed in section 2.3.5. Moreover, the fact that the fishnet action is recovered from the strong twist limit of the γ-deformed $\mathcal{N}=4$ SYM action provides hope that its Yangian symmetry may survive this limit.

Fishnet Feynman Integrals. In [172] and [162] Yangian symmetry was proven for a large class of fishnet Feynman graphs. In this thesis we will focus on the case of the square fishnet integrals, discussed in section 2.3.5:

$$\tilde{I}_{\alpha\beta} = \tag{3.3.34}$$



which represents the following correlator in the fishnet theory:

$$\left\langle \mathrm{tr}\big(Z(x_1)\cdots Z(x_\alpha)\bar{X}(x_{\alpha+1})\cdots\bar{X}(x_{\alpha+\beta})\bar{Z}(x_{\alpha+\beta+1})\cdots\bar{Z}(x_{2\alpha+\beta})X(x_{2\alpha+\beta+1})\cdots X(x_{2\alpha+2\beta})\big)\right\rangle \tag{3.3.35}$$

in the planar limit. The Yangian symmetry is formulated as follows:

$$\mathrm{J}^A \tilde{I}_{\alpha\beta} = 0, \qquad \widehat{\mathrm{J}}^A \tilde{I}_{\alpha\beta} = 0, \tag{3.3.36}$$

where $\mathrm{J}^A \in \{\mathrm{P}^\mu, \mathrm{L}^{\mu\nu}, \mathrm{D}, \mathrm{K}^\mu\}$ are level-zero generators, explicitly realised as differential operators in the coordinates $x_1, \ldots, x_{2(\alpha+\beta)}$. They are constructed as an $n = 2(\alpha+\beta)$ site representation of the conformal algebra $\mathfrak{so}(1,5)$:

$$\mathrm{J}^A = \sum_{j=1}^{n} \mathrm{J}_j^A, \tag{3.3.37}$$

where the densities J_j^A read

$$\begin{aligned} \mathrm{P}_j^\mu &= -i\partial_j^\mu, & \mathrm{L}_j^{\mu\nu} &= i x_j^\mu \partial_j^\nu - i x_j^\nu \partial_j^\mu, \\ \mathrm{D}_j &= -i x_{j\mu}\partial_j^\mu - i\Delta_j, & \mathrm{K}_j^\mu &= -i\big(2x_j^\mu x_j^\nu - \eta^{\mu\nu} x_j^2\big)\partial_{j,\nu} - 2i\Delta_j x_j^\mu, \end{aligned} \tag{3.3.38}$$

with $\partial_j^\mu \coloneqq \partial_{x_j}^\mu$. These densities also furnish the building blocks for the corresponding level-one generators, which are realised in the evaluation representation (3.1.104) as

$$\widehat{\mathrm{J}}^A = \tfrac{1}{2} f^A{}_{BC} \sum_{k=1}^{n} \sum_{j=1}^{k-1} \mathrm{J}_j^C \mathrm{J}_k^B + \sum_{j=1}^{n} s_j \mathrm{J}_j^A. \tag{3.3.39}$$

[19] One can also consider the theory in Minkowski space, where the Yangian $Y[\mathfrak{so}(2,4)]$ appears.

Here the $f^A{}_{BC}$ denote the structure constants of the conformal algebra $\mathfrak{so}(1,5)$. The symbols s_j represent the evaluation parameters of the representation and are tuned to guarantee Yangian invariance (3.3.36) according to the rules of [172]:

$$s_j = (\underbrace{0,\ldots,0}_{\alpha},\underbrace{-1,\ldots,-1}_{\beta},\underbrace{-2,\ldots,-2}_{\alpha},\underbrace{-3,\ldots,-3}_{\beta})_j, \qquad j=1,\ldots,n. \tag{3.3.40}$$

Notably, the Basso–Dixon graphs (2.3.67) have different properties under the Yangian generators, i.e. they do not satisfy (3.3.36) and develop a non-zero right hand side. The study of the extension of (3.3.36) to the Basso–Dixon graphs represents a large part of this thesis, and is discussed in chapter 5.

The Yangian invariance of the correlators (3.3.35) was proven in [172] using the RTT realisation of the conformal Yangian, where the Yangian generators are packaged into a monodromy matrix

$$T(u) \simeq \mathbb{I} + \frac{1}{u}\mathrm{J} + \frac{1}{u^2}\widehat{\mathrm{J}} + \cdots \tag{3.3.41}$$

which satisfies a Yang–Baxter like (RTT) equation

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v). \tag{3.3.42}$$

The proof of Yangian invariance relies on the so-called 'lasso' method, which involves moving products of Lax matrices through integration vertices. In (3.3.42) $R(u)=\mathbb{I}+u\mathbb{P}$ is the R-matrix, and $\mathbb{P}$ is the permutation operator. Here the auxiliary space V_a is the defining representation of $\mathfrak{so}(1,5)$, and so is isomorphic to $\mathbb{C}^6$. The physical space representations are principal series representation of the conformal group, and can roughly be viewed as a space of functions of $x\in\mathbb{R}^4$, on which the differential operators (3.3.38) can act. $T_a(u)$ can be viewed as a matrix in the auxiliary space, whose entries are differential operators in the external coordinates. It can be built as a product of Lax matrices

$$L_{\alpha\beta}(u) = u\,\mathbb{I}_{\alpha\beta} + \frac{1}{2}S^{ab}_{\alpha\beta}\mathrm{J}^{\Delta}_{ab}, \tag{3.3.43}$$

where $S^{ab}_{\alpha\beta}$ is a spinor representation of the conformal group [70] and J_{ab} contain the generators J^A defined explicitly in (3.3.38). The indices $\alpha,\beta=1,\ldots,6$ are matrix indices in the auxiliary space. The level-one generators built from this monodromy matrix via (3.3.41) are equivalent to the ones defined in (3.3.39) up to the addition of extra level-zero generators, see [172].

Yangian Bootstrap. It is natural to ask what constraints the Yangian invariance equations (3.3.36) impose on the fishnet integrals $\tilde{I}_{\alpha\beta}$, given that integrability is typically very constraining on physical quantities.

Conformal symmetry, i.e. invariance under the level-zero generators J^A, implies that the above integrals can be written in the form

$$\tilde{I}_{\alpha\beta} = V_{\alpha\beta}\,\tilde{\phi}_{\alpha\beta}(u_j), \tag{3.3.44}$$

where $V_{\alpha\beta}$ is the conformal weight of the integral and depends only on the kinematics x_{ij}^2. $\tilde{\phi}_{\alpha\beta}$ denotes a conformal function which only depends on a number of conformal cross ratios u_j, cf. section 2.2.3.

The level-one invariance equation $\widehat{\mathrm{J}}^A \tilde{I}_{\alpha\beta} = 0$ leads to differential constraints for the conformal functions $\tilde{\phi}_{\alpha\beta}$. Since $\widehat{\mathrm{J}}^A$ is a second order differential operator in the external coordinates $x_1, \ldots, x_n$, and $\tilde{I}_{\alpha\beta} = V_{\alpha\beta}\, \tilde{\phi}_{\alpha\beta}(u_j)$ depends on these coordinates through the cross ratios u_j, one can use the chain rule to determine PDEs in terms of the cross ratios which the function $\tilde{\phi}_{\alpha\beta}$ should satisfy. As a toy example of this procedure, for $n = 4$ points with two cross ratios u and v defined as in (2.2.57), the derivative $\partial^\mu_{x_1}$ acts on functions of u, v as

$$\begin{aligned} \frac{\partial}{\partial x_1^\mu}\phi(u,v) &= \frac{\partial u}{\partial x_1^\mu}\frac{\partial \phi}{\partial u} + \frac{\partial v}{\partial x_1^\mu}\frac{\partial \phi}{\partial v} \\ &= 2u\left(\frac{x_{12}^\mu}{x_{12}^2} - \frac{x_{13}^\mu}{x_{13}^2}\right)\partial_u\phi + 2v\left(\frac{x_{14}^\mu}{x_{14}^2} - \frac{x_{13}^\mu}{x_{13}^2}\right)\partial_v\phi. \end{aligned} \tag{3.3.45}$$

The calculations to determine the full differential constraints on $\tilde{\phi}_{\alpha\beta}$ are more tedious versions of (3.3.45), although they are straightforward. To see the structure in more detail, let us specify the level-one generator $\widehat{\mathrm{J}}^A$ to the level-one momentum generator:

$$\widehat{\mathrm{P}}^\mu = \tfrac{i}{2}\sum_{j<k=1}^{n}\big(\mathrm{P}^\mu_j \mathrm{D}_k + \mathrm{P}_{j\nu}\mathrm{L}^{\mu\nu}_k - (j \leftrightarrow k)\big) + \sum_{j=1}^{n} s_j \mathrm{P}^\mu_j \coloneqq \sum_{j<k=1}^{n}\widehat{\mathrm{P}}^\mu_{jk} + \sum_{j=1}^{n} s_j \mathrm{P}^\mu_j. \tag{3.3.46}$$

Here, using the densities for the conformal generators given in (3.3.38), one finds

$$\widehat{\mathrm{P}}^\mu_{jk} = \tfrac{i}{2}\big(T^{\nu\mu\rho}\partial_{j,\rho}\partial_{k,\nu} + \Delta_j\partial^\mu_k - \Delta_k\partial^\mu_j\big), \qquad T^{\nu\mu\rho} = x^\nu_{jk}\eta^{\mu\rho} + x^\rho_{jk}\eta^{\mu\nu} - x^\mu_{jk}\eta^{\nu\rho}. \tag{3.3.47}$$

Now the invariance of the above diagrams under the level-one generators can be written as

$$0 = \widehat{\mathrm{P}}^\mu \tilde{I}_{\alpha\beta} = V_{\alpha\beta}\sum_{j<k=1}^{n}\frac{x^\mu_{jk}}{x^2_{jk}}\mathrm{PDE}_{jk}\,\tilde{\phi}_{\alpha\beta}(u_j), \tag{3.3.48}$$

where PDE_{jk} denotes some differential operators in the conformal cross ratios u_j. At least for lower numbers of points it can be argued that the vectors x^μ_{jk}/x^2_{jk} are independent such that Yangian invariance implies a system of differential equations for the conformal function [10]:

$$\mathrm{PDE}_{jk}\,\tilde{\phi}_{\alpha\beta} = 0, \qquad 1 \leq j < k \leq n. \tag{3.3.49}$$

Specifying to level-one generators different from $\widehat{\mathrm{P}}^\mu$ will lead to equations like (3.3.48) with different tensor structures. However, the resulting PDEs in the cross ratios will not be independent from (3.3.49). This is because all the other level one generators can be accessed from $\widehat{\mathrm{P}}^\mu$ by commuting it with appropriate level-zero generators, via (3.1.87).

Notably, many more examples of conformal Feynman integrals in various dimensions D have been shown to be invariant under the appropriate conformal Yangian $Y[\mathfrak{so}(1, D+1)]$. This includes the conformal n-gons in $D = n$ dimensions discussed in section 2.3.4, as well as examples with generic conformal propagator powers. This was further extended to some classes of massive Feynman diagrams [65, 163] which represent correlators in a massive extension of the fishnet theory [173]. In all of these cases Yangian invariance implies PDEs in the cross ratios (or massive extensions thereof), analogous to (3.3.49), for the respective conformal functions.

The *Yangian bootstrap* approach to computing Yangian invariant Feynman integrals was initiated in [10]. The goal is to compute the conformal function ϕ of a Yangian invariant integral by solving the PDEs in the cross ratios resulting from Yangian invariance, analogous to (3.3.49). The solution space to these equations is spanned by a set of *Yangian invariants* $f_i(u_j)$, which are functions of the cross ratios. The conformal function can then be expanded as

$$\phi = \sum_i c_i f_i(u_j), \tag{3.3.50}$$

where c_i are some numerical coefficients. The precise linear combination which represents the Feynman integral in question must be fixed by some extra input, for example permutation symmetry in the external legs or numerical input. Systems of PDEs in several variables are in general very complicated, and there is no general method to solve them. Nevertheless, a few examples of Yangian invariant integrals depending on a small number of conformal invariants have been successfully bootstrapped. The first was the conformal box integral $\tilde{I}_{11}$, the bootstrap of which we discuss in section 4.1.1. The box was also generalised to the case with generic conformal propagator powers, and the Yangian bootstrap succeeded to fix this conformal function to a linear combination of Appell F_4 functions. In the massless case the next simplest example is the double box $\tilde{I}_{21}$, which is simply related to the conformal hexagon in six dimensions via (2.3.65). In this case Yangian invariance implies a system of 15 PDEs in 9 cross ratios $u_1, \ldots, u_9$, which were argued to be solved by a set of generalised Lauricella functions [10]. However, the problem of identifying the integral as a linear combination of these functions remains unsolved. It is an interesting question whether there exists a change of variables in cross ratio space which renders this system solvable in terms of the appropriate elliptic polylogarithms. In the massless case, the number of independent cross ratios goes up quickly with the number of points $2 \to 5 \to 9 \to 14 \to \cdots$. When some propagators are turned massive, it is possible to smooth out these transitions and study integrals with three cross ratios or even one. Several examples were studied in these cases, allowing for a better understanding of what is possible with the Yangian bootstrap approach [163].

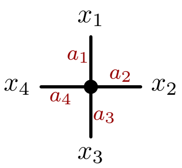
This ends our review of integrability in the conformal fishnet theory. We proceed to describe the results of this thesis, beginning with the box integral in Minkowski space.

Chapter 4

Conformal Box Integral in Minkowski Space

An important class of Feynman integrals are the box integrals with massless internal lines. These have the general form

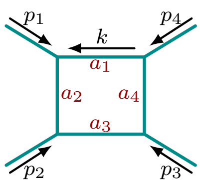
$$I^{\boldsymbol{a},D}_{\text{box}}(x_{ij}^2) =$$



$$= \int \frac{\mathrm{d}^D x_a}{\pi^{D/2}} \frac{1}{x_{a1}^{2a_1} x_{a2}^{2a_2} x_{a3}^{2a_3} x_{a4}^{2a_4}}, \tag{4.0.1}$$

where $x_i \in \mathbb{R}^D$. In the dual space this can be written

$$I^{\boldsymbol{a},D}_{\text{box}}(p_i^2, s, t) =$$



$$= \int \frac{\mathrm{d}^D k}{\pi^{D/2}} \frac{1}{k^{2a_1}(k+p_1)^{2a_2}(k+p_{12})^{2a_3}(k+p_{123})^{2a_4}}, \tag{4.0.2}$$

where $x_i - x_{i+1} = p_i, k = x_{a1}$, and $s = x_{13}^2, t = x_{24}^2$. When this is interpreted as momentum space, the on-shell conditions for the external particles are $p_i^2 = x_{i,i+1}^2 = 0$. In D dimensions the integral (4.0.1) has not been evaluated as an explicit function of the external kinematics.

It has, however, been computed for conformal propagator powers $a_1 + a_2 + a_3 + a_4 = D$ in terms of Appel F_4 functions [10, 174], and for on-shell kinematics $p_i^2 = 0$ in terms of hypergeometric ${}_3F_2$ [175]. Interestingly, the integral (4.0.1) with $D = 4$ and unit propagator powers $a_1 = a_2 = a_3 = a_4 = 1$ represents non-perturbatively a single correlator in the fishnet theory:

$$\langle \mathrm{tr}(Z(x_1)\bar{X}(x_2)\bar{Z}(x_3)X(x_4))\rangle. \tag{4.0.3}$$

In Minkowski space, we consider the time-ordered correlator, where the propagators are regularised via the Feynman prescription $1/x_{jk}^2 \to 1/(x_{jk}^2 + i\epsilon)$. In Euclidean space, the integral is conformal and can be represented in terms of the famous Bloch–Wigner function of the conformal variable z. In Minkowski space, the precise branch of the Bloch–Wigner function on which the box integral lies depends sensitively on the signs of the external kinematic invariants x_{ij}^2. As mentioned around (2.2.67), global special conformal transformations can change the signs of these invariants, spoiling the global conformal invariance.

In this chapter we provide a detailed overview of the conformal box integral in four dimensions. We provide a brief historical overview and outline a few ways in which the Euclidean box integral can be calculated. We give an overview of the Bloch–Wigner function $D(z)$, and review its main properties in relation to the box integral. We study in detail the breaking of conformal invariance in Minkowski space, at the level of several representations of the Minkowskian integral. To quantify the breaking of conformal invariance, we provide a full classification of conformally equivalent configurations of four points in Minkowski space. To this end, we introduce the notion of *Minkowskian conformal plane configurations*, which are generalisations of the Euclidean conformal plane configuration discussed in section 2.2.4. By explicitly rewriting a result in the literature for the Minkowski box integral in terms of z and $\bar{z}$ in each kinematic region, we find that there are up to four values for the Minkowski box integral on any conformal trajectory. This is a convenient alternative to simply analytically continuing the Euclidean box integral via the Osterwalder–Schrader prescription (2.1.34).

We further propose a new method to calculate the box integral in Minkowski space, by introducing *double infinity configurations* in Minkowski space. These configurations make good use of the fact that the conformal boundary of Minkowski space is a three-dimensional manifold, as opposed to Euclidean space where it is a single point ∞. By using conformal transformations to place two points on the conformal boundary, the Minkowski box integral simplifies substantially, such that it can be calculated by simple contour integration techniques. We discuss the extent to which these double infinity configurations cover the full kinematic space.

Finally, we explore the possibility that the Minkowski box is fully fixed by integrability, namely its Yangian symmetry. We find that Yangian symmetry, together with permutation symmetry, is sufficient to fix the functional form of the box integral in all kinematic regions up to four undetermined constants. These constants can be fixed by analytic continuation from the Euclidean sheet.

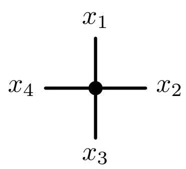
4.1 Euclidean Box Integral

One of the simplest four-point correlators in the fishnet theory

$$\langle \mathrm{tr}(Z(x_1)\bar{X}(x_2)\bar{Z}(x_3)X(x_4))\rangle \tag{4.1.1}$$

is represented by the conformal box integral in four dimensions:

$$I =$$



$$= \int \frac{\mathrm{d}^4 x_a}{\pi^2} \frac{1}{x_{a1}^2 x_{a2}^2 x_{a3}^2 x_{a4}^2} = \frac{1}{x_{13}^2 x_{24}^2}\phi(z,\bar{z}), \tag{4.1.2}$$

which is the integral (4.0.1) for $a_i = 1$ and $D = 4$. (4.1.2) appears ubiquitously in four-dimensional conformal field theory. In this section we describe a few ways this integral can be calculated, and discuss properties of the Bloch–Wigner function, to which it evaluates.

4.1.1 Methods of Calculation

The box integral was calculated first in [176] for general Minkowskian kinematics, by a careful integration of the Feynman parametrisation in each kinematic region. There some features under conformal transformations were already described, although full invariance under these transformations was first discussed in [177]. Several calculations have followed [178–180, 67, 10, 1, 2], in both Euclidean and Minkowskian kinematics. In this section we collect a few methods of calculating the integral (4.1.2).

Direct Integration. One approach to calculating a Feynman integral is the direct integration of its Feynman parametrisation. In section 2.3.3 we proved the conformal invariance of (4.1.2) and derived the conformal Feynman parametrisation, see (2.3.51):

$$\begin{aligned} I &= \frac{1}{x_{13}^2 x_{24}^2} \int_0^\infty \mathrm{d}\beta_1 \mathrm{d}\beta_3 \frac{1}{(\beta_1 + \beta_1\beta_3 + \beta_3)(\beta_1 v + 1 + \beta_3 u)} \\ &= \frac{1}{x_{13}^2 x_{24}^2} \int_0^\infty \mathrm{d}\beta_1 \mathrm{d}\beta_3 \frac{1}{u(1+\beta_1)(\beta_3 + \frac{\beta_1}{1+\beta_1})(\beta_3 + \frac{1+\beta_1 v}{u})}, \end{aligned} \tag{4.1.3}$$

where u and v are the usual four-point cross ratios defined in (2.2.57). The integral over β_3 can be performed to recover

$$I = \frac{1}{x_{13}^2 x_{24}^2} \int_0^\infty \mathrm{d}\beta_1 \frac{\log(1+\frac{1}{\beta_1}) + \log(1+\beta_1 v) - \log(u)}{v(\beta_1 + \frac{1}{1-z})(\beta_1 + \frac{1}{1-\bar{z}})}, \tag{4.1.4}$$

where $z\bar{z} = u, (1-z)(1-\bar{z}) = v$. The remaining β_1 integral can be computed using the integral identities

$$\int_0^\infty \mathrm{d}\beta \frac{1}{(\beta+\kappa)(\beta+\bar{\kappa})} = \frac{\log\kappa - \log\bar{\kappa}}{\kappa - \bar{\kappa}}, \tag{4.1.5}$$

$$\int_0^\infty \mathrm{d}\beta \frac{\log(1+\lambda\beta)}{(\beta+\kappa)(\beta+\bar{\kappa})} = -\frac{\mathrm{Li}_2(1-\lambda\kappa) - \mathrm{Li}_2(1-\lambda\bar{\kappa})}{\kappa-\bar{\kappa}}, \tag{4.1.6}$$

where $\lambda > 0$ and $\kappa \in \mathbb{C}, \bar{\kappa} = \kappa^*$. In (4.1.5) and (4.1.6) the logarithms have the usual branch cut on the negative real axis, and the dilogarithms have a cut on $(1,\infty)$. They can be proven easily by using an appropriate keyhole contour around the branch cuts and the residue theorem. Using (4.1.6) the first two integrals in (4.1.4) both evaluate to

$$\frac{1}{x_{13}^2 x_{24}^2} \frac{\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})}{z - \bar{z}}, \tag{4.1.7}$$

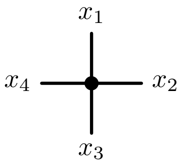
where for the first integral the substitution $\beta_1 \to 1/\beta_1$ was required. Using (4.1.5) the third integral in (4.1.4) evaluates to

$$\frac{1}{x_{13}^2 x_{24}^2} \frac{\log u(\log(1-z) - \log(1-\bar{z}))}{z-\bar{z}}. \tag{4.1.8}$$

Putting all of these results together, we arrive at

$$I = \frac{1}{x_{13}^2 x_{24}^2} \frac{2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})(\log(1-z) - \log(1-\bar{z}))}{z-\bar{z}}. \tag{4.1.9}$$

Yangian Bootstrap. An alternative approach for calculating (4.1.2) was proposed in [10], which exploits the Yangian invariance of this integral

$\widehat{\mathrm{J}}^a$ 

$$= \widehat{\mathrm{J}}^a \frac{1}{x_{13}^2 x_{24}^2} \phi(z, \bar{z}) = 0, \tag{4.1.10}$$

discussed in section 3.3.3. (4.1.10) implies that the conformal function $\phi(z, \bar{z})$ satisfies the differential equations

$$[D_j(z) - D_j(\bar{z})]\phi(z, \bar{z}) = 0, \qquad j = 1, 2, \tag{4.1.11}$$

where differential operators D_j are defined

$$D_1(z) = z(z-1)^2 \partial_z^2 + (3z-1)(z-1)\partial_z + z, \tag{4.1.12}$$
$$D_2(z) = z^2(z-1)\partial_z^2 + (3z-2)z\partial_z + z. \tag{4.1.13}$$

(4.1.11) is a system of two PDEs in two variables $z, \bar{z}$. Such systems are difficult to solve in general. The authors of [10] first solved the equations on the boundary $z = \bar{z}$, where they identified a four-dimensional space of functions consistent with (4.1.11). These could be extended away from this boundary, to yield four Yangian invariants $f_i = \tilde{f}_i/(z - \bar{z})$, where

$$\begin{aligned}
\tilde{f}_1 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log z\bar{z}(\log(1-z) - \log(1-\bar{z})), \qquad (4.1.14)\\
\tilde{f}_2 &= \log z - \log \bar{z},\\
\tilde{f}_3 &= \log(1-z) - \log(1-\bar{z}),\\
\tilde{f}_4 &= 1.
\end{aligned}$$

Therefore the conformal function can be expanded

$$\phi(z, \bar{z}) = \sum_{i=1}^{4} c_i f_i(z, \bar{z}). \tag{4.1.15}$$

c_i are constants which can be partially fixed using the permutation invariance of the full integral. Let us denote by (ij) the transposition which exchanges $x_i \leftrightarrow x_j$. While (ij) leaves the integral I invariant, it does generate a transformation of the conformal invariants $z \to z'$, $\bar{z} \to \bar{z}'$. For example, we have

$$(13) : u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \to \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} = v, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \to \frac{x_{34}^2 x_{12}^2}{x_{13}^2 x_{24}^2} = u. \tag{4.1.16}$$

Since $u = z\bar{z}, v = (1-z)(1-\bar{z})$ the transposition (13) maps $\{z, \bar{z}\} \to \{1-z, 1-\bar{z}\}$. Similarly, the transposition (14) is found to map $\{z, \bar{z}\} \to \{1/z, 1/\bar{z}\}$. There is a subtlety in this action however. z and $\bar{z}$ are recovered from u and v as the roots of a quadratic equation

$$z, \bar{z} = \frac{1 + u - v \pm \sqrt{(1-u-v)^2 - 4uv}}{2}, \tag{4.1.17}$$

and so there is an ambiguity of which of the solutions we call z, and which we call $\bar{z}$. We showed in section 2 that in Euclidean space we have that $z \in \mathbb{C}, \bar{z} = z^*$. For definiteness we will always take z to be the solution of (4.1.17) with positive imaginary part. Then the transpositions (13) and (14) act in a well-defined manner on $z, \bar{z}$:

$$(13) : z \to 1 - \bar{z}, \bar{z} \to 1 - z, \qquad (14) : z \to \frac{1}{\bar{z}}, \bar{z} \to \frac{1}{z}. \tag{4.1.18}$$

The invariance of I under these transposition leads to functional equations for the conformal function ϕ:

$$\phi(1 - \bar{z}, 1 - z) = \phi(z, \bar{z}), \qquad \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) = z\bar{z}\phi(z, \bar{z}). \tag{4.1.19}$$

The Yangian invariants are mapped into each other under these transpositions. For example

$$(13) : f_2 = \frac{\log z - \log \bar{z}}{z - \bar{z}} \to \frac{\log(1 - \bar{z}) - \log(1 - z)}{(1 - \bar{z} - (1 - z))} = \frac{\log(1 - \bar{z}) - \log(1 - z)}{z - \bar{z}} = -f_3. \tag{4.1.20}$$

Overall we have

$$(13) : f_1 \to f_1, \quad f_2 \leftrightarrow -f_3, \quad f_4 \to f_4, \tag{4.1.21}$$

$$(14) : f_1 \to z\bar{z}f_1, \quad f_2 \to z\bar{z}f_2, \quad f_3 \to z\bar{z}(f_2 - f_3 - 2\pi i f_4), \quad f_4 \to z\bar{z}f_4. \tag{4.1.22}$$

Using (4.1.21) the first constraint in (4.1.19) fixes $c_2 = -c_3 := c$ in the expansion (4.1.15). Using (4.1.22) the second constraint in (4.1.19) can be expanded to give

$$c_1 f_1 + c(f_2 - f_3) + c_4 f_4 = c_1 f_1 + c(f_2 - f_2 + f_3 + 2\pi i f_4) + c_4 f_4 \tag{4.1.23}$$

$$= c_1 f_1 + c f_3 + (2\pi i c + c_4) f_4 \tag{4.1.24}$$

for all $z, \bar{z}$, which implies that $c = 0$. Using the star-triangle relation the remaining constants can be fixed $c_1 = 1, c_4 = 0$ [10].

Orthogonal Polynomials and Quaternions. We briefly mention a cute calculation of the box integral, based on its representation in terms of quaternions [181]. Any vector $x = (a, b, c, d) \in \mathbb{R}^4$ can be represented as a quaternion

$$x = a + bi + cj + dk, \tag{4.1.25}$$

where the symbols i, j, k obey the algebra $i^2 = j^2 = k^2 = ijk = -1$. The norm squared of x is still given by x^2 in this representation. Representing the external points of the box integral as such, the box integral can be transformed into the form

$$I = \frac{1}{x_{13}^2 x_{24}^2} \int \frac{\mathrm{d}^4 x_a}{\pi^2} \frac{1}{x_a^2 (x_a - 1)^2 (x_a - z)^2}. \tag{4.1.26}$$

The integral $\int \mathrm{d}^4 x_a$ is interpreted as a fourfold real integral over the quaternionic components of $x_a = x_a^1 + x_a^2 i + x_a^3 j + x_a^4 k$. z is a quaternionic conformal variable related to the usual z seen in the rest of this thesis. The denominators can be expanded over the Gegenbauer polynomials

$$\frac{1}{(x-y)^2} = \frac{1}{|x||y|}\sum_{n=0}^{\infty} C_n(x,y)\left(\frac{|x|}{|y|}\right)_<, \qquad x_< := \min(x, 1/x), \tag{4.1.27}$$

and the integral (4.1.26) performed using orthogonality properties of $C_n(x,y)$ to recover a result equivalent to (4.1.9).

4.1.2 Bloch–Wigner Function and Single-Valuedness

The box integral in Euclidean space is simply related to the *Bloch–Wigner function* $D(z)$. This function has numerous fascinating mathematical properties, and was discovered first by Bloch in the context of algebraic K-theory [182]. Its properties are reviewed in depth in [183], and here we describe a few. $D(z)$ is a real-valued function of a complex variable z, and is defined by

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|. \tag{4.1.28}$$

Since $\bar{z} = z^*$ in Euclidean space, see section 2.2.4, the result for the box integral can be rewritten

$$\phi = \frac{2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(z^*) + \log zz^*(\log(1-z) - \log(1-z^*))}{z - z^*} = \frac{2D(z)}{\mathrm{Im}(z)}. \tag{4.1.29}$$

While typical expressions involving polylogarithms have branch cuts (and discontinuities across these cuts), $D(z)$ is a continuous function of z on $\mathbb{C}\setminus\{0,1\}$. To see this, we investigate the possible discontinuities of $D(z)$ and show that they are 0. For definiteness we define the complex logarithm with the usual branch cut on the negative real axis, so that the corresponding dilogarithm has a cut on $(1,\infty)$ (see figure 4.1).

For these choices of branch cuts we define the discontinuity operations

$$\mathrm{disc}_0 f(z) := f(z+i\epsilon) - f(z-i\epsilon), \qquad z \in (-\infty, 0), \tag{4.1.30}$$
$$\mathrm{disc}_1 f(z) := f(z+i\epsilon) - f(z-i\epsilon), \qquad z \in (1, \infty), \tag{4.1.31}$$

where $\epsilon \ll 1$, so that $\mathrm{disc}_0 \log z = 2\pi i$ and $\mathrm{disc}_1 \mathrm{Li}_2(z) = 2\pi i \log z$. If a function f is continuous across a branch cut then we clearly have $\mathrm{disc} f = 0$ for the appropriate discontinuity. When z crosses a branch cut in a clockwise manner, $\bar{z} = z^*$ crosses the same cut anticlockwise. For functions $f(z)$ satisfying $f(z^*) = f(z)^*$ (like polylogarithms), we thus have

$$\mathrm{disc}_0 f(z^*) = -\mathrm{disc}_0 f(z), \qquad \mathrm{disc}_1 f(z^*) = -\mathrm{disc}_1 f(z). \tag{4.1.32}$$

Since $D(z)$ is built from $\mathrm{Li}_2(z), \log z$, and $\log(1-z)$ and their complex conjugates, the only possible branch cuts of $D(z)$ are $(-\infty, 0)$ and $(1,\infty)$, with corresponding branch points 0 and 1 respectively.[1] We thus compute the discontinuities

$$\mathrm{disc}_0 D(z) = \frac{1}{2i}\mathrm{disc}_0(\mathrm{Li}_2(z) - \mathrm{Li}_2(z^*) + \tfrac{1}{2}\log zz^*(\log(1-z) - \log(1-z^*))) \tag{4.1.33}$$
$$= \frac{1}{4i}(\log(1-z) - \log(1-z^*))(\mathrm{disc}_0 \log z + \mathrm{disc}_0 \log z^*) = 0 \tag{4.1.34}$$

[1] In both cases there is also a branch point at ∞.

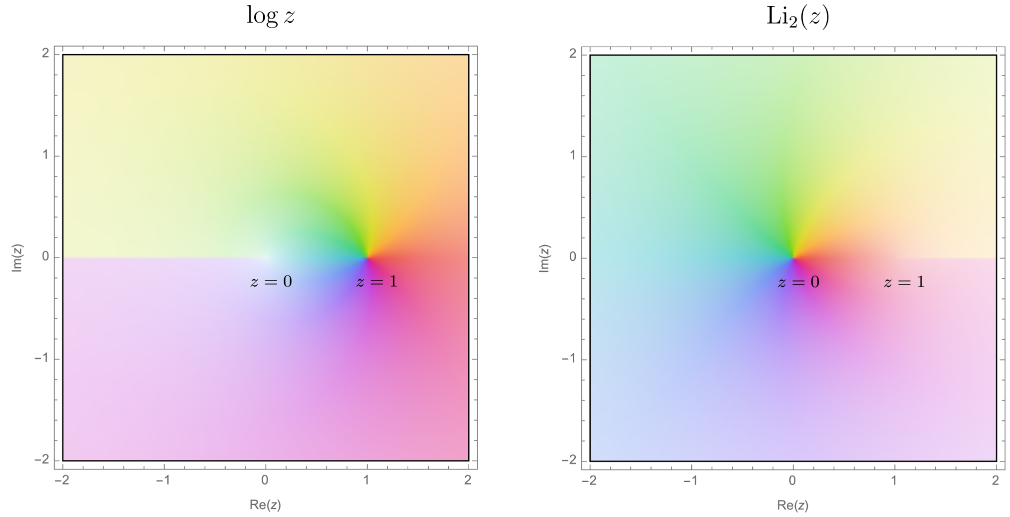


Figure 4.1: Complex plots of $\log z$ and $\mathrm{Li}_2(z)$. Colour indicates argument of function value, with red indicating $\mathrm{Arg}(f(z)) = 0$ and blue indicating $\mathrm{Arg}(f(z)) = \pi$. Intensity indicates magnitude of function value, with very bright spots indicating singularities, and black spots indicating zeros. The respective branch cuts on $(-\infty, 0)$ and $(1, \infty)$ are easily seen.

because of (4.1.32). Similarly, we have

$$2i\mathrm{disc}_1 D(z) = \mathrm{disc}_1\mathrm{Li}_2(z) - \mathrm{disc}_1\mathrm{Li}_2(z^*) + \tfrac{1}{2}\log zz^*(\mathrm{disc}_1 \log(1-z) - \mathrm{disc}_1\log(1-z^*)) \tag{4.1.35}$$

$$= 2\mathrm{disc}_1\mathrm{Li}_2(z) + \log z^2 \mathrm{disc}_1 \log(1-z) = 0, \tag{4.1.36}$$

where we used $z = z^*$ on the cut, and $\mathrm{disc}_1\mathrm{Li}_2(z) = -\log z\, \mathrm{disc}_1 \log(1-z) = 2\pi i \log z$. Since the only possible discontinuities of $D(z)$ vanish, it is continuous throughout the complex plane. We will also refer to it as being *single-valued*, because it takes unambiguous values throughout the complex plane, independently of the chosen logarithm branch. This single-valuedness is illustrated in figure 4.2, where we plot $\phi(z) = 2D(z)/\mathrm{Im}(z)$.

$D(z)$ is also a real analytic function of z, except at the points $z = 0$ and $z = 1$ where it has logarithmic singularities. At the level of the box integral, these singularities correspond to coincident points in Euclidean space, and lightlike separated points in Minkowski space. Furthermore, $D(z)$ satisfies the six-fold symmetry

$$D(z) = -D(1-z) = D\left(1 - \frac{1}{z}\right) = -D\left(\frac{1}{z}\right) = D\left(\frac{1}{1-z}\right) = D\left(\frac{z}{z-1}\right), \tag{4.1.37}$$

which at the level of the box integral ensures its invariance under permutations of the external points. The Bloch–Wigner function is also known to compute the volume of an ideal tetrahedron in hyperbolic 3-space. This is exposed at the level of the box integral in a momentum-twistor formulation [67].

The fact that the box integral is represented by a single-valued function of the conformal variable z is not an accident. It is a consequence of the Steinmann relations applied to the momentum space interpretation of the integral, where it can represent massive scattering [87]. The Steinmann relations forbid double discontinuities in overlapping channels.

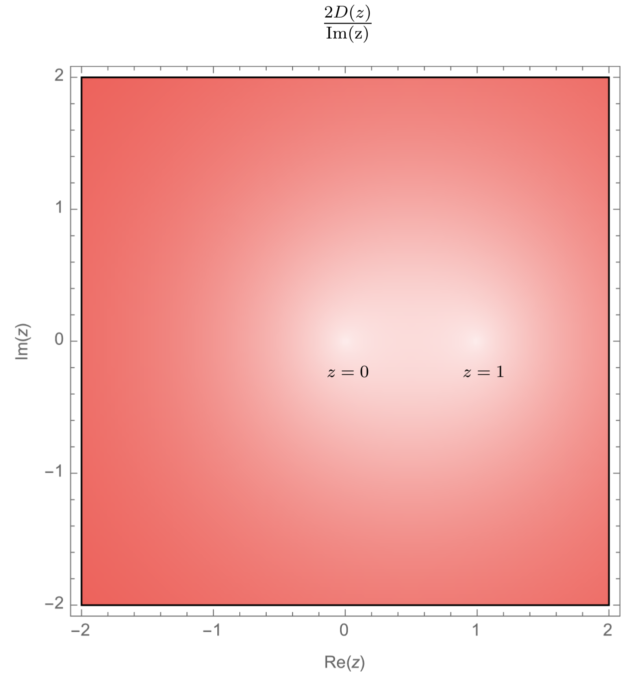


Figure 4.2: Complex plot of the box conformal function $\phi(z) = \frac{2D(z)}{\mathrm{Im}(z)}$. The function is manifestly positive and continuous throughout the complex plane, with singularities at $z = 0$ and $z = 1$.

The Bloch–Wigner function is perhaps the simplest example of a single-valued combination of polylogarithms representing a Feynman integral. In section 2.3.5 we introduced the ladder functions

$$L_\alpha(z, \bar{z}) = \sum_{r=0}^{\alpha} \frac{(-1)^r (2\alpha - r)!}{r!(\alpha - r)!\alpha!} \log(z\bar{z})^r (\mathrm{Li}_{2\alpha-r}(z) - \mathrm{Li}_{2\alpha-r}(\bar{z})), \tag{4.1.38}$$

which are similarly single-valued in the complex variable z, as are the Basso–Dixon functions (2.3.70). The class of single-valued polylogarithms can be generalised to a wider class of *single-valued harmonic polylogarithms*, reviewed in [86] in the context of scattering amplitudes in the multi-Regge limit. How to understand single-valuedness in more kinematical variables is a much more difficult question, and currently not very well understood. For example, it would be interesting to constrain the functional form of the six-point double box integral (2.3.64) using considerations of single-valuedness. An investigation of single-valued polylogarithms in two complex variables was initiated in [184].

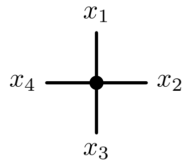
4.2 Minkowski Box Integral

We consider the correlator (4.1.1) again, but this time as a time-ordered correlation function for the fishnet theory in Minkowski space:

$$\langle \mathrm{tr}(\mathbb{T}[Z(x_1)\bar{X}(x_2)\bar{Z}(x_3)X(x_4)]) \rangle, \tag{4.2.1}$$

where $\mathbb{T}$ denotes the time-ordering of fields. This correlator is represented by the box integral in Minkowski space, where the Feynman $i\epsilon$ prescription is now applied to the propagators[2]

$$I =$$



$$= \int \frac{\mathrm{d}^4 x_a}{i\pi^2} \frac{1}{(x_{a1}^2 + i\epsilon)(x_{a2}^2 + i\epsilon)(x_{a3}^2 + i\epsilon)(x_{a4}^2 + i\epsilon)}. \tag{4.2.2}$$

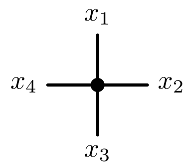
In this case the $i\epsilon$ prescription breaks global conformal invariance of the integral, as mentioned at the end of section 2.2.3 and further described below. This leads to the integral depending on not only the conformal variables z and $\bar{z}$, but also the precise *kinematic region* k, which is defined by the signs of the Poincaré invariants x_{ij}^2. We will first discuss the breaking of conformal invariance at the level of three representations of the integral (4.2.2), namely its original representation, its Feynman parametrisation, and its Mellin–Barnes representation. We then describe precisely the set of four-point configurations in Minkowski space which can be mapped into each other under conformal transformations, using the notion of Minkowskian conformal planes. Unlike the Euclidean case, we find that the conformal group does not act transitively on the set of four-point configurations with the same $z, \bar{z}$. We then specialise the result of [178] for the Minkowski box integral to write it explicitly as a function of $z, \bar{z}$ in each kinematic region k. Using this result and the classification of conformally equivalent configurations, we conclude that the box integral can take at most four values on any conformal trajectory.

4.2.1 Breaking of Global Conformal Invariance

Here we discuss the conformal invariance properties of the Minkowski box integral for three different representations.

Position Space Representation. The position space representation for the conformal function $\phi = x_{13}^2 x_{24}^2 I$ is

$$\phi =$$



$$= \int \frac{\mathrm{d}^4 x_a}{i\pi^2} \frac{x_{13}^2 x_{24}^2}{(x_{a1}^2 + i\epsilon)(x_{a2}^2 + i\epsilon)(x_{a3}^2 + i\epsilon)(x_{a4}^2 + i\epsilon)}. \tag{4.2.3}$$

Let us consider its properties under translations, rotations, dilatations, and special conformal transformations. Clearly it is invariant under translations and Lorentz rotations, and the $i\epsilon$ prescription has no effect. Under dilatations $x_i \to c x_i$, we also make the change of variables on the integration variable $x_a \to c x_a$ and find

$$\phi \to \int \frac{\mathrm{d}^4 x_a}{i\pi^2} \frac{x_{13}^2 x_{24}^2}{(x_{a1}^2 + i\epsilon/c^2)(x_{a2}^2 + i\epsilon/c^2)(x_{a3}^2 + i\epsilon/c^2)(x_{a4}^2 + i\epsilon/c^2)} = \phi, \tag{4.2.4}$$

because $c > 0$ and ϵ is an infinitesimal regulator. Therefore the Minkowski box integral is invariant under dilatations. The situation is trickier with special conformal transformations

$$x_i^\mu \to \frac{x_i^\mu - x_i^2 b^\mu}{b(x_i)}, \tag{4.2.5}$$

[2]For notational convenience we still refer to the integral as I in this section.

where $b(x) = 1 - 2b \cdot x + b^2 x^2$. In this case we have

$$\phi \to x_{13}^2 x_{24}^2 \int \frac{\mathrm{d}^4 x_a}{i\pi^2} \prod_{j=1}^{4} \frac{1}{(x_{aj}^2 + i\epsilon_{aj})}, \qquad \epsilon_{ij} := b(x_i)b(x_j)\epsilon. \tag{4.2.6}$$

We mention that $b(x)$ can be positive or negative depending on b and x. From (4.2.6) we see that depending on the configuration $\{x_1, x_2, x_3, x_4\}$ the signs of the infinitesimal regulators ϵ_{aj} can change, and they even change throughout the integration because of the $b(x_a)$ factor. This makes it clear that there are subtleties with the invariance of the Minkowski box integral when it comes to special conformal transformations, although the exact details are better exposed in the Mellin–Barnes representation.

Feynman Parametrisation. The Feynman parametrisation exposes the dependence of the box integral on the kinematics[3]

$$\phi = \phi(x_{ij}^2) = \int [\mathrm{d}^3\alpha] \frac{x_{13}^2 x_{24}^2}{(\sum_{j<k} \alpha_i \alpha_k x_{jk}^2 + i\epsilon)^2}, \tag{4.2.7}$$

where $\int [\mathrm{d}^3\alpha]$, denotes a projective integral over the Feynman parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. We also notice that complex conjugation reverses the signs of the kinematics:

$$\phi(x_{ij}^2)^* = \phi(-x_{ij}^2). \tag{4.2.8}$$

For Euclidean kinematics $x_{ij}^2 < 0$ there are no singularities on the integration contour. Then the $i\epsilon$ regulator can be safely ignored and the integral is manifestly real-valued. However if any of the $x_{ij}^2 > 0$ then this regulator is important and the integral can take complex values. The representation (4.2.7) makes manifest the invariance of the integral under translations, Lorentz rotations, and dilatations. However in this representation the behaviour under special conformal transformations is obscured. Indeed, even for the Euclidean kinematics it is not clear how to show invariance under special conformal transformations in this representation. It is of course not possible to obtain a conformal Feynman parametrisation analogous to (4.1.3), as the calculation to obtain this representation depends on the signs of the kinematics x_{ij}^2. One would have to proceed on a case-by-case basis for each possible sign assignment.

Mellin–Barnes Representation. The most useful representation for our analysis is the Mellin–Barnes representation [178]

$$\phi = \frac{1}{(2\pi i)^2} \int_{C_1} ds \int_{C_2} ds' (\mathit{\Gamma}_{s,s'})^2 \left(\frac{x_{12}^2 + i\epsilon}{x_{13}^2 + i\epsilon}\right)^{-s} \left(\frac{x_{34}^2 + i\epsilon}{x_{24}^2 + i\epsilon}\right)^{-s} \left(\frac{x_{14}^2 + i\epsilon}{x_{13}^2 + i\epsilon}\right)^{-s'} \left(\frac{x_{23}^2 + i\epsilon}{x_{24}^2 + i\epsilon}\right)^{-s'}, \tag{4.2.9}$$

where $\mathit{\Gamma}_{x,y} := \mathit{\Gamma}(x)\mathit{\Gamma}(y)\mathit{\Gamma}(1-x-y)$. The complex power is defined as $z^s := \exp(s \log z)$, where the logarithm has the usual branch cut on the negative real axis. C_j are infinite imaginary lines in the complex plane

$$C_j = \{\gamma_j + it \,|\, t \in (-\infty, \infty)\}, \tag{4.2.10}$$

[3] We refer to the x_{ij}^2 interchangeably as Poincaré invariants and kinematics.

where γ_1, γ_2 satisfy $0 < \gamma_1, \gamma_2 < 1$, $0 < \gamma_1 + \gamma_2 < 1$. Under special conformal transformations (4.2.5) we have $\phi \to \phi' =$

$$\frac{1}{(2\pi i)^2}\int_{C_1} ds \int_{C_2} ds' (\Gamma_{s,s'})^2 \left(\frac{x_{12}^2+i\epsilon_{12}}{x_{13}^2+i\epsilon_{13}}\right)^{-s} \left(\frac{x_{34}^2+i\epsilon_{34}}{x_{24}^2+i\epsilon_{24}}\right)^{-s} \left(\frac{x_{14}^2+i\epsilon_{14}}{x_{13}^2+i\epsilon_{13}}\right)^{-s'} \left(\frac{x_{23}^2+i\epsilon_{23}}{x_{24}^2+i\epsilon_{24}}\right)^{-s'} \tag{4.2.11}$$

where the ϵ_{ij} were defined in (4.2.6). From (4.2.11) we see the extent to which conformal invariance is broken is encoded in the signs of the infinitesimal imaginary parts $i\epsilon_{ij}$. Given a configuration of four points in Minkowski space $w = \{x_1, x_2, x_3, x_4\}$, there is a possible branch jumping of the integral under *finite* special conformal transformations C_b for which at least one of the $\epsilon_{ij} < 0$. ϵ_{ij} changes from positive to negative when $C_b x_k$ crosses infinity for some $k \in \{i, j\}$ (see (2.2.92)) and the kinematics x_{kl}^2 for $l \neq k$ change sign, i.e. the kinematic region changes. In computing (4.2.11) the $i\epsilon_{ij}$ select the correct branch of the logs/dilogs which appear in the computation, after which the result depends only on the cross ratios u and v, or equivalently z and $\bar{z}$. The computation of the integral (4.2.9) was completed for arbitrary signs of kinematics in [178], and we specialise it in section 4.2.3 to give the box integral as a function of z and $\bar{z}$ in each kinematic region $\text{sgn}(k(w))$ (defined below). While from (4.2.11) there are naively eight values of the integral which can be reached under SCTs, corresponding to the eight possible sign assignments for ϵ_{ij}, it is actually up to four. The dependence of the box integral on only the conformal invariants z and $\bar{z}$ as well as the kinematic region $\text{sgn}(k(w))$ will be referred to as *pseudo*-conformal invariance.

4.2.2 Minkowskian Conformal Planes

In this section we develop the theory of conformal plane configurations in Minkowski space. The purpose of this is to classify which configurations of four points in Minkowski space can be mapped into each other under conformal transformations. It was shown in section 2.2.4 that in Euclidean space the conformal invariants $z, \bar{z}$ are sufficient to determine whether two configurations can be conformally mapped into each other. The situation is more subtle in Minkowski space $\mathbb{R}^{1,3}$, where the squared differences x_{ij}^2 can be positive or negative[4]. The conformal structure is much richer, and we will see that z and $\bar{z}$ can be complex conjugate pairs, or independent real numbers. As such we will be more mathematically formal in this section. We define the set of non-singular four-point configurations

$$V = \{\{x_1, x_2, x_3, x_4\} \quad | \quad x_i \in \mathbb{R}^{1,3}, x_{ij}^2 \neq 0\}. \tag{4.2.12}$$

and the set of configurations with conformal variables $z, \bar{z}$:

$$V_{z,\bar{z}} = \left\{\{x_1, x_2, x_3, x_4\} \in V \quad | \quad u = z\bar{z}, v = (1-z)(1-\bar{z})\right\}. \tag{4.2.13}$$

Since z and $\bar{z}$ appear as the roots of a quadratic equation (4.1.17), without loss of generality we will always take $z \geq \bar{z}$ when $z, \bar{z} \in \mathbb{R}$ and $\text{Im}(z) > 0$ when $z, \bar{z} \in \mathbb{C} \setminus \mathbb{R}$.

[4] In [1] we took care to perform this analysis in conformally compactified Minkowski space $\mathbb{R}_c^{1,3}$, where the conformal group is well-defined. For ease of notation/discussion we refer here only to Minkowski space, with the understanding that special conformal transformations can map us through infinity in a manner which is well-defined in the compactification, see section 2.2.6.

Based on the possible values for z and $\bar{z}$, we have the decomposition[5]

$$V = \bigcup_{\substack{z,\bar{z}\in\mathbb{R}\setminus\{0,1\} \\ z\geq\bar{z}}} V_{z,\bar{z}} \cup \bigcup_{\substack{z\in\mathbb{H} \\ \bar{z}=z^*}} V_{z,\bar{z}} \coloneqq V_{\mathbb{R}} \cup V_{\mathbb{C}}, \tag{4.2.14}$$

where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane $\mathrm{Im}\, z > 0$. Note that the 'boundary' between $V_{\mathbb{R}}$ and $V_{\mathbb{C}}$ occurs when $z = \bar{z}$, and we have chosen to include it in $V_{\mathbb{R}}$.

Given a configuration $w = \{x_1, x_2, x_3, x_4\} \in V$, we define the *kinematics* $k(w)$ of w to be the 6-tuple[6]

$$k(w) \coloneqq \begin{bmatrix} x_{12}^2, x_{34}^2 \\ x_{23}^2, x_{14}^2 \\ x_{13}^2, x_{24}^2 \end{bmatrix}. \tag{4.2.15}$$

We define the *sign* of the kinematics

$$\mathrm{sgn}(k(w)) \coloneqq \mathrm{sgn} \begin{bmatrix} x_{12}^2, x_{34}^2 \\ x_{23}^2, x_{14}^2 \\ x_{13}^2, x_{24}^2 \end{bmatrix}, \tag{4.2.16}$$

where $\mathrm{sgn}(x) = +$ if $x > 0$ and $\mathrm{sgn}(x) = -$ if $x < 0$. Since we consider non-singular configurations (4.2.12) none of the x_{ij}^2 vanish and so $\mathrm{sgn}(x_{ij}^2) \in \{-,+\}$ for each $i < j$. Therefore depending on $w \in V$ there are $2^6 = 64$ possibilities for $\mathrm{sgn}(k(w))$. We denote by P_{ij} the operator which flips the sign of x_{ij}^2, so for example $P_{13}\left[\begin{smallmatrix}++\\++\\++\end{smallmatrix}\right] = \left[\begin{smallmatrix}++\\++\\-+\end{smallmatrix}\right]$. Moreover we define $P \coloneqq \prod_{i<j} P_{ij}$ as the operator which flips the sign of all kinematics, so for example $P\left[\begin{smallmatrix}+-\\+-\\+-\end{smallmatrix}\right] = \left[\begin{smallmatrix}-+\\-+\\-+\end{smallmatrix}\right]$.

We ask the question: Does $\mathrm{Conf}(\mathbb{R}^{1,3})$ act transitively on each $V_{z,\bar{z}}$, as was the case in Euclidean space? In this case the answer is *no* in general, and the existence of $A \in \mathrm{Conf}(\mathbb{R}^{1,3})$ connecting configurations $w, y \in V_{z,\bar{z}}$ depends on $k(w)$ and $k(y)$. Inversions and special conformal transformations change the kinematics of a configuration, in such a way that the cross ratios are fixed. However they can only change the *signs* of the kinematics in a restricted way. For the rest of this section we refine (4.2.14) such that $\mathrm{Conf}(\mathbb{R}^{1,3})$ acts transitively on each set in the decomposition. This is possible provided the discrete transformations $\mathcal{P}, \mathcal{T}$ are also used.

Let $b \in \mathbb{R}^{1,3}$, $w = \{x_1, x_2, x_3, x_4\} \in V$. We saw in (2.2.52) that under a special conformal transformation $w \to C_b w$ the kinematics transform

$$x_{ij}^2 \to \frac{x_{ij}^2}{b(x_i)b(x_j)}, \qquad \mathrm{sgn}(x_{ij}^2) \to \frac{\mathrm{sgn}(x_{ij}^2)}{\mathrm{sgn}(b(x_i)b(x_j))}, \tag{4.2.17}$$

where $b(x) = 1 - 2b\cdot x + b^2x^2$. As b changes, $b(x_i)$ passing through 0 corresponds to x_i 'crossing infinity' (2.2.92). For a fixed configuration $w \in V$ and $b \in \mathbb{R}^{1,3}$ there are up to $2^4 = 16$ possibilities[7] for

$$\mathrm{sgn}(b(w)) \coloneqq \mathrm{sgn}(b(x_1), b(x_2), b(x_3), b(x_4)), \tag{4.2.18}$$

[5]In Euclidean space we showed that only $z \in \bar{\mathbb{H}} \setminus \{0, 1\}$ was possible, here the cross ratios can also be independent real numbers.

[6]This notation underlines the different roles of the three rows in the block, cf. the definition (2.2.57) of the conformal cross-ratios

[7]Note that it is not always 16, e.g. if the configuration contains $x = 0$ then $b(x) = 1$ for all b.

and up to eight possibilities for the 6-tuple $\operatorname{sgn}(k_b(w))$, where

$$k_b(w) \coloneqq \begin{bmatrix} b(x_1)b(x_2), b(x_3)b(x_4) \\ b(x_2)b(x_3), b(x_1)b(x_4) \\ b(x_1)b(x_3), b(x_2)b(x_4) \end{bmatrix}. \tag{4.2.19}$$

These eight possibilities can be deduced easily by calculating $\operatorname{sgn}(k_b(w))$ for each of the sixteen possibilities for (4.2.18). Letting $S \coloneqq \left[\begin{smallmatrix} ++ \\ ++ \\ ++ \end{smallmatrix}\right]$, the possibilities are

$$S, \qquad P_{12}P_{14}P_{13}S, \qquad P_{12}P_{23}P_{24}S, \qquad P_{34}P_{23}P_{13}S, \qquad P_{34}P_{14}P_{24}S, \tag{4.2.20}$$

$$P_{23}P_{14}P_{13}P_{24}S, \qquad P_{12}P_{34}P_{23}P_{14}S, \qquad P_{12}P_{34}P_{13}P_{24}S.$$

For example, if $\operatorname{sgn}(b(w)) = (+-+-)$ or $(-+-+)$ then $\operatorname{sgn}(k_b(w)) = \left[\begin{smallmatrix} -- \\ -- \\ ++ \end{smallmatrix}\right] = P_{12}P_{34}P_{23}P_{14}S$. Denoting these eight elements in the order they appear in (4.2.20) by g_i for $i = 0, 1, \ldots, 7$, collecting them in a set $G = \{g_i\}_{i=0}^{7}$, and introducing the composition law

$$\operatorname{sgn} \begin{bmatrix} b_1, b_2 \\ b_3, b_4 \\ b_5, b_6 \end{bmatrix} \cdot \operatorname{sgn} \begin{bmatrix} d_1, d_2 \\ d_3, d_4 \\ d_5, d_6 \end{bmatrix} \coloneqq \operatorname{sgn} \begin{bmatrix} b_1d_1, b_2d_2 \\ b_3d_3, b_4d_4 \\ b_5d_5, b_6d_6 \end{bmatrix} \tag{4.2.21}$$

then $(G, \cdot)$ is an abelian group of order eight, with identity element $g_0 = S$. Note that $g_i^2 = g_0$ for all $i = 0, \ldots, 7$, and so we conclude that $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. This composition law is compatible with the action of SCTs on any initial kinematics, in the sense that

$$\operatorname{sgn}(k(C_b w)) = \operatorname{sgn}(k_b(w)) \cdot \operatorname{sgn}(k(w)).$$

Therefore the group G encodes the action of SCTs C_b on $\operatorname{sgn}(k(w))$. We define the eight sets

$$K_1 \coloneqq G, \qquad K_2 \coloneqq P_{12}G, \qquad K_3 \coloneqq P_{23}G, \qquad K_4 \coloneqq P_{13}G, \qquad \bar{K}_i \coloneqq PK_i, \tag{4.2.22}$$

where $P_{12}G \coloneqq \{P_{12}g_i\}_{i=0}^{7}$, and similarly for the others. Each of these sets has eight elements, and together they account for the $8 \times 8 = 64$ different possibilities for $\operatorname{sgn}(k(w))$ for a configuration $w \in V$. They are listed explicitly in appendix B.1. Furthermore they are each invariant under the action[8] of G, and so the eight sets of signs of kinematics $K_i, \bar{K}_i$ are invariant under SCTs[9]. In other words, given a configuration $w \in V$ such that $\operatorname{sgn}(k(w)) \in K_i$ (or $\bar{K}_i$) then $\operatorname{sgn}(k(C_b w)) \in K_i$ (or $\bar{K}_i$) for any $b \in \mathbb{R}^{1,3}$. We define the refined subsets of $V_{z,\bar{z}}$

$$V_{i,z,\bar{z}} \coloneqq \{w \in V_{z,\bar{z}} \mid \operatorname{sgn}(k(w)) \in K_i\}, \qquad \bar{V}_{i,z,\bar{z}} \coloneqq \{w \in V_{z,\bar{z}} \mid \operatorname{sgn}(k(w)) \in \bar{K}_i\}, \tag{4.2.23}$$

which are each separately invariant under the conformal group by the above discussion. Investigating the sets $V_{i,z,\bar{z}}$ and $\bar{V}_{i,z,\bar{z}}$ in detail we find that there are different constraints

[8]The action of G on these sets is defined analogously to the action of G on itself.

[9]A similar argument holds for inversions.

i	$z, \bar{z}$ for non-empty $V_{i,z,\bar{z}}$, $\bar{V}_{i,z,\bar{z}}$
1	$z, \bar{z} \in (-\infty, 0)$ OR $(0, 1)$ OR $(1, \infty)$
2	$\bar{z} \in (-\infty, 0), z \in (0, 1)$
3	$\bar{z} \in (0, 1), z \in (1, \infty)$
4	$\bar{z} \in (-\infty, 0), z \in (1, \infty)$

Table 4.1: $z, \bar{z}$ such that $V_{i,z,\bar{z}}$ is non-empty for $z, \bar{z} \in \mathbb{R}$. Note for $i = 1$ we always take $z \geq \bar{z}$.

on z and $\bar{z}$ for each i. The first fact is that a configuration $w \in V$ can only have $z, \bar{z} \in \mathbb{C} \setminus \mathbb{R}$ if $w \in \bar{V}_{1,z,\bar{z}}$, and so

$$V_{\mathbb{C}} = \bigcup_{\substack{z \in \mathbb{H} \\ \bar{z} = z^*}} \bar{V}_{1,z,\bar{z}}. \tag{4.2.24}$$

Note that $\bar{K}_1$ includes the Euclidean sign assignment $PS = \left[\begin{smallmatrix} - & - \\ - & - \\ - & - \end{smallmatrix}\right]$, and the seven other possible signs of kinematics are found by acting with G on PS. (4.2.24) is proven using Minkowskian conformal plane configurations, introduced below, in appendix B.2. Furthermore, given $w \in V_{i,z,\bar{z}}$ (or $\bar{V}_{i,z,\bar{z}}$) with $z, \bar{z} \in \mathbb{R}$ then z and $\bar{z}$ are constrained to different intervals of $\mathbb{R}$ depending on i (table 4.1). For example, let $w \in V_{4,z,\bar{z}}$ so that $\mathrm{sgn}(k(w)) \in K_4$. Going through all the possibilities for $\mathrm{sgn}(k(w))$ in table B.1 we deduce that the cross ratios satisfy $u < 0$ and $v < 0$. Since $u = z\bar{z}$, $v = (1 - z)(1 - \bar{z})$, and we take $z \geq \bar{z}$ we conclude that $z \in (1, \infty)$, $\bar{z} \in (-\infty, 0)$. Conversely, given a configuration $w \in V$ with a known z and $\bar{z}$ we immediately have information about the signs of the kinematics of w. For example, if we are told w has $\bar{z} = -1, z = 2$, we see from table 4.1 that there are two possibilities $w \in V_{4,2,-1}$ or $\bar{V}_{4,2,-1}$, and so $\mathrm{sgn}(k(w)) \in K_4$ or $\bar{K}_4$. We define V_i ($\bar{V}_i$) as the union of all $V_{i,z,\bar{z}}$ ($\bar{V}_{i,z,\bar{z}}$) over the possible real $z, \bar{z}$ in table 4.1. For example

$$V_1 = \bigcup_{\substack{z,\bar{z} \in (-\infty,0) \\ z,\bar{z} \in (0,1) \\ z,\bar{z} \in (1,\infty) \\ z \geq \bar{z}}} V_{1,z,\bar{z}}, \quad V_2 = \bigcup_{\substack{z \in (0,1) \\ \bar{z} \in (\infty,0)}} V_{2,z,\bar{z}}. \tag{4.2.25}$$

The question remains: Does $\mathrm{Conf}(\mathbb{R}^{1,3})$ act transitively on each non-empty $V_{i,z,\bar{z}}$ (and $\bar{V}_{i,z,\bar{z}}$)? The answer is yes *up to the discrete transformations* $\mathcal{P}, \mathcal{T}$. The proof relies on Minkowskian conformal planes. We define the five *Minkowskian conformal plane* configurations $w_{\mathbb{C}}(r, \phi)$ and $w_{bc}(a, \eta)$ for $b, c \in \{+, -\}$ in table 4.2. They are expressed in terms of the unit vectors $e_0 = (1, 0, 0, 0)$ and $e_3 = (0, 0, 0, 1)$. Similarly to the Euclidean case we can always find $A \in \mathrm{Conf}(\mathbb{R}^{1,3})$ and possibly a discrete transformation $\mathcal{P}$, $\mathcal{T}$, or $\mathcal{PT}$ to map configurations in $V_{\mathbb{C}}, V_i$, and $\bar{V}_i$ to one of the five configurations in table 4.2.

The case of $V_{\mathbb{C}}$ is totally analogous to the Euclidean case, and indeed any configuration $w \in V_{\mathbb{C}}$ with $z = re^{i\phi}, \bar{z} = re^{-i\phi}$ can be conformally mapped (no need for discrete transformations) to the pseudo-Euclidean configuration $w_{\mathbb{C}}(r, \phi) \in \bar{V}_{1,z,\bar{z}}$ for $r > 0, \phi \in (0, \pi)$, see appendix B.2 for the proof.

Configuration	$\{x_1, x_2, x_3, x_4\}$	$z, \bar{z}$
$w_{\mathbb{C}}(r,\phi)$	$(0,\ r(0,\sin\phi,0,\cos\phi),\ e_3,\ \iota)$	$re^{i\phi}, re^{-i\phi}$
$w_{++}(a,\eta)$	$(0,\ a(\cosh\eta,0,0,\sinh\eta),\ e_0,\ \iota)$	$ae^{-\eta}, ae^{\eta}$
$w_{--}(a,\eta)$	$(0,\ a(\sinh\eta,0,0,\cosh\eta),\ e_3,\ \iota)$	$ae^{-\eta}, ae^{\eta}$
$w_{-+}(a,\eta)$	$(0,\ a(\sinh\eta,0,0,\cosh\eta),\ e_0,\ \iota)$	$-ae^{-\eta}, ae^{\eta}$
$w_{+-}(a,\eta)$	$(0,\ a(\cosh\eta,0,0,\sinh\eta),\ e_3,\ \iota)$	$-ae^{-\eta}, ae^{\eta}$

Table 4.2: Minkowskian conformal plane configurations. ι is the image of $x = 0$ under inversions, see figure 2.3.

The eight remaining sets $V_i, \bar{V}_i$ can be conformally mapped to the four sets $w_{bc}(a,\eta)$. This is a bit more subtle, and care must be taken to restrict the range of a and η in such a way that each $w \in V_{i,z,\bar{z}}$ (or $\bar{V}_{i,z,\bar{z}}$) is mapped to exactly one configuration $w_{bc}(a,\eta)$ for the appropriate b, c. This can be done provided the discrete transformations $\mathcal{P}$, $\mathcal{T}$ are used. The results are summarised in table 4.3.

V	Mapped to	Configuration bounds	$z, \bar{z}$ bounds
$V_{\mathbb{C}}$	$w_{\mathbb{C}}(r,\phi)$	$r > 0, \phi \in (0,\pi)$	$z \in \mathbb{H},\ \bar{z} = z^*$
V_1	$w_{++}(a,\eta)$	$a \in (0,1),\ e^{\eta} \in (1, 1/a)$	$0 < \bar{z} \leq z < 1$
		$a \in (1,\infty),\ e^{\eta} \in (1, a)$	$1 < \bar{z} \leq z < \infty$
		$a \in (-\infty,0),\ e^{\eta} \in (1,\infty)$	$-\infty < \bar{z} \leq z < 0$
V_2	$w_{-+}(a,\eta)$	$a \in (0,\infty), e^{\eta} \in (0, 1/a)$	$-\infty < \bar{z} < 0 < z < 1$
V_3	$w_{--}(a,\eta)$	$a \in (0,\infty), e^{\eta} \in (\max(a, 1/a), \infty)$	$0 < \bar{z} < 1 < z < \infty$
V_4	$w_{+-}(a,\eta)$	$a \in (0,\infty), e^{\eta} \in (1/a, \infty)$	$-\infty < \bar{z} < 0 < 1 < z < \infty$

Table 4.3: Conformal plane structure for $V_{\mathbb{C}}$ and V_i. For $\bar{V}_i$ the only change is that the signs on $w_{bc}(a,\eta)$ reverse. For example, $\bar{V}_1$ gets mapped to $w_{--}(a,\eta)$, and all other columns with respect to V_1 are unchanged.

We explain the argument for V_2, so let $w = \{x_1, x_2, x_3, x_4\} \in V_2$. As in the Euclidean case we can translate each point by $-x_4$ and make a inversion $\mathcal{I}$ to map

$$w \to w_1 := \{0, y_2, y_3, \iota\}, \tag{4.2.26}$$

where $y_i := \mathcal{I}(x_i - x_4)$ for $i = 2, 3$. The stabiliser of 0 and ι is generated by dilatations and Lorentz rotations $SO^+(1,3) \subset SO^+(2,4)$. How we proceed next depends on the signs of the remaining free kinematics. We compute

$$(y_2^2, y_3^2, y_{23}^2) = \left(\frac{x_{12}^2}{x_{14}^2 x_{24}^2}, \frac{x_{13}^2}{x_{14}^2 x_{34}^2}, \frac{x_{23}^2}{x_{24}^2 x_{34}^2}\right), \tag{4.2.27}$$

where we used the property that under inversions the kinematics transform

$$\mathcal{I} : x_{ij}^2 \longrightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}. \tag{4.2.28}$$

Since $w \in V_2$ we have information about the signs of the kinematics, namely that $\mathrm{sgn}(k(w)) \in K_2$ (see table B.1). For example, we could have $\mathrm{sgn}(k(w)) = P_{12}S = \left[\begin{smallmatrix} - & + \\ + & + \\ + & + \end{smallmatrix}\right]$. In this case we have that

$$\mathrm{sgn}(y_2^2, y_3^2, y_{23}^2) = (- + +). \tag{4.2.29}$$

In fact going through all the cases in table B.1 it turns out that (4.2.29) holds for any $\mathrm{sgn}(k(w)) \in K_2$, and hence $w \in V_2$. We have that $y_3^2 = c^2 > 0$, so we can use a Lorentz rotation L to map $y_3 \to (\pm c, 0, 0, 0)$ and then rescale to $\pm e_0$. This will map the point y_2 to some point $r_2 := \frac{1}{c} L y_2$, and the configuration

$$w_1 \to w_2^{\pm} := \{0,\ r_2,\ \pm e_0,\ \iota\}. \tag{4.2.30}$$

The stabiliser of $0, \pm e_0$, and ι is generated by rotations $SO(3)$ acting on the Euclidean coordinates of $\mathbb{R}^{1,3}$. We use an $SO(3)$ rotation R to eliminate the second and third component of r_2, so $r_2 \to (r_2^0, 0, 0, r_2^3)$. We know from (4.2.29) that $(r_2^0)^2 - (r_2^3)^2 < 0$, so we map

$$w_2^{\pm} \to \bar{w}^{\pm}(a, \eta) := R w_2^{\pm} = \{0,\ a(\sinh\eta, 0, 0, \cosh\eta),\ \pm e_0,\ \iota\}, \tag{4.2.31}$$

for some $a \in \mathbb{R} \setminus \{0\}, \eta \in \mathbb{R}$. We compute the conformal invariants of $\bar{w}^{\pm}(a, \eta)$ to be

$$z = \max(-a e^{\mp\eta}, a e^{\pm\eta}), \qquad \bar{z} = \min(-a e^{\mp\eta}, a e^{\pm\eta}). \tag{4.2.32}$$

Note we have the issue that starting with a configuration $w \in V_2$ we have two possible configurations (4.2.31) we can end up with, $\bar{w}^+(a, \eta)$ and $\bar{w}^-(a, \eta)$. Moreover we know from table 4.1 that $z \in (0, 1), \bar{z} \in (-\infty, 0)$. We need our final configuration $w_{-+}(a, \eta)$ to contain each possible set of conformal invariants exactly once. This is because we want to be sure we are mapping all $w \in V_{2,z,\bar{z}}$ to the exact same configuration, to prove that $\mathrm{Conf}(\mathbb{R}^{1,3})$ acts transitively on $V_{2,z,\bar{z}}$. A priori there could be multiple configurations with the same $z, \bar{z}$ in our final $w_{-+}(a, \eta)$. To resolve these issues we need to use the discrete transformations $\mathcal{P}$ and $\mathcal{T}$. We first note that $\bar{w}^+(a, \eta) = \mathcal{T}\bar{w}^-(a, -\eta)$. Therefore if we end up at $\bar{w}^-(a, \eta)$ we simply apply a time-reversal transformation and relabel $\eta \to \eta' := -\eta$ to arrive at $\bar{w}^+(a, \eta)$. We can invert (4.2.32) to recover a and η for $\bar{w}^+(a, \eta)$

$$a = \pm\sqrt{-z\bar{z}}, \qquad \eta = \pm\frac{1}{2}\log(-\bar{z}/z). \tag{4.2.33}$$

We see from (4.2.33) that indeed multiple configurations $\bar{w}^+(a, \eta)$ give the same $z, \bar{z}$, which we want to avoid. We note that $\bar{w}^+(a, \eta) = \mathcal{P}\bar{w}^+(-a, -\eta)$. Therefore any configuration with $a < 0$ we can apply a parity transformation and relabel $\eta \to \eta' = -\eta$ at no cost, so we can assume $a > 0$. In this case we have that

$$z = a e^{\eta}, \qquad \bar{z} = -a e^{-\eta}, \qquad a = \sqrt{-z\bar{z}}, \qquad \eta = \frac{1}{2}\log(-z/\bar{z}). \tag{4.2.34}$$

From (4.2.34) we see that we have exactly one configuration $\bar{w}^+(a > 0, \eta) := w_{-+}(a, \eta)$ for each $z, \bar{z}$, as required. To get $z, \bar{z}$ in the correct range $-\infty < \bar{z} < 0 < z < 1$,

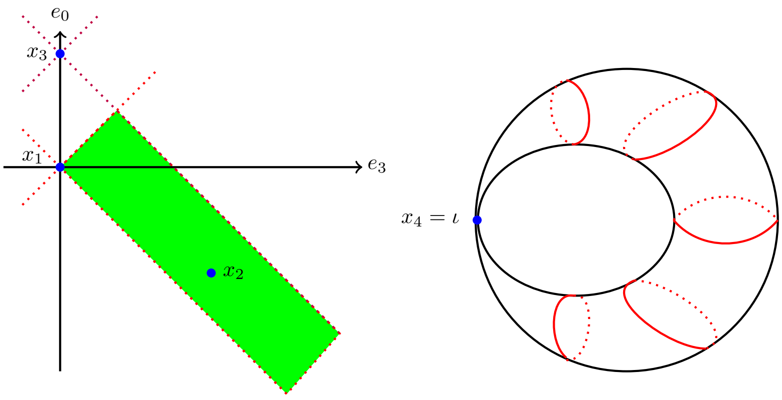


Figure 4.3: Minkowskian conformal plane configuration $w_{-+}(a, \eta)$ for V_2, $z \in (0,1)$, $\bar{z} \in (-\infty, 0)$. x_2 can be anywhere in green region depending on $z, \bar{z}$. We show the full conformal infinity to emphasise that this is a configuration with a single point at infinity.

or equivalently enforce the $y_{23}^2 > 0$ condition of (4.2.29), there is a further constraint $e^\eta < 1/a$. The final configuration $w_{-+}(a, \eta)$ in terms of $z, \bar{z}$ is

$$w_{-+}(z, \bar{z}) = \left\{0,\ \left(\frac{\bar{z}+z}{2}, 0, 0, \frac{z-\bar{z}}{2}\right),\ e_0,\ \iota\right\}, \tag{4.2.35}$$

and is shown in figure 4.3.

These arguments can be repeated analogously for the other sets $V_{i,z,\bar{z}}$ and $\bar{V}_{i,z,\bar{z}}$ to show that up to the discrete transformations $\mathcal{P}, \mathcal{T}$ all configurations in these sets are conformally equivalent to a single Minkowskian conformal plane configuration $w_{b,c}(a, \eta)$ or $w_{\mathbb{C}}(r, \phi)$, as summarised in table 4.3. Note that for $\bar{V}_{1,z,\bar{z}}$ there are two cases, one where $z, \bar{z} \in \mathbb{R}$ and one where $z, \bar{z} \in \mathbb{C} \setminus \mathbb{R}$. Therefore up to the discrete transformations $\mathcal{P}, \mathcal{T}$ $\mathrm{Conf}(\mathbb{R}^{1,3})$ acts transitively on each non-empty $V_{i,z,\bar{z}}$ and $\bar{V}_{i,z,\bar{z}}$. Given configurations $w_1, w_2 \in V_{i,z,\bar{z}}$ (or $\bar{V}_{i,z,\bar{z}}$) we deduce from the above the existence of $A_i \in \mathrm{Conf}(\mathbb{R}^{1,3})$ and discrete transformations $\mathcal{D}_i \in \{\mathbb{I}, \mathcal{P}, \mathcal{T}, \mathcal{PT}\}$ for $i = 1, 2$ such that $\mathcal{D}_1 A_1 w_1 = \mathcal{D}_2 A_2 w_2 = \bar{w}(z, \bar{z})$ for exactly one Minkowskian conformal plane configuration $\bar{w}(z, \bar{z})$ given in table 4.2. Then if we let $A := (\mathcal{D}_1 A_1)^{-1} \mathcal{D}_2 A_2$ we see that $w_1 = A w_2$.

Overall we have shown that all configurations $w \in V_{z,\bar{z}}$ with $z, \bar{z} \in \mathbb{C} \setminus \mathbb{R}$ are conformally equivalent, and that for $w \in V_{z,\bar{z}}$ with $z, \bar{z} \in \mathbb{R}$ there are two conformal equivalence classes of configurations (up to $\mathcal{P}$, $\mathcal{T}$) exchanged by a reversal of kinematics $V_{i,z,\bar{z}} \to \bar{V}_{i,z,\bar{z}} = P V_{i,z,\bar{z}}$ for a fixed i.

4.2.3 Functional Form in Different Regions

In [178] the Minkowskian box integral (4.2.2) is calculated as a function of the kinematics x_{ij}^2. We rewrite this result in each kinematic region, where it can be expressed explicitly in terms of the conformal invariants $z, \bar{z}$. This is straightforward to do, but a bit tedious. To reduce our work, we first note that given a configuration $w = \{x_1, x_2, x_3, x_4\} \in V$ with real $z, \bar{z}$ it is always possible to find a permutation $\sigma \in S_4$ such that σw either has $z, \bar{z} \in (0,1)$, or $\bar{z} \in (-\infty, 0), z \in (0,1)$. Since the box integral I is invariant under permutations, we can focus our attention on *restricted* configurations w with $z, \bar{z} \in \mathbb{C}$, $z, \bar{z} \in (0,1)$, or $\bar{z} \in (-\infty, 0), z \in (0,1)$ so that w is either in $V_{\mathbb{C}}, V_1, \bar{V}_1, V_2$, or $\bar{V}_2$ as defined in section 4.2.2. We recall the four Yangian invariant functions $\tilde{f}_i(z, \bar{z})$ defined in (4.1.14):

$$
\begin{aligned}
\tilde{f}_1 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log z\bar{z}(\log(1-z) - \log(1-\bar{z})), \\
\tilde{f}_2 &= \log z - \log \bar{z}, \\
\tilde{f}_3 &= \log(1-z) - \log(1-\bar{z}), \\
\tilde{f}_4 &= 1.
\end{aligned}
\tag{4.2.36}
$$

Remarkably, the result for the box integral can be expressed in each kinematic region as a linear combination of these Yangian invariants. This suggests that it may be possible to bootstrap the Minkowskian result directly from Yangian symmetry: this direction is explored in section 4.4. We evaluate arguments on the branch cut of the logarithms (the negative real axis) slightly above the cut, so we implicitly take $\log z \to \log(z + i\epsilon)$. Due to the way we organised things with permutations, we never need to consider arguments on the branch cut $(1, \infty)$ of $\mathrm{Li}_2(z)$. In section 4.4 we will be more systematic with the branch choices, but here this is sufficient. Note that $\tilde{f}_1$ is proportional to the Bloch–Wigner function when $\bar{z} = z^*$, but is slightly more general since we also allow for the possibility of real $z, \bar{z}$ with $z \neq \bar{z}$. In this section, given a configuration $w \in V$, we abbreviate the kinematic region $k := \mathrm{sgn}(k(w))$. For any restricted configuration we can write the box integral (4.2.2) in terms of $f_i(z, \bar{z})$:

$$
\phi(z, \bar{z}, k) = \sum_{i=1}^{4} \frac{c_i^k \tilde{f}_i(z, \bar{z})}{z - \bar{z}} = \sum_{i=1}^{4} c_i^k f_i(z, \bar{z}),
\tag{4.2.37}
$$

where $c_i^k \in \mathbb{C}$ depend on the kinematic region k. We specialise the result of [178] to restricted configurations, i.e. those in $V_{\mathbb{C}}$, V_1 with $z, \bar{z} \in (0,1)$, $\bar{V}_1$ with $z, \bar{z} \in (0,1)$, V_2, and $\bar{V}_2$.

We take a configuration $w \in V_1$ with $0 < \bar{z} \leq z < 1$, so that $k \in K_1$ (see table B.1). Firstly we conjecture that such configurations w cannot have $k = k^* := \left[\begin{smallmatrix} -- \\ -- \\ ++ \end{smallmatrix}\right]$, when there is a priori no reason this is not possible. We demonstrate this fact numerically in appendix B.3. We call this the 'missing' kinematic region. Note it is possible for a configuration $w \in V_1$ to have $k = k^*$ if $z, \bar{z} \in (-\infty, 0)$ or $z, \bar{z} \in (1, \infty)$. It is also possible to pick kinematics x_{ij}^2 such that $k = k^*$ and $z, \bar{z} \in (0,1)$, however we conjecture such kinematics cannot be realised by configurations $w \in V$. While the Minkowskian integral can be defined in the missing region k^* via the Feynman parametrisation (4.2.7), we omit this case in table 4.4 because these kinematical configurations cannot be accessed by applying a conformal transformation to physical configurations.

We list the results for the box integral $\phi(z, \bar{z})$ as a function of the kinematic region $k \in K_1$ in table 4.4. Since $\mathrm{Conf}(\mathbb{R}^{1,3})$ acts transitively[10] on each $V_{1,z,\bar{z}}$ we see that given any configuration $w \in V_1$ with $z, \bar{z} \in (0,1)$ one can reach **three** branches of the box integral using SCTs. If $w \in V_{\mathbb{C}}$ or $\bar{V}_1$ with $0 < \bar{z} \leq z < 1$, then $k \in \bar{K}_1$. In this case the kinematics reversed version of the missing region $Pk^* = \left[\begin{smallmatrix} ++ \\ ++ \\ -- \end{smallmatrix}\right]$ with $z, \bar{z} \in (0,1)$ is realisable, although numerically it is found to be much rarer than the other regions (appendix B.3). We list the results for the box integral $\phi(z, \bar{z})$ as a function of the

[10] Up to $\mathcal{P}$, $\mathcal{T}$ which doesn't change $I(w)$.

kinematic region $k \in \bar{K}_1$ in table 4.5. In this case starting with any configuration $w \in V_{\mathbb{C}}$ or $\bar{V}_1$ with $0 < \bar{z} \leq z < 1$ one can reach **four** branches of the box integral with SCTs. For a fixed $z, \bar{z} \in (0,1)$ there are **six** possible values of the box integral, corresponding to the four functions in table 4.5 and the two differing functions in table 4.4.

We list the corresponding results for configurations $w \in V$ with $\bar{z} \in (-\infty, 0), z \in (0,1)$, so that $k \in K_2$ or $k \in \bar{K}_2$, in tables 4.6 and 4.7 respectively. In both cases starting with any finite configuration $w \in V_2$ (or $\bar{V}_2$) one can reach **four** branches of the box integral using SCTs. Since the functions in tables 4.6 and 4.7 overlap given any finite configuration with $\bar{z} \in (-\infty, 0), z \in (0,1)$ there are **four** possible values for the box integral.

k	$\phi(z, \bar{z}, k)$
$\begin{bmatrix}+&+\\+&+\\+&+\end{bmatrix}, \begin{bmatrix}-&+\\+&-\\-&+\end{bmatrix}, \begin{bmatrix}-&+\\-&+\\+&-\end{bmatrix}, \begin{bmatrix}+&-\\-&+\\-&+\end{bmatrix}, \begin{bmatrix}+&-\\+&-\\+&-\end{bmatrix}$	f_1
$\begin{bmatrix}+&+\\-&-\\-&-\end{bmatrix}$	$f_1 - 2\pi i f_3$
$\begin{bmatrix}-&-\\+&+\\-&-\end{bmatrix}$	$f_1 + 2\pi i f_2$
$\begin{bmatrix}-&-\\-&-\\+&+\end{bmatrix}$	Missing region

Table 4.4: $\phi(z, \bar{z}, k)$ for $0 < \bar{z} \leq z < 1$, $k \in K_1$.

k	$\phi(z, \bar{z}, k)$
$\begin{bmatrix}-&-\\-&-\\-&-\end{bmatrix}, \begin{bmatrix}+&-\\-&+\\+&-\end{bmatrix}, \begin{bmatrix}+&-\\+&-\\-&+\end{bmatrix}, \begin{bmatrix}-&+\\+&-\\+&-\end{bmatrix}, \begin{bmatrix}-&+\\-&+\\-&+\end{bmatrix}$	f_1
$\begin{bmatrix}-&-\\+&+\\+&+\end{bmatrix}$	$f_1 + 2\pi i f_3$
$\begin{bmatrix}+&+\\-&-\\+&+\end{bmatrix}$	$f_1 - 2\pi i f_2$
$\begin{bmatrix}+&+\\+&+\\-&-\end{bmatrix}$	$f_1 + 2\pi i(f_2 - f_3 + 2\pi i f_4)$

Table 4.5: $\phi(z, \bar{z}, k)$ for $0 < \bar{z} \leq z < 1$ or $z \in \mathbb{H}, \bar{z} = z^*$, $k \in \bar{K}_1$.

k	$\phi(z, \bar{z}, k)$
$\begin{bmatrix}-&+\\+&+\\+&+\end{bmatrix}, \begin{bmatrix}-&-\\-&+\\-&+\end{bmatrix}, \begin{bmatrix}-&-\\+&-\\+&-\end{bmatrix}$	f_1
$\begin{bmatrix}+&+\\+&-\\-&+\end{bmatrix}, \begin{bmatrix}+&+\\-&+\\+&-\end{bmatrix}, \begin{bmatrix}-&+\\-&-\\-&-\end{bmatrix}$	$f_1 - 2\pi i f_3$
$\begin{bmatrix}+&-\\-&-\\+&+\end{bmatrix}$	$f_1 - 2\pi i f_2$
$\begin{bmatrix}+&-\\+&+\\-&-\end{bmatrix}$	$f_1 + 2\pi i(f_2 - f_3 + 2\pi i f_4)$

Table 4.6: $\phi(z, \bar{z}, k)$ for $\bar{z} \in (-\infty, 0)$, $z \in (0,1)$, $k \in K_2$.

4.3 Double Infinity Configurations

In this section we introduce the *double infinity configurations*. These are a family of configurations of four points in Minkowski space, where we place two points on the

k	$\phi(z,\bar{z},k)$
$\begin{bmatrix} + & - \\ - & - \\ - & - \end{bmatrix}$, $\begin{bmatrix} + & + \\ + & - \\ + & - \end{bmatrix}$, $\begin{bmatrix} + & + \\ - & + \\ - & + \end{bmatrix}$	$f_1 - 2\pi i f_3$
$\begin{bmatrix} - & - \\ - & + \\ + & - \end{bmatrix}$, $\begin{bmatrix} - & - \\ + & - \\ - & + \end{bmatrix}$, $\begin{bmatrix} + & - \\ + & + \\ + & + \end{bmatrix}$	f_1
$\begin{bmatrix} - & + \\ + & + \\ - & - \end{bmatrix}$	$f_1 + 2\pi i(f_2 - f_3 + 2\pi i f_4)$
$\begin{bmatrix} - & + \\ - & - \\ + & + \end{bmatrix}$	$f_1 - 2\pi i f_2$

Table 4.7: $\phi(z,\bar{z},k)$ for $\bar{z} \in (-\infty, 0)$, $z \in (0,1)$, $k \in \bar{K}_2$.

conformal boundary $\partial\mathbb{R}_c^{1,3}$, defined in section 2.2.6. These provide another canonical position for four points in Minkowski space, in addition to the single infinity configurations defined in table 4.2. We analyse these configurations in the context of the classification of the previous section, and find that they cover the vast majority of the kinematic space where $z, \bar{z}$ are real. Interestingly, the box integral (4.2.2) simplifies vastly in these configurations, and can be calculated by a simple application of the residue theorem.

4.3.1 Two Points at Infinity

Here we define the double infinity configurations. We consider eight configurations $\boldsymbol{X}^{ab}, \boldsymbol{Y}^{ab}$ of four points in Minkowski space, where $a, b \in \{+,-\}$, defined by the Hermitian matrices (related to usual Minkowski vectors by (2.2.95))

$$X_1^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; X_2^{ab} = \begin{pmatrix} \xi_+ & 0 \\ 0 & \eta_b \end{pmatrix}, \; X_3^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; X_4^{ab} = \begin{pmatrix} \eta_a & 0 \\ 0 & \xi_- \end{pmatrix}, \tag{4.3.1}$$

$$Y_1^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; Y_2^{ab} = \begin{pmatrix} \xi_+ & 0 \\ 0 & -\eta_b \end{pmatrix}, \; Y_3^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; Y_4^{ab} = \begin{pmatrix} \eta_a & 0 \\ 0 & -\xi_- \end{pmatrix}, \tag{4.3.2}$$

where $\xi_+, \xi_- \in \mathbb{R}\backslash\{0,1\}$ and we take the limit $\eta_\pm \to \pm\infty$. For each a, b we see that X_1^{ab} and Y_1^{ab} correspond to the origin in Minkowski space, while X_3^{ab} and Y_3^{ab} correspond to the unit vectors $e_0 = (1,0,0,0)$ and $e_3 = (0,0,0,1)$ respectively. The remaining points live on $\mathcal{I}^+$ or $\mathcal{I}^-$ depending on the choice of a or b, and are parametrised by a degree of freedom $\xi_\pm$. The configurations $\boldsymbol{X}^{+-}$ and $\boldsymbol{X}^{--}$ are visualised on an extended Penrose diagram in figure 4.4. In the limit $\eta_\pm \to \pm\infty$ $\boldsymbol{X}^{ab}$ correspond to the same configuration in terms of unitary matrices $\boldsymbol{U}_{\boldsymbol{X}}$ (see section 2.2.6), and similarly $\boldsymbol{Y}^{ab}$ correspond to the same configuration $\boldsymbol{U}_{\boldsymbol{Y}}$

$$\boldsymbol{U}_{\boldsymbol{X}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{i-\xi_+}{i+\xi_+} & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & \frac{i-\xi_-}{i+\xi_-} \end{pmatrix} \right\}, \tag{4.3.3}$$

$$\boldsymbol{U}_{\boldsymbol{Y}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{i-\xi_+}{i+\xi_+} & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & \frac{i+\xi_-}{i-\xi_-} \end{pmatrix} \right\}. \tag{4.3.4}$$

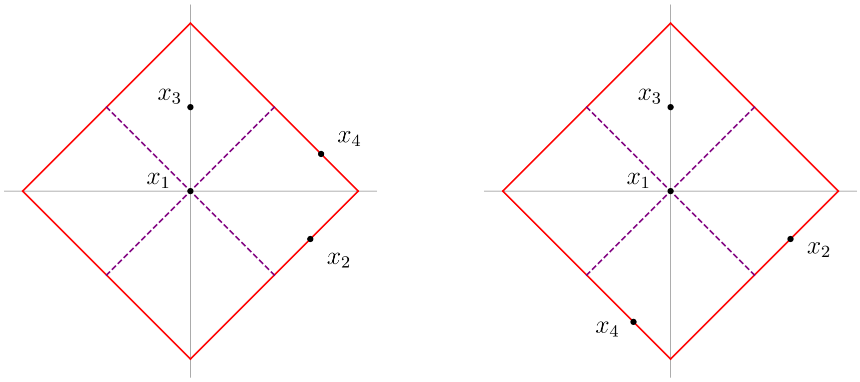


Figure 4.4: Left: Configuration $\boldsymbol{X}^{+-}$ — x_4 and x_2 on $\mathcal{I}^+(\theta = 0)$ and $\mathcal{I}^-(\theta = 0)$. Right: Configuration $\boldsymbol{X}^{--}$ — x_4 and x_2 on $\mathcal{I}^-(\theta = \pi)$ and $\mathcal{I}^-(\theta = 0)$. In terms of unitary matrices $\boldsymbol{X}^{--}$ and $\boldsymbol{X}^{+-}$ coincide due to the identification $\mathcal{I}^+ \sim \mathcal{A}\mathcal{I}^-$.

Note that even though each $\boldsymbol{X}^{ab}$ corresponds to the same configuration in terms of unitary matrices $\boldsymbol{U}_{\boldsymbol{X}}$, the value of the box integral $\phi(\boldsymbol{X}^{ab})$ depends on a, b, i.e. the direction X_2 and X_4 are sent to $\mathcal{I}$ (and similarly for $\boldsymbol{Y}^{ab}$). This is because the box integral is singular for configurations with points on $\mathcal{I}$, and we should think of X_2, Y_2 and X_4, Y_4 as being slightly off $\mathcal{I}$ so that $\mathcal{I}^+$ and $\mathcal{I}^-$ are distinguished. We take the kinematics to be large, but finite. We compute the kinematics $k(\boldsymbol{X}^{ab})$ and $k(\boldsymbol{Y}^{ab})$, defined in (4.2.15), in the limit $\eta_\pm \to \pm\infty$ to be

$$k(\boldsymbol{X}^{ab}) = \begin{bmatrix} \xi_+\eta_b, & \eta_a(\xi_- - 1) \\ \eta_b(\xi_+ - 1), & \xi_-\eta_a \\ 1, & -\eta_a\eta_b \end{bmatrix}, \tag{4.3.5}$$

$$k(\boldsymbol{Y}^{ab}) = -k(\boldsymbol{X}^{ab}). \tag{4.3.6}$$

We also compute the conformal cross ratios in the limit

$$u(\boldsymbol{X}^{ab}) = u(\boldsymbol{Y}^{ab}) = \xi_+(1 - \xi_-), \qquad v(\boldsymbol{X}^{ab}) = v(\boldsymbol{Y}^{ab}) = \xi_-(1 - \xi_+), \tag{4.3.7}$$

so that the cross-ratios u and v are the same for each of the eight configurations $\boldsymbol{X}^{ab}$ and $\boldsymbol{Y}^{ab}$. Written in terms of z and $\bar{z}$ the cross-ratios for each configuration are

$$z = \max(\xi_+, 1 - \xi_-), \qquad \bar{z} = \min(\xi_+, 1 - \xi_-), \tag{4.3.8}$$

since by convention we take $z \geq \bar{z}$. Note that since $\xi_+, \xi_- \in \mathbb{R} \setminus \{0, 1\}$, the configurations (4.3.1) and (4.3.2) can be tuned to give any real z and $\bar{z}$ with $z \geq \bar{z}$. Depending on ξ_+, ξ_- and the choice of configuration $\boldsymbol{X}^{ab}$ or $\boldsymbol{Y}^{ab}$ the kinematics of the configuration can vary widely. In fact most configurations w with real z and $\bar{z}$ and a given $\mathrm{sgn}(k(w))$ can be conformally mapped to one of the double infinity configurations w_∞, i.e. (4.3.1) or (4.3.2) (possibly permuted) with the same z and $\bar{z}$ and the *same signs of kinematics* so that $\phi(w) = \phi(w_\infty)$ by pseudo-conformal invariance. Specialising to the restricted kinematics considered in tables 4.4–4.7, the only kinematic regions which cannot be realised by a double infinity configuration are the missing kinematic region k^* described in section section 4.2.3, and the corresponding region obtained by

Kinematic region k	w_∞	ξ_+, ξ_- range
$\begin{bmatrix} - & + \\ + & + \\ + & + \end{bmatrix}$	$\boldsymbol{X}^{+-}$	$\xi_+ \in (0,1)$, $\xi_- \in (1,\infty)$
$\begin{bmatrix} + & + \\ + & - \\ - & + \end{bmatrix}$	$\boldsymbol{Y}^{++}$	$\xi_+ \in (-\infty,0)$, $\xi_- \in (0,1)$
$\begin{bmatrix} + & + \\ - & + \\ + & - \end{bmatrix}$	$\boldsymbol{X}^{++}$	$\xi_+ \in (0,1)$, $\xi_- \in (1,\infty)$
$\begin{bmatrix} - & - \\ - & + \\ - & + \end{bmatrix}$	$\boldsymbol{Y}^{--}$	$\xi_+ \in (-\infty,0)$, $\xi_- \in (0,1)$
$\begin{bmatrix} - & - \\ + & - \\ + & - \end{bmatrix}$	$\boldsymbol{X}^{--}$	$\xi_+ \in (0,1)$, $\xi_- \in (1,\infty)$
$\begin{bmatrix} - & + \\ - & - \\ - & - \end{bmatrix}$	$\boldsymbol{Y}^{+-}$	$\xi_+ \in (-\infty,0)$, $\xi_- \in (0,1)$
$\begin{bmatrix} + & - \\ - & - \\ + & + \end{bmatrix}$	$\boldsymbol{X}^{-+}$	$\xi_+ \in (0,1)$, $\xi_- \in (1,\infty)$
$\begin{bmatrix} + & - \\ + & + \\ - & - \end{bmatrix}$	$\boldsymbol{Y}^{-+}$	$\xi_+ \in (-\infty,0)$, $\xi_- \in (0,1)$

Table 4.8: How to realise all kinematic regions in V_2 ($z \in (-\infty,0), \bar{z} \in (0,1)$) with double infinity configurations. No need for permutations in this case.

reversing all kinematics Pk^* (which can be realised by physical configurations). For example, we show how to realise all kinematic regions in V_2 in table 4.8. What makes these w_∞ configurations particularly interesting is that the box integral (4.2.2) with these configurations can be calculated directly in Minkowski space, without reference to Feynman parameters or a Mellin-Barnes representation. For each configuration the integral depends only on the choice of ξ_+ and ξ_-, and so we define

$$\phi^{ab}_{\boldsymbol{X}}(\xi_+, \xi_-) \coloneqq \phi(\boldsymbol{X}^{ab}), \qquad \phi^{ab}_{\boldsymbol{Y}}(\xi_+, \xi_-) \coloneqq \phi(\boldsymbol{Y}^{ab}), \tag{4.3.9}$$

where we recall $\phi(w) \coloneqq x_{13}^2 x_{24}^2 I(w)$ is the conformal function of the box integral. How we make sense of the limit $\eta_\pm \to \pm\infty$ in the calculation will be described shortly. First we note that we can use symmetries to reduce the number of integrals to compute from eight to two. We can immediately combine (4.2.8) and (4.3.6) to conclude

$$\phi^{ab}_{\boldsymbol{Y}}(\xi_+, \xi_-) = \phi^{ab}_{\boldsymbol{X}}(\xi_+, \xi_-)^*. \tag{4.3.10}$$

We can also note relations between the configurations (4.3.1) using permutations, discrete transformations, and translations. We have

$$\begin{aligned} \mathcal{P} \circ (24)\boldsymbol{X}^{+-} &= \boldsymbol{X}^{-+}(\xi_+ \leftrightarrow \xi_-), && (4.3.11)\\ T_{\mathbb{I}_2} \circ \mathcal{PT} \circ (13)\boldsymbol{X}^{--} &= \boldsymbol{X}^{++}(\xi_\pm \to 1 - \xi_\pm). && (4.3.12) \end{aligned}$$

Since the conformal function is fully invariant under each of the transformations used in (4.3.11) and (4.3.12) we see that

$$\phi^{-+}_{\boldsymbol{X}}(\xi_+, \xi_-) = \phi^{+-}_{\boldsymbol{X}}(\xi_-, \xi_+), \qquad \phi^{++}_{\boldsymbol{X}}(\xi_+, \xi_-) = \phi^{--}_{\boldsymbol{X}}(1 - \xi_+, 1 - \xi_-). \tag{4.3.13}$$

In section 4.3.2 we describe our results for $\phi^{+-}_{\boldsymbol{X}}(\xi_+, \xi_-)$ and $\phi^{--}_{\boldsymbol{X}}(\xi_+, \xi_-)$, leaving the details of the calculation to appendix B.4. We can then use (4.3.10) and (4.3.13) to recover the value of the integral for each of the configurations (4.3.1) and (4.3.2).

4.3.2 Calculation of Minkowski Box Integral

We describe how to compute the conformal box integral (4.2.2) in the double infinity configurations (4.3.1) and (4.3.2). The conformal function takes the form

$$\phi = \int \frac{\mathrm{d}^4 x_a}{i\pi^2} \frac{x_{13}^2 x_{24}^2}{(x_{a1}^2 + i\epsilon)(x_{a2}^2 + i\epsilon)(x_{a3}^2 + i\epsilon)(x_{a4}^2 + i\epsilon)}. \tag{4.3.14}$$

We write the integration variable x_a in spherical coordinates

$$x_a = (t, r\sin\theta\cos\alpha, r\sin\theta\sin\alpha, r\cos\theta), \qquad X_a = \begin{pmatrix} t + r\cos\theta & re^{-i\alpha}\sin\theta \\ re^{i\alpha}\sin\theta & t - r\cos\theta \end{pmatrix}. \tag{4.3.15}$$

Integration over all of Minkowski space corresponds to integration over $-\infty < t < \infty, r > 0, \alpha \in [0, 2\pi)$, and $\theta \in [0, \pi]$. For definiteness we consider the configuration $\boldsymbol{X}^{+-}$ defined in (4.3.1), although the calculation proceeds similarly for the others. The Lorentz squares x_{ai}^2 for $i = 1, 2, 3, 4$ in the denominator of (4.3.14) are calculated in the appropriate limit, for example:

$$\begin{aligned} x_{a2}^2 = |X_a - X_2| &= \det \begin{pmatrix} \xi_+ - t - r\cos\theta & -re^{-i\alpha}\sin\theta \\ -re^{i\alpha}\sin\theta & \eta_- - t + r\cos\theta \end{pmatrix} \\ &= \eta_-(\xi_+ - t - r\cos\theta) + O(\eta_-^0). \end{aligned} \tag{4.3.16}$$

Overall we compute all of the quantities appearing in (4.3.14) in the limit to be

$$x_{a1}^2 = t^2 - r^2, \tag{4.3.17}$$
$$x_{a2}^2 = \eta_-(\xi_+ - t - r\cos\theta), \tag{4.3.18}$$
$$x_{a3}^2 = (t-1)^2 - r^2, \tag{4.3.19}$$
$$x_{a4}^2 = \eta_+(\xi_- - t + r\cos\theta), \tag{4.3.20}$$
$$x_{13}^2 = 1, \quad x_{24}^2 = -\eta_+\eta_-. \tag{4.3.21}$$

Letting $x = \cos\theta$, for this configuration the box integral becomes $\phi_{\boldsymbol{X}}^{+-}(\xi_+, \xi_-) \simeq$

$$\frac{2i}{\pi} \int_{x,r,t} \frac{r^2 \eta_+ \eta_-}{(t^2 - r^2 + i\epsilon)((t-1)^2 - r^2 + i\epsilon)\eta_-(\xi_+ - t - rx + \frac{i\epsilon}{\eta_-})\eta_+(\xi_- - t + rx + \frac{i\epsilon}{\eta_+})}, \tag{4.3.22}$$

where $\int_{x,r,t} := \int_{-1}^{1} dx \int_0^\infty dr \int_{-\infty}^\infty dt$ and the trivial α integration has been performed. Note that in the limit $\eta_\pm \to \pm\infty$ the integral is well-defined, as factors of $\eta_\pm$ cancel in numerator and denominator. We define an ϵ' by $\frac{\epsilon}{\eta_\pm} \sim \pm\epsilon'$, and throughout the calculation we treat $\epsilon' \ll \epsilon$ in the limit $\eta_\pm \to \pm\infty$. This is essential to properly regulate the poles of the integrand.

We note that integrand of (4.3.22) is a meromorphic function of t, with 6 poles in the complex t-plane. This integrand also decreases sufficiently quickly as $t \to \infty$, and thus the t integral can be computed using a large semicircular contour (closed in the upper or the lower half-plane) and the residue theorem. The r integral can then computed using a Hankel contour, which produces logarithms. The θ integral is then simply a sum of integrals which can be written in terms of dilogarithms. We plot these contours and describe the full details of this calculation in appendix B.4. The answer can be expressed in terms of four integrals

$$h_1(a,b) \coloneqq (a-b)\int_0^1 dx \frac{1}{(x-a)(x-b)}, \quad h_2(a,b) \coloneqq (a-b)\int_0^1 dx \frac{\log(x)}{(x-a)(x-b)}, \tag{4.3.23}$$

$$h_3(a,b) \coloneqq (a-b)\int_0^1 dx \frac{\log(1-x)}{(x-a)(x-b)}, \quad h_4(a,b) \coloneqq (a-b)\int_0^1 dx \frac{\log(1+x)}{(x-a)(x-b)},$$

which are all expressible as simple combinations of logs and dilogs of a and b. These are all well-defined in our case as each a, b will have a small imaginary part. The final result for the integral is

$$\begin{aligned}
&\phi_{\boldsymbol{X}}^{+-}(\xi_+,\xi_-) = \qquad (4.3.24)\\
&\frac{1}{\Box\xi - 1}\Bigg(\log(1/2)(h_1(r_{1+},r_{2+}) - h_1(-r_{1-},-r_{2-}))\\
&+\log(-1/2-i\epsilon)(-h_1(-r_{1+},-r_{2+}) + h_1(r_{1-},r_{2-})) - \log(-\Delta\xi/2+i\epsilon)h_1(s_{1+}^2,s_{2+}^2)\\
&+\log(\Delta\xi/2-i\epsilon)h_1(s_{1-}^2,s_{2-}^2) + \frac{1}{2}(h_2(s_{1+}^2,s_{2+}^2) - h_2(s_{1-}^2,s_{2-}^2)) + \Big(\log(\xi_+ - i\epsilon)\times\\
&(h_1(-r_{1-},-s_{1-}) - h_1(r_{1+},s_{1+})) + \log(-\xi_- - i\epsilon)(h_1(r_{2-},s_{1-}) - h_1(-r_{2+},-s_{1+}))\\
&+h_3(r_{1+},s_{1+}) + h_3(-r_{2+},-s_{1+}) - h_4(-r_{1-},-s_{1-}) - h_4(r_{2-},s_{1-}) - (\xi_\pm \to 1-\xi_\mp)\Big)\Bigg),
\end{aligned}$$

where $\Box\xi = \xi_+ + \xi_-$, $\Delta\xi = \xi_+ - \xi_-$, and $r_{i\pm}$ and $s_{i\pm}$ for $i = 1, 2$ are functions of ξ_+, ξ_- defined in (B.4.7) and (B.4.8). (4.3.24) is invariant under $\xi_\pm \to 1 - \xi_\mp$. Note that the overall prefactor agrees with our expectation

$$\frac{1}{\Box\xi - 1} = \pm\frac{1}{z-\bar{z}}. \tag{4.3.25}$$

For the configuration $\boldsymbol{X}^{--}$ the integral becomes $\phi_{\boldsymbol{X}}^{--}(\xi_+,\xi_-) =$

$$\frac{2i}{\pi}\int_{x,r,t}\frac{r^2}{(t^2-r^2+i\epsilon)((t-1)^2-r^2+i\epsilon)(\xi_+ - t - rx - i\epsilon')(\xi_- - t + rx - i\epsilon')}. \tag{4.3.26}$$

The calculation proceeds similarly, and the final result for the integral is

$$\begin{aligned}
&\phi_{\boldsymbol{X}}^{--}(\xi_+,\xi_-) = \qquad (4.3.27)\\
&\frac{1}{\Box\xi - 1}\Bigg(\log(1/2)(h_1(r_{1+},r_{2+}) - h_1(-r_{1-},-r_{2-})) + \log(-1/2-i\epsilon)\times\\
&(h_1(r_{1-},r_{2-}) - h_1(-r_{1+},-r_{2+})) + \Big(-\log(\xi_+ - i\epsilon)h_1(r_{1+},s_{1+}) + \log(\xi_- - i\epsilon)h_1(r_{2+},s_{1+})\\
&+\log(-1+\xi_+ - i\epsilon)h_1(-r_{1+},s_{2+}) + \log(-1+\xi_- - i\epsilon)h_1(-r_{2+},s_{2+})\\
&+h_3(r_{1+},s_{1+}) - h_3(-r_{1+},s_{2+}) + h_4(-r_{2+},s_{2+}) - h_4(r_{2+},s_{1+}) + (\xi_+ \leftrightarrow \xi_-)\Big)\Bigg),
\end{aligned}$$

and is invariant under $\xi_+ \leftrightarrow \xi_-$. Using (4.3.10) and (4.3.13) an expression for the box integral for each of the eight configurations in (4.3.1) and (4.3.2) can be recovered from (4.3.24) and (4.3.27). These results were numerically verified in each accessible kinematic region with tables 4.4–4.7 as well as with the package [185]. Note there are numerical issues when $\Delta\xi = 0, \Box\xi = 0, 1, 2$, $\xi_+ = 1/2$, or $\xi_- = 1/2$. These are easily remedied however by adding a small correction to ξ_+ or ξ_-.

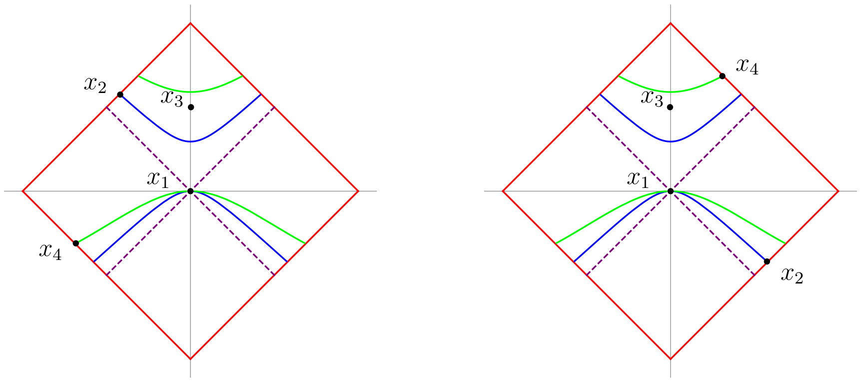


Figure 4.5: Configurations w (left) and y (right). The blue and green curves show the trajectories of x_2 and x_4 respectively under the SCT $C_{(0,0,0,b)}$ for $b \in \mathbb{R}$. x_4 crosses infinity again at $b = 3/2 + \epsilon$.

Numerical Example. As a concrete example, consider the configuration $w := \boldsymbol{X}^{-+}$ which has $\xi_+ = 1/4, \xi_- = 2/3$. For this configuration we have $\bar{z} = 1/4, z = 1/3$, $\mathrm{sgn}(k(w)) = \left[\begin{smallmatrix}+&+\\-&-\\+&+\end{smallmatrix}\right]$, and $w \in \bar{V}_1$. The value of the box integral is $\phi_{\boldsymbol{X}}^{-+}(1/4, 2/3) \simeq 5.88 - 21.69i$, which can be calculated using (4.3.24) and (4.3.13), table 4.5, or for example the package [185]. Under the infinitesimal SCT $C_{(0,0,0,\epsilon)}$ w is mapped to $y := \boldsymbol{X}^{+-}(\xi_+ = 1/4, \xi_- = 2/3)$ with $\mathrm{sgn}(k(y)) = \left[\begin{smallmatrix}-&-\\+&+\\+&+\end{smallmatrix}\right]$ and $\phi_{\boldsymbol{X}}^{+-}(1/4, 2/3) \simeq 5.88 - 8.88i$. Under the SCT $C_{(0,0,0,3/2+\epsilon)}$ w is mapped to a configuration r with $\mathrm{sgn}(k(r)) = \left[\begin{smallmatrix}-&+\\-&+\\-&+\end{smallmatrix}\right]$ and $I(r) \simeq 5.88$. This example shows three of the four branches of the box integral accessible in $\bar{V}_{1,1/4,1/3}$ (table 4.5). In figure 4.5 we show the configurations w and y and the orbits of x_2 and x_4 under the SCT parametrised by $(0, 0, 0, b)$.

4.4 Yangian Bootstrap for the Minkowski Box

In section 4.1.1 we showed that Yangian symmetry implies that the conformal function for the Euclidean box integral ϕ is a linear combination of four Yangian invariant functions $f_i = \tilde{f}_i/(z - \bar{z})$, where

$$\begin{aligned}
\tilde{f}_1 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log z\bar{z}(\log(1-z) - \log(1-\bar{z})), \\
\tilde{f}_2 &= \log z - \log \bar{z}, \\
\tilde{f}_3 &= \log(1-z) - \log(1-\bar{z}), \\
\tilde{f}_4 &= 1.
\end{aligned} \tag{4.4.1}$$

We then argued using permutation symmetry and some mild extra input that the integral is just equal to f_1. Remarkably, we saw in section 4.2.3 that in each kinematic region the Minkowski box integral is still a linear combination of these invariants. We thus naturally investigate to which extent the box integral in Minkowski space can be calculated, starting with only the assumption of Yangian invariance and a few further discrete symmetries.

In this section, we choose to work with a slightly different function basis

$$g_i(z, \bar{z}) = \frac{\tilde{g}_i(z, \bar{z})}{z - \bar{z}}, \tag{4.4.2}$$

with

$$\begin{aligned} \tilde{g}_1 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + (\log z + \log \bar{z})\big(\log(1-z) - \log(1-\bar{z})\big), \\ \tilde{g}_2 &= \log z - \log \bar{z}, \\ \tilde{g}_3 &= \log(1-z) - \log(1-\bar{z}), \\ \tilde{g}_4 &= \log(1-\tfrac{1}{z}) - \log(1-\tfrac{1}{\bar{z}}). \end{aligned} \tag{4.4.3}$$

This basis is motivated by the fact that the three functions g_2, g_3, g_4 can be understood as single discontinuities of the highest transcendentality function g_1.[11] Note that while we argued in section 4.1.2 that the discontinuities of the function $\tilde{g}_1$ vanish, this was for Euclidean kinematics where $\bar{z} = z^*$. Here z and $\bar{z}$ can also be independent real numbers. Therefore in this case we mean discontinuities in one of the conformal variables, with the other one held fixed. In (4.4.3) we also explicitly split up the $\log z\bar{z}$ factor in g_1, which is convenient for taking discontinuities.

In this section we use the Yangian plus further discrete symmetries to severely constrain the final result for the box integral in Minkowski space. Since there are $2^6 = 64$ kinematic regions, labelled by the signs of the six kinematical invariants x_{jk}^2, there are a priori $64 \times 4 = 256$ undetermined parameters, as each region comes with a specific linear combination of the four-dimensional solution space (4.4.3). We end up with twelve parameters that are currently undetermined by our use of integrability and symmetry. These still need to be fixed by other means. We show here how these can be fixed by analytic continuation in the x_{jk}^2, although they can also be fixed by numerical input. Finally we present a compact formula (4.4.32), from which the result for the box integral can be easily read off in all kinematic regions. This is analogous to tables 4.4-4.7, but it fits in a single equation and is valid beyond the restricted configurations defined in section 4.2.3.

4.4.1 Yangian Invariant Ansatz and Discrete Symmetries

Given the Yangian invariance of the box integral in Minkowski space and local conformal invariance in each kinematic region k, we can expand the solution in terms of the invariants[12]

$$\phi(z, \bar{z}, k) = \sum_{i=1}^{4} c_i^k g_i(z, \bar{z}, k). \tag{4.4.4}$$

In the Euclidean region $k = \left[\begin{smallmatrix} -- \\ -- \end{smallmatrix}\right]$, any permutation of the external points will result in a configuration also in the Euclidean region. This implies that the integral can be

[11] The function g_4 can be understood as a single discontinuity of g_1 about $z = \infty$, with $\bar{z}$ fixed.

[12] There is still a subtlety in defining the functions g_i when z or $\bar{z}$ lies on a branch cut of these functions, and so we refine this ansatz appropriately in section 4.4.2.

fully bootstrapped using its Yangian and permutation symmetries to be proportional to *one* of the above Yangian invariants:

$$\phi\left(\begin{smallmatrix}-&-\\-&-\\-&-\end{smallmatrix}\right) = g_4(z, \bar{z}). \tag{4.4.5}$$

Clearly, at least parts of the permutation symmetries are broken once the kinematic region k takes a less symmetric form than in the Euclidean region, and thus, imposing permutation symmetries will be less restrictive. For example, if we take a configuration $w_1 = \{x_1, x_2, x_3, x_4\}$ with kinematic region $k = \left[\begin{smallmatrix}+&+\\+&+\\-&-\end{smallmatrix}\right]$, then applying the transposition $(14) : w_1 \to w_2 \coloneqq \{x_4, x_2, x_3, x_1\}$, we see that w_2 is in the kinematic region $k' = \left[\begin{smallmatrix}-&-\\+&+\\+&+\end{smallmatrix}\right]$. Therefore rather than imposing a condition on the full conformal function, as in (4.1.19), the transposition (14) establishes a relation between the conformal function in regions k and k':

$$z\bar{z}\phi(z, \bar{z}, k) = \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}, k'\right). \tag{4.4.6}$$

We will use three different types of discrete symmetries to fix the constants in (4.4.4). For this purpose it is useful to repeat the the Feynman parametrisation and Mellin–Barnes representation of ϕ, given also in (4.2.7) and (4.2.9). These take the form

$$\phi(x_{ij}^2) = \int [\mathrm{d}^3\alpha] \frac{x_{13}^2 x_{24}^2}{(\sum_{j<k} \alpha_i \alpha_k x_{jk}^2 + i\epsilon)^2}, \tag{4.4.7}$$

and

$$\phi = \frac{1}{(2\pi i)^2} \int_{C_1} ds \int_{C_2} ds' (\Gamma_{s,s'})^2 \left(\frac{x_{12}^2 + i\epsilon}{x_{13}^2 + i\epsilon}\right)^{-s} \left(\frac{x_{34}^2 + i\epsilon}{x_{24}^2 + i\epsilon}\right)^{-s} \left(\frac{x_{14}^2 + i\epsilon}{x_{13}^2 + i\epsilon}\right)^{-s'} \left(\frac{x_{23}^2 + i\epsilon}{x_{24}^2 + i\epsilon}\right)^{-s'}. \tag{4.4.8}$$

Shuffling Symmetry. There is a large redundancy in the space of kinematic regions k, which we will denote as 'shuffling' symmetry. The box integral has a symmetry under the separate exchange of the kinematic variables

$$x_{12}^2 \leftrightarrow x_{34}^2, \quad x_{23}^2 \leftrightarrow x_{14}^2, \quad x_{13}^2 \leftrightarrow x_{24}^2, \tag{4.4.9}$$

which is clear from the representation (4.4.8). This means that given a kinematic region k, represented by a sign block (4.2.16), the box integral is invariant under the operation of swapping the signs in any row of the sign block, for example $\left[\begin{smallmatrix}-&+\\-&+\\+&-\end{smallmatrix}\right] \to \left[\begin{smallmatrix}+&-\\-&+\\+&+\end{smallmatrix}\right]$. If k and k' are related by such a shuffle, then we have

$$\phi(z, \bar{z}|k) = \phi(z, \bar{z}|k'). \tag{4.4.10}$$

Conjugation Symmetry. It is clear from (4.4.7) that simultaneous reversal of the signs of the kinematic invariants x_{ij}^2 is equivalent to complex conjugation of the integral:

$$\phi(z, \bar{z}|k) = \phi(z, \bar{z}| - k)^*. \tag{4.4.11}$$

Permutation Symmetry. As is clear from (4.4.7), the box integral ϕ is covariant under permutations $\sigma \in S_4$ of the external points

$$\sigma : (x_1, x_2, x_3, x_4) \to (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}). \tag{4.4.12}$$

σ induces a few transformations. First of all the conformal weight $x_{13}^2 x_{24}^2$ transforms $x_{13}^2 x_{24}^2 \to f_\sigma(u,v) x_{13}^2 x_{24}^2$. The conformal invariants transform $z \to h_\sigma(z), \bar{z} \to h_\sigma(\bar{z})$. We note that since we always take $z \geq \bar{z}$ if $z, \bar{z} \in \mathbb{R}\backslash\{0,1\}$ and $\operatorname{Im} z > 0$ if $\bar{z} = z^* \in \mathbb{C}\backslash\mathbb{R}$, the conformal invariants may swap roles after the permutation i.e. $z \to h_\sigma(\bar{z}), \bar{z} \to h_\sigma(z)$. Finally, modulo shuffling, permutations permute the rows of the sign block: $k \to k_\sigma$. For example, $\sigma = (12)$ swaps rows 2 and 3 of the sign block:

$$\operatorname{sign}\begin{bmatrix} x_{12}^2, x_{34}^2 \\ x_{23}^2, x_{14}^2 \\ x_{13}^2, x_{24}^2 \end{bmatrix} \to \operatorname{sign}\begin{bmatrix} x_{12}^2, x_{34}^2 \\ x_{13}^2, x_{24}^2 \\ x_{23}^2, x_{14}^2 \end{bmatrix}. \tag{4.4.13}$$

These facts are summarised in table 4.9. The overall constraint permutations impose on the conformal function in different kinematic regions is

$$\phi(z, \bar{z}|k) = \begin{cases} f_\sigma^{-1}\phi(h_\sigma(z), h_\sigma(\bar{z})|k_\sigma) & \text{if} \quad z' = h_\sigma(z) \\ f_\sigma^{-1}\phi(h_\sigma(\bar{z}), h_\sigma(z)|k_\sigma) & \text{if} \quad z' = h_\sigma(\bar{z}) \end{cases}. \tag{4.4.14}$$

σ	f_σ	$h_\sigma(z)$	k_σ
$(\,), (12)(34), (13)(24), (14)(23)$	1	z	k
$(24), (13), (1234), (1432)$	1	$1 - z$	row 1 $\leftrightarrow$ row 2
$(23), (14), (1243), (1342)$	u	$\frac{1}{z}$	row 1 $\leftrightarrow$ row 3
$(12), (34), (1423), (1324)$	v	$\frac{z}{z-1}$	row 2 $\leftrightarrow$ row 3
$(234), (124), (132), (143)$	v	$\frac{1}{1-z}$	row 1,2,3 $\to$ row 2,3,1
$(243), (123), (134), (142)$	u	$1 - \frac{1}{z}$	row 1,2,3 $\to$ row 3,1,2

Table 4.9: Transformation of conformal invariants and kinematic region under permutations.

4.4.2 Constraints From Symmetries

In this section we impose the constraints of the above symmetries. We will find that this reduces the computation of the box integral in all kinematic regions to fixing only twelve constant parameters. These will be determined in section 4.4.3.

Yangian Symmetry. Although we already wrote the constraint on $\phi(z, \bar{z}, k)$ from Yangian symmetry in (4.4.4), there are a few technicalities to mention. In Euclidean space the conformal invariants are always constrained via $\bar{z} = z^*$. However, in Minkowski space z and $\bar{z}$ can also be independent real numbers. The possible range of z and $\bar{z}$ depends on the kinematic region, and is summarised above in table 4.1. The functions g_j have branch cuts on various intervals of the real axis, which are fixed after specifying the usual branch of the logarithm on the negative real axis. In order to consistently define the functions g_j appearing in the ansatz (4.4.4) for all possible values of $z, \bar{z}$, we regularise them according to

$$g_j^-(z, \bar{z}) = g_j(z - i\delta, \bar{z} + i\delta), \qquad j = 1, 2, 3, 4, \tag{4.4.15}$$

where δ is an infinitesimal positive real number. We similarly define regularised functions g_j^+ identically to the above, but with the replacement $\delta \to -\delta$. Note that such regularisations break the antisymmetry of the functions g_j in z and $\bar{z}$. As such, we explicitly specify z and $\bar{z}$ in terms of the cross-ratios u and v as

$$\begin{aligned} z &= \tfrac{1}{2}(1 + u - v) + \tfrac{1}{2}\sqrt{(1 - u - v)^2 - 4uv}, \\ \bar{z} &= \tfrac{1}{2}(1 + u - v) - \tfrac{1}{2}\sqrt{(1 - u - v)^2 - 4uv}, \end{aligned} \tag{4.4.16}$$

so that in particular $z \geq \bar{z}$ when $z, \bar{z} \in \mathbb{R}\backslash\{0, 1\}$ and $\mathrm{Im}(z) > 0$ when $z \in \mathbb{C}\backslash\mathbb{R}, \bar{z} = z^*$. Then the exchange $z \leftrightarrow \bar{z}$ is equivalent to $g_j^+ \leftrightarrow g_j^-$. As already mentioned, fixing $z, \bar{z}$ as (4.4.16) can lead to them swapping roles after a permutation of the external points, see the discussion above table 4.9. For example, $\sigma = (13)$ generates the transformation of conformal invariants $z \to z' = 1 - \bar{z}$ and $\bar{z} \to \bar{z}' = 1 - z$. In general, whether z and $\bar{z}$ swap after a permutation depends on both the range of z and $\bar{z}$ and the permutation. For example, under the permutation $\sigma = (14)$ with $h_\sigma(z) = 1/z$, the conformal variables swap roles if $z, \bar{z} > 0$, but not if $z > 0, \bar{z} < 0$.

Since we are imposing permutation symmetry on our final function ϕ, we allow for the appearance of both functions g_j^+ and g_j^- in our ansatz derived from Yangian invariance. In the end we can always express the functions $g_j^\mp$ in terms of $g_j^\pm$ to get an expression for the integral in terms of just four regularised functions g_j^- or g_j^+, see Appendix B.5. Our refined ansatz is

$$\phi(k) = \sum_{i=1}^{4} \left(\alpha_i^k g_i^- + \beta_i^k g_i^+ \right), \tag{4.4.17}$$

where α_i^k and β_i^k are complex numbers depending on the kinematic region k. In total there are $64 \times 4 = 256$ constants α_i^k to fix. β_i^k can be expressed in terms of α_i^k using the transition matrix (B.5.1).

Permutations, Shuffles, and Conjugation. A priori we have $2^6 = 64$ functions $\phi(k)$ to fix, one for each of the possible kinematic regions k. However, permutation, shuffling, and conjugation symmetry already give very large constraints on the linear combination (4.4.17). Under these three operations there are *six* equivalence classes of sign blocks. We list a representative from each equivalence class:

$$k_1 = \begin{bmatrix} -- \\ -- \\ -- \end{bmatrix}, \quad k_2 = \begin{bmatrix} -+ \\ -+ \\ -+ \end{bmatrix}, \quad k_3 = \begin{bmatrix} -+ \\ ++ \\ ++ \end{bmatrix}, \quad k_4 = \begin{bmatrix} -- \\ -+ \\ -+ \end{bmatrix}, \quad k_5 = \begin{bmatrix} -- \\ ++ \\ ++ \end{bmatrix}, \quad k_6 = \begin{bmatrix} -+ \\ ++ \\ -- \end{bmatrix}. \tag{4.4.18}$$

Using shuffling symmetry, we can identify $-+ \sim +-$ in any row of the sign block: we will always choose the order $-+$ when possible. This already restricts the number of independent signatures to $3^3 = 27$. These remaining 27 signatures organise themselves into six equivalence classes $\Lambda_1, \Lambda_2, \ldots, \Lambda_6$ under permutations and conjugations. Λ_1 contains k_1 and $-k_1$, and Λ_2 contains only k_2. Explicitly, these equivalence classes read

$$
\begin{aligned}
\Lambda_1 &= \left\{ \begin{bmatrix} -- \\ -- \\ -- \end{bmatrix}, \begin{bmatrix} ++ \\ ++ \\ ++ \end{bmatrix} \right\}, \\
\Lambda_2 &= \left\{ \begin{bmatrix} -+ \\ -+ \\ -+ \end{bmatrix} \right\}, \\
\Lambda_3 &= \left\{ \begin{bmatrix} -+ \\ ++ \\ ++ \end{bmatrix}, \begin{bmatrix} ++ \\ -+ \\ ++ \end{bmatrix}, \begin{bmatrix} ++ \\ ++ \\ -+ \end{bmatrix}, \begin{bmatrix} -+ \\ -- \\ -- \end{bmatrix}, \begin{bmatrix} -- \\ -+ \\ -- \end{bmatrix}, \begin{bmatrix} -- \\ -- \\ -+ \end{bmatrix} \right\}, \\
\Lambda_4 &= \left\{ \begin{bmatrix} -- \\ -+ \\ -+ \end{bmatrix}, \begin{bmatrix} -+ \\ -- \\ -+ \end{bmatrix}, \begin{bmatrix} -+ \\ -+ \\ -- \end{bmatrix}, \begin{bmatrix} ++ \\ -+ \\ -+ \end{bmatrix}, \begin{bmatrix} -+ \\ ++ \\ -+ \end{bmatrix}, \begin{bmatrix} -+ \\ -+ \\ ++ \end{bmatrix} \right\}, \\
\Lambda_5 &= \left\{ \begin{bmatrix} -- \\ ++ \\ ++ \end{bmatrix}, \begin{bmatrix} ++ \\ -- \\ ++ \end{bmatrix}, \begin{bmatrix} ++ \\ ++ \\ -- \end{bmatrix}, \begin{bmatrix} ++ \\ -- \\ -- \end{bmatrix}, \begin{bmatrix} -- \\ ++ \\ -- \end{bmatrix}, \begin{bmatrix} -- \\ -- \\ ++ \end{bmatrix} \right\}, \\
\Lambda_6 &= \left\{ \begin{bmatrix} -+ \\ ++ \\ -- \end{bmatrix}, \begin{bmatrix} ++ \\ -+ \\ -- \end{bmatrix}, \begin{bmatrix} -- \\ ++ \\ -+ \end{bmatrix}, \begin{bmatrix} -+ \\ -- \\ ++ \end{bmatrix}, \begin{bmatrix} -- \\ -+ \\ ++ \end{bmatrix}, \begin{bmatrix} ++ \\ -- \\ -+ \end{bmatrix} \right\}.
\end{aligned} \tag{4.4.19}
$$

When the box integral is known in the representative kinematic region k_i, it can be deduced for each of the remaining signatures in Λ_i using (4.4.14) and (4.4.11). Therefore we only need to fix $4 \times 6 = 24$ constants, 4 for each of the k_i. We can eliminate some of these constants by using the fact that some of the k_i are invariant under a subgroup of S_4.

Region $\boldsymbol{k_1}$. k_1 is fully invariant under permutations. For example, invariance under the transposition (13) gives a constraint on the ansatz (4.4.17):

$$\alpha_i^{k_1} g_i^-(z, \bar{z}) + \beta_i^{k_1} g_i^+(z, \bar{z}) = \alpha_i^{k_1} g_i^+(1-z, 1-\bar{z}) + \beta_i^{k_1} g_i^-(1-z, 1-\bar{z}), \tag{4.4.20}$$

where summation over i is assumed. In (B.5.5)-(B.5.8) we list the behaviour of the regularised functions $g_i^\pm$ under permutations. Only g_1 and g_4 are compatible with the functional equation (4.4.20), which forces $\alpha_2^{k_1} = \beta_2^{k_1} = \alpha_3^{k_1} = \beta_3^{k_1} = 0$. Furthermore, invariance of k_1 under the transposition (14) forces $\alpha_4^{k_1} = \beta_4^{k_1} = 0$. Therefore (4.4.17) reduces to

$$\phi(k_1) = \alpha_1^{k_1} g_1^- + \beta_1^{k_1} g_1^+. \tag{4.4.21}$$

For the possible values of $z, \bar{z}$ in the kinematic region k_1 (see table 4.1), we could use (B.5.1) for example to deduce $g_1^+ = g_1^-$. Therefore we can rewrite (4.4.21) as

$$\phi(k_1) = a_1 g_1^- = a_1 g_1^+, \tag{4.4.22}$$

where $a_1 = \alpha_1^{k_1} + \beta_1^{k_1} \in \mathbb{C}$. The box integral for the other signature in Λ_1 can be calculated using (4.4.11):

$$\phi(-k_1) = a_1^* g_1^+ = a_1^* g_1^-. \tag{4.4.23}$$

Therefore, for the equivalence class Λ_1, there is only a single constant a_1 left to be fixed.

Region $\boldsymbol{k_2}$. k_2 is also fully invariant under permutations, and similarly to k_1 we constrain

$$\phi(k_2) = a_2 g_1^- = a_2 g_1^+. \tag{4.4.24}$$

Therefore, in the equivalence class Λ_2 there is also just a single constant a_2 to fix.

Region $\boldsymbol{k_3}$. k_3 is not completely invariant under permutations, and so we expect the functional form of ϕ to be less restricted. We do have invariance under the transposition (12) however. Under this permutation we have $z \to z' = \frac{\bar{z}}{\bar{z}-1}$ and $\bar{z} \to \bar{z}' = \frac{z}{z-1}$. Since $\sigma k_3 = k_3$ we have the constraint on the expansion (4.4.17):

$$\alpha_i^{k_3} g_i^-(z,\bar{z}) + \beta_i^{k_3} g_i^+(z,\bar{z}) = \frac{1}{v}\alpha_i^{k_3} g_i^+\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right) + \frac{1}{v}\beta_i^{k_3} g_i^-\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right). \tag{4.4.25}$$

Because of (B.5.5) and (B.5.7) this gives no constraint on $\alpha_1, \beta_1, \alpha_3, \beta_3$, however we do have $\alpha_2^{k_3} = \beta_2^{k_3} = \alpha_4^{k_3} = \beta_4^{k_3} = 0$. Therefore we have

$$\phi(k_3) = \alpha_1^{k_3} g_1^- + \alpha_3^{k_3} g_3^- + \beta_1^{k_3} g_1^+ + \beta_3^{k_3} g_3^+. \tag{4.4.26}$$

For the kinematic region k_3 we have $z \in (0,1), \bar{z} \in (-\infty, 0)$, and

$$g_1^- = g_1^+ + 2\pi i g_3^+, \qquad g_3^- = g_3^+. \tag{4.4.27}$$

Therefore we can write (4.4.26) in terms of either g_i^- or g_i^+:

$$\phi(k_3) = a_3 g_1^- + 2\pi i c_3 g_3^- = a_3 g_1^+ + 2\pi i(c_3 + a_3) g_3^+, \tag{4.4.28}$$

where we take $a_3 = \alpha_1^{k_3} + \beta_1^{k_3}$ and $2\pi i c_3 = \alpha_3^{k_3} + \beta_3^{k_3} - 2\pi i \beta_1^{k_3}$. The rest of the sector Λ_3 can be reached from k_3 via permutations and conjugation. Therefore in Λ_3 there are two constants a_3 and c_3 to fix.

Region $\boldsymbol{k_4}$. k_4 is also invariant under the transposition (12). Analogously to k_3 this leads to

$$\phi(k_4) = a_4 g_1^- + 2\pi i c_4 g_3^- = a_4 g_1^+ + 2\pi i(c_4 + a_4) g_3^+, \tag{4.4.29}$$

so that there are two constants a_4 and c_4 to fix in Λ_4.

Region $\boldsymbol{k_5}$. k_5 is also invariant under the transposition (12), which leads to

$$\begin{aligned}\phi(k_5) &= a_5 g_1^- + 2\pi i c_5 g_3^- \\ &= a_5 g_1^+ + 2\pi i c_5 g_3^+ + 4\pi i c_5 \theta_1 \bar{\theta}_1 (g_2^+ - g_3^+ + g_4^+),\end{aligned} \tag{4.4.30}$$

where $\theta_1 = \theta(z-1)$ and $\bar{\theta}_1 = \theta(\bar{z}-1)$. Therefore there are two constants a_5 and c_5 to fix in Λ_5. Note the appearance of the theta functions in the second line of (4.4.30) renders the g_i^- basis perhaps slightly more natural.

Region $\boldsymbol{k_6}$. The region k_6 has no symmetry under non-trivial permutations, and therefore we cannot derive a constraint as easily as in the previous cases. We thus have

$$\begin{aligned}\phi(k_6) &= a_6 g_1^- + 2\pi i b_6 g_2^- + 2\pi i c_6 g_3^- + 2\pi i d_6 g_4^- \\ &= \alpha_6 g_1^+ - 2\pi i d_6 g_2^+ + 2\pi i \bar{c}_6 g_3^+ - 2\pi i b_6 g_4^+,\end{aligned} \tag{4.4.31}$$

where $\bar{c}_6 = a_6 + b_6 + c_6 + d_6$. Therefore there are still four constants a_6, b_6, c_6, d_6 to fix in Λ_6.

Combined Symmetries. To summarise, combining the above symmetries yields the following form for the box integral that depends on twelve unfixed parameters:

$$
\begin{aligned}
\phi(z,\bar{z}|k) = {} & + a_k g_1^- \\
& + 2\pi i g_2^- \big(- c_3\theta_{\substack{++\\-+\\++}} + \hat{c}_3^*\theta_{\substack{--\\-+\\--}} - c_4\theta_{\substack{-+\\--\\-+}} + \hat{c}_4^*\theta_{\substack{-+\\++\\-+}} - c_5\theta_{\substack{++\\--\\++}} + c_5\theta_{\substack{--\\++\\--}} \\
& \qquad + b_6\theta_{\substack{-+\\++\\--}} + d_6\theta_{\substack{-+\\--\\++}} - c_6\theta_{\substack{++\\-+\\--}} - d_6\theta_{\substack{--\\++\\-+}} - b_6\theta_{\substack{++\\--\\-+}} - c_6\theta_{\substack{--\\-+\\++}}\big) \\
& + 2\pi i g_3^- \big(+ c_3\theta_{\substack{-+\\++\\++}} - \hat{c}_3^*\theta_{\substack{-+\\--\\--}} + c_4\theta_{\substack{--\\-+\\-+}} - \hat{c}_4^*\theta_{\substack{++\\-+\\-+}} + c_5\theta_{\substack{--\\++\\++}} - c_5\theta_{\substack{++\\--\\--}} \\
& \qquad + c_6\theta_{\substack{-+\\++\\--}} + c_6\theta_{\substack{-+\\--\\++}} - b_6\theta_{\substack{++\\-+\\--}} + b_6\theta_{\substack{--\\++\\-+}} + d_6\theta_{\substack{++\\--\\-+}} - d_6\theta_{\substack{--\\-+\\++}}\big) \\
& + 2\pi i g_4^- \big(- \hat{c}_3\theta_{\substack{++\\++\\-+}} + c_3^*\theta_{\substack{-+\\--\\--}} - \hat{c}_4\theta_{\substack{-+\\-+\\--}} + c_4^*\theta_{\substack{-+\\-+\\++}} - c_5\theta_{\substack{++\\++\\--}} + c_5\theta_{\substack{--\\--\\++}} \\
& \qquad + d_6\theta_{\substack{-+\\++\\--}} + b_6\theta_{\substack{-+\\--\\++}} + d_6\theta_{\substack{++\\-+\\--}} - \bar{c}_6\theta_{\substack{--\\++\\-+}} - \bar{c}_6\theta_{\substack{++\\--\\-+}} + b_6\theta_{\substack{--\\-+\\++}}\big).
\end{aligned}
\tag{4.4.32}
$$

Here $a_k \in \{a_i, a_i^*\}$ depends on the kinematic region, e.g. $a_{k_2} = a_2$, and we abbreviate $\hat{c}_i = c_i + a_i$. Moreover, we have introduced the above theta-functions such that

$$
\theta k'(k) = \begin{cases} 1 & \text{if} \quad k' = k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.33}
$$

For instance we have

$$
\theta_{\substack{++\\-+\\++}}\left(\substack{++\\-+\\++}\right) = 1, \qquad\qquad \theta_{\substack{++\\-+\\++}}\left(\substack{-+\\--\\++}\right) = 0. \tag{4.4.34}
$$

4.4.3 Analytic Continuation

After exhausting the available symmetries of the box integral, we are left with twelve independent constants that remain to be fixed. For instance, we could calculate the box integral for a set of arbitrary numerical configurations in the relevant regions to fix these numbers. In the present section we explicitly demonstrate how to obtain the twelve parameters using analytic continuation.

To connect the box integral in different kinematic regions, we note that it is always represented by the same Feynman parametrised integral (4.4.7) which gives a natural analytic continuation beyond real kinematics. In particular, this tells us that away from its poles, (4.4.7) is a continuous function of the x_{ij}^2. Hence, we can relate the value of the box integral in different regions by connecting them via paths in x_{ij}^2 space on which the integral is regular. Since the integral diverges at points where one of the x_{ij}^2 vanishes, to change the signature of the kinematics on a regular path, we will have to continue the function through the complex plane. In this process, for generic complex x_{ij}^2, z and $\bar{z}$ will cross branch cuts of the function basis g_i. Carefully tracking the movement of z and $\bar{z}$ and adding or subtracting the corresponding discontinuities will therefore allow us to deduce the functional representation of the box integral in any of the regions.

As a practical definition of the discontinuity of a function we use

$$
\mathrm{disc}_{z=a} f(z) = \lim_{\epsilon \to 0} \left(f(\gamma(\epsilon)) - f(\gamma(1-\epsilon)) \right), \tag{4.4.35}
$$

where $\gamma(t)$ is a complex contour that encircles the branch point a once on a clockwise path and starts and ends at z, i.e. $\gamma(0) = \gamma(1) = z$. For branch cuts of f on the real

Function	Branch cut in z, $\bar{z}$
g_1	$(-\infty, 0]$, $[1, \infty)$
g_2	$(-\infty, 0]$
g_3	$[1, \infty)$
g_4	$[0, 1]$

Table 4.10: Location of branch cuts for the function basis g_i.

axis, this definition implies for $a, z \in \mathbb{R}$ and z on a branch cut starting at a:

$$\mathrm{disc}_{z=a} f(z) = \lim_{\epsilon \to 0} \left(f(z \pm i\epsilon) - f(z \mp i\epsilon) \right) , \tag{4.4.36}$$

Here the sign of the $i\epsilon$ depends on the ordering of a and z. This expression is easy to evaluate and sufficiently general for the set of functions g_i. We choose the branch cuts of our function basis to lie on the real axis, which is consistent with taking the principal value of the appearing logarithm and dilogarithm functions. To be precise, we have listed the branch cuts in table 4.10.

For convenience, we explicitly note the non-vanishing discontinuities of the g_i around their branch points

$$\begin{aligned}
&\mathrm{disc}_{z=0} g_1 = +\mathrm{disc}_{\bar{z}=0} g_1 = +2\pi i g_3, &&\mathrm{disc}_{z=1} g_1 = +\mathrm{disc}_{\bar{z}=1} g_1 = -2\pi i g_2, \\
&\mathrm{disc}_{z=0} g_2 = -\mathrm{disc}_{\bar{z}=0} g_2 = +2\pi i, &&\mathrm{disc}_{z=1} g_3 = -\mathrm{disc}_{\bar{z}=1} g_3 = +2\pi i, \\
&\mathrm{disc}_{z=1} g_4 = -\mathrm{disc}_{\bar{z}=1} g_4 = -2\pi i, &&\mathrm{disc}_{z=0} g_4 = -\mathrm{disc}_{\bar{z}=0} g_4 = -2\pi i.
\end{aligned} \tag{4.4.37}$$

In the Euclidean region $k_1 = \left[\begin{smallmatrix} - & - \\ - & - \\ - & - \end{smallmatrix}\right]$ the remaining coefficient a_1 is fixed by the star-triangle relation for generic propagator powers [10], such that

$$\phi(k_1) = g_1^+ = g_1^- , \tag{4.4.38}$$

i.e. $a_1 = 1$. We will use this region as a starting point of the paths leading into the five other equivalence classes. Since for real kinematics, z and $\bar{z}$ are either real or a pair of complex conjugates in the Euclidean region, we can always set up an entirely real path in kinematics space that sends all x_{ij}^2 to -1 without picking up any discontinuities: any possible branch cut passage will happen simultaneously for z and $\bar{z}$ and give cancelling contributions. Hence to connect regions where some of the x_{ij}^2 differ in sign, for simplicity we can restrict ourselves to paths of the form

$$x_{ij}^2 = e^{i\varphi_{ij}} . \tag{4.4.39}$$

To ensure that we do not encounter any poles of the integrand of (4.4.7) we further demand $0 < \varphi_{ij} < \pi$, i.e. we always rotate the x_{ij}^2 through the upper half of the complex plane. This prescription is inherited from the positive $i\epsilon$-shift in the original expression (4.2.2) for the box integral, which in turn translates into a positive $i\epsilon$-shift of the x_{ij}^2 in the Feynman parametrisation (4.4.7), c.f. [186]. Then, the paths on which we analytically continue from a region k_m to another region k_n, can be parametrised by

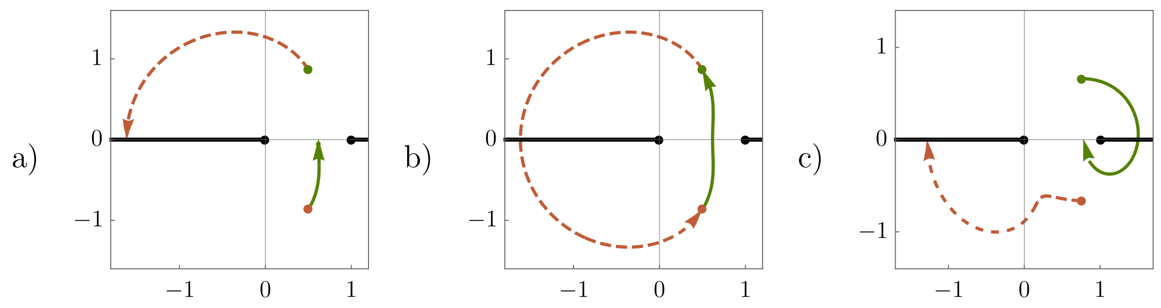


Figure 4.6: The path of z (solid green) and $\bar{z}$ (dashed red) under the continuation from k_1 to (a) k_3 or k_4, (b) k_5 and (c) k_6. In the last case we further introduce a factor of $1/2$, i.e. we set $x_{23}^2 = e^{i\varphi_{23}}/2$, to avoid the degenerate point $z = \bar{z}$. We denote the point where the paths begin by a dot in the respective color. Note that for (a) and (b) the branch cut of the square root in (4.4.16) is passed immediately at the beginning of the path. Since away from the real line, $g_i^{\pm}(z, \bar{z}) = g_i^{\pm}(\bar{z}, z)$ this does not change the result for ϕ. We also include the branch cuts of g_1 as solid black lines.

$$\varphi_{ij} = \begin{cases} 0, & \text{if } \mathrm{sgn}_m(x_{ij}^2) = +\mathrm{sgn}_n(x_{ij}^2) = +1 \\ \pi, & \text{if } \mathrm{sgn}_m(x_{ij}^2) = +\mathrm{sgn}_n(x_{ij}^2) = -1 \\ \varphi, & \text{if } \mathrm{sgn}_m(x_{ij}^2) = -\mathrm{sgn}_n(x_{ij}^2) = +1 \\ \pi - \varphi, & \text{if } \mathrm{sgn}_m(x_{ij}^2) = -\mathrm{sgn}_n(x_{ij}^2) = -1 \end{cases}, \tag{4.4.40}$$

where $\varphi \in [0, \pi]$ and $\mathrm{sgn}_m(x_{ij}^2)$ is the sign of x_{ij}^2 in the region k_m.

Region k_2. For $k_2 = \left[\begin{smallmatrix} - & + \\ - & + \\ - & + \end{smallmatrix}\right]$ we have

$$\varphi_{34} = \varphi_{14} = \varphi_{24} = \pi - \varphi \tag{4.4.41}$$

and all other φ_{ij} equal to π. Hence, u and v are actually inert under this analytical continuation and so are z and $\bar{z}$. Therefore, we find

$$\phi(k_2) = g_1^+ = g_1^-, \tag{4.4.42}$$

fixing $a_2 = 1$.

Region k_3. For $k_3 = \left[\begin{smallmatrix} - & + \\ + & + \\ + & + \end{smallmatrix}\right]$ we have

$$\varphi_{12} = \pi \tag{4.4.43}$$

as the only constant phase and all other equal to $\pi - \varphi$. The path that z and $\bar{z}$ trace out under this continuation is shown in Figure 4.6a). Since the path for $\bar{z}$ ends on the negative real axis, the concrete expression in terms of g_i depends on the regularisation procedure. For the g_i^+, the branch cut in $\bar{z}$ lies slightly above the negative real axis, whereas for g_i^- it lies slightly below. Therefore, we introduce the operators $\theta^{\pm}$ that act according to

$$\theta^{\pm} g_i^{\pm} = g_i^{\pm}, \qquad \theta^{\pm} g_i^{\mp} = 0. \tag{4.4.44}$$

Then, we can compactly write the result in the region k_3 as

$$\phi(k_3) = (1 + \theta^+ \mathrm{disc}_{\bar{z}=0})\phi(k_1) = g_1^- = g_1^+ + 2\pi i g_3^+, \tag{4.4.45}$$

fixing $a_3 = 1 + c_3 = 1$.

Region $\boldsymbol{k_4}$. In order to move from $k_1 = \left[\begin{smallmatrix}-&-\\-&-\\-&-\end{smallmatrix}\right]$ to $k_4 = \left[\begin{smallmatrix}-&-\\-&+\\-&+\end{smallmatrix}\right]$, we choose

$$\varphi_{14} = \varphi_{24} = \pi - \varphi, \tag{4.4.46}$$

and all other angles equal to π. Interestingly, this induces the very same path for u and v (and hence for z and $\bar{z}$) as the continuation in the previous paragraph. Hence, we conclude

$$\phi(k_4) = \phi(k_3) = g_1^- = g_1^+ + 2\pi i g_3^+, \tag{4.4.47}$$

fixing $a_4 = 1 + c_4 = 1$.

Region $\boldsymbol{k_5}$. For the region $k_5 = \left[\begin{smallmatrix}-&-\\+&+\\+&+\end{smallmatrix}\right]$ we find

$$\varphi_{13} = \varphi_{24} = \varphi_{23} = \varphi_{41} = \pi - \varphi, \tag{4.4.48}$$

and all other angles equal to π. From the z, $\bar{z}$ path shown in Figure 4.6b), we conclude

$$\phi(k_5) = (1 + \mathrm{disc}_{\bar{z}=0})\phi(k_1) = g_1^{\pm} + 2\pi i g_3^{\pm}, \tag{4.4.49}$$

independently of the regularisation.

Region $\boldsymbol{k_6}$. Finally, for $k_6 = \left[\begin{smallmatrix}-&+\\+&+\\-&-\end{smallmatrix}\right]$ the non-constant phases are

$$\varphi_{34} = \varphi_{23} = \varphi_{14} = \pi - \varphi. \tag{4.4.50}$$

The corresponding path for $z, \bar{z}$ in Figure 4.6c) implies

$$\begin{aligned}\phi(k_6) &= (1 - \theta^- \mathrm{disc}_{\bar{z}=0})(1 - \mathrm{disc}_{z=1})\phi(k_1) \\ &= g_1^+ + 2\pi i g_2^+ = g_1^- + 2\pi i (g_2^- - g_3^- + 2\pi i) \\ &= g_1^- - 2\pi i g_4^-,\end{aligned} \tag{4.4.51}$$

fixing $a_6 = -d_6 = 1$ and $b_6 = c_6 = \bar{c}_6 = 0$.

Summary. In summary, we find

$$a_R = 1, \qquad c_6 = \bar{c}_6 = b_6 = 0, \qquad 1 + c_3 = 1 + c_4 = c_5 = -d_6 = 1, \tag{4.4.52}$$

such that the ansatz (4.4.32) yields the full result for the box integral:

$$\begin{aligned}\phi(R) = &+ g_1^- \\ &+ 2\pi i g_2^- \big(+\theta_{\begin{smallmatrix}-&-\\-&+\\-&-\end{smallmatrix}} + \theta_{\begin{smallmatrix}-&+\\+&+\\-&+\end{smallmatrix}} - \theta_{\begin{smallmatrix}+&+\\-&-\\+&+\end{smallmatrix}} + \theta_{\begin{smallmatrix}-&-\\+&+\\-&-\end{smallmatrix}} - \theta_{\begin{smallmatrix}-&+\\-&-\\+&+\end{smallmatrix}} + \theta_{\begin{smallmatrix}-&-\\+&+\\-&+\end{smallmatrix}}\big) \\ &+ 2\pi i g_3^- \big(-\theta_{\begin{smallmatrix}-&+\\-&-\\-&-\end{smallmatrix}} - \theta_{\begin{smallmatrix}+&+\\-&+\\-&+\end{smallmatrix}} + \theta_{\begin{smallmatrix}-&-\\+&+\\+&+\end{smallmatrix}} - \theta_{\begin{smallmatrix}+&+\\-&-\\-&-\end{smallmatrix}} - \theta_{\begin{smallmatrix}+&+\\-&-\\-&+\end{smallmatrix}} + \theta_{\begin{smallmatrix}-&-\\-&+\\+&+\end{smallmatrix}}\big) \\ &+ 2\pi i g_4^- \big(-\theta_{\begin{smallmatrix}+&+\\+&+\\-&+\end{smallmatrix}} - \theta_{\begin{smallmatrix}-&+\\-&+\\-&-\end{smallmatrix}} - \theta_{\begin{smallmatrix}+&+\\+&+\\-&-\end{smallmatrix}} + \theta_{\begin{smallmatrix}-&-\\-&-\\+&+\end{smallmatrix}} - \theta_{\begin{smallmatrix}-&+\\+&+\\-&-\end{smallmatrix}} - \theta_{\begin{smallmatrix}+&+\\-&+\\-&-\end{smallmatrix}}\big).\end{aligned} \tag{4.4.53}$$

This compact formula for the box integral is valid in all 64 kinematic regions of Minkowski space, as verified by explicit comparison with the one-loop evaluation package [185]. This compact formula is one of the main results of this thesis; it is a computationally efficient result which makes manifest the breaking of global conformal invariance of the integral. We conclude our discussion of the conformal box integral in Minkowski space and proceed to discuss one of our other main directions: the study of Yangian Ward identities for fishnet four-point integrals.

Chapter 5

Yangian Ward Identities for Basso–Dixon Correlators

In section 3.3.3 we explained that the fishnet Feynman integrals $\tilde{I}_{\alpha\beta}$, which represent specific correlators in the fishnet theory, are Yangian invariant. The four-point limit of these correlators are represented by the Basso–Dixon integrals $I_{\alpha\beta}$. However, it turns out that in taking the coincidence limit $\tilde{I}_{\alpha\beta} \to I_{\alpha\beta}$ the level-one Yangian invariance is destroyed. Nevertheless, we study this coincidence limit to determine the imprint that integrability (in this case level-one Yangian invariance) leaves on the Basso–Dixon conformal functions $\phi_{\alpha\beta}(u, v)$. Their functional representation (2.3.70) as a determinant of ladder integrals is so simple that it is tempting to conjecture that it can be determined by methods of integrability. In order to understand this coincidence limit, we interpret the Yangian invariance of the fishnet integrals $\tilde{I}_{\alpha\beta}$ as Ward identities for the corresponding correlators. Since the Yangian generators are second order differential operators in the external points, there are subtleties which come from identifying points corresponding to different fields. By performing this somewhat technical limit, we find that the Yangian level-one momentum generator $\widehat{\mathrm{P}}^\mu$ does not annihilate the Basso–Dixon correlators, but rather returns a specific linear combination of analogous correlators, where one of the fields is replaced by a descendant $\varPhi \to \partial^\mu_{x_i}\varPhi$. At the level of Feynman integrals, this leads to vector integrals contributing on the right hand side of the Yangian Ward identity. We devised a conformal tensor reduction to express the Yangian equations as formal identities for the conformal Basso–Dixon functions $\phi_{\alpha\beta}(u, v)$. Schematically these take the form

$$\mathcal{D}_{uv}\phi_{\alpha\beta} = \mathrm{d}^+\mathcal{A}\phi_{\alpha\beta}. \tag{5.0.1}$$

Here $\mathcal{D}_{uv}$ is a differential operator in the conformal cross ratios u and v (or alternatively $z, \bar{z}$), d^+ is a dimension-raising operator which shifts the dimension $D \to D + 2$, and $\mathcal{A}$ is a linear combination of operators which raise specific propagator powers of the integral. For the ladders we provide an algorithmic way to describe the operators $\mathcal{A}$. For the case of the box integral ($\alpha = \beta = 1$) the inhomogeneity on the right hand side of the above Ward identity vanishes.

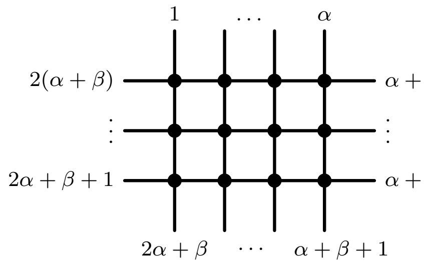
This chapter is organised as follows. We first formulate the Yangian invariance of the fishnet integrals $\tilde{I}_{\alpha\beta}$ as a Ward identity for correlation functions in fishnet theory, and describe how to take the four-point limit of this Ward identity. We then explain what this identity means at the level of vector Feynman integrals and the corresponding conformal functions, giving several explicit examples. Then we use the technology introduced in section 2.3.2 to rewrite the appearing vector integral coefficients in terms of higher dimensional scalar integrals, and describe a method to conformalise the resulting expressions. Using this, we can then express the Yangian Ward identities as single operatorial equations (5.0.1) for the Basso–Dixon functions. We explain the natural generalisation of our equations to the D-dimensional fishnet theory [11]. Finally, we describe the separability of the equations in $D = 2$.

5.1 From Fishnets to Basso–Dixon Correlators

5.1.1 Yangian Invariant Fishnets

Our starting point is the Yangian symmetry of fishnet Feynman integrals $\tilde{I}_{\alpha\beta}$ [172,162], which we described in section 3.3.3. We recall that these integrals are represented by square fishnets, with lattice of propagators with $\alpha \times \beta$ integration vertices:

$$\tilde{I}_{\alpha\beta} = \tag{5.1.1}$$



which represents the following correlator in the fishnet theory:

$$\big\langle \mathrm{tr}(Z(x_1)\cdots Z(x_\alpha)\bar{X}(x_{\alpha+1})\cdots \bar{X}(x_{\alpha+\beta})\bar{Z}(x_{\alpha+\beta+1})\cdots \bar{Z}(x_{2\alpha+\beta})X(x_{2\alpha+\beta+1})\cdots X(x_{2\alpha+2\beta}))\big\rangle \tag{5.1.2}$$

The integrals (5.1.1) are annihilated by the conformal Yangian level-zero and level-one generators J^A and $\widehat{\mathrm{J}}^A$, which act as differential operators on the coordinates $x_j^\mu \in \mathbb{R}^4$ for $j = 1, 2, \ldots, n \coloneqq 2(\alpha+\beta)$:

$$\mathrm{J}^A \tilde{I}_{\alpha\beta} = 0, \qquad\qquad \widehat{\mathrm{J}}^A \tilde{I}_{\alpha\beta} = 0. \tag{5.1.3}$$

Explicit expressions for the level-zero and level-one generators are given in (3.3.39) and (3.3.39). For convenience we repeat them here:

$$\mathrm{J}^A = \sum_{j=1}^{n} \mathrm{J}_j^A, \tag{5.1.4}$$

$$\begin{aligned} \mathrm{P}_j^\mu &= -i\partial_j^\mu, & \mathrm{L}_j^{\mu\nu} &= ix_j^\mu \partial_j^\nu - ix_j^\nu \partial_j^\mu, \\ \mathrm{D}_j &= -ix_{j\mu}\partial_j^\mu - i\Delta_j, & \mathrm{K}_j^\mu &= -i\big(2x_j^\mu x_j^\nu - \eta^{\mu\nu}x_j^2\big)\partial_{j,\nu} - 2i\Delta_j x_j^\mu, \end{aligned} \tag{5.1.5}$$

$$\widehat{\mathrm{J}}^A = \tfrac{1}{2} f^A{}_{BC} \sum_{k=1}^{n}\sum_{j=1}^{k-1} \mathrm{J}_j^C \mathrm{J}_k^B + \sum_{j=1}^{n} s_j \mathrm{J}_j^A, \tag{5.1.6}$$

where $f^A{}_{BC}$ denote the structure constants of the conformal algebra $\mathfrak{so}(1,5)$, and the evaluation parameters s_j are chosen as

$$s_j = (\underbrace{0,\dots,0}_{\alpha},\underbrace{-1,\dots,-1}_{\beta},\underbrace{-2,\dots,-2}_{\alpha},\underbrace{-3,\dots,-3}_{\beta})_j, \qquad j=1,\dots,n. \tag{5.1.7}$$

We recall that invariance under the level-zero generators J^A implies that the above integrals take the form

$$\tilde{I}_{\alpha\beta} = V_{\alpha\beta}\,\tilde{\phi}_{\alpha\beta}(u_j), \tag{5.1.8}$$

with $V_{\alpha\beta}$ being the conformal weight of the integral and $\tilde{\phi}_{\alpha\beta}$ denoting a conformal function which depends on the cross ratios u_j. After specifying the level-one invariance condition to the level-one momentum generator

$$\widehat{\mathrm{P}}^\mu = \tfrac{i}{2}\sum_{j<k=1}^{n}\big(\mathrm{P}_j^\mu \mathrm{D}_k + \mathrm{P}_{j\nu}\mathrm{L}_k^{\mu\nu} - (j\leftrightarrow k)\big) + \sum_{j=1}^{n} s_j \mathrm{P}_j^\mu = \sum_{j<k=1}^{n}\widehat{\mathrm{P}}_{jk}^\mu + \sum_{j=1}^{n} s_j\mathrm{P}_j^\mu. \tag{5.1.9}$$

the structure is

$$0 = \widehat{\mathrm{P}}^\mu \tilde{I}_{\alpha\beta} = V_{\alpha\beta}\sum_{j<k=1}^{n}\frac{x_{jk}^\mu}{x_{jk}^2}\mathrm{PDE}_{jk}\,\tilde{\phi}_{\alpha\beta}(u_j), \tag{5.1.10}$$

where PDE_{jk} denotes differential operators in u_j. One should then argue that the vectors x_{jk}^μ/x_{jk}^2 are independent such that Yangian invariance implies a system of differential equations for the conformal function [10]:

$$\mathrm{PDE}_{jk}\,\tilde{\phi}_{\alpha\beta} = 0, \qquad 1\le j<k\le n. \tag{5.1.11}$$

We stress that these are *homogeneous* partial differential equations, a fact that will change when considering coincidence limits of the external points.

We choose to use the level-one momentum generator $\widehat{\mathrm{P}}^\mu$ because calculations with this generator are easiest. However, acting with the other level-one generators will provide no independent PDEs in the cross-ratios, i.e. no new constraints on the conformal functions $\tilde{\phi}_{\alpha\beta}$. For instance, we could extract the action of $\widehat{\mathrm{D}}$ and $\widehat{\mathrm{L}}^{\mu\nu}$ from contractions of the following algebra relation evaluated on $\tilde{I}_{\alpha\beta}$:

$$2i\Big(\eta^{\mu\nu}\widehat{\mathrm{D}} - \widehat{\mathrm{L}}^{\mu\nu}\Big)\tilde{I}_{\alpha\beta} = [\mathrm{K}^\mu,\widehat{\mathrm{P}}^\nu]\tilde{I}_{\alpha\beta} = \mathrm{K}^\mu\widehat{\mathrm{P}}^\nu\tilde{I}_{\alpha\beta}. \tag{5.1.12}$$

Here we have used the Yangian algebra (3.1.87) and $\mathrm{K}^\mu\tilde{I}_{\alpha\beta}=0$. Hence, for example the analogue of the invariance equation (5.1.10) for the level-one dilatation operator $\widehat{\mathrm{D}}$ can be obtained from the $\widehat{\mathrm{P}}^\nu$ identity by action with K^μ and contraction with $\eta^{\mu\nu}$. It is clear that this cannot modify the PDEs which $\tilde{\phi}_{\alpha\beta}$, only the coefficient functions.

5.1.2 Four-Point Coincidence Limit

Now we investigate the imprint that the Yangian invariance equation (5.1.10) for the fishnet correlators leaves on Basso–Dixon correlators, which emerge from the former in a coincident point limit. While the limit can be carried out straightforwardly for the correlators, there is a subtlety in the limit of the invariance equation, which is due to the fact that the Yangian generators in their differential form do not commute with the coincidence limit.

In order to tackle these issues with the coincidence limit, it is useful to interpret the above Yangian symmetry of Feynman integrals (5.1.3) as Ward identities for the corresponding correlation functions (5.1.2) in the bi-scalar fishnet theory:

$$\begin{aligned}&\widehat{\mathbb{J}}^A \left\langle Z(x_1)\cdots Z(x_\alpha)\bar{X}(x_{\alpha+1})\cdots\bar{X}(x_{\alpha+\beta})\bar{Z}(x_{\alpha+\beta+1})\cdots\bar{Z}(x_{2\alpha+\beta})X(x_{2\alpha+\beta+1})\cdots X(x_{2\alpha+2\beta})\right\rangle\\ &\quad = 0,\end{aligned} \tag{5.1.13}$$

where for brevity we have omitted the trace from the correlation function (5.1.2). Note that (5.1.13) is a somewhat formal identity, since Yangian symmetry of the Fishnet action has not yet been proven. However, it is still a true identity at the level of Feynman integrals.

In order to elucidate the coincidence limit, it is necessary to distinguish the representation of the conformal algebra on coordinates J^A and the representation on the fields $\mathbb{J}^A$. On a single leg, the field and coordinate representation are essentially related by a minus sign, [1]

$$\mathbb{J}^A\Phi(x) = -\mathrm{J}^A(\Delta)\Phi(x), \tag{5.1.14}$$

where the scaling dimension Δ in the coordinate representation generator is dictated by the scaling dimension of the field Φ.

However, when acting on multiple fields, the field representation carries an additional label that encodes on which of the *fields* the operator acts (in contrast to the coordinate representation which carries a label that encodes which *coordinate* the operator acts on), i.e.

$$\mathbb{J}_k^A(\Phi_1(x_1)\dots\Phi_n(x_n)) = -\Phi_1(x_1)\dots(\mathrm{J}_k^A\Phi_k(x_k))\dots\Phi_n(x_n), \tag{5.1.15}$$

where $\Phi_i \in \{X, Z, \bar{X}, \bar{Z}\}$. The distinction between these two representations becomes non-trivial as soon as we consider products of fields that are evaluated at the same coordinate. As an example, consider the action of an operator in the field representation on the product of two fields that depend on the same coordinate:

$$\mathbb{J}_1^A(\Phi_1(x_1)\Phi_2(x_1)) = -(\mathrm{J}_1^A\Phi_1(x_1))\Phi_2(x_1). \tag{5.1.16}$$

In contrast, an operator in the coordinate representation cannot distinguish between the two fields in this product and by the Leibniz rule naturally acts on both fields[2]

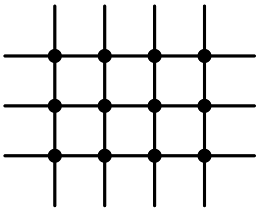
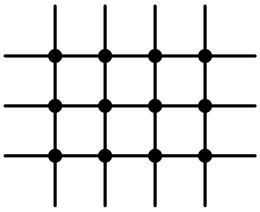
$$\mathrm{J}_1^A(\Delta_1+\Delta_2)(\Phi_1(x_1)\Phi_2(x_1)) = (\mathrm{J}_1^A(\Delta_1)\Phi_1(x_1))\Phi_2(x_1)+\Phi_1(x_1)(\mathrm{J}_1^A(\Delta_2)\Phi_2(x_1)). \tag{5.1.17}$$

Therefore, the field representation allows us to act separately on different fields that depend on the same coordinate. As we will discuss next, this provides us with a natural extension of the above Yangian symmetry of square fishnet diagrams to diagrams with coincident external legs.

[1] This ensures consistent commutation relations due to $\mathbb{J}^A\mathbb{J}^B\Phi = -\mathbb{J}^A\mathrm{J}^B\Phi = \mathrm{J}^B\mathrm{J}^A\Phi$ (see [187] for more details).

[2] Note that the scaling dimensions only ever appear multiplicatively in the conformal generators (5.1.5). Hence, while they could be shuffled around on the right hand side of this equation, we have picked a physically sensible representation for this identity.

Before the coincidence limit, the Yangian level-one symmetry may be written in two equivalent forms of the following level-one Ward identity for the fishnet correlator:

$\widehat{\mathbb{J}}^A$  $= \widehat{\mathrm{J}}^A$  $= 0.$ (5.1.18)

Here the level-one generator in the field representation takes the form

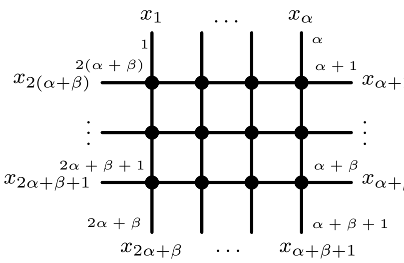
$$\widehat{\mathbb{J}}^A = \widehat{\mathbb{J}}^A_{\mathrm{bi}} - \sum_{j=1}^{n} s_j \mathbb{J}^A_j = \tfrac{1}{2} f^A{}_{BC} \sum_{k=1}^{n} \sum_{j=1}^{k-1} \mathbb{J}^C_j \mathbb{J}^B_k - \sum_{j=1}^{n} s_j \mathbb{J}^A_j. \tag{5.1.19}$$

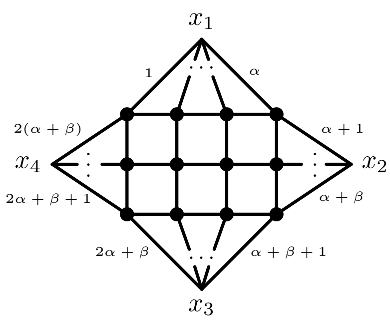
Naturally, since all fields in the fishnet correlators depend on separate coordinates, the field and coordinate representation are virtually indistinguishable.

We now take the limit of the Yangian Ward identity for fishnet correlators (5.1.18) to a Ward identity for Basso–Dixon correlators. This limit consists of taking the external points x_j on each of the four sides of the square fishnet to be coincident according to

$$\begin{aligned} &x_j \to x_1, && \text{for} \quad j = 1, \ldots, \alpha, \\ &x_j \to x_2, && \text{for} \quad j = \alpha+1, \ldots, \alpha+\beta, \\ &x_j \to x_3, && \text{for} \quad j = \alpha+\beta+1, \ldots, 2\alpha+\beta, \\ &x_j \to x_4, && \text{for} \quad j = 2\alpha+\beta+1, \ldots, 2(\alpha+\beta). \end{aligned} \tag{5.1.20}$$

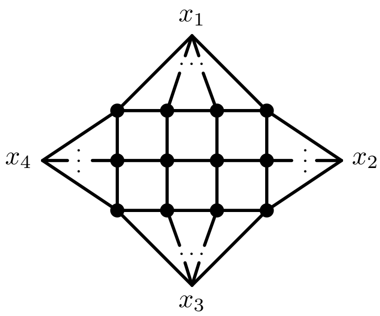
Graphically the limit is illustrated by the following figure:



$\to$


. (5.1.21)

Importantly, while the limit of the Yangian Ward identity in the coordinate representation suffers from the subtleties due to the identification of points, it trivially commutes with the level-one generators in the field representation:

$\widehat{\mathbb{J}}^A$ 

$= 0.$ (5.1.22)

However, translating the Ward identity into a differential equation for Feynman integrals, analogously to (5.1.10), becomes more subtle. To illustrate this point, consider the level-one generator acting on the product of two fields whose coordinate arguments are taken to be coincident:

$$\widehat{\mathbb{J}}^A\Phi_1(x_1)\Phi_2(x_1) = \tfrac{1}{2}f^A{}_{BC}[\mathbb{J}_1^C\Phi_1(x_1)][\mathbb{J}_2^B\Phi_2(x_1)] \neq \widehat{\mathrm{J}}^A\Phi_1(x_1)\Phi_2(x_1). \tag{5.1.23}$$

The last inequality implies that we cannot simply replace the field representation $\mathbb{J}$ of the conformal (Yangian) generators by the coordinate representation J. However, we can replace the generators in the middle term of the above equation, where each conformal generator acts on a single field:

$$\tfrac{1}{2}f^A{}_{BC}[\mathbb{J}_1^C\Phi_1(x_1)][\mathbb{J}_2^B\Phi_2(x_1)] = \tfrac{1}{2}f^A{}_{BC}[\mathrm{J}_1^C\Phi_1(x_1)][\mathrm{J}_1^B\Phi_2(x_1)]. \tag{5.1.24}$$

When inserting the above identities into a correlator, this will lead to correlators including descendant fields. Since the conformal generators are represented by first-order differential operators with vector indices, we will thus find true vector integrals contributing to the Ward identity, where these derivatives act on single propagators, but cannot be pulled in front of the whole integral.

Notably, in other contributions to the Yangian Ward identity, the field representation of the level-one generators can be replaced by the coordinate representation, for instance

$$\begin{aligned}&\tfrac{1}{2}f^A{}_{BC}[\mathbb{J}_1^C\Phi_1(x_1)]\Phi_2(x_1)[\mathbb{J}_3^B\Phi_3(x_2)] + \tfrac{1}{2}f^A{}_{BC}\Phi_1(x_1)[\mathbb{J}_2^C\Phi_2(x_1)][\mathbb{J}_3^B\Phi_3(x_2)]\\ &\qquad= \tfrac{1}{2}f^A{}_{BC}[\mathrm{J}_1^C\Phi_1(x_1)\Phi_2(x_1)][\mathrm{J}_2^B\Phi_3(x_2)],\end{aligned} \tag{5.1.25}$$

which is due to the fact that the generators J_j^A are first order differential operators and where we assume that the Δ_j are equal for fields evaluated at equal points.

Similarly, local contributions to the Yangian level-one generator, like the terms multiplying the evaluation parameters s_a, can be replaced by the coordinate space generators as e.g.

$$\sum_{a=1}^{\alpha+1} s_a\mathbb{J}_a^A\Phi_1(x_1)\dots\Phi_\alpha(x_1)\Phi_{\alpha+1}(x_2) = \big(s_1\mathrm{J}_1^A + s_2\mathrm{J}_2^A\big)\Phi_1(x_1)\dots\Phi_\alpha(x_1)\Phi_{\alpha+1}(x_2), \tag{5.1.26}$$

as long as the evaluation parameters associated with fields situated at the same point x_1 are equal, i.e. in the above example $s_1 = \dots = s_\alpha$.

With these points in mind, we can now write the above Ward identity for an $\alpha\times\beta$ Basso–Dixon graph in the form[3]

$$\begin{aligned}0 &= \widehat{\mathbb{J}}^A|_{1n}\langle\Phi_1(x_1)\Phi_2(x_1)\dots\Phi_{n-1}(x_4)\Phi_n(x_4)\rangle = \widehat{\mathrm{J}}^A|_{14}\langle\Phi_1(x_1)\Phi_2(x_1)\dots\Phi_{n-1}(x_4)\Phi_n(x_4)\rangle\\ &+\tfrac{1}{2}\sum_{a<b=1}^{\alpha} f^A{}_{BC}\langle\Phi_1(x_1)\dots[\mathrm{J}_1^C\Phi_a(x_1)]\dots[\mathrm{J}_1^B\Phi_b(x_1)]\dots\Phi_\alpha(x_1)\Phi_{\alpha+1}(x_2)\dots\Phi_n(x_4)\rangle\\ &+\tfrac{1}{2}\sum_{a<b=\alpha+1}^{\alpha+\beta} f^A{}_{BC}\langle\Phi_1(x_1)\dots\Phi_\alpha(x_1)\Phi_{\alpha+1}(x_2)\dots[\mathrm{J}_2^C\Phi_a(x_2)]\dots[\mathrm{J}_2^B\Phi_b(x_2)]\dots\Phi_n(x_4)\rangle\\ &+(\text{two similar}).\end{aligned} \tag{5.1.27}$$

[3]We always consider single-trace correlation functions. Here we omit the trace for brevity.

Here the notation $|_{1n}$ indicates the range of the summations in the definition of the level-one generator (5.1.19). This is the consequence of Yangian symmetry for the infinite class of Basso–Dixon integrals.

In order to understand what this identity means explicitly, let us again specify the generators $\widehat{\mathbb{J}}^A$ and $\widehat{\mathrm{J}}^A$ to the level-one momentum operator $\widehat{\mathbb{P}}^\mu$ and $\widehat{\mathrm{P}}^\mu$, respectively, see (5.1.9). With these expressions the above level-one Ward identity becomes

$$\begin{aligned}
0 &= \widehat{\mathrm{P}}^\mu|_{14}\langle \mathit{\Phi}_1(x_1)\mathit{\Phi}_2(x_1)\dots\mathit{\Phi}_{n-1}(x_4)\mathit{\Phi}_n(x_4)\rangle \\
&+\tfrac{i}{2}\sum_{a<b=1}^{\alpha}\big\{\langle \mathit{\Phi}_1(x_1)\dots[\mathit{\Delta}_a\mathit{\Phi}_a(x_1)]\dots[\partial_1^\mu\mathit{\Phi}_b(x_1)]\dots\mathit{\Phi}_\alpha(x_1)\mathit{\Phi}_{\alpha+1}(x_2)\dots\mathit{\Phi}_n(x_4)\rangle \\
&\qquad -\langle \mathit{\Phi}_1(x_1)\dots[\partial_1^\mu\mathit{\Phi}_a(x_1)]\dots[\mathit{\Delta}_b\mathit{\Phi}_b(x_1)]\dots\mathit{\Phi}_\alpha(x_1)\mathit{\Phi}_{\alpha+1}(x_2)\dots\mathit{\Phi}_n(x_4)\rangle\big\} \\
&+\tfrac{i}{2}\sum_{a<b=\alpha+1}^{\alpha+\beta}\big\{\langle \mathit{\Phi}_1(x_1)\dots\mathit{\Phi}_\alpha(x_1)\mathit{\Phi}_{\alpha+1}(x_2)\dots[\mathit{\Delta}_a\mathit{\Phi}_a(x_2)]\dots[\partial_2^\mu\mathit{\Phi}_b(x_2)]\dots\mathit{\Phi}_n(x_4)\rangle \\
&\qquad -\langle \mathit{\Phi}_1(x_1)\dots\mathit{\Phi}_\alpha(x_1)\mathit{\Phi}_{\alpha+1}(x_2)\dots[\partial_2^\mu\mathit{\Phi}_a(x_2)]\dots[\mathit{\Delta}_b\mathit{\Phi}_b(x_2)]\dots\mathit{\Phi}_n(x_4)\rangle\big\} \\
&+(\text{two similar}).
\end{aligned} \tag{5.1.28}$$

Notably, due to the anti-symmetrisation $j \leftrightarrow k$ in (5.1.9) and the symmetry of $T^{\nu\mu\rho}$ in ν and ρ, the $T^{\nu\mu\rho}$-contribution to $\widehat{\mathrm{P}}^\mu$ drops out of the last four lines of the above equations.

If we assume that the above fields have distinct scaling dimensions $\mathit{\Delta}_a$, the above is the final form of our Ward identity. Note, however, that this does not correspond to the above bi-scalar fishnet theory. However, this finds application in the case of the D-dimensional generalisation of the fishnet theory of [11], as we describe later.

In the four-dimensional fishnet theory we consider ordinary scalar fields with $\mathit{\Delta}_a = 1$. Then the coincidence limit implies that the scaling dimensions $\mathit{\Delta}_j$ entering the coordinate representation $\widehat{\mathrm{P}}^\mu|_{14}$ in the above expression take the values

$$\mathit{\Delta}_1 = \mathit{\Delta}_3 = \sum_{a=1}^{\alpha}\mathit{\Delta}_a = \alpha, \qquad\qquad \mathit{\Delta}_2 = \mathit{\Delta}_4 = \sum_{a=1}^{\beta}\mathit{\Delta}_a = \beta. \tag{5.1.29}$$

This is consistent with the choice of scaling dimensions that implies level-zero invariance of the Basso–Dixon integrals:

$$\mathrm{J}^A(\mathit{\Delta}_j)I_{\alpha\beta} = 0. \tag{5.1.30}$$

Using $\mathit{\Delta}_a = 1$ (and multiplying by an overall factor $2i$), we evaluate one sum of each double sum to find

$$\begin{aligned}
&2i\widehat{\mathrm{P}}^\mu|_{14}\langle \mathit{\Phi}_1(x_1)\mathit{\Phi}_2(x_1)\dots\mathit{\Phi}_{n-1}(x_4)\mathit{\Phi}_n(x_4)\rangle = \\
&\sum_{a=1}^{\alpha}(2a-\alpha-1)\big\{\langle \mathit{\Phi}_1(x_1)\dots[\partial_1^\mu\mathit{\Phi}_a(x_1)]\dots\mathit{\Phi}_\alpha(x_1)\mathit{\Phi}_{\alpha+1}(x_2)\dots\mathit{\Phi}_n(x_4)\rangle \\
&\qquad +\langle \mathit{\Phi}_1(x_1)\dots\mathit{\Phi}_{\alpha+\beta+1}(x_3)\dots[\partial_3^\mu\mathit{\Phi}_{a+\alpha+\beta}(x_3)]\dots\mathit{\Phi}_{2\alpha+\beta}(x_3)\dots\mathit{\Phi}_n(x_4)\rangle\big\} \\
&+\sum_{a=1}^{\beta}(2a-\beta-1)\big\{\langle \mathit{\Phi}_1(x_1)\dots\mathit{\Phi}_\alpha(x_1)\mathit{\Phi}_{\alpha+1}(x_2)\dots[\partial_2^\mu\mathit{\Phi}_{a+\alpha}(x_2)]\dots\mathit{\Phi}_{\alpha+\beta}(x_2)\dots\mathit{\Phi}_n(x_4)\rangle \\
&\qquad +\langle \mathit{\Phi}_1(x_1)\dots\mathit{\Phi}_{2\alpha+\beta}(x_3)\mathit{\Phi}_{2\alpha+\beta+1}(x_4)\dots[\partial_4^\mu\mathit{\Phi}_{a+2\alpha+\beta}(x_4)]\dots\mathit{\Phi}_{2(\alpha+\beta)}(x_4)\big\}.
\end{aligned} \tag{5.1.31}$$

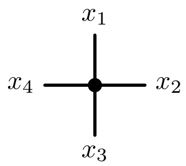
Hence, the Yangian Ward identity implies that acting with the four-point coordinate space level-one momentum generator on a Basso–Dixon graph yields a combination of correlators of $n-1$ scalar fields Φ_a and a single descendent field $\partial^\mu \Phi_b(x)$. We will refer to the left hand side of (5.1.31) as the *differential* contribution to the Ward identity, and the right hand side as the *vector* contribution. In section 5.2 we explicitly evaluate both sides of the equation as a constraint on conformal functions.

5.1.3 Examples

To illustrate (5.1.31) let us display a few simple cases graphically.

Box. In the simplest case of the box integral with $\alpha = \beta = 1$ the vector contribution on the right hand side of (5.1.31) vanishes identically and we recover the statement [10]

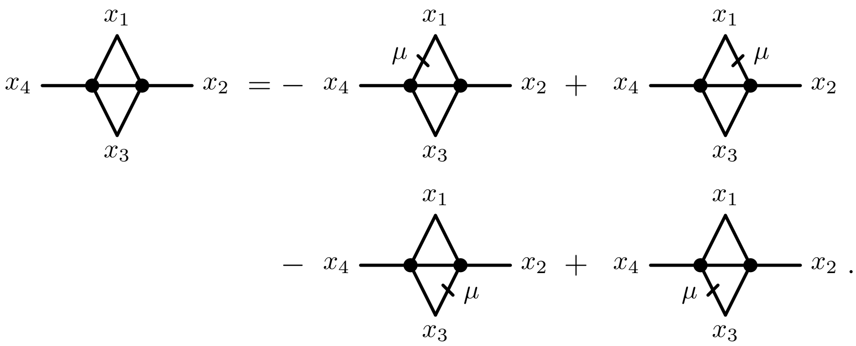
$$\widehat{\mathrm{P}}^\mu|_{14}$$



$$= 0. \qquad (5.1.32)$$

Double Ladder. The next simplest example is the double ladder integral with $\alpha = 2, \beta = 1$, for which the Yangian Ward identity reads

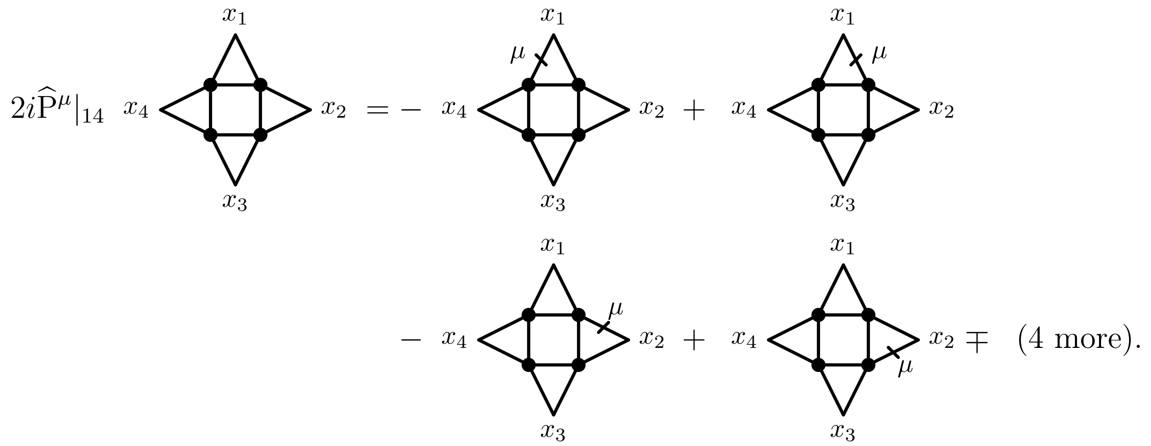
$$2i\widehat{\mathrm{P}}^\mu|_{14}$$



$$(5.1.33)$$

Here we use slashed lines to denote propagators that carry an additional derivative.

Window. As a final example, the Yangian Ward identity for the simplest integral with a two-dimensional lattice of integration points, the window integral,[4] reads

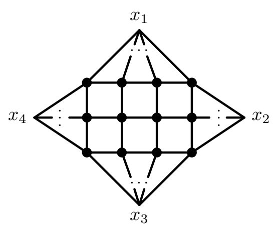


$$(5.1.34)$$

5.2 From Correlators to Feynman Integrals

In order to understand the above relations in terms of Feynman integrals, let us explicitly evaluate the different contributions. Due to conformal level-zero symmetry, we can write the Basso–Dixon correlator in the form

$$I_{\alpha\beta} = \left\langle \mathrm{tr}(Z^{\alpha}(x_1)\bar{X}^{\beta}(x_2)\bar{Z}^{\alpha}(x_3)X^{\beta}(x_4))\right\rangle =$$



$$= \frac{\xi^{2\alpha\beta}}{x_{13}^{2\alpha}x_{24}^{2\beta}}\phi_{\alpha\beta}(u,v), \tag{5.2.1}$$

where u and v are the four-point conformal cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{5.2.2}$$

$I_{\alpha\beta}$ can be expressed as a single $\alpha\beta$-loop Feynman integral

$$I_{\alpha\beta} = \frac{\xi^{2\alpha\beta}}{\pi^{2\alpha\beta}} \int \prod_{l,m=1}^{\alpha,\beta} \mathrm{d}^4 x_{l,m} \left(\prod_{j=1}^{\alpha}\prod_{k=0}^{\beta} \frac{1}{(x_{j,k}-x_{j,k+1})^2} \prod_{j=0}^{\alpha}\prod_{k=1}^{\beta} \frac{1}{(x_{j,k}-x_{j+1,k})^2} \right), \tag{5.2.3}$$

where $x_{i,0} = x_1, x_{\alpha+1,j} = x_2, x_{i,\beta+1} = x_3, x_{0,j} = x_4$, for $i = 1, 2, \ldots, \alpha$ and $j = 1, 2, \ldots, \beta$. As described in section 2.3.5, the functions $\phi_{\alpha\beta}$ are known in closed form, via the Basso–Dixon formula (2.3.70).

[4] This diagram is referred to as the window because of its form in dual position space. In position space it resembles a shuriken.

In the Yangian Ward identity we also encounter a version of this integral where one external leg carries an additional derivative, i.e.

$$I_{\alpha\beta}^{\mu,n} = \frac{\xi^{2\alpha\beta}}{\pi^{2\alpha\beta}} \int \prod_{l,m=1}^{\alpha,\beta} \mathrm{d}^4 x_{l,m} \left(\frac{(x_{n,1} - x_1)^\mu}{(x_{n,1} - x_1)^2} \prod_{j=1}^{\alpha} \prod_{k=0}^{\beta} \frac{1}{(x_{j,k} - x_{j,k+1})^2} \prod_{j=0}^{\alpha} \prod_{k=1}^{\beta} \frac{1}{(x_{j,k} - x_{j+1,k})^2} \right) \tag{5.2.4}$$

where $n = 1, 2, \ldots, \alpha$. Integrals containing derivatives acting on points different from x_1 are related to $I_{\alpha\beta}^{\mu,n}$ via permutations of the external points. For convenience we henceforth omit the dependence on the fishnet theory coupling constant ξ^2 in expressions for $I_{\alpha\beta}$ and $I_{\alpha\beta}^{\mu,n}$.

5.2.1 Ward Identity in Terms of Feynman Integrals

Here we analyse in detail the left and right hand side of the Yangian Ward identity for generic Basso–Dixon integrals (5.1.31).

Differential Part of Ward Identity (Left Hand Side). Using the following values for the scaling dimensions and evaluation parameters entering the level-one momentum generator

$$\Delta_j = (\alpha, \beta, \alpha, \beta)_j, \qquad s_j = -(0, 1, 2, 3)_j, \qquad j = 1, \ldots, 4, \tag{5.2.5}$$

we act with $\widehat{\mathrm{P}}^\mu$, defined in (5.1.9), on (5.2.1) and find the general expression

$$2i\widehat{\mathrm{P}}^\mu I_{\alpha\beta} = \frac{-4}{x_{13}^{2\alpha} x_{24}^{2\beta}} \left[\left(\frac{x_{12}^\mu}{x_{12}^2} + \frac{x_{34}^\mu}{x_{34}^2} \right) u \mathcal{D}_{uv}^{\alpha\beta} \phi_{\alpha\beta}(u, v) + \left(\frac{x_{23}^\mu}{x_{23}^2} + \frac{x_{41}^\mu}{x_{41}^2} \right) v \mathcal{D}_{vu}^{\alpha\beta} \phi_{\alpha\beta}(u, v) \right], \tag{5.2.6}$$

with the differential operator

$$\mathcal{D}_{uv}^{\alpha\beta} = \alpha\beta + (\alpha + \beta + 1)v\partial_v + \left((\alpha + \beta + 1)u - \tfrac{\alpha+\beta}{2}\right)\partial_u + v^2\partial_v^2 + (u - 1)u\partial_u^2 + 2uv\partial_u\partial_v. \tag{5.2.7}$$

This result, if tedious to derive, is a simple consequence of the chain rule. In the following we will use the shorthand

$$H_{\alpha\beta}^\mu \coloneqq 2i\widehat{\mathrm{P}}^\mu I_{\alpha\beta}. \tag{5.2.8}$$

Using conformal symmetry one can argue that the coefficients of the vectors x_{jk}^μ / x_{jk}^2 are in fact independent in the Yangian Ward identity [10]. Let us thus investigate these coefficients on the right hand side of the equation (5.1.31) in the following. For completeness we note that $\mathcal{D}_{uv}^{\alpha\beta}$ and $\mathcal{D}_{vu}^{\alpha\beta}$ are the special cases $\gamma = \gamma' = (\alpha + \beta)/2$ of the differential operators that are known to annihilate the Appell hypergeometric function F_4:

$$\begin{aligned}
\mathcal{D}_{vu}^{\alpha\beta\gamma\gamma'} &= \big(\alpha\beta + (\alpha + \beta + 1)u\partial_u + ((\alpha + \beta + 1)v - \gamma')\partial_v + u^2\partial_u^2 + (v - 1)v\partial_v^2 + 2vu\partial_u\partial_v\big), \\
\mathcal{D}_{uv}^{\alpha\beta\gamma\gamma'} &= \big(\alpha\beta + (\alpha + \beta + 1)v\partial_v + ((\alpha + \beta + 1)u - \gamma)\partial_u + v^2\partial_v^2 + (u - 1)u\partial_u^2 + 2vu\partial_v\partial_u\big).
\end{aligned} \tag{5.2.9}$$

Notably, the conformal box integral for generic propagator powers satisfies homogenous Ward identities, and so is annihilated by $\mathcal{D}_{vu}^{\alpha\beta\gamma\gamma'}$ and $\mathcal{D}_{uv}^{\alpha\beta\gamma\gamma'}$. Therefore it can be expressed in terms of F_4, with the parameters $\alpha, \beta, \gamma, \gamma'$ relating to the four propagator powers [10].

Vector Part of Ward Identity (Right Hand Side). Just like the scalar integrals (5.2.3), all vector integrals of the form (5.2.4) satisfy (level-zero) conformal Ward identities:

$$\mathrm{D} I^{\mu,n}_{\alpha\beta}(x_1,\ldots,x_4) = 0, \tag{5.2.10}$$

$$\big(\mathrm{K}^\mu \eta_{\nu\rho} + 2i(\delta^\mu_\nu x_{1,\rho} - \delta^\mu_\rho x_{1,\nu})\big)\, I^{\rho,n}_{\alpha\beta}(x_1,\ldots,x_4) = -2i\delta^\mu_\nu I_{\alpha\beta}(x_1,\ldots,x_4), \tag{5.2.11}$$

with scaling dimensions $\Delta_i = (\alpha+1,\beta,\alpha,\beta)$. These identities can be derived straightforwardly by commuting the homogeneous Ward identities for the scalar integrals with the extra derivatives contained in the vector integrals. The general solution to (5.2.10) and (5.2.11) takes the form

$$x_{13}^{2\alpha} x_{24}^{2\beta} I^{\mu,n}_{\alpha\beta} = -\frac{x_{12}^\mu}{x_{12}^2} F^{\alpha\beta}_{2,n}(u,v) - \frac{x_{13}^\mu}{x_{13}^2} F^{\alpha\beta}_{3,n}(u,v) - \frac{x_{14}^\mu}{x_{14}^2} F^{\alpha\beta}_{4,n}(u,v), \tag{5.2.12}$$

where the coefficient functions further need to satisfy

$$F^{\alpha\beta}_{2,n}(u,v) + F^{\alpha\beta}_{3,n}(u,v) + F^{\alpha\beta}_{4,n}(u,v) = I_{\alpha\beta}(u,v), \tag{5.2.13}$$

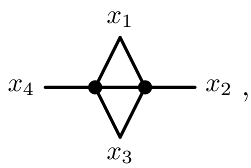
for each $n = 1, 2, \ldots, \alpha$.

5.2.2 Examples

We present here explicitly the form of the Yangian Ward identity for all Basso–Dixon graphs up to four loops.

Double Ladder. We consider the correlator (5.2.1) for $\alpha = 2, \beta = 1$

$$\big\langle \mathrm{tr}(Z(x_1)Z(x_1)\bar{X}(x_2)\bar{Z}(x_3)\bar{Z}(x_3)X(x_4))\big\rangle = \tag{5.2.14}$$



which is represented by the well-known double ladder integral

$$I_{21} = \int \frac{\mathrm{d}^4 x_a}{\pi^2}\frac{\mathrm{d}^4 x_b}{\pi^2}\frac{1}{(x_{a1}^2 x_{a3}^2 x_{a4}^2) x_{ab}^2 (x_{b1}^2 x_{b2}^2 x_{b3}^2)} = \frac{1}{x_{13}^4 x_{24}^2}\phi_{21}(u,v). \tag{5.2.15}$$

We would like to understand what the Yangian Ward identities (5.1.18) imply for the double ladder integral. Therefore we specialise (5.1.31) to compute the action of the level-one generator $\widehat{\mathrm{P}}^\mu$ on the four-point correlator (5.2.14):

$$H_{21}^\mu := 2i\widehat{\mathrm{P}}^\mu|_{14}\left\langle \mathrm{tr}(Z^2(x_1)\bar{X}(x_2)\bar{Z}^2(x_3)X(x_4))\right\rangle. \tag{5.2.16}$$

The Yangian Ward identity implies that

$$\begin{aligned} H_{21}^\mu = &+ \left\langle \mathrm{tr}(Z(x_1)[\partial_1^\mu Z(x_1)]\bar{X}(x_2)\bar{Z}^2(x_3)X(x_4))\right\rangle \\ &- \left\langle \mathrm{tr}([\partial_1^\mu Z(x_1)]Z(x_1)\bar{X}(x_2)\bar{Z}^2(x_3)X(x_4))\right\rangle \\ &+ \left\langle \mathrm{tr}(Z^2(x_1)\bar{X}(x_2)\bar{Z}(x_3)[\partial_3^\mu \bar{Z}(x_3)]X(x_4))\right\rangle \\ &- \left\langle \mathrm{tr}(Z^2(x_1)\bar{X}(x_2)[\partial_3^\mu \bar{Z}(x_3)]\bar{Z}(x_3)X(x_4))\right\rangle, \end{aligned} \tag{5.2.17}$$

which is another representation of (5.1.33). Explicitly, we see that evaluating the four-point coordinate space level-one momentum generator on the four-point correlator (5.2.14) yields a linear combination of correlation functions involving a single descendant field. Let us focus on the single contribution

$$2I_{21}^{\mu,2} = \left\langle \mathrm{tr}(Z(x_1)[\partial_1^\mu Z(x_1)]\bar{X}(x_2)\bar{Z}^2(x_3)X(x_4))\right\rangle, \tag{5.2.18}$$

where the vector integral $I_{21}^{\mu,2}$ is a specialisation of (5.2.4):

$$\begin{aligned} 2I_{21}^{\mu,2} &= \int \frac{\mathrm{d}^4x_a}{\pi^2}\frac{\mathrm{d}^4x_b}{\pi^2}\frac{1}{x_{a1}^2x_{a3}^2x_{a4}^2x_{ab}^2}\left(\partial_1^\mu\frac{1}{x_{b1}^2}\right)\frac{1}{x_{b2}^2x_{b3}^2} \\ &= 2\int \frac{\mathrm{d}^4x_a}{\pi^2}\frac{\mathrm{d}^4x_b}{\pi^2}\frac{x_{b1}^\mu}{x_{a1}^2x_{a3}^2x_{a4}^2x_{ab}^2x_{b1}^4x_{b2}^2x_{b3}^2}. \end{aligned} \tag{5.2.19}$$

We can then represent H_{21}^μ in terms of antisymmetrisations of $I_{21}^{\mu,2}$:

$$H_{21}^\mu = (2I_{21}^{\mu,2} - x_2 \leftrightarrow x_4) - x_1 \leftrightarrow x_3. \tag{5.2.20}$$

Moreover, using (5.2.12) we find the following vector decomposition for $I_{21}^{\mu,2}$:

$$x_{13}^4x_{24}^2I_{21}^{\mu,2} = -\frac{x_{12}^\mu}{x_{12}^2}F_2(u,v) - \frac{x_{13}^\mu}{x_{13}^2}F_3(u,v) - \frac{x_{14}^\mu}{x_{14}^2}F_4(u,v). \tag{5.2.21}$$

By an explicit Feynman parametrisation one can obtain integral expressions for the conformal functions $F_i(u,v)$ which were useful for numerical checks, see appendix A. From (5.2.13) the vector coefficients $F_i(u,v)$ further satisfy

$$\phi_{21}(u,v) = F_2(u,v) + F_3(u,v) + F_4(u,v). \tag{5.2.22}$$

This can also be seen by contracting both sides of (5.2.21) with $-x_1^\mu$ and sending $x_1 \to \infty$ with a conformal transformation. Under the transpositions of points $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$ the cross ratios are exchanged $u \leftrightarrow v$. Using this fact and (5.2.21) we calculate H_{21}^μ in terms of the F_i as

$$\begin{aligned} x_{13}^4x_{24}^2H_{21}^\mu = &-2\left(\frac{x_{12}^\mu}{x_{12}^2}+\frac{x_{34}^\mu}{x_{34}^2}\right)[F_2(u,v)-F_4(v,u)] \\ &-2\left(\frac{x_{23}^\mu}{x_{23}^2}+\frac{x_{41}^\mu}{x_{41}^2}\right)[F_2(v,u)-F_4(u,v)]. \end{aligned} \tag{5.2.23}$$

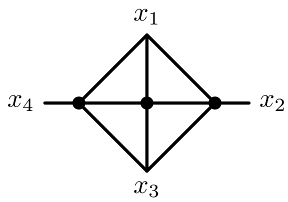
Comparing (5.2.23) and (5.2.6) we recover the following constraint on the components of the vector decomposition F_i:

$$2u\mathcal{D}_{uv}^{21}\phi_{21} = F_2(u,v) - F_4(v,u), \tag{5.2.24}$$

and the same equation with u and v swapped. So the Yangian differential operator acting on the conformal double ladder is a combination of the coefficient functions in the vector decomposition (5.2.21). The left hand side of (5.2.24) can be calculated exactly by acting with $2u\mathcal{D}_{uv}^{21}$ on the ladder function (2.3.68). We obtained explicit Feynman parametrisations of F_i in order to check agreement with the right hand side numerically, see (A.2.13) and (A.2.15).

Triple Ladder. We consider the correlator (5.2.1) for $\alpha = 3, \beta = 1$

$$\left\langle \mathrm{tr}(Z^3(x_1)\bar{X}(x_2)\bar{Z}^3(x_3)X(x_4))\right\rangle = \tag{5.2.25}$$



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which is represented by the triple ladder integral

$$I_{31} = \int \frac{\mathrm{d}^4 x_a}{\pi^2}\frac{\mathrm{d}^4 x_b}{\pi^2}\frac{\mathrm{d}^4 x_c}{\pi^2}\frac{1}{(x_{a1}^2 x_{a3}^2 x_{a4}^2)x_{ab}^2(x_{b1}^2 x_{b3}^2)x_{bc}^2(x_{c1}^2 x_{c2}^2 x_{c3}^2)}. \tag{5.2.26}$$

Again specialising (5.1.31) to this case will lead to information about certain vector Feynman integrals, which represent correlators containing a descendant field. In this case there are a priori two independent vector integrals which can appear

$$I_{31}^{\mu,3} = \int \frac{\mathrm{d}^4 x_a}{\pi^2}\frac{\mathrm{d}^4 x_b}{\pi^2}\frac{\mathrm{d}^4 x_c}{\pi^2}\frac{x_{c1}^{\mu}}{(x_{a1}^2 x_{a3}^2 x_{a4}^2)x_{ab}^2(x_{b1}^2 x_{b3}^2)x_{bc}^2(x_{c1}^4 x_{c2}^2 x_{c3}^2)}, \tag{5.2.27}$$

$$I_{31}^{\mu,2} = \int \frac{\mathrm{d}^4 x_a}{\pi^2}\frac{\mathrm{d}^4 x_b}{\pi^2}\frac{\mathrm{d}^4 x_c}{\pi^2}\frac{x_{b1}^{\mu}}{(x_{a1}^2 x_{a3}^2 x_{a4}^2)x_{ab}^2(x_{b1}^4 x_{b3}^2)x_{bc}^2(x_{c1}^2 x_{c2}^2 x_{c3}^2)}. \tag{5.2.28}$$

These are independent in the sense that they cannot be mapped into each other under permutations of the external points. However upon computing

$$H_{31}^{\mu} = 2i\widehat{\mathrm{P}}^{\mu}|_{14}\left\langle \mathrm{tr}(Z^3(x_1)\bar{X}(x_2)\bar{Z}^3(x_3)X(x_4))\right\rangle \tag{5.2.29}$$

using (5.1.31) we find that only $I_{31}^{\mu,3}$ contributes

$$H_{31}^{\mu} = (4I_{31}^{\mu,3} - x_2 \leftrightarrow x_4) - x_1 \leftrightarrow x_3. \tag{5.2.30}$$

We expand $I_{31}^{\mu,3}$ in a vector decomposition (5.2.12)

$$x_{13}^6 x_{24}^2 I_{31}^{\mu,3} = -\frac{x_{12}^{\mu}}{x_{12}^2}G_2(u,v) - \frac{x_{13}^{\mu}}{x_{13}^2}G_3(u,v) - \frac{x_{14}^{\mu}}{x_{14}^2}G_4(u,v). \tag{5.2.31}$$

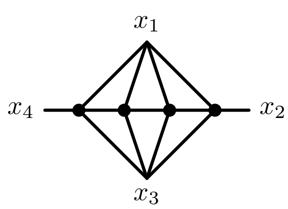
Comparing (5.2.30) and (5.2.6) we find the constraint

$$u\mathcal{D}_{uv}^{31}\phi_{31} = G_2(u,v) - G_4(v,u), \tag{5.2.32}$$

and the same equation with u and v swapped. Integral expressions for $G_i(u,v)$ are given in appendix A, which were used to numerically confirm (5.2.32).

Quadruple Ladder. For $\alpha = 4, \beta = 1$ we have the correlator

$$\left\langle \mathrm{tr}(Z^4(x_1)\bar{X}(x_2)\bar{Z}^4(x_3)X(x_4))\right\rangle = \tag{5.2.33}$$



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which is represented by the quadruple ladder integral

$$I_{41} = \int \frac{\mathrm{d}^4x_a}{\pi^2}\frac{\mathrm{d}^4x_b}{\pi^2}\frac{\mathrm{d}^4x_c}{\pi^2}\frac{\mathrm{d}^4x_d}{\pi^2}\frac{1}{(x_{a1}^2x_{a3}^2x_{a4}^2)x_{ab}^2(x_{b1}^2x_{b3}^2)x_{bc}^2(x_{c1}^2x_{c3}^2)x_{cd}^2(x_{d1}^2x_{d2}^2x_{d3}^2)}. \quad (5.2.34)$$

In this case there are also two independent vector integrals which can appear:

$$I_{41}^{\mu,4} = \int \frac{\mathrm{d}^4x_a}{\pi^2}\frac{\mathrm{d}^4x_b}{\pi^2}\frac{\mathrm{d}^4x_c}{\pi^2}\frac{\mathrm{d}^4x_d}{\pi^2}\frac{x_{d1}^\mu}{(x_{a1}^2x_{a3}^2x_{a4}^2)x_{ab}^2(x_{b1}^2x_{b3}^2)x_{bc}^2(x_{c1}^2x_{c3}^2)x_{cd}^2(x_{d1}^4x_{d2}^2x_{d3}^2)}, \quad (5.2.35)$$

$$I_{41}^{\mu,3} = \int \frac{\mathrm{d}^4x_a}{\pi^2}\frac{\mathrm{d}^4x_b}{\pi^2}\frac{\mathrm{d}^4x_c}{\pi^2}\frac{\mathrm{d}^4x_d}{\pi^2}\frac{x_{c1}^\mu}{(x_{a1}^2x_{a3}^2x_{a4}^2)x_{ab}^2(x_{b1}^2x_{b3}^2)x_{bc}^2(x_{c1}^4x_{c3}^2)x_{cd}^2(x_{d1}^2x_{d2}^2x_{d3}^2)}. \quad (5.2.36)$$

We compute

$$H_{41}^\mu = 2i\widehat{\mathrm{P}}^\mu|_{14}\left\langle \mathrm{tr}(Z^4(x_1)\bar{X}(x_2)\bar{Z}^4(x_3)X(x_4))\right\rangle \quad (5.2.37)$$

using (5.1.31) and this time we find that both vector integrals contribute

$$\begin{aligned} H_{41}^\mu &= (6I_{41}^{\mu,4} + 2I_{41}^{\mu,3} + (2I_{41}^{\mu,3} + x_2 \leftrightarrow x_4) - x_2 \leftrightarrow x_4) - x_1 \leftrightarrow x_3 \\ &= (6I_{41}^{\mu,4} + 2I_{41}^{\mu,3} - x_2 \leftrightarrow x_4) - x_1 \leftrightarrow x_3. \end{aligned} \quad (5.2.38)$$

We decompose (5.2.35) and (5.2.36) as (5.2.12)

$$x_{13}^8x_{24}^2I_{41}^{\mu,4} = -\frac{x_{12}^\mu}{x_{12}^2}V_2(u,v) - \frac{x_{13}^\mu}{x_{13}^2}V_3(u,v) - \frac{x_{14}^\mu}{x_{14}^2}V_4(u,v), \quad (5.2.39)$$

$$x_{13}^8x_{24}^2I_{41}^{\mu,3} = -\frac{x_{12}^\mu}{x_{12}^2}\bar{V}_2(u,v) - \frac{x_{13}^\mu}{x_{13}^2}\bar{V}_3(u,v) - \frac{x_{14}^\mu}{x_{14}^2}\bar{V}_4(u,v). \quad (5.2.40)$$

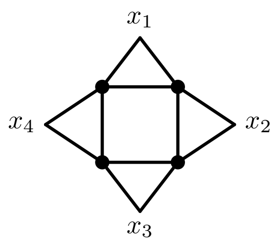
Then comparing (5.2.38) and (5.2.6) leads to the following constraint between $V_i(u,v)$ and $\bar{V}_i(u,v)$

$$2u\mathcal{D}_{uv}^{41}\phi_{41} = 3(V_2(u,v) - V_4(v,u)) + \bar{V}_2(u,v) - \bar{V}_4(v,u), \quad (5.2.41)$$

and the same equation with u and v swapped. This was numerically verified using the Feynman parametrisations in appendix A.

Window. For $\alpha = \beta = 2$, we consider the correlator (5.2.1) which reads

$$\left\langle \mathrm{tr}(Z^2(x_1)\bar{X}^2(x_2)\bar{Z}^2(x_3)X^2(x_4))\right\rangle =$$



. (5.2.42)

The corresponding window integral takes the form

$$\begin{aligned} I_{22} &= \int \frac{\mathrm{d}^4x_a}{\pi^2}\frac{\mathrm{d}^4x_b}{\pi^2}\frac{\mathrm{d}^4x_c}{\pi^2}\frac{\mathrm{d}^4x_d}{\pi^2}\frac{1}{(x_{a1}^2x_{a4}^2)x_{ab}^2(x_{b1}^2x_{b2}^2)x_{bc}^2(x_{c2}^2x_{c3}^2)x_{cd}^2(x_{d3}^2x_{d4}^2)x_{da}^2} \\ &= \frac{1}{x_{13}^4x_{24}^4}\phi_{22}(u,v). \end{aligned} \quad (5.2.43)$$

Let us write (5.1.31) specialised to this case in full:

$$
\begin{aligned}
&H_{22}^{\mu} = 2i\widehat{\mathrm{P}}^{\mu}|_{14} \left\langle Z^2(x_1)\bar{X}^2(x_2)\bar{Z}^2(x_3)X^2(x_4)\right\rangle \qquad (5.2.44)\\
&= \left\langle Z(x_1)[\partial_1^{\mu} Z(x_1)]\bar{X}^2(x_2)\bar{Z}^2(x_3)X^2(x_4)\right\rangle - \left\langle [\partial_1^{\mu} Z(x_1)]Z(x_1)\bar{X}^2(x_2)\bar{Z}^2(x_3)X^2(x_4)\right\rangle\\
&+ \left\langle Z^2(x_1)\bar{X}(x_2)[\partial_2^{\mu}\bar{X}(x_2)]\bar{Z}^2(x_3)X^2(x_4)\right\rangle - \left\langle Z^2(x_1)[\partial_2^{\mu}\bar{X}(x_2)]\bar{X}(x_2)\bar{Z}^2(x_3)X^2(x_4)\right\rangle\\
&+ \left\langle Z^2(x_1)\bar{X}^2(x_2)\bar{Z}(x_3)[\partial_3^{\mu}\bar{Z}(x_3)]X^2(x_4)\right\rangle - \left\langle Z^2(x_1)\bar{X}^2(x_2)[\partial_3^{\mu}\bar{Z}(x_3)]\bar{Z}(x_3)X^2(x_4)\right\rangle\\
&+ \left\langle Z^2(x_1)\bar{X}^2(x_2)\bar{Z}^2(x_3)X(x_4)[\partial_4^{\mu} X(x_4)]\right\rangle - \left\langle Z^2(x_1)\bar{X}^2(x_2)\bar{Z}^2(x_3)[\partial_4^{\mu} X(x_4)]X(x_4)\right\rangle,
\end{aligned}
$$

where in this equation we omitted traces for brevity. We focus on the first correlators in the second and third lines of (5.2.44) which read

$$
\begin{aligned}
2I_{22}^{\mu,2} &= \left\langle \mathrm{tr}(Z(x_1)[\partial_1^{\mu} Z(x_1)]\bar{X}^2(x_2)\bar{Z}^2(x_3)X^2(x_4))\right\rangle, &(5.2.45)\\
2\tau I_{22}^{\mu,2} &= \left\langle \mathrm{tr}(Z^2(x_1)\bar{X}(x_2)[\partial_2^{\mu}\bar{X}(x_2)]\bar{Z}^2(x_3)X^2(x_4))\right\rangle. &(5.2.46)
\end{aligned}
$$

Here $I_{22}^{\mu,2}$ is the vector integral

$$
I_{22}^{\mu,2} = \int \frac{\mathrm{d}^4x_a}{\pi^2}\frac{\mathrm{d}^4x_b}{\pi^2}\frac{\mathrm{d}^4x_c}{\pi^2}\frac{\mathrm{d}^4x_d}{\pi^2}\frac{x_{b1}^{\mu}}{(x_{a1}^2x_{a4}^2)x_{ab}^2(x_{b1}^4x_{b2}^2)x_{bc}^2(x_{c2}^2x_{c3}^2)x_{cd}^2(x_{d3}^2x_{d4}^2)x_{da}^2}, \qquad (5.2.47)
$$

and τ is the cycle which maps $x_i \to x_{i+1}$, where we identify $x_5 \equiv x_1$. In fact, all integrals appearing in (5.2.44) can be recovered from $I_{22}^{\mu,2}$ by a permutation of points, which at most exchanges u and v. Computing H_{22}^{μ} in full we find

$$
H_{22}^{\mu} = 2(I_{22}^{\mu,2} + \tau I_{22}^{\mu,2} - x_2 \leftrightarrow x_4) - x_1 \leftrightarrow x_3. \qquad (5.2.48)
$$

We expand $I_{22}^{\mu,2}$ in terms of its vector components

$$
x_{13}^4x_{24}^4I_{22}^{\mu} = -\frac{x_{12}^{\mu}}{x_{12}^2}W_2(u,v) - \frac{x_{13}^{\mu}}{x_{13}^2}W_3(u,v) - \frac{x_{14}^{\mu}}{x_{14}^2}W_4(u,v). \qquad (5.2.49)
$$

Under the cycle τ and transpositions $x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4$ we have $u \leftrightarrow v$. Using this and (5.2.49) we compute (5.2.48) in terms of the $W_i(u,v)$ to be

$$
x_{13}^4x_{24}^4H_{22}^{\mu} = -4\left(\frac{x_{12}^{\mu}}{x_{12}^2} + \frac{x_{34}^{\mu}}{x_{34}^2}\right)(W_2(u,v) - W_4(v,u)) + (x_1 \leftrightarrow x_3). \qquad (5.2.50)
$$

Comparing (5.2.50) and (5.2.6) we have the constraint

$$
u\mathcal{D}_{uv}^{22}\phi_{22} = W_2(u,v) - W_4(v,u), \qquad (5.2.51)
$$

and the same equation with $u \leftrightarrow v$. The structure of (5.2.24) and (5.2.51) persists for general Basso–Dixon correlators: $u\mathcal{D}_{uv}^{\alpha\beta}\phi_{\alpha\beta}$ is equal to a linear combination of coefficients in the vector decomposition of vector integrals related to $\phi_{\alpha\beta}$. For higher α, β more vector integrals $I_{\alpha\beta}^{\mu,n}$, and hence more independent vector coefficients, contribute to the Ward Identity.

5.3 Yangian Ward Identities

In the previous section we saw that the Yangian differential operator $u\mathcal{D}_{uv}^{\alpha\beta}$ acting on the conformal function $\phi_{\alpha\beta}$ is equal to a combination of coefficients of the conformal expansion (5.2.12) of certain vector integrals. As described in section 2.3.2, such coefficients can be written as linear combinations of higher dimensional scalar integrals with modified propagator powers. We exploit this fact to rewrite the vector coefficients as such, and reformulate the Yangian equations in the form of (5.2.24) as a formal identity for $\phi_{\alpha\beta}$.

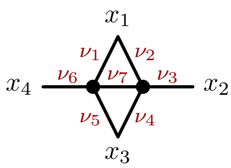
Doing this for the double ladder vector integral (5.2.19), the vector integral coefficients F_2, F_3, F_4 defined in (5.2.21) can be expanded as

$$F_2(u,v) = x_{13}^4 x_{24}^2 x_{12}^2 \big(I^6_{2,2,2,1,1,1,1} + I^6_{1,2,2,1,2,1,1} + I^6_{1,2,2,1,1,2,1} + I^6_{1,2,2,1,1,1,2}\big), \tag{5.3.1}$$

$$F_3(u,v) = x_{13}^6 x_{24}^2 \big(I^6_{1,2,1,1,2,1,2} + I^6_{2,2,1,2,1,1,1} + I^6_{1,2,1,2,2,1,1} + I^6_{1,2,1,2,1,2,1} + I^6_{1,2,1,2,1,1,2}\big), \tag{5.3.2}$$

$$F_4(u,v) = x_{13}^4 x_{24}^2 x_{14}^2 I^6_{1,2,1,1,1,2,2}, \tag{5.3.3}$$

where propagator powers are assigned to the integral $I^6_{\nu_1,\ldots,\nu_7}$ as follows:



. (5.3.4)

Note that essentially the same calculation to arrive at (5.3.1)-(5.3.3) was described in section 2.3.2, see (2.3.35)-(2.3.37). Interestingly, $F_4(u,v)$ is given by a single manifestly conformal 6-dimensional scalar integral. Note that $F_2(u,v)$ and $F_3(u,v)$ are not manifestly conformal when expressed as (5.3.1) and (5.3.2).

5.3.1 Conformalisation

In general, performing a vector decomposition on the integrals appearing on the right hand side of the Yangian Ward identities will lead to expressions for the vector coefficients which are not manifestly conformal. However, they can always be rewritten using manifestly conformal 6-dimensional integrals, as will be described in the following. To explain the method, let us consider some conformal function of four external points

$$f(x_1,x_2,x_3,x_4) = f(u,v), \tag{5.3.5}$$

with the usual conformal four-point cross ratios (5.2.2). If we consider the limit of one point going to infinity, e.g. x_1, we see that the cross ratios remain finite and the function is given by

$$\lim_{x_1\to\infty} f(x_1,x_2,x_3,x_4) = f(\tilde{u},\tilde{v}) \tag{5.3.6}$$

with the reduced cross ratios

$$\tilde{u} = \frac{x_{34}^2}{x_{24}^2}, \qquad \tilde{v} = \frac{x_{23}^2}{x_{24}^2}. \tag{5.3.7}$$

The same function of reduced cross ratios can be reached by performing a translation

$$f(x_1, x_2, x_3, x_4) = f(0, x_2 - x_1, x_3 - x_1, x_4 - x_1) \coloneqq f(0, y_2, y_3, y_4), \tag{5.3.8}$$

and then an inversion on y_2, y_3, y_4

$$f(0, y_2, y_3, y_4) \to f\left(0, \frac{y_2}{y_2^2}, \frac{y_3}{y_3^2}, \frac{y_4}{y_4^2}\right) = f(\tilde{u}, \tilde{v}), \tag{5.3.9}$$

where the last equivalence uses the identity

$$\left(\frac{y_i^\mu}{y_i^2} - \frac{y_j^\mu}{y_j^2}\right)^2 = \frac{y_{ij}^2}{y_i^2 y_j^2}. \tag{5.3.10}$$

Since all of these operations are linear, we can perform them for all integrals in (5.3.1) and (5.3.2) separately, keeping in mind that they will lead to an identity between integrals only when summed together. To turn this observation into a recipe for conformalisation, note that the steps in (5.3.8) and (5.3.9) are invertible, whereas sending one point to infinity is not. Hence, starting from a non-manifestly conformal sum of integrals, we can reach a manifestly conformal expression for the same quantity via the following steps:

- Send one of the points x_j to infinity.
- Perform an inversion on the remaining points and the integration variables.
- Restore the eliminated point by translating the remaining external points by x_j.

Example: Double Ladder Conformalisation. As an example, consider the integral $x_{13}^4 x_{24}^2 x_{12}^2 I^6_{1,2,2,1,2,1,1}$, which appears in the sum (5.3.1). Taking the limit $x_1 \to \infty$, we are left with

$$\lim_{x_1 \to \infty} x_{13}^4 x_{24}^2 x_{12}^2 I^6_{1,2,2,1,2,1,1} = \int \mathrm{d}^6 x_a \mathrm{d}^6 x_b \frac{x_{24}^2}{x_{2b}^4 x_{3b}^2 x_{3a}^4 x_{4a}^2 x_{ab}^2}. \tag{5.3.11}$$

Following the above recipe, we now perform an inversion

$$x_j^\mu \to \frac{x_j^\mu}{x_j^2} \tag{5.3.12}$$

on this integral. To rewrite it in terms of squared differences of x-variables, we also perform the substitution $x_{a,b}^\mu \to \frac{x_{a,b}^\mu}{x_{a,b}^2}$ for which

$$\mathrm{d}^6 x_{a,b} \to \frac{\mathrm{d}^6 x_{a,b}}{x_{a,b}^{12}}. \tag{5.3.13}$$

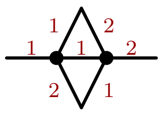
Then using (5.3.10) the integral becomes

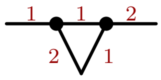
$$\int \mathrm{d}^6 x_a \mathrm{d}^6 x_b \frac{x_{24}^2}{x_{2b}^4 x_{3b}^2 x_{3a}^4 x_{4a}^2 x_{ab}^2} \to \int \mathrm{d}^6 x_a \mathrm{d}^6 x_b \frac{x_2^2 x_3^6 x_{24}^2}{x_a^4 x_b^4 x_{2b}^4 x_{3b}^2 x_{3a}^4 x_{4a}^2 x_{ab}^2}. \tag{5.3.14}$$

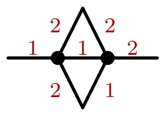
Now we restore the point x_1 by substituting $x_j \to x_j - x_1$ and arrive at the final replacement

$$x_{13}^4 x_{24}^2 x_{12}^2 I^6_{1,2,2,1,2,1,1} \to x_{13}^6 x_{24}^2 x_{12}^2 I^6_{2,2,2,1,2,1,1}. \tag{5.3.15}$$

Graphically, the steps we have performed can be represented by

$x_{13}^4 x_{24}^2 x_{12}^2$ 

$\longrightarrow x_{24}^2$ 

$\longrightarrow x_{13}^6 x_{24}^2 x_{12}^2$ 

. (5.3.16)

Let us stress again that this is not an identity between integrals but only valid if the integral on the left-hand side appears as a summand in some conformal expression. The full conformalised result for F_2 and F_3 in (5.3.1) and (5.3.2) reads

$$F_2(u,v) = x_{12}^2 x_{24}^2 x_{13}^4 (I^6_{2,1,2,1,1,1,2} + x_{13}^2 I^6_{2,2,2,1,2,1,1} + x_{14}^2 I^6_{2,2,2,1,1,2,1}), \tag{5.3.17}$$
$$F_3(u,v) = x_{24}^2 x_{13}^6 (I^6_{1,2,1,1,2,1,2} + I^6_{2,1,1,2,1,1,2} + x_{13}^2 I^6_{2,2,1,2,2,1,1} + x_{14}^2 I^6_{2,2,1,2,1,2,1}). \tag{5.3.18}$$

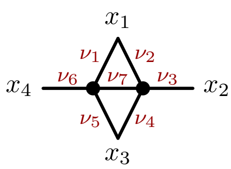
5.3.2 Double Ladder Ward Identity

In the previous section we discussed how to make a tensor decomposition of the vector integrals appearing in the right hand side of the Yangian Ward identities, and conformalise the resulting expressions. We proceed to present the refined Yangian Ward identities for the Basso–Dixon integrals up to four loops, starting with the double ladder.

Let us introduce operators A_j which raise the propagator power ν_j by 1, and also a dimension-raising operator d^+ which increases the dimension of an integral $D \to D+2$ and adds a factor of $1/\pi$ per loop. Then using (5.3.1)-(5.3.3) the Yangian constraint (5.2.24) for the double ladder can be written as

$$\begin{aligned}
&\left[2u\mathcal{D}^{21}_{uv} - x_{12}^2 \mathrm{d}^+ A_3(A_2(A_1 + A_5 + A_6 + A_7) - A_1 A_7)\right] \phi_{21} = 0, \\
&\left[2v\mathcal{D}^{21}_{vu} - x_{14}^2 \mathrm{d}^+ A_6(A_1(A_2 + A_3 + A_4 + A_7) - A_2 A_7)\right] \phi_{21} = 0,
\end{aligned} \tag{5.3.19}$$

where we recall the propagators are labelled as



. (5.3.20)

The above equations (5.3.19) are mapped into each other under $x_2 \leftrightarrow x_4$, which corresponds to $u \leftrightarrow v$. One can obtain two more equations of this type by swapping $x_1 \leftrightarrow x_3$. There are two ways to proceed to simplify (5.3.19). Firstly, one can use the manifestly conformal version (5.3.17) of $F_2(u,v)$. This leads directly to

$$\begin{aligned}
&[2u\mathcal{D}^{21}_{uv} - x_{12}^2 \mathrm{d}^+ A_{1,2,3}(x_{13}^2 A_5 + x_{14}^2 A_6)]\phi_{21} = 0, \\
&[2v\mathcal{D}^{21}_{vu} - x_{14}^2 \mathrm{d}^+ A_{1,2,6}(x_{13}^2 A_4 + x_{12}^2 A_3)]\phi_{21} = 0.
\end{aligned} \tag{5.3.21}$$

Secondly, (5.3.19) can be simplified by acting with $\partial_{x_{12}^2}$ on the Schwinger parametrisation (2.3.22)

$$\partial_{x_{12}^2}\left(\prod_{j=1}^{7}\int_0^\infty \frac{\mathrm{d}\alpha_j\alpha_j^{\nu_j-1}}{\Gamma_{\nu_j}}\right)\frac{\exp(-\mathcal{F}/\mathcal{U})}{\mathcal{U}^2} = -\left(\prod_{j=1}^{7}\int_0^\infty \frac{\mathrm{d}\alpha_j\alpha_j^{\nu_j-1}}{\Gamma_{\nu_j}}\right)\frac{\partial_{x_{12}^2}\mathcal{F}\exp(-\mathcal{F}/\mathcal{U})}{\mathcal{U}^3} \tag{5.3.22}$$

$$= -\left(\prod_{j=1}^{7}\int_0^\infty \frac{\mathrm{d}\alpha_j\alpha_j^{\nu_j-1}}{\Gamma_{\nu_j}}\right)\frac{(\alpha_1\alpha_2\alpha_3+\alpha_2\alpha_3\alpha_5+\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_3\alpha_7+\alpha_2\alpha_3\alpha_7)\exp(-\mathcal{F}/\mathcal{U})}{\mathcal{U}^3},$$

Where $\mathcal{U}$ and $\mathcal{F}$ are the Symanzik polynomials for the double ladder, given in (2.3.30) and (2.3.31). Using (5.3.22) for $\nu_j = 1$ we can write (5.3.19) as

$$\begin{aligned}\left[2u\mathcal{D}_{uv}^{21} + x_{12}^2\partial_{x_{12}^2} + 2x_{12}^2\mathrm{d}^+A_1A_3A_7\right]\phi_{21} &= 0,\\ \left[2v\mathcal{D}_{vu}^{21} + x_{14}^2\partial_{x_{14}^2} + 2x_{14}^2\mathrm{d}^+A_2A_6A_7\right]\phi_{21} &= 0.\end{aligned} \tag{5.3.23}$$

Here one should take the x_{ij}^2 in the Schwinger parametrisation to be all independent. Replacing $x_{12}^2\partial_{x_{12}^2}\phi(u,v) \to u\partial_u\phi$ and $x_{14}^2\partial_{x_{14}^2}\phi(u,v) \to v\partial_v\phi$ and dividing by 2 we have finally

$$\begin{aligned}\left[u\mathcal{D}_{uv}^{21} + \frac{u}{2}\partial_u + x_{12}^2\mathrm{d}^+A_1A_3A_7\right]\phi_{21} &= 0,\\ \left[v\mathcal{D}_{vu}^{21} + \frac{v}{2}\partial_v + x_{14}^2\mathrm{d}^+A_2A_6A_7\right]\phi_{21} &= 0.\end{aligned} \tag{5.3.24}$$

Interestingly, (5.3.24) reveals that $F_4(v,u) = x_{12}^2\mathrm{d}^+A_1A_3A_7\phi_{21}$, a 6-dimensional conformal integral, can be expressed as a (shifted) Yangian differential operator acting on the double ladder function ϕ_{21}. Both (5.3.21) and (5.3.24) are manifestly conformal representations of the Yangian Ward identities for the double ladder. We find that in general acting with ∂_u on $\phi_{\alpha\beta}$ leads to the most compact Ward identity. However, we still find the representation (5.3.21) useful in the context of separability in two dimensions, see section 5.4.

5.3.3 ℓ-Ladder

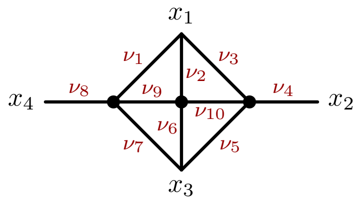
We can write down the Ward identities for the general ladders algorithmically. Recall the first double ladder equation with the insertion of a derivative ∂_u reads

$$\left[u\mathcal{D}_{uv}^{21} + \frac{u}{2}\partial_u + x_{12}^2\mathrm{d}^+A_{1,3,7}\right]\phi_{21} = 0, \tag{5.3.25}$$

where we abbreviate $A_{j_1,j_2,\dots,j_n} := A_{j_1}A_{j_2}\dots A_{j_n}$. We define the generalised triple ladder as

$$I_{31}^{\nu,D} = \int \frac{\mathrm{d}^Dx_a}{\pi^{D/2}}\frac{\mathrm{d}^Dx_b}{\pi^{D/2}}\frac{\mathrm{d}^Dx_c}{\pi^{D/2}}\frac{1}{(x_{a1}^{2\nu_1}x_{a3}^{2\nu_7}x_{a4}^{2\nu_8})x_{ab}^{2\nu_9}(x_{b1}^{2\nu_2}x_{b3}^{2\nu_6})x_{bc}^{2\nu_{10}}(x_{c1}^{2\nu_3}x_{c2}^{2\nu_4}x_{c3}^{2\nu_5})}, \tag{5.3.26}$$

which can be represented in diagrammatic form as



(5.3.27)

We define the conformal function in the unit-propagator four-dimensional limit as

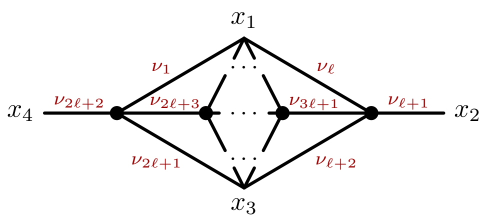
$$\phi_{31}(u,v) = \int \frac{\mathrm{d}^4 x_a}{\pi^2} \frac{\mathrm{d}^4 x_b}{\pi^2} \frac{\mathrm{d}^4 x_c}{\pi^2} \frac{x_{13}^6 x_{24}^2}{(x_{a1}^2 x_{a3}^2 x_{a4}^2) x_{ab}^2 (x_{b1}^2 x_{b3}^2) x_{bc}^2 (x_{c1}^2 x_{c2}^2 x_{c3}^2)}. \tag{5.3.28}$$

Applying the same methods on the Ward identity (5.2.32), the triple ladder equation takes the form

$$\left[u\mathcal{D}_{uv}^{31} + u\partial_u + 2x_{12}^2 \mathrm{d}^+ A_{1,4,9,10} + x_{12}^2 \mathrm{d}^+ A_{2,4,10}(A_1 + A_7 + A_8 + A_9)\right] \phi_{31} = 0. \tag{5.3.29}$$

The general ℓ-ladder has $3\ell + 1$ propagators, to which we assign propagator powers analogously to (5.3.20) and (5.3.27):

$I_{\ell 1}^{\nu,D} =$



(5.3.30)

from which the conformal function is defined as

$$\phi_{\ell 1} = x_{13}^{2\ell} x_{24}^2 I_{\ell 1}^{\nu_j = 1, D=4}. \tag{5.3.31}$$

The quadruple ladder equation takes the form

$$\begin{aligned} &\Big[u\mathcal{D}_{uv}^{41} + \frac{3}{2}u\partial_u + 3x_{12}^2 \mathrm{d}^+ A_{1,5,11,12,13} + 2x_{12}^2 \mathrm{d}^+ A_{2,5,12,13}(A_1 + A_9 + A_{10} + A_{11}) \\ &+ x_{12}^2 \mathrm{d}^+ A_{3,5,13}\big(A_1 A_2 + A_9 A_2 + A_{10} A_2 + A_{11} A_2 + A_1 A_8 + A_8 A_9 + A_8 A_{10} + \\ &A_1 A_{11} + A_8 A_{11} + A_9 A_{11} + A_{10} A_{11} + A_1 A_{12} + A_9 A_{12} + A_{10} A_{12} + A_{11} A_{12}\big)\Big] \phi_{41} = 0. \end{aligned} \tag{5.3.32}$$

Note the 15-term expression on the second two lines of (5.3.32) is the $\mathcal{U}$ polynomial for a double ladder with Feynman parameters $(\alpha_1, \alpha_2, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12})$, cf. (2.3.30). If $\mathcal{U}^{(j)}$ is the $\mathcal{U}$ polynomial for the j-ladder where we replace $\alpha \to A$ then we can rewrite (5.3.32) as

$$\left[u\mathcal{D}_{uv}^{41} + \frac{3}{2}u\partial_u + 3x_{12}^2 \mathrm{d}^+ A_{1,5,11,12,13} + 2x_{12}^2 \mathrm{d}^+ A_{2,5,12,13}\mathcal{U}_{1,9,10,11}^{(1)} + x_{12}^2 \mathrm{d}^+ A_{3,5,13}\mathcal{U}_{1,2,8,\dots,12}^{(2)}\right] \phi_{41} = 0 \tag{5.3.33}$$

where $\mathcal{U}_{1,9,10,11}^{(1)} = A_1 + A_9 + A_{10} + A_{11}$ is the $\mathcal{U}$ polynomial for the box, or 1-ladder. This pattern persists for the higher ladders

$$\left[u\mathcal{D}_{uv}^{\ell 1} + \frac{\ell - 1}{2}u\partial_u + x_{12}^2 \mathrm{d}^+ \sum_{j=0}^{\ell-2} (\ell - 1 - j) A_{j+1,\ell+1,2\ell+3+j,\dots,3\ell+1} \mathcal{U}_{1,\dots,j,2\ell-j+2,\dots,2\ell+j+2}^{(j)}\right] \phi_{\ell 1} = 0, \tag{5.3.34}$$

where we take $\mathcal{U}^{(0)} = 1$. This has been verified explicitly up to the sextuple ladder. One can apply the algorithm presented in section 5.3.1 to render the expression manifestly conformal and more compact. For example, the triple ladder equation (5.3.29) becomes

$$\left[u\mathcal{D}_{uv}^{31} + u\partial_u + 3x_{12}^2 \mathrm{d}^+ A_{1,4,9,10} + x_{12}^2 \mathrm{d}^+ A_{1,2,4,10}(x_{13}^3 A_7 + x_{14}^2 A_8)\right] \phi_{31} = 0, \tag{5.3.35}$$

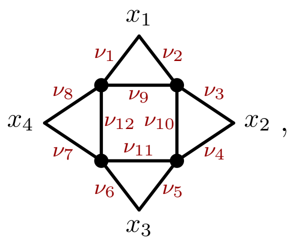
and the quadruple ladder equation (5.3.33) becomes

$$\Big[u\mathcal{D}_{uv}^{41} + \frac{3}{2}u\partial_u + 6x_{12}^2 \mathrm{d}^+ A_{1,5,11,12,13} + 3x_{12}^2 \mathrm{d}^+ A_{1,2,5,12,13}(x_{13}^2 A_9 + x_{14}^2 A_{10}) \tag{5.3.36}$$
$$+ x_{12}^2 \mathrm{d}^+ A_{3,5,13}(x_{13}^2 A_{2,9,11} + x_{13}^2 A_{1,8,11} + x_{14}^2 A_{2,10,11} + x_{13}^4 A_{1,2,8,9} + x_{13}^2 x_{14}^2 A_{1,2,8,10})\Big] \phi_{41} = 0.$$

5.3.4 Window

We also present the Yangian Ward identities for the conformal window integral, given by

$$I_{22}^{\nu,D} = \tag{5.3.37}$$



with the conformal function defined as

$$\phi_{22}(u,v) = x_{13}^4 x_{24}^4 I_{22}^{\nu_j=1,D=4}. \tag{5.3.38}$$

The Yangian equation takes the form

$$\begin{aligned}&[u\mathcal{D}_{uv}^{22} + u\partial_u + 2x_{12}^2 \mathrm{d}^+(A_{1,4,11,12}\mathcal{U}_{2,3,9,10}^{(1)} + A_{1,4,9,10}\mathcal{U}_{6,7,11,12}^{(1)}) + \\ &x_{12}^2 \mathrm{d}^+(\mathcal{U}_{6,7,11,12}^{(1)}(A_{1,3,9}(A_4 + A_5 + A_{10}) + A_{2,4,10}(A_1 + A_8 + A_9)) + \\ &(A_{1,3} + A_{2,4})(A_{11}(A_9(A_6 + A_7) + A_{12}(A_9 + A_{10}))) \\ &+ A_{2,4}(A_6 + A_7)(A_{10,12} - A_{9,11}))]\phi_{22} = 0.\end{aligned} \tag{5.3.39}$$

5.3.5 Momentum Space Conformal Anomaly

In this section we give a brief overview of how the above Yangian Ward identities can be mapped to the dual position space. Here the dual x- and p-coordinates are related by

$$p_j^\mu = x_j^\mu - x_{j+1}^\mu. \tag{5.3.40}$$

We will also refer to the p coordinates as momentum space coordinates, since that is how they are most commonly seen. To translate the Yangian generators into momentum space it suffices to follow the arguments of [163] with masses set to zero. There it was shown that the bi-local Yangian level-one generator $\widehat{\mathrm{P}}^\mu$ as given in (5.1.9), maps to the local special conformal generator in momentum space, i.e. on quantities obeying momentum conservation we have

$$\widehat{\mathrm{P}}^\mu \simeq \tfrac{i}{2}\bar{\mathrm{K}}^\mu = \tfrac{i}{2}\sum_{j=1}^{n} \bar{\mathrm{K}}_j^\mu. \tag{5.3.41}$$

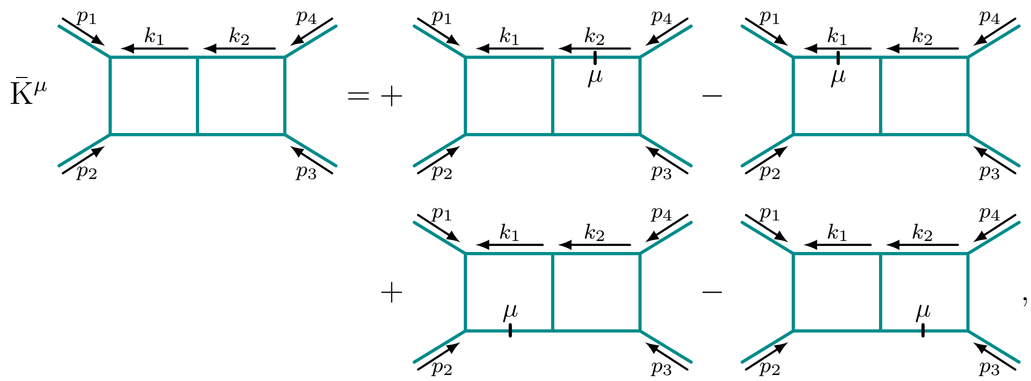
Here the $\simeq$ implies that by use of momentum conservation the momentum p_n is eliminated in the quantity acted on, which allows to drop derivatives ∂_{p_n}.[5] The generator $\bar{\mathrm{K}}^\mu_j$ forms part of the following momentum space representation of the conformal algebra:

$$\begin{aligned} \bar{\mathrm{P}}^\mu_j &= p^\mu_j\,, & \bar{\mathrm{K}}^\mu_j &= p^\mu_j\,\partial_{p_j}\cdot\partial_{p_j} - 2\,p_j\cdot\partial_{p_j}\partial^\mu_{p_j} - 2\bar{\Delta}_j\partial^\mu_{p_j}\,,\\ \bar{\mathrm{D}}_j &= p_j\cdot\partial_{p_j} + \bar{\Delta}_j\,, & \bar{\mathrm{L}}^{\mu\nu}_j &= p^\mu_j\partial^\nu_{p_j} - p^\nu_j\partial^\mu_{p_j}\,. \end{aligned} \tag{5.3.42}$$

Hence, we can alternatively understand the Yangian Ward identities for $\widehat{\mathrm{P}}^\mu$ as anomaly equations for a momentum space special conformal symmetry. Note that according to the prescription given in [163], with (5.2.5) for the Basso–Dixon integrals, we here use[6]

$$\bar{\Delta}_j = 1 + \tfrac{\alpha+\beta}{2}, \qquad j = 1,\dots,4. \tag{5.3.43}$$

Example: Double Ladder. The Yangian identities for the double ladder Feynman integral were given in (5.1.33). Due to the above arguments we can formulate them in momentum space:



, (5.3.44)

where the momentum flowing through unlabelled lines can be determined by momentum conservation. Here the respective momentum space Feynman integrals (green lines) are explicitly given by

$$I_{21} = \int\frac{\mathrm{d}^4k_1}{\pi^2}\int\frac{\mathrm{d}^4k_2}{\pi^2}\frac{1}{k_2^2(k_2+p_{12})^2(k_2+p_{123})^2(k_2-k_1)^2k_1^2(k_1+p_1)^2(k_1+p_{12}^2)}, \tag{5.3.45}$$

$$I^{\mu,2}_{21} = \int\frac{\mathrm{d}^4k_1}{\pi^2}\int\frac{\mathrm{d}^4k_2}{\pi^2}\frac{k_1^\mu}{k_2^2(k_2+p_{12})^2(k_2+p_{123})^2(k_2-k_1)^2k_1^4(k_1+p_1)^2(k_1+p_{12}^2)}, \tag{5.3.46}$$

where $p_{ij} := p_i + p_j$ and $p_{ijk} := p_i + p_j + p_k$. Note that in order to act with $\bar{\mathrm{K}}^\mu$ on the Feynman integral, we eliminate one (arbitrary) momentum using momentum

[5] Note that using the definition (5.1.9) of $\widehat{\mathrm{P}}$, the nth momentum is distinguished on the left hand side.

[6] Note that from [163] we deduce the formula $\bar{\Delta}_j = (\Delta_j + \Delta_{j+1})/2 + s_j - s_{j+1}$, which using (5.2.5) yields $\bar{\Delta}_j = (1+\frac{\alpha+\beta}{2}, 1+\frac{\alpha+\beta}{2}, 1+\frac{\alpha+\beta}{2}, -3+\frac{\alpha+\beta}{2})_j$. Eliminating the last momentum using momentum conservation, $\bar{\Delta}_4$ does not contribute to the action of $\bar{\mathrm{K}}^\mu$ and we can use the homogeneous prescription (5.3.43).

conservation, e.g. p_4. Then using (5.3.43) we find the momentum space version of (5.2.6),

$$\bar{\mathrm{K}}^\mu I_{\alpha\beta} = \frac{4}{p_{12}^{2\alpha} p_{23}^{2\beta}} \left[\left(\frac{p_1^\mu}{p_1^2} + \frac{p_3^\mu}{p_3^2} \right) u \mathcal{D}_{uv}^{\alpha\beta} \phi_{\alpha\beta}(u,v) + \left(\frac{p_2^\mu}{p_2^2} + \frac{p_4^\mu}{p_4^2} \right) v \mathcal{D}_{vu}^{\alpha\beta} \phi_{\alpha\beta}(u,v) \right], \quad (5.3.47)$$

where p_4 is understood to be replaced by $-p_1 - p_2 - p_3$. This can then be compared with the right hand side of (5.3.44).

5.3.6 *D*–Dimensional Generalisation

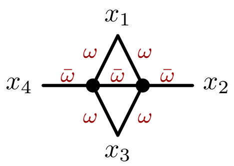
Above we have derived Yangian Ward identities for Basso–Dixon integrals with unit propagators in $D = 4$. In this section we consider a natural generalisation of that derivation for integrals with generalised propagator powers in D spacetime dimensions. Following the derivation of (5.1.31) naturally leads to a two-parameter family of D-dimensional Basso–Dixon integrals. Remarkably, these integrals can be identified with Basso–Dixon correlators in the D-dimensional fishnet theory proposed in [11], defined by the Lagrangian

$$\mathcal{L}_{\mathrm{FN}}^{\omega D} = N_c \,\mathrm{tr}\Big[-X(-\partial_\mu \partial^\mu)^\omega \bar{X} - Z(-\partial_\mu \partial^\mu)^{\frac{D}{2}-\omega} \bar{Z} + \xi^2 X Z \bar{X} \bar{Z} \Big], \quad (5.3.48)$$

with the anisotropy $\omega \in (0, D/2)$. Like the original fishnet theory, this is a non-unitary theory of scalars X, Z. It is also non-local, except for very special choices of D, ω, namely when $D \in 4\mathbb{N}$ for isotropic $\omega = D/4$. Despite this, the theory appears to be integrable, and the corresponding fishnet Feynman integrals still enjoy a Yangian symmetry. At the level of Feynman integrals, the vertical propagator powers are ω, and the horizontal ones are $\bar{\omega} := D/2 - \omega$.

We show how the derivation goes for the generalised double ladder integral. We consider the four-point correlator

$$\big\langle \mathrm{tr}(Z^2(x_1)\bar{X}(x_2)\bar{Z}^2(x_3)X(x_4)) \big\rangle_{\omega D} = \quad (5.3.49)$$



which is represented by the modified double ladder integral

$$I_{21}^{\omega D} = \int \frac{\mathrm{d}^D x_a}{\pi^{D/2}} \frac{\mathrm{d}^D x_b}{\pi^{D/2}} \frac{1}{(x_{a1}^{2\omega} x_{a3}^{2\omega} x_{a4}^{D-2\omega}) x_{ab}^{D-2\omega} (x_{b1}^{2\omega} x_{b2}^{D-2\omega} x_{b3}^{2\omega})} = \frac{1}{x_{13}^{4\omega} x_{24}^{D-2\omega}} \phi_{21}^{\omega D}(u,v). \quad (5.3.50)$$

We use the generic expression (5.1.28) with $\Delta_a = \omega$ to obtain the action of $\widehat{\mathrm{P}}^\mu$ on the four-point correlator (5.3.49):

$$\begin{aligned} H_{21,\omega D}^\mu :=& 2i\widehat{\mathrm{P}}^\mu|_{14} \big\langle \mathrm{tr}(Z^2(x_1)\bar{X}(x_2)\bar{Z}^2(x_3)X(x_4)) \big\rangle_{\omega D} \\ =& + \omega \big\langle \mathrm{tr}(Z(x_1)[\partial_1^\mu Z(x_1)]\bar{X}(x_2)\bar{Z}^2(x_3)X(x_4)) \big\rangle_{\omega D} \\ & - \omega \big\langle \mathrm{tr}([\partial_1^\mu Z(x_1)]Z(x_1)\bar{X}(x_2)\bar{Z}^2(x_3)X(x_4)) \big\rangle_{\omega D} \\ & + \omega \big\langle \mathrm{tr}(Z^2(x_1)\bar{X}(x_2)\bar{Z}(x_3)[\partial_3^\mu \bar{Z}(x_3)]X(x_4)) \big\rangle_{\omega D} \\ & - \omega \big\langle \mathrm{tr}(Z^2(x_1)\bar{X}(x_2)[\partial_3^\mu \bar{Z}(x_3)]\bar{Z}(x_3)X(x_4)) \big\rangle_{\omega D} . \end{aligned} \quad (5.3.51)$$

Let us first evaluate the right hand side of this expression. The correlator appearing in the second line of (5.3.51) can be represented by the integral

$$2\omega I^{\mu}_{21,\omega D} = 2\omega \int \frac{\mathrm{d}^D x_a}{\pi^{D/2}} \frac{\mathrm{d}^D x_b}{\pi^{D/2}} \frac{x^{\mu}_{b1}}{(x^{2\omega}_{a1} x^{2\omega}_{a3} x^{D-2\omega}_{a4}) x^{D-2\omega}_{ab} (x^{2(\omega+1)}_{b1} x^{D-2\omega}_{b2} x^{2\omega}_{b3})} \tag{5.3.52}$$

$$= -\frac{2\omega}{x^{4\omega}_{13} x^{D-2\omega}_{24}} \left(\frac{x^{\mu}_{12}}{x^2_{12}} F^{\omega D}_2(u,v) + \frac{x^{\mu}_{13}}{x^2_{13}} F^{\omega D}_3(u,v) + \frac{x^{\mu}_{14}}{x^2_{14}} F^{\omega D}_4(u,v) \right), \tag{5.3.53}$$

where the conformal vector decomposition (5.2.12) still holds in this case. The other three correlators are related to this expression by the transpositions $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$. Computing $H^{\mu}_{21,\omega D}$ in full we thus find

$$x^{4\omega}_{13} x^{D-2\omega}_{24} H^{\mu}_{21,\omega D} = -2\omega^2 \left(\frac{x^{\mu}_{12}}{x^2_{12}} + \frac{x^{\mu}_{34}}{x^2_{34}} \right) \left(F^{\omega D}_2(u,v) - F^{\omega D}_4(v,u)\right) - x_1 \leftrightarrow x_3. \tag{5.3.54}$$

In order to obtain the left hand side of (5.3.51) we act with $2i\widehat{\mathrm{P}}^{\mu}$ on the generic conformal integral of the form

$$I^{\omega D}_{21} = \frac{1}{x^{4\omega}_{13} x^{D-2\omega}_{24}} \phi^{\omega D}_{21}(u,v), \tag{5.3.55}$$

which yields

$$2i\widehat{\mathrm{P}}^{\mu} I^{\omega D}_{21} = \frac{-4}{x^{4\omega}_{13} x^{D-2\omega}_{24}} \left[\left(\frac{x^{\mu}_{12}}{x^2_{12}} + \frac{x^{\mu}_{34}}{x^2_{34}} \right) u \mathcal{D}^{21,\omega D}_{uv} + \left(\frac{x^{\mu}_{23}}{x^2_{23}} + \frac{x^{\mu}_{41}}{x^2_{41}} \right) v \mathcal{D}^{21,\omega D}_{vu} \right] \phi^{\omega D}_{21}(u,v). \tag{5.3.56}$$

Here the differential operator $\mathcal{D}^{21,\omega D}_{uv}$ takes the form

$$\mathcal{D}^{21,\omega D}_{uv} = \alpha\beta + (\alpha+\beta+1) v\partial_v + ((\alpha+\beta+1)u - \gamma)\partial_u + v^2 \partial_v^2 + (u-1)u\partial_u^2 + 2vu\partial_v\partial_u \tag{5.3.57}$$

with

$$\alpha = 2\omega, \qquad \beta = \tfrac{D}{2} - \omega, \qquad \gamma = 1 + \tfrac{\omega}{2}. \tag{5.3.58}$$

Comparing (5.3.54) and (5.3.56) we thus obtain the following Yangian Ward identity for the (ω, D) generalisation for the double ladder:

$$2u\mathcal{D}^{21,\omega D}_{uv} \phi^{\omega D}_{21} = \omega^2 (F^{\omega D}_2(u,v) - F^{\omega D}_4(v,u)), \tag{5.3.59}$$

and the same equation with $u \leftrightarrow v$. Performing a tensor decomposition as in section 2.3.2, this equation can be written in the form

$$\left(u\mathcal{D}^{21,\omega D}_{uv} + \frac{\omega}{2} u\partial_u + x^2_{12} \mathrm{d}^+ \omega^2 \left(\tfrac{D}{2} - \omega\right)^2 A_{1,3,7} \right) \phi^{\omega D}_{21} = 0. \tag{5.3.60}$$

Currently there is no known functional form for the conformal function $\phi^{\omega D}_{21}$. However, we can use the conformal Feynman parametrisations (A.1.3) and (A.1.5) to verify (5.3.60) numerically. There is a further representation of the Ward identity which is analogous to (5.3.21):

$$[2u\mathcal{D}^{21,\omega D}_{uv} - x^2_{12} \mathrm{d}^+ \omega^2 \left(\tfrac{D}{2} - \omega\right) A_{1,2,3} (\omega x^2_{13} A_5 + \left(\tfrac{D}{2} - \omega\right) x^2_{14} A_6)] \phi^{\omega D}_{21} = 0, \tag{5.3.61}$$

and the same with $x_2 \leftrightarrow x_4$.

Generalised ℓ-Ladder. The Yangian Ward identity (5.3.34) for the ladders generalises to the D-dimensional theory (5.3.48) as follows:

$$\Big[u\mathcal{D}^{\ell 1,\omega D}_{uv}+\frac{\ell-1}{2}\omega u\partial_u+\omega x_{12}^2\mathrm{d}^+\sum_{j=1}^{\ell-1}(\ell-j)\mathcal{A}_{j,\ell+1,2\ell+2+j,\dots,3\ell+1}\tilde{\mathcal{U}}^{(j-1)}_{1,\dots,j-1,2\ell-j+3,\dots,2\ell+j+1}\Big]\phi^{\omega D}_{\ell 1}=0 \tag{5.3.62}$$

where $\tilde{\mathcal{U}}^{(j)}$ is the $\mathcal{U}$ polynomial for the j-ladder where we replace $\alpha_j \to \mathcal{A}_j = \nu_j A_j$, and

$$\mathcal{D}^{\ell 1,\omega D}_{uv} = \alpha\beta + (\alpha+\beta+1)v\partial_v + ((\alpha+\beta+1)u-\gamma)\partial_u + v^2\partial_v^2 + (u-1)u\partial_u^2 + 2vu\partial_v\partial_u \tag{5.3.63}$$

with

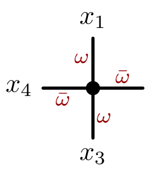
$$\alpha = \ell\omega, \qquad \beta = \tfrac{D}{2}-\omega, \qquad \gamma = 1+(\ell-1)\tfrac{\omega}{2}. \tag{5.3.64}$$

5.4 Separation of Variables in 2 Dimensions

In the previous section we derived the Yangian Ward identities for the ladders in the generalised fishnet theory $\mathcal{L}^{\omega D}_{\mathrm{FN}}$. Here we present the fact that these equations separate in two dimensions.[7] We begin with the Ward identity for the 2D box integral with anisotropy ω and conformal function $\phi_\omega(z,\bar{z})$, which has an exact Yangian symmetry. The separated Yangian equations for this function represent ordinary differential equations (ODEs), which can be solved straightforwardly. We then present the separated inhomogeneous equations for the two-dimensional double ladder.

5.4.1 Separated Ward Identities for the 2D Box

We consider the 2D Yangian invariant box integral with anisotropy ω:



$$= \int \frac{\mathrm{d}^2x_a}{\pi}\frac{1}{x_{a1}^{2\omega}x_{a2}^{2-2\omega}x_{a3}^{2\omega}x_{a4}^{2-2\omega}} = \frac{1}{x_{13}^{2\omega}x_{24}^{2-2\omega}}\phi_\omega(z,\bar{z}), \tag{5.4.1}$$

where here $\bar{\omega}=1-\omega$. The homogeneous Yangian Ward identities can be derived by using (5.3.62) for $\ell=1$. In terms of $z,\bar{z}$ these are

$$[\mathcal{D}^\omega_{z,i}-\mathcal{D}^\omega_{\bar{z},i}]\phi_\omega(z,\bar{z})=0, \qquad i=1,2. \tag{5.4.2}$$

Here the differential operators $\mathcal{D}^\omega_{z,i}$ take the explicit form

$$\begin{aligned}\mathcal{D}^\omega_{z,1} &= \omega(1-\omega)z + z(2z-1)\partial_z + z^2(z-1)\partial_z^2,\\ \mathcal{D}^\omega_{z,2} &= \omega(1-\omega)z + (2z^2-3z+1)\partial_z + z(z-1)^2\partial_z^2.\end{aligned} \tag{5.4.3}$$

These operators can be linearly combined into a separated form

$$\mathcal{D}^\omega_z := \frac{\bar{z}-1}{z-\bar{z}}\big[\mathcal{D}^\omega_{z,1}-\mathcal{D}^\omega_{\bar{z},1}\big] - \frac{\bar{z}}{z-\bar{z}}\big[\mathcal{D}^\omega_{z,2}-\mathcal{D}^\omega_{\bar{z},2}\big] = \omega(\omega-1)+(1-2z)\partial_z + z(1-z)\partial_z^2, \tag{5.4.4}$$

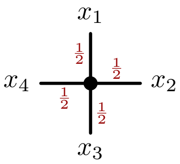
[7] The reason for this can be traced to the fact that the Euclidean conformal algebra in two dimensions splits $\mathfrak{so}(1,3)\simeq\mathfrak{sl}(2)\times\mathfrak{sl}(2)$, such that the corresponding Yangian algebra factorises [188].

and analogously for $z \leftrightarrow \bar{z}$. The function ϕ_ω thus satisfies the ODEs

$$\mathcal{D}_z^\omega \phi_\omega = 0, \qquad \mathcal{D}_{\bar{z}}^\omega \phi_\omega = 0. \tag{5.4.5}$$

5.4.2 Bootstrapping the 2D Box for $\omega = 1/2$: Elliptic K

The next goal is to solve the above differential equations, first for $\omega = 1/2$ and subsequently for general ω, in order to determine the 2D box integral. In two dimensions the isotropic box is given by (5.4.1) with $\omega = 1/2$:[8]



$$= \int \frac{\mathrm{d}^2 x_a}{\pi} \frac{1}{|x_{a1}||x_{a2}||x_{a3}||x_{a4}|} = \frac{1}{|x_{13}||x_{24}|}\phi(z,\bar{z}). \tag{5.4.6}$$

Here we have $|x_{ij}| = \sqrt{x_{ij}^2}$. Yangian invariance implies that $\phi(z,\bar{z})$ satisfies the separated equations (5.4.5) with $\omega = 1/2$:

$$\mathcal{D}_z\phi = \mathcal{D}_{\bar{z}}\phi = 0, \qquad \mathcal{D}_z = 1 + 4(2z-1)\partial_z + 4z(z-1)\partial_z^2. \tag{5.4.7}$$

Note that these equations are manifestly symmetric under $z \to 1-z$, $\bar{z} \to 1-\bar{z}$. These equations are ordinary differential equations which can be solved straightforwardly to find

$$\phi(z,\bar{z}) = f_1(\bar{z})K(z) + f_2(\bar{z})K(1-z), \tag{5.4.8}$$

and the same with $z \leftrightarrow \bar{z}$. Here the complete elliptic integral of the first kind is defined by

$$K(z) = \int_0^{\pi/2} \mathrm{d}\theta \frac{1}{\sqrt{1 - z\sin^2\theta}}. \tag{5.4.9}$$

Acting on (5.4.8) with $\mathcal{D}_{\bar{z}}$, and solving the resulting ordinary differential equation for $f_1(\bar{z})$ and $f_2(\bar{z})$, we conclude that in general $\phi(z,\bar{z})$ is a linear combination of four factorised Yangian invariants:

$$\phi(z,\bar{z}) = c_1K(z)K(\bar{z}) + c_2K(z)K(1-\bar{z}) + c_3K(1-z)K(\bar{z}) + c_4K(1-z)K(1-\bar{z}). \tag{5.4.10}$$

Fixing Constants. We can fix the constant prefactors c_i by using reality conditions and the permutation covariance of the integral (5.4.6). We first note some properties of the function $K(z)$. $K(z)$ has branch points at $z = 1$ and $z = \infty$, and we take the usual branch cut on $(1,\infty)$. It satisfies the identities[9]

$$K(1 - \tfrac{1}{z}) = \sqrt{z}K(1-z), \tag{5.4.11}$$

$$K(\tfrac{1}{z}) = \sqrt{z}(K(z) \mp iK(1-z)), \tag{5.4.12}$$

[8] Although the propagators $|x_{aj}|^{-1}$ are not those arising from two-dimensional quantum field theory, this integral still appears in conformal field theory contexts [32, 189].

[9] We consider the case where $\bar{z} = z^*$ and $\mathrm{Im}(z) \neq 0$, so that in particular $z \neq \bar{z}$.

where we take the negative sign if $\mathrm{Im}(z) > 0$ and the positive sign if $\mathrm{Im}(z) < 0$. We also have $K(\bar{z}) = K(z)^*$ if $\bar{z} = z^*$. For definiteness we take z to be the solution of $z\bar{z} = u$ and $(1-z)(1-\bar{z}) = v$ with positive imaginary part.

We now proceed to fix the constants c_i. Firstly, since (5.4.6) is a Euclidean, real-valued integral we must have $c_2 = c_3 \coloneqq c$. This is because $K(z)K(1-\bar{z}) = (K(1-z)K(\bar{z}))^*$, and this choice ensures that the imaginary parts in these terms cancel. Furthermore, (5.4.6) is invariant under the transposition of points $x_1 \leftrightarrow x_3$. This transposition maps $z \to 1-\bar{z}$ and $\bar{z} \to 1-z$. Therefore we should have

$$\phi(z,\bar{z}) = \phi(1-\bar{z}, 1-z), \tag{5.4.13}$$

which fixes $c_1 = c_4 \coloneqq \tilde{c}$. The transpositon $x_1 \leftrightarrow x_4$ implies the condition

$$\sqrt{u}\phi(z,\bar{z}) = \phi(\tfrac{1}{\bar{z}}, \tfrac{1}{z}). \tag{5.4.14}$$

This equation can be expanded

$$\begin{aligned}&\sqrt{z\bar{z}}\big[\tilde{c}(K(z)K(\bar{z}) + K(1-z)K(1-\bar{z})\big] + c\big[K(z)K(1-\bar{z}) + K(1-z)K(\bar{z}))\big] \\ &= \tilde{c}\big[K(\tfrac{1}{z})K(\tfrac{1}{\bar{z}}) + K(1-\tfrac{1}{z})K(1-\tfrac{1}{\bar{z}})\big] + c\big[K(\tfrac{1}{z})K(1-\tfrac{1}{\bar{z}}) + K(1-\tfrac{1}{z})K(\tfrac{1}{\bar{z}})\big],\end{aligned} \tag{5.4.15}$$

and using (5.4.11), (5.4.12) this gives the constraint

$$\tilde{c}(K(1-z)K(1-\bar{z}) + iK(1-z)K(\bar{z}) - iK(z)K(1-\bar{z})) = 0, \tag{5.4.16}$$

which is only consistent for generic $z, \bar{z}$ if $\tilde{c} = 0$. Overall we have fixed the solution up to an overall constant c. We fix the constant $c = 4/\pi$ using numerical input, so that the final result is

$$\phi(z,\bar{z}) = \frac{4}{\pi}[K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})]. \tag{5.4.17}$$

This result agrees with the one given in [190], obtained by a direct simplification of hypergeometric functions arising from their separation of variables approach, see also [189,191]. They use a slightly different conformal variable η, related to our variable z by

$$\eta = 1 - \frac{1}{z}, \qquad \bar{\eta} = 1 - \frac{1}{\bar{z}}. \tag{5.4.18}$$

We chose to use z, $\bar{z}$ since the expression for the conformal function (5.4.17) takes a slightly simpler form in terms of these variables.

Cuts and Discontinuities. Let us consider discontinuities of the solution (5.4.17), which are related to cuts of the integral [186]. Cuts of the integral obey the same Yangian equations [172, 2], and so these discontinuities can be expressed as a linear combination of the four Yangian invariants $K(z)K(\bar{z}), K(1-z)K(\bar{z}), K(z)K(1-\bar{z})$, and $K(1-z)K(1-\bar{z})$.

The conformal function (5.4.17) is single-valued in $z, \bar{z}$ if $\bar{z} = z^*$, which is true for Euclidean kinematics. It can be thought of as a 2D version of the Bloch–Wigner function (4.1.28), which is similarly single-valued. However, as discussed is chapter 4,

z and $\bar{z}$ can also be independent real numbers in Minkowskian kinematics. Conformal invariance is broken globally in this case, and the result for the integral in kinematic regions away from the Euclidean sheet can be obtained by taking discontinuities of (5.4.17).

We proceed to calculate these discontinuities. For this purpose we only need to calculate the discontinuity of $K(z)$ across the cut on $(1,\infty)$. We have

$$\mathrm{disc}_1 K(z) := K(z+i\epsilon) - K(z-i\epsilon) = 2iK(1-z), \quad z>1. \tag{5.4.19}$$

Using this we can fix $\bar{z} \in \mathbb{C}, z>1$ and calculate

$$\mathrm{disc}_1 \phi(z,\bar{z}) := \phi(z+i\epsilon,\bar{z}) - \phi(z-i\epsilon,\bar{z}) = -\frac{8}{\pi i} K(1-z)K(1-\bar{z}). \tag{5.4.20}$$

Similarly fix $z<0$ and $\bar{z} \in \mathbb{C}$. Then

$$\mathrm{disc}_0 \phi(z,\bar{z}) := \phi(z+i\epsilon,\bar{z}) - \phi(z-i\epsilon,\bar{z}) = \frac{8}{\pi i} K(z)K(\bar{z}). \tag{5.4.21}$$

Notably, both $\mathrm{disc}_1\phi(z,\bar{z})$ and $\mathrm{disc}_0\phi(z,\bar{z})$ are solutions to the Yangian PDEs (5.4.7), as expected. We can also consider double discontinuities of $\phi(z,\bar{z})$. Fix now $\bar{z}<0, z>1$. Then we have

$$\mathrm{disc}_{\bar{0}}\mathrm{disc}_1 \phi(z,\bar{z}) := \mathrm{disc}_1\phi(z,\bar{z}+i\epsilon) - \mathrm{disc}_1\phi(z,\bar{z}-i\epsilon) = \frac{16}{\pi} K(1-z)K(\bar{z}), \tag{5.4.22}$$

and similarly if we fix $\bar{z}>1, z<0$, we can find

$$\mathrm{disc}_{\bar{1}}\mathrm{disc}_0 \phi(z,\bar{z}) = \frac{16}{\pi} K(1-\bar{z})K(z). \tag{5.4.23}$$

Again, the double discontinuities are solutions to the Yangian PDEs (5.4.7).

Transcendentality. For the isotropic box in 2D we identified four Yangian invariants g_i:

$$\begin{aligned} g_1 &= K(z)K(\bar{z}), & g_2 &= K(1-z)K(\bar{z}), \\ g_3 &= K(z)K(1-\bar{z}), & g_4 &= K(1-z)K(1-\bar{z}). \end{aligned} \tag{5.4.24}$$

In comparison, for the 4D conformal box we have the four Yangian invariants $f_i := \tilde{f}_i/(z-\bar{z})$:

$$\begin{aligned} \tilde{f}_1 &= 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + (\log z + \log\bar{z})(\log(1-z) - \log(1-\bar{z})), \\ \tilde{f}_2 &= \log z - \log\bar{z}, \\ \tilde{f}_3 &= \log(1-z) - \log(1-\bar{z}), \\ \tilde{f}_4 &= 1. \end{aligned} \tag{5.4.25}$$

The respective conformal functions are proportional to the Yangian invariants as follows:

$$\phi_{\mathrm{2D}} \sim \frac{g_2+g_3}{\pi}, \qquad \phi_{\mathrm{4D}} \sim f_1. \tag{5.4.26}$$

	$\phi_{2\mathrm{D}}$	$\phi_{4\mathrm{D}}$
$\mathrm{disc}_1\phi$	$-\frac{1}{\pi i}g_4$	$+\pi i f_2$
$\mathrm{disc}_0\phi$	$+\frac{1}{\pi i}g_1$	$+\pi i f_3$
$\mathrm{disc}_{\bar{0}}\mathrm{disc}_1\phi$	$+\frac{1}{\pi}g_2$	$-(\pi i)^2 f_1$
$\mathrm{disc}_{\bar{1}}\mathrm{disc}_0\phi$	$+\frac{1}{\pi}g_3$	$+(\pi i)^2 f_1$

Table 5.1: Discontinuities of isotropic box functions $\phi_{2\mathrm{D}}$ and $\phi_{4\mathrm{D}}$ in two and four dimensions, ignoring numerical constants.

We note the differences in the transcendentality of the Yangian invariants in each case. Here π is assigned transcendentality 1. While polylogs of order n have transcendentality n, the elliptic K integral has been argued in [192] to have transcendentality 1, because

$$\lim_{x\to 0} K(x) = \frac{\pi}{2}. \tag{5.4.27}$$

Therefore each g_i has transcendentality 2, while the f_i have transcendentality ranging from 0 to 2. Furthermore, the conformal functions $\phi_{2\mathrm{D}}$ and $\phi_{4\mathrm{D}}$ have transcendentality 1 and 2 respectively. In table 5.1 we present the discontinuities of these functions. We notice a curious difference between the two cases. Taking discontinuities of $\phi_{4\mathrm{D}}$ reduces the functional transcendentality by 1 in each case, such that any further discontinuities will simply vanish. However, since all the Yangian invariants for the 2D case have the same transcendentality 2, one can continue to take discontinuities of $\phi_{2\mathrm{D}}$, and remain within the family of functions g_i. This shows, at least in this very simplified setting, a fundamental difference between taking discontinuities of polylogs and elliptic integrals.

5.4.3 Bootstrapping the 2D Box for Generic ω: Legendre P and Q

For generic ω the separated differential equations for the box conformal function ϕ_ω read

$$0 = \big[\omega(\omega-1) + (1-2z)\partial_z + z(1-z)\partial_z^2\big]\phi_\omega, \tag{5.4.28}$$
$$0 = \big[\omega(\omega-1) + (1-2\bar{z})\partial_{\bar{z}} + \bar{z}(1-\bar{z})\partial_{\bar{z}}^2\big]\phi_\omega, \tag{5.4.29}$$

which are still symmetric under $z \to 1-z$ and $\bar{z} \to 1-\bar{z}$, respectively. The above ODEs are solved by the Legendre functions

$$\phi_\omega(z,\bar{z}) = c_1(\bar{z})P_{\omega-1}(2z-1) + c_2(\bar{z})Q_{\omega-1}(2z-1), \tag{5.4.30}$$

and similarly for $z \leftrightarrow \bar{z}$. We note that

$$P_{\omega-1}(2z-1)\big|_{\omega\to\frac{1}{2}} = \tfrac{2}{\pi}K(1-z), \tag{5.4.31}$$
$$Q_{\omega-1}(2z-1)\big|_{\omega\to\frac{1}{2}} = K(z), \tag{5.4.32}$$

from which we see that Legendre P has an apparent lower transcendentality than Legendre Q. With the above solutions for the separated equations, the generic ansatz for the full parametric box reads

$$\begin{aligned}\phi_\omega(z,\bar{z}) = &+ c_1 P_{\omega-1}(2z-1)P_{\omega-1}(2\bar{z}-1) + c_2 P_{\omega-1}(2z-1)Q_{\omega-1}(2\bar{z}-1)\\ &+ c_3 Q_{\omega-1}(2z-1)P_{\omega-1}(2\bar{z}-1) + c_4 Q_{\omega-1}(2z-1)Q_{\omega-1}(2\bar{z}-1).\end{aligned} \tag{5.4.33}$$

Using numerical input we can fix these constants, and find

$$\phi_\omega = \frac{4}{\pi}\Big[Q_{\omega-1}(\tau)\tilde{P}_{\omega-1}(\bar{\tau}) + Q_{\omega-1}(\bar{\tau})\tilde{P}_{\omega-1}(\tau) - 2\cot(\pi\omega)\tilde{P}_{\omega-1}(\tau)\tilde{P}_{\omega-1}(\bar{\tau})\Big], \tag{5.4.34}$$

where $\tau \coloneqq 2z-1$ and

$$\tilde{P}_{\omega-1}(\tau) \coloneqq \frac{\pi}{2}P_{\omega-1}(\tau). \tag{5.4.35}$$

Using (5.4.31) and (5.4.32) this reduces to (5.4.17) in the limit $\omega \to 1/2$. This representation differs from the hypergeometric one given in [190], and this one is perhaps slightly more natural since it expresses a one-parameter integral in terms of one-parameter functions $P_{\omega-1}$ and $Q_{\omega-1}$.

Discontinuities. Again we can verify that the discontinuities of the solution (5.4.34) are given by linear combinations of Yangian invariants. We first give the discontinuities of $\tilde{P}_{\omega-1}(2z-1)$ and $Q_{\omega-1}(2z-1)$:

$$\begin{aligned}&\mathrm{disc}_1 Q_{\omega-1}(2z-1) = 2i\tilde{P}_{\omega-1}(2z-1),\\ &\mathrm{disc}_1 \tilde{P}_{\omega-1}(2z-1) = 0,\\ &\mathrm{disc}_0 Q_{\omega-1}(2z-1) = 2i\cos(\pi\omega)(\cos(\pi\omega)\tilde{P}_{\omega-1}(2z-1) - \sin(\pi\omega)Q_{\omega-1}(2z-1)),\\ &\mathrm{disc}_0 \tilde{P}_{\omega-1}(2z-1) = 2i\sin(\pi\omega)(\cos(\pi\omega)\tilde{P}_{\omega-1}(2z-1) - \sin(\pi\omega)Q_{\omega-1}(2z-1)).\end{aligned} \tag{5.4.36}$$

Using these we have

$$\mathrm{disc}_1\phi_\omega(z,\bar{z}) = -\frac{8}{\pi i}\tilde{P}_{\omega-1}(2z-1)\tilde{P}_{\omega-1}(2\bar{z}-1), \tag{5.4.37}$$

and

$$\begin{aligned}\mathrm{disc}_0\phi_\omega(z,\bar{z}) = \frac{8}{\pi i}\Big[&\beta_{1,\omega}\tilde{P}_{\omega-1}(2z-1)\tilde{P}_{\omega-1}(2\bar{z}-1) + \beta_{2,\omega}\tilde{P}_{\omega-1}(2z-1)Q_{\omega-1}(2\bar{z}-1)\\ &+ \beta_{3,\omega}Q_{\omega-1}(2z-1)\tilde{P}_{\omega-1}(2\bar{z}-1) + \beta_{4,\omega}Q_{\omega-1}(2z-1)Q_{\omega-1}(2\bar{z}-1)\Big],\end{aligned} \tag{5.4.38}$$

where the coefficient functions $\beta_{i,\omega}$ are given by

$$\beta_{1,\omega} = \cos^2(\pi\omega), \qquad \beta_{2,\omega} = \beta_{3,\omega} = -\sin(\pi\omega)\cos(\pi\omega), \qquad \beta_{4,\omega} = \sin^2(\pi\omega). \tag{5.4.39}$$

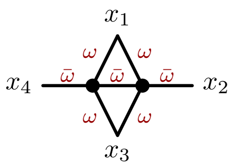
In the limit $\omega \to 1/2$, equations (5.4.37) and (5.4.38) reduce to (5.4.20) and (5.4.21), respectively. The double discontinuities take the form

$$\begin{aligned}\mathrm{disc}_{\bar{0},1}\phi_\omega(z,\bar{z}) &= -\tfrac{16\sin(\pi\omega)}{\pi}\tilde{P}_{\omega-1}(2z-1)\Big[\cos(\pi\omega)\tilde{P}_{\omega-1}(2\bar{z}-1) - \sin(\pi\omega)Q_{\omega-1}(2\bar{z}-1)\Big],\\ \mathrm{disc}_{\bar{1},0}\phi_\omega(z,\bar{z}) &= -\tfrac{16\sin(\pi\omega)}{\pi}\tilde{P}_{\omega-1}(2\bar{z}-1)\Big[\cos(\pi\omega)\tilde{P}_{\omega-1}(2z-1) - \sin(\pi\omega)Q_{\omega-1}(2z-1)\Big],\end{aligned}$$

where $\mathrm{disc}_{\bar{i},j} \coloneqq \mathrm{disc}_{\bar{i}}\mathrm{disc}_j$, which reduce to (5.4.22) and (5.4.23) as $\omega \to \frac{1}{2}$.

5.4.4 2D Double Ladder

Consider the Yangian equations for the double ladder



(5.4.40)

as given in (5.3.61) in $D = 2$ with generic ω:

$$\left[2u\mathcal{D}_{uv}^{21,\omega D} - A_{uv}\mathrm{d}^+\right]\phi_{21}^{\omega D} = 0, \tag{5.4.41}$$

$$\left[2v\mathcal{D}_{vu}^{21,\omega D} - A_{vu}\mathrm{d}^+\right]\phi_{21}^{\omega D} = 0. \tag{5.4.42}$$

Here we have introduced the shorthand

$$A_{uv} := x_{12}^2\omega^2(\tfrac{D}{2} - \omega)A_{1,2,3}\big(\omega x_{13}^2 A_5 + (\tfrac{D}{2} - \omega)x_{14}^2 A_6\big), \tag{5.4.43}$$

$$A_{vu} := x_{14}^2\omega^2(\tfrac{D}{2} - \omega)A_{1,2,6}\big(\omega x_{13}^2 A_4 + (\tfrac{D}{2} - \omega)x_{12}^2 A_3\big). \tag{5.4.44}$$

Now we switch to $z, \bar{z}$ such that

$$\left[4\nu(1-\nu) + (\nu+2)(2z-1)\partial_z + 2z(z-1)\partial_z^2\right]\phi(z,\bar{z}) = \big(\tfrac{1}{1-z}A_{vu} + \tfrac{1}{z}A_{uv}\big)\mathrm{d}^+\phi(z,\bar{z}), \tag{5.4.45}$$

$$\left[4\nu(1-\nu) + (\nu+2)(2\bar{z}-1)\partial_{\bar{z}} + 2\bar{z}(\bar{z}-1)\partial_{\bar{z}}^2\right]\phi(z,\bar{z}) = \big(\tfrac{1}{1-\bar{z}}A_{vu} + \tfrac{1}{\bar{z}}A_{uv}\big)\mathrm{d}^+\phi(z,\bar{z}). \tag{5.4.46}$$

Again we see that the equations separate into a z and $\bar{z}$ dependent piece. However, the operators A_{uv} and A_{vu} including the shifts of propagator powers are a priori only defined on a Feynman integral. It is thus not obvious how to solve or interpret these equations as purely mathematical identities for some generic functions in analogy to the above box equations in 2D. We leave the further exploration of this interesting point for future endeavours, and conclude this chapter.

Chapter 6

The Spectral Problem for Dynamical Fishnet Theory

In this chapter we summarise the recent progress made in the spectral problem of strongly-twisted $\mathcal{N}=4$ super Yang–Mills theory. This problem is well-understood in the case of full $\mathcal{N}=4$ SYM, where the one-loop dilatation operator takes the form of a $\mathfrak{psu}(2,2|4)$ spin chain [135]. At higher loops the dilatation operator can be expressed in terms of higher-range spin chains, see for example [193, 142]. The operator for the strongly-twisted theory was first investigated in [194]. There it was calculated in the fishnet case by an explicit Feynman diagrammatic calculation up to four loops for a diagonalisable sector of operators. The anomalous dimensions can be calculated by solving a set of asymptotic Bethe ansatz equations.

The study of the one-loop dilatation operator in strongly-twisted $\mathcal{N}=4$ SYM was initiated in [9]. This operator was calculated by explictly taking the double-scaling limit of the dilatation operator of β-twisted $\mathcal{N}=4$ [195, 169]. It was found that this operator is non-diagonalisable in many operator sectors. As described in section 2.2.5, this leads to logarithms in the corresponding two-point functions for these sectors, which reflects the fact that the strongly-twisted theory is a logarithmic CFT. In the non-diagonalisable sectors, the dilatation operator has a rich structure of Jordan cells. The sizes and multiplicities of these cells determine the exact form of the logarithms appearing in the two-point functions, cf. (2.2.88) and (2.2.89).

In [9] several diagonalisable sectors of operators were identified, and the corresponding dilatation operator could be diagonalised by means of a Bethe ansatz. However, several sectors of non-diagonalisable operators were also found, and the description of the corresponding Jordan block spectra by means of integrability or combinatorics was unclear. To elucidate this problem, the authors of [6] studied a simple non-diagonalisable sector of operators in the full strongly γ-twisted theory, i.e. the dynamical fishnet theory (6.2.1), namely the sector of three holomorphic scalars ϕ_1, ϕ_2, ϕ_3. In this sector the one-loop dilatation operator takes the form of a generically non-diagonalisable spin chain depending on three couplings ξ_1, ξ_2, ξ_3, which was dubbed the *eclectic* spin chain. In [6] it was pointed out that taking a naive limit of the eigenvectors of the finitely γ-twisted model obtained from the Bethe ansatz fails to describe the (generalised) eigenvectors of the eclectic spin chain. In fact, this limit only reproduces the most trivial eigenvector, called the *locked* state. The limit can be modified to exhaust more of the generalised eigenvectors [196, 197], by taking careful linear combinations of eigenvectors of the finitely-twisted model, although this approach does not directly appeal to integrability.

The eclectic Hamiltonian is still integrable, however, at least in the sense that it can be derived from an R-matrix which satisfies the quantum Yang–Baxter equation. It is still an open problem how to derive the generalised eigenvectors of the eclectic spin chain directly from integrability. It is not even clear that the eclectic model deserves to be called integrable. Even though the transfer matrix $t(u)$ built from its R-matrix constitutes a one-parameter family of commuting operators, the expansion of $\log t(u)$ around a special point (the point where the R-matrix reduces to the permutation operator, in this case $u = 0$) is finite, i.e. it is simply a polynomial in u. Therefore it is not clear that there are 'enough' commuting operators to guarantee its integrability.

It was conjectured in [6] that a simpler version of the eclectic spin chain, namely the case with $\xi_1 = \xi_2 = 0$, reproduces the Jordan block spectrum of the full eclectic model for special 'filling conditions' i.e. constraints on the occupancy numbers of the fields ϕ_1, ϕ_2, ϕ_3. Looking towards its solution this is encouraging, as the resulting *hypereclectic* spin chain is combinatorially much less intricate than the eclectic spin chain. A detailed combinatorial analysis of the hypereclectic spin chain was carried out in [3], where we found an exact solution for the Jordan block spectrum of the hypereclectic model. We obtained an elegant generating function for the spectrum of Jordan blocks. It is reminiscent of a partition function, since it can be obtained by computing a trace over the state space

$$\mathcal{Z}(q) = \operatorname{tr} q^{\hat{S}'} , \tag{6.0.1}$$

where $\hat{S}'$ is a certain counting operator, which is diagonal in the canonical basis of tensor product states of the spin chain. It uniquely encodes in full generality the sizes and multiplicities of the Hamiltonian's Jordan block decomposition:

$$\mathcal{Z}(q) = \sum_j N_j [j]_q = N_1\, q^0 + N_2 \left(q^{-\frac{1}{2}} + q^{\frac{1}{2}}\right) + N_3 \left(q^{-1} + q^0 + q^1\right) + \dots , \tag{6.0.2}$$

where N_j is the number of Jordan blocks of length j, and $[j]_q$ is a q-analog of j, cf. (6.3.41). It is easy to see that the $\{N_j\}$ are indeed uniquely fixed once one knows $\mathcal{Z}(q)$. We also derive formulas expressing $\mathcal{Z}(q)$ more explicitly than (6.0.1) in terms of q-binomial coefficients. For example, for the case corresponding to the fishnet interaction Lagrangian (3.3.2), with $L-M$ fields ϕ_1, $M-1$ fields ϕ_2, and a single, non-interacting third field ϕ_3, we find for the one-loop spectrum of Jordan blocks in the cyclic sector the (shifted) q-binomial coefficients

$$Z_{L,M}(q) = \begin{bmatrix} L-1 \\ M-1 \end{bmatrix}_q = \prod_{k=1}^{M-1} \frac{q^{\frac{L-k}{2}} - q^{-\frac{L-k}{2}}}{q^{\frac{k}{2}} - q^{-\frac{k}{2}}} \,. \tag{6.0.3}$$

In this section we describe in detail this combinatorial solution of the eclectic spin chain, which represents the most recent progress in understanding the one-loop dilatation operator in strongly twisted $\mathcal{N} = 4$ SYM. We firstly review non-diagonalisable matrices and Jordan normal form. We then introduce the (hyper)eclectic spin chain, discussing its basic structure and integrability. We finally describe our approach to fully enumerate the Jordan block spectrum of the hypereclectic spin chain in terms of a generating function $\mathcal{Z}(q)$.

6.1 Jordan Blocks and Jordan Normal Form

An important class of $n \times n$ complex matrices $M \in \mathrm{Mat}_n(\mathbb{C})$ are Hermitian matrices, which satisfy

$$M^\dagger = M, \tag{6.1.1}$$

where $M^\dagger$ is the Hermitian conjugate of M. Such matrices are especially relevant in the context of (finite-dimensional) quantum mechanics, where they represent observables. Via the spectral theorem, there exists an invertible matrix $B \in \mathrm{Mat}_n(\mathbb{C})$, such that

$$B^{-1}MB = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \tag{6.1.2}$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the (possibly degenerate) eigenvalues of M. In other words, M is diagonalisable by a change of basis B. The vector space $\mathbb{C}^n$ splits into one-dimensional subspaces, each spanned by an eigenvector $v_i \in \mathbb{C}^n$, $i = 1, 2, \ldots, n$, which satisfy

$$Mv_i = \lambda_i v_i. \tag{6.1.3}$$

In non-unitary theories such as strongly twisted $\mathcal{N} = 4$ SYM, non-hermitian operators can appear, which are not necessarily diagonalisable. For example, the matrix

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{6.1.4}$$

is not diagonalisable. M_1 has a single eigenvalue $\lambda = 1$ with algebraic multiplicity 2. However, the eigenspace corresponding to this eigenvalue is only one-dimensional:

$$\dim \ker(M_1 - \mathbb{I}) = 1. \tag{6.1.5}$$

Therefore there is only a single eigenvector, given by

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (M_1 - \mathbb{I})e_1 = 0, \tag{6.1.6}$$

and so M_1 is not diagonalisable. The standard basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ is a *Jordan chain* of length two, corresponding to the eigenvalue $\lambda = 1$. This is because

$$(M_1 - \mathbb{I})e_2 = e_1, \qquad (M_1 - \mathbb{I})e_1 = 0. \tag{6.1.7}$$

As such the matrix M_1 is already in *Jordan normal form*, and is a Jordan block of size two $M_1 = J_2(1)$, see the definition (6.1.9) below. The number of Jordan blocks corresponding to an eigenvalue λ of a complex matrix M is called the *geometric multiplicity* of λ, and is equivalently calculated as $\dim \ker(M - \lambda\mathbb{I})$.

In general, any matrix $M \in \mathrm{Mat}_n(\mathbb{C})$ can be brought uniquely to Jordan normal form by a change of basis:

$$B^{-1}MB = \begin{pmatrix} J_{l_1}(\lambda_{i_1}) & & \\ & \ddots & \\ & & J_{l_p}(\lambda_{i_p}) \end{pmatrix} := J_{l_1}(\lambda_{i_1}) \oplus \cdots \oplus J_{l_p}(\lambda_{i_p}), \tag{6.1.8}$$

where $l_1 + \cdots + l_p = n$ and each $J_l(\lambda)$ is the Jordan block of size l corresponding to the eigenvalue λ:

$$J_l(\lambda) = \begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix}. \tag{6.1.9}$$

We note that for each Jordan block there is a Jordan chain $v_1, \ldots, v_l$, such that

$$(J_l(\lambda) - \lambda\mathbb{I})v_k = v_{k-1}, \qquad k = 2, \ldots, l, \tag{6.1.10}$$

$$(J_l(\lambda) - \lambda\mathbb{I})v_1 = 0. \tag{6.1.11}$$

Henceforth we will refer to v_1 as the *top state* of the Jordan block. We note that the top state of $J_l(\lambda)$ is an eigenvector with eigenvalue λ. $v_2, \ldots, v_l$ are *generalised* eigenvectors of rank $2, \ldots, l$. Generalised eigenvectors of rank 1 are simply eigenvectors. If all of the Jordan blocks are of size $l = 1$, then M is diagonalisable.

There is a well-established algorithm for computing the Jordan normal form of matrices, found in any textbook on linear algebra, for example [198]. It consists of finding, for each eigenvalue λ, the vectors in the kernel of $(M - \lambda\mathbb{I})^k$ but not in $(M - \lambda\mathbb{I})^{k-1}$, until $(M - \lambda\mathbb{I})^k$ becomes the zero matrix. In particular, the generalised eigenvectors of rank k span the space

$$U_k(\lambda)/U_{k-1}(\lambda), \qquad U_k(\lambda) \coloneqq \ker(M - \lambda\mathbb{I})^k, \tag{6.1.12}$$

where / refers to the usual quotient of vector spaces. From this we can easily deduce that the number of Jordan blocks of length l corresponding to the eigenvalue λ can be computed as

$$N_l(\lambda) = 2\dim U_l(\lambda) - \dim U_{l+1}(\lambda) - \dim U_{l-1}(\lambda). \tag{6.1.13}$$

Nilpotent Matrices. In the next section we are particularly interested in *nilpotent* matrices. A matrix $M \in \mathrm{Mat}_n(\mathbb{C})$ is nilpotent if there exists some positive integer k such that

$$M^k = 0. \tag{6.1.14}$$

In this case all the eigenvalues of M are zero. Indeed, let v be an eigenvector of M with eigenvalue λ:

$$Mv = \lambda v. \tag{6.1.15}$$

Then applying M to (6.1.15) $k-1$ times we see that

$$M^k v = \lambda^k v = 0. \tag{6.1.16}$$

Since eigenvectors are non-zero by definition, this implies that $\lambda = 0$. If M is non-zero this implies that it is non-diagonalisable. Therefore given any non-zero nilpotent matrix $M \in \mathrm{Mat}_n(\mathbb{C})$, there exists an invertible matrix B such that

$$B^{-1}MB = \bigoplus_{i=1}^{l} J_{n_i}(0), \tag{6.1.17}$$

where $n_1 + \cdots + n_l = n$ and $l \neq n$.

6.2 The (Hyper)eclectic Spin Chain

In this section we define the one-loop dilatation operator for the dynamical fishnet theory, with interaction Lagrangian

$$\begin{aligned}\mathcal{L}^{\text{int}}_{\text{DFN}} = N_c \text{tr}\left(\xi_1^2 \phi_2^\dagger \phi_3^\dagger \phi_2 \phi_3 + \xi_2^2 \phi_3^\dagger \phi_1^\dagger \phi_3 \phi_1 + \xi_3^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\right) \\ + N_c \text{tr}\left(i\sqrt{\xi_2 \xi_3}(\psi^3 \phi_1 \psi^2 + \bar{\psi}_3 \phi_1^\dagger \bar{\psi}_2) + \text{cyclic}\right),\end{aligned} \tag{6.2.1}$$

for a particular 3-scalar sector of operators. This model becomes the bi-scalar fishnet theory when $\xi_1 = \xi_2 = 0, \xi_3 \equiv \xi$, cf. section 3.3. The one-loop dilatation operator for this sector has been dubbed the *eclectic* spin chain H_{ec} [9]. As described below, this Hamiltonian commutes with the translation operator U, and therefore it can be diagonalised in separate cyclicity classes, defined by their eigenvalue under the translation operator. Notably, the hypereclectic spin chain is non-diagonalisable when we consider sectors of operators containing all three scalar fields. Therefore the spectral problem is replaced with the problem of counting and classifying the Jordan blocks of the model. The eclectic Hamiltonian is integrable, in the sense that it can be derived from an R-matrix which satisfies the Yang–Baxter equation. We demonstrate this fact, and discuss the notion of an integrable, non-diagonalisable model.

6.2.1 Hamiltonian

We consider local single-trace operators in the holomorphic 3-scalar sector of the theory (6.2.1)

$$\mathcal{O}_{j_1, j_2, \ldots, j_L}(x) = \text{tr}\left(\phi_{j_1} \phi_{j_2} \cdots \phi_{j_L}(x)\right), \qquad j_i \in \{1, 2, 3\}. \tag{6.2.2}$$

In $\mathcal{N} = 4$ SYM the one-loop dilatation operator in the analogous sector can be written as a sum over permutation operators and enjoys an $\mathfrak{su}(3)$ symmetry [136]. In the strongly twisted theory (6.2.1) this symmetry is broken and the one-loop dilatation operator $H_{\text{ec}} : (\mathbb{C}^3)^{\otimes L} \to (\mathbb{C}^3)^{\otimes L}$ is a sum over *chiral* permutation operators [6]

$$H_{\text{ec}} = H_1 + H_2 + H_3 = \sum_{i=1}^{L} \left(\xi_1 \mathcal{P}_1^{i,i+1} + \xi_2 \mathcal{P}_2^{i,i+1} + \xi_3 \mathcal{P}_3^{i,i+1}\right). \tag{6.2.3}$$

The chiral permutation operators $\mathcal{P}_i : \mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathbb{C}^3 \otimes \mathbb{C}^3$ act as follows:

$$\mathcal{P}_1 |32\rangle = |23\rangle, \qquad \mathcal{P}_2 |13\rangle = |31\rangle, \qquad \mathcal{P}_3 |21\rangle = |12\rangle, \tag{6.2.4}$$

and annihilate all other pairs of states. Periodic boundary conditions are implemented, i.e. $\mathcal{P}_i^{L,L+1} \equiv \mathcal{P}_i^{L,1}$. We have simplified the notation for the states of the spin chain by

$$|\phi_{j_1} \phi_{j_2} \cdots \phi_{j_L}\rangle \to |j_1 j_2 \cdots j_L\rangle. \tag{6.2.5}$$

Therefore the Hamiltonian (6.2.3) scans a state for neighboring fields in *chiral order* $|32\rangle, |13\rangle$, or $|21\rangle$, and swaps them to *anti-chiral order* $|23\rangle, |31\rangle$, and $|12\rangle$ respectively. For example we have

$$\begin{aligned}H_{\text{ec}} |321\rangle &= \xi_1 |231\rangle + \xi_3 |312\rangle + \xi_2 |123\rangle, \\ H_{\text{ec}} |123\rangle &= 0.\end{aligned} \tag{6.2.6}$$

Setting $\xi_1 = \xi_2 = 0$ we recover the *hypereclectic* model

$$H_{\text{hec}} = \xi_3 \sum_{i=1}^{L} \mathcal{P}_3^{i,i+1}. \tag{6.2.7}$$

The Hamiltonians (6.2.3) and (6.2.7) are block diagonal with respect to sectors of fixed numbers K of ϕ_3 fields, $M - K$ of ϕ_2 fields, and $L - M$ of ϕ_1 fields. We define $V^{L,M,K}$ to be the subspace of $(\mathbb{C}^3)^{\otimes L}$ corresponding to these numbers of fields. Clearly we have

$$\dim V^{L,M,K} = \frac{L!}{(L-M)!(M-K)!K!}. \tag{6.2.8}$$

H_3 corresponds to the one-loop dilatation operator in the fishnet theory, where we consider K non-dynamical insertions ϕ_3, which act as walls. For $K = 0$ this operator, although non-Hermitian, is diagonalisable via a coordinate Bethe ansatz [9]. It corresponds essentially to a chiral version of the XY-model [199].

6.2.2 Translation Operator and Cyclicity Classes

We can further reduce the state space by considering the translation invariance of these Hamiltonians. Each H_i commutes with the translation operator U

$$[H_i, U] = 0, \qquad i = 1, 2, 3, \tag{6.2.9}$$

where U generates a shift along the chain

$$U|j_1 j_2 \cdots j_{L-1} j_L\rangle = |j_L j_1 j_2 \cdots j_{L-1}\rangle. \tag{6.2.10}$$

This further implies $[H_{\text{ec}}, U] = 0$. Therefore we can choose to work in a basis where U is diagonal. U has L distinct eigenvalues given by the L^{th} roots of unity

$$\omega_L^k = e^{2\pi i k/L}, \qquad k = 0, 1, \ldots, L-1. \tag{6.2.11}$$

The U-eigenstates in $V^{L,M,K}$ with eigenvalue ω_L^k are said to be in the k^{th} *cyclicity class* $V_k^{L,M,K}$. The $k = 0$ cyclicity class $V_{k=0}^{L,M,K}$ is known as the cyclic sector. The states in the k^{th} cyclicity class are easily generated by acting repeatedly on a reference elementary state[1] with $\omega_L^{-k} U$. For example, given $|123\rangle \in V^{3,2,1}$ we can form the cyclic state

$$|123\rangle + U|123\rangle + U^2|123\rangle = |123\rangle + |312\rangle + |231\rangle, \tag{6.2.12}$$

and states with $k = 1$ or $k = 2$

$$|123\rangle + \omega_3^{-1} U|123\rangle + \omega_3^{-2} U^2|123\rangle = |123\rangle + e^{-2\pi i/3}|312\rangle + e^{-4\pi i/3}|231\rangle, \tag{6.2.13}$$

$$|123\rangle + \omega_3^{-2} U|123\rangle + \omega_3^{-4} U^2|123\rangle = |123\rangle + e^{-4\pi i/3}|312\rangle + e^{-8\pi i/3}|231\rangle. \tag{6.2.14}$$

For a given L, M, K counting the number of states in $V^{L,M,K}$ with a given cyclicity k requires Pólya counting, see for example [200]. We denote the states in the k^{th} cyclicity class by

$$|j_1 j_2 \cdots j_L\rangle_k \coloneqq \sum_{l=0}^{L-1} (w^{-k} U)^l |j_1 j_2 \cdots j_L\rangle \coloneqq \mathcal{C}_k |j_1 j_2 \cdots j_L\rangle, \tag{6.2.15}$$

[1] We call single ket states $|j_1 j_2 \ldots j_L\rangle$ elementary. General states are linear combinations of these.

where $\mathcal{C}_k$ is an (unnormalised) projector[2] $\mathcal{C}_k^2 \propto \mathcal{C}_k$ onto the k^{th} cyclicity class $V_k^{L,M,K}$. For the hypereclectic spin chain we find it more natural to consider a so-called *static* basis, which we describe at the beginning of section 6.3.1.

6.2.3 Spectrum

The eclectic Hamiltonian (6.2.3) is nilpotent in sectors where all three particles are present i.e. in $V^{L,M,K}$ when $L-M, M-K, K \neq 0$. This fact is not immediately obvious, although it follows straightforwardly from the combinatorial considerations we introduce in the following sections. Therefore, by the discussion at the end of section 6.1, in these sectors all the eigenvalues of H_{ec} are zero. Furthermore, there exists a change of basis matrix B such that H_{ec} acts as a direct sum of Jordan blocks (6.1.9) with eigenvalue zero.

Let $H_{\text{ec}}^{L,M,K}$ be the restriction of the eclectic Hamiltonian to cyclic ($k=0$) states in $V^{L,M,K}$. Then, by the above discussion, there exists a change of basis matrix $B_{L,M,K}$ such that

$$B_{L,M,K}^{-1} H_{\text{ec}}^{L,M,K} B_{L,M,K} = \bigoplus_{i=1}^{l} J_{n_i}(0). \tag{6.2.16}$$

For example, consider the cyclic sector of $V^{L,M,K}$ for $L=5, M=3, K=1$. This sector is spanned by six states

$$\begin{aligned} &|11223\rangle_0, \quad |12123\rangle_0, \quad |12213\rangle_0, \\ &|21123\rangle_0, \quad |21213\rangle_0, \quad |22113\rangle_0, \end{aligned} \tag{6.2.17}$$

where $|\cdots\rangle_0$ is defined by (6.2.15), for example

$$|22113\rangle_0 = |22113\rangle + |32211\rangle + |13221\rangle + |11322\rangle + |21132\rangle. \tag{6.2.18}$$

The eclectic Hamiltonian can be calculated explicitly by acting with H_{ec} on each of the states in (6.2.17). For example

$$H_{\text{ec}}|22113\rangle_0 = \xi_1|21123\rangle_0 + \xi_2|12213\rangle_0 + \xi_3|21213\rangle_0. \tag{6.2.19}$$

The full matrix is calculated to be

$$H_{\text{ec}}^{531} = \begin{pmatrix} 0 & \xi_3 & \xi_2 & \xi_1 & 0 & 0 \\ 0 & 0 & \xi_3 & \xi_3 & \xi_1+\xi_2 & 0 \\ 0 & 0 & 0 & 0 & \xi_3 & \xi_2 \\ 0 & 0 & 0 & 0 & \xi_3 & \xi_1 \\ 0 & 0 & 0 & 0 & 0 & \xi_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.2.20}$$

where the basis is ordered as in (6.2.17). One can verify that defining the change of

[2] Note that this projection may also result in the zero vector.

basis matrix

$$B_{531} = \begin{pmatrix} -\frac{(\xi_1+\xi_2)\left(\xi_1^2+18\xi_2\xi_1+\xi_2^2\right)}{4\xi_3^3} & 2\xi_3^4 & 3\left(\xi_1+\xi_2\right)\xi_3^2 & \xi_1^2+\xi_2^2 & 0 & 0 \\ \frac{(\xi_1-\xi_2)^2}{2\xi_3^2} & 0 & 2\xi_3^3 & 2\left(\xi_1+\xi_2\right)\xi_3 & 0 & 0 \\ \frac{\xi_1-\xi_2}{2\xi_3} & 0 & 0 & \xi_3^2 & \xi_2 & 0 \\ \frac{\xi_2-\xi_1}{2\xi_3} & 0 & 0 & \xi_3^2 & \xi_1 & 0 \\ 0 & 0 & 0 & 0 & \xi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.2.21}$$

we have that

$$B_{531}^{-1} H_{\text{ec}}^{531} B_{531} = \left(\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = J_1(0) \oplus J_5(0). \tag{6.2.22}$$

Therefore the eclectic Hamiltonian H_{ec}^{531} is similar to a direct sum of Jordan blocks of length one and five. We write this as

$$\text{JNF}_{531}^{k=0} = 1 \oplus 5. \tag{6.2.23}$$

The Jordan normal form is identical in the other cyclicity classes $k = 1, 2, 3, 4$. Therefore the full Jordan normal form in the sector $L = 5, M = 3, K = 1$ is

$$\text{JNF}_{531} = 1^5 \oplus 5^5. \tag{6.2.24}$$

Note that the change of basis (6.2.21) is not well-defined at $\xi_3 = 0$. This reflects the fact that the couplings of the eclectic model can be fine tuned to give different Jordan block decompositions. Indeed, if $\xi_3 = 0$ then the Jordan normal form of H_{ec}^{531} is $1 \oplus 2 \oplus 3$. Therefore, when referring to 'the' Jordan normal form of H_{ec} in a specific L, M, K sector, we mean the Jordan normal form for generic couplings, away from these special points.

The 'spectral problem' for the eclectic Hamiltonian is the following: given a particle sector labelled by L, M, K, what is the Jordan normal form i.e. the integers n_i in (6.2.16). This Jordan spectrum is exactly what determines the logarithmic structure of the two-point functions in the operator sector (6.2.2), c.f. (2.2.88) and (2.2.89). In [6] various patterns were noticed in the Jordan block spectra by explicit computation, for example

$$\text{JNF}_{731}^{k=0} = 1 \oplus 5 \oplus 9, \tag{6.2.25}$$

$$\text{JNF}_{931}^{k=0} = 1 \oplus 5 \oplus 9 \oplus 13. \tag{6.2.26}$$

Together with (6.2.23), these lead to the natural conjecture

$$\text{JNF}_{5+2n,3,1}^{k=0} = \bigoplus_{j=0}^{n} (4j+1), \tag{6.2.27}$$

however these and more complicated patterns were unexplained by methods of integrability. In section 6.3 we show how to understand these patterns by combinatorial methods. In fact, we do this for the case of the hypereclectic model $H_{\text{hec}} = H_{\text{ec}}(\xi_1 = \xi_2 = 0, \xi_3 = 1)$. In [6] a certain *universality* was hypothesised; namely that the Jordan normal form of the hypereclectic Hamiltonian for particular filling conditions is precisely that of the eclectic Hamiltonian for generic couplings. We discuss and provide evidence for this hypothesis in section 6.3.3. Assuming the universality hypothesis, solving the hypereclectic spin chain is equivalent to solving the eclectic chain for generic values of the couplings ξ_i.

6.2.4 Integrability

In section 3.1.2 we discussed the integrable Heisenberg $\mathfrak{su}(2)$ spin chain. The key to its integrability was that the Hamiltonian could be derived from a transfer matrix $t(u)$ which constitutes a one-parameter family of commuting operators on the spin chain. This commutativity stemmed from the fact that $t(u)$ could be built from an R-matrix which satisfies the quantum Yang–Baxter equation. The transfer matrix could be diagonalised by means of an integrability construction know as the algebraic Bethe ansatz.

Eclectic $\boldsymbol{R}$-matrix. In fact, the eclectic Hamiltonian (6.2.3) is also integrable, in the sense that it can also be derived from a transfer matrix $t(u)$ built from such an R-matrix. In this case the Hilbert space is $\mathcal{H} = (\mathbb{C}^3)^{\otimes L}$, and we take an auxiliary space $V_a \simeq \mathbb{C}^3$. The R-matrix can be built from the the *chiral projectors* $\mathfrak{p}_i : \mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathbb{C}^3 \otimes \mathbb{C}^3$, $i = 1, 2, 3$. These operators each have a single non-trivial action:

$$\mathfrak{p}_3|21\rangle = \xi_3|21\rangle, \qquad \mathfrak{p}_1|32\rangle = \xi_1|32\rangle, \qquad \mathfrak{p}_2|13\rangle = \xi_2|13\rangle, \tag{6.2.28}$$

and annihilate all other pairs of states. The sum of these operators

$$\mathfrak{p}^{ij} := \mathfrak{p}_1^{ij} + \mathfrak{p}_2^{ij} + \mathfrak{p}_3^{ij} \tag{6.2.29}$$

tests whether a pair of fields are in chiral order. The chiral projectors $\mathfrak{p}_i$ are related to the chiral permutation operators $\mathcal{P}_i$ (also defined in (6.2.4))

$$\mathcal{P}_1|32\rangle = |23\rangle, \qquad \mathcal{P}_2|13\rangle = |31\rangle, \qquad \mathcal{P}_3|21\rangle = |12\rangle, \tag{6.2.30}$$

via the standard permutation operator $\mathbb{P}^{ij} : V^i \otimes V^j \to V^i \otimes V^j$, which simply swaps vectors in a tensor product $\mathbb{P}^{ij} v_i \otimes v_j = v_j \otimes v_i$. The precise relation is

$$\xi_k \mathcal{P}_k^{ij} = \mathbb{P}^{ij} \mathfrak{p}_k^{ij} = \mathfrak{p}_k^{ji} \mathbb{P}^{ij}, \tag{6.2.31}$$

or equivalently

$$\mathfrak{p}_k^{ij} = \xi_k \mathbb{P}^{ij} \mathcal{P}_k^{ij} = \xi_k \mathcal{P}_k^{ji} \mathbb{P}^{ij}. \tag{6.2.32}$$

The fact that the order of the indices in $\mathcal{P}_k^{ij}$ and $\mathfrak{p}_k^{ij}$ are of strict importance in (6.2.31) and (6.2.32) reflects their chirality. Note that the chiral permutation operators $\mathcal{P}_i$ are the densities with appear in the eclectic Hamiltonian (6.2.3).

In terms of $\mathbb{P}$ and $\mathfrak{p}$ the eclectic R-matrix $R^{ij}(u) : V_i \otimes V_j \to V_i \otimes V_j$ is defined

$$R^{ij}(u) = u\,\mathfrak{p}^{ij} + \mathbb{P}^{ij}, \tag{6.2.33}$$

where u is the spectral parameter. In matrix form this is

$$R(u) = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_2 u & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & \xi_3 u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \xi_1 u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{6.2.34}$$

This matrix satisfies the quantum Yang–Baxter equation:

$$R^{12}(u-u')R^{13}(u)R^{23}(u') = R^{23}(u')R^{13}(u)R^{12}(u-u'). \tag{6.2.35}$$

Let us give a few more details on how to verify this equation explicitly. (6.2.35) is an identity between linear operators in $V^1 \otimes V^2 \otimes V^3$, where each $V^i \simeq \mathbb{C}^3$. Each R^{ij} acts non-trivially on V^i and V^j via the definition (6.2.33). They can be defined explicitly as

$$R^{12}(u) = R(u) \otimes \mathbb{I}_3, \tag{6.2.36}$$

$$R^{23}(u) = \mathbb{I}_3 \otimes R(u), \tag{6.2.37}$$

$$R^{13}(u) = (\mathbb{I} \otimes \mathbb{P})R^{12}(u)(\mathbb{I} \otimes \mathbb{P}). \tag{6.2.38}$$

Each R^{ij} can be defined as a 27×27 matrix in Mathematica using (6.2.36)-(6.2.38) and the *KroneckerProduct* function, and the equation (6.2.35) can be easily verified symbolically.

The transfer matrix is built from the R-matrix (6.2.34) via

$$t(u) = \mathrm{tr}_a(R^{aL}R^{a,L-1}\cdots R^{a1}). \tag{6.2.39}$$

At $u = 0$, the transfer matrix evaluates to

$$t(0) = \mathrm{tr}_a(\mathbb{P}^{aL}\mathbb{P}^{a,L-1}\cdots \mathbb{P}^{a1}). \tag{6.2.40}$$

The permutation operator obeys the identity

$$\mathbb{P}^{ia}\mathbb{P}^{ja} = \mathbb{P}^{ij}\mathbb{P}^{ia}. \tag{6.2.41}$$

In particular, if $a = j$ we see that

$$(\mathbb{P}^{ij})^2 = \mathbb{P}^{ij}\mathbb{P}^{ji} = \mathbb{I}. \tag{6.2.42}$$

Applying (6.2.41) repeatedly to (6.2.40), we find

$$\begin{aligned} t(0) &= \mathrm{tr}_a(\mathbb{P}^{L,L-1}\mathbb{P}^{L,L-2}\cdots\mathbb{P}^{L1}\mathbb{P}^{aL}) \\ &= \mathbb{P}^{L,L-1}\mathbb{P}^{L,L-2}\cdots\mathbb{P}^{L1} \\ &= \mathbb{P}^{12}\mathbb{P}^{23}\cdots\mathbb{P}^{L-1,L} = U, \end{aligned} \tag{6.2.43}$$

where we also used that $\mathrm{tr}_a\,\mathbb{P}^{aL} = \mathbb{I}^L$. Therefore the transfer matrix evaluated at $u = 0$ coincides with the shift operator $U : |j_1 j_2 \cdots j_{L-1} j_L\rangle \to |j_L j_1 j_2 \cdots j_{L-1}\rangle$.

Hamiltonian. The transfer matrix and its logarithm can be expanded in powers of u:

$$t(u) = \sum_j u^j S_{j+1} = S_1 + uS_2 + u^2 S_3 + \cdots\,, \tag{6.2.44}$$

$$\log t(u) = \sum_j u^j Q_{j+1} = Q_1 + uQ_2 + u^2 Q_3 + \cdots\,, \tag{6.2.45}$$

where $S_j : \mathcal{H} \to \mathcal{H}$ and $Q_j : \mathcal{H} \to \mathcal{H}$ are operators on the spin chain. We have already established that $S_1 = U$. The calculation of S_2 is similar. We expand the transfer matrix (6.2.39) to linear order in u using (6.2.33), and find

$$S_2 = \mathrm{tr}_a(\mathfrak{p}^{aL}\mathbb{P}^{a,L-1}\cdots\mathbb{P}^{a1}) + \mathrm{tr}_a(\mathbb{P}^{aL}\mathfrak{p}^{a,L-1}\cdots\mathbb{P}^{a1}) + \cdots + \mathrm{tr}_a(\mathbb{P}^{aL}\mathbb{P}^{a,L-1}\cdots\mathfrak{p}^{a1}). \tag{6.2.46}$$

The first term in (6.2.46) can be calculated as

$$\begin{aligned}
&\mathrm{tr}_a(\mathfrak{p}^{aL}\mathbb{P}^{a,L-1}\cdots\mathbb{P}^{a1}) = \mathrm{tr}_a(\mathbb{P}^{aL}\mathcal{P}^{aL}\mathbb{P}^{a,L-1}\cdots\mathbb{P}^{a1}) \\
=&\mathrm{tr}_a(\mathbb{P}^{aL}\mathbb{P}^{a,L-1}\mathcal{P}^{L-1,L}\mathbb{P}^{a,L-1}\mathbb{P}^{a,L-1}\cdots\mathbb{P}^{a1}) = \mathrm{tr}_a(\mathbb{P}^{aL}\mathbb{P}^{a,L-1}\mathcal{P}^{L-1,L}\mathbb{P}^{a,L-2}\cdots\mathbb{P}^{a1}) \\
=&\mathrm{tr}_a(\mathbb{P}^{a,L-2}\cdots\mathbb{P}^{a1}\mathbb{P}^{aL}\mathbb{P}^{a,L-1}\mathcal{P}^{L-1,L}) = \mathrm{tr}_a(\mathbb{P}^{a,L-1}\mathbb{P}^{L-1,L-2}\cdots\mathbb{P}^{L-1,1}\mathbb{P}^{L-1,L}\mathcal{P}^{L-1,L}) \\
=&\mathbb{P}^{L-1,L-2}\cdots\mathbb{P}^{L-1,1}\mathbb{P}^{L-1,L}\mathcal{P}^{L-1,L} = \mathbb{P}^{12}\mathbb{P}^{23}\cdots\mathbb{P}^{L-1,L}\mathcal{P}^{L-1,L} \\
=&U\mathcal{P}^{L-1,L},
\end{aligned} \tag{6.2.47}$$

where we denoted

$$\mathcal{P} \coloneqq \xi_1\mathcal{P}_1 + \xi_2\mathcal{P}_2 + \xi_3\mathcal{P}_3. \tag{6.2.48}$$

In this calculation we repeatedly used (6.2.41), as well as (6.2.32) and (6.2.42). Using cyclicity of the trace each term in (6.2.46) can be calculated analogously, so we have

$$S_2 = U\sum_{i=1}^{L}\mathcal{P}^{i,i+1} = U\,H_{\mathrm{ec}}. \tag{6.2.49}$$

Knowing S_2, one can easily calculate Q_2 as

$$Q_2 = \left.\frac{d\log t(u)}{du}\right|_{u=0} = t^{-1}(0)\left.\frac{dt(u)}{du}\right|_{u=0} = U^{-1}S_2 = H_{\mathrm{ec}}, \tag{6.2.50}$$

so that Q_2 is precisely the eclectic Hamiltonian.

Higher Charges. The calculation of the higher charges becomes slightly more involved. For example, for $L = 4$ we calculate

$$S_3 = \mathrm{tr}_a(\mathfrak{p}^{a4}\mathfrak{p}^{a3}\mathbb{P}^{a2}\mathbb{P}^{a1}) + \mathrm{tr}_a(\mathfrak{p}^{a4}\mathbb{P}^{a3}\mathfrak{p}^{a2}\mathbb{P}^{a1}) + 4\text{ terms}. \tag{6.2.51}$$

We calculate the two written terms in (6.2.51), as they are the two qualitatively different terms which can appear. The first term is where both $\mathfrak{p}$'s are beside each other

$$\begin{aligned}
&\mathrm{tr}_a(\mathfrak{p}^{a4}\mathfrak{p}^{a3}\mathbb{P}^{a2}\mathbb{P}^{a1}) = \mathrm{tr}_a(\mathbb{P}^{a4}\mathcal{P}^{a4}\mathbb{P}^{a3}\mathcal{P}^{a3}\mathbb{P}^{a2}\mathbb{P}^{a1}) \\
&= \mathrm{tr}_a(\mathbb{P}^{a4}\mathbb{P}^{a3}\mathbb{P}^{a3}\mathcal{P}^{a4}\mathbb{P}^{a3}\mathbb{P}^{a2}\mathbb{P}^{a2}\mathcal{P}^{a3}\mathbb{P}^{a2}\mathbb{P}^{a1}) \\
&= \mathrm{tr}_a(\mathbb{P}^{a4}\mathbb{P}^{a3}\mathcal{P}^{34}\mathbb{P}^{a2}\mathcal{P}^{23}\mathbb{P}^{a1}) = \mathrm{tr}_a(\mathbb{P}^{a1}\mathbb{P}^{14}\mathbb{P}^{13}\mathbb{P}^{12}\mathcal{P}^{34}\mathcal{P}^{23}) \\
&= \mathbb{P}^{14}\mathbb{P}^{13}\mathbb{P}^{12}\mathcal{P}^{34}\mathcal{P}^{23} = U\mathcal{P}^{34}\mathcal{P}^{23}.
\end{aligned} \tag{6.2.52}$$

We see that this term is a range 3 term, because it connects site 2 and site 4 of the spin chain. Such calculations can be sanity checked by explicitly calculating the trace for a given state. For example, the operator $U\mathcal{P}^{34}\mathcal{P}^{23}$ should map the state $|1211\rangle$ to $\xi_3^2|2111\rangle$. We can calculate the first term in (6.2.51) on this state explicitly:

$$\mathrm{tr}_a(\mathfrak{p}^{a4}\mathfrak{p}^{a3}\mathbb{P}^{a2}\mathbb{P}^{a1})|1211\rangle = \sum_{i=1}^{3} {}_a\langle i|\mathfrak{p}^{a4}\mathfrak{p}^{a3}\mathbb{P}^{a2}\mathbb{P}^{a1}|1211\rangle \otimes |i\rangle_a \tag{6.2.53}$$

$$\begin{aligned} &= \sum_{i=1}^{3} {}_a\langle i|\mathfrak{p}^{a4}\mathfrak{p}^{a3}|i111\rangle \otimes |2\rangle_a = \xi_3^2 \sum_{i=1}^{3} |i111\rangle \otimes \langle i|2\rangle_a \\ &= \xi_3^2 \sum_{i=1}^{3} \delta_{i2}|i111\rangle = \xi_3^2|2111\rangle, \end{aligned} \tag{6.2.54}$$

as expected. The second term in (6.2.51) is similarly calculated to be

$$\mathrm{tr}_a(\mathfrak{p}^{a4}\mathbb{P}^{a3}\mathfrak{p}^{a2}\mathbb{P}^{a1}) = U\mathcal{P}^{34}\mathcal{P}^{12}, \tag{6.2.55}$$

which is a bilocal operator, with each piece being range 2. Overall, S_3 splits into a local range 3 piece, and a bilocal piece:

$$\begin{aligned} S_3 &= U(\mathcal{P}^{34}\mathcal{P}^{23} + \mathcal{P}^{23}\mathcal{P}^{12} + \mathcal{P}^{12}\mathcal{P}^{41} + \mathcal{P}^{41}\mathcal{P}^{34}) + U(\mathcal{P}^{34}\mathcal{P}^{12} + \mathcal{P}^{23}\mathcal{P}^{41}) \\ &:= U(S_3^{\mathrm{loc}} + S_3^{2-\mathrm{loc}}). \end{aligned} \tag{6.2.56}$$

Interestingly, Q_3 is actually a local operator. Computing the second logarithmic derivative, we find that

$$2Q_3 = t^{-1}(0)\frac{d^2t(u)}{du^2}\Big|_{u=0} - \left(t^{-1}(0)\frac{dt(u)}{du}\Big|_{u=0}\right)^2 = 2U^{-1}S_3 - U^{-1}S_2U^{-1}S_2. \tag{6.2.57}$$

Terms in $U^{-1}S_2U^{-1}S_2 = H_{\mathrm{ec}}^2$ term cancel the bilocal terms in S_3, and overall we find that

$$2Q_3 = \mathcal{P}^{34}\mathcal{P}^{23} - \mathcal{P}^{23}\mathcal{P}^{34} + \mathcal{P}^{41}\mathcal{P}^{34} - \mathcal{P}^{34}\mathcal{P}^{41} + \mathcal{P}^{12}\mathcal{P}^{41} - \mathcal{P}^{41}\mathcal{P}^{12} + \mathcal{P}^{23}\mathcal{P}^{12} - \mathcal{P}^{12}\mathcal{P}^{23}. \tag{6.2.58}$$

For general L this pattern persists, and we have

$$Q_3 = \frac{1}{2}\sum_{i=1}^{L}[\mathcal{P}^{i,i+1}, \mathcal{P}^{i-1,i}]. \tag{6.2.59}$$

The calculation is essentially the same for the Heisenberg spin chain discussed in section 3.1.2, however in the eclectic case one needs to be careful with the chirality of the operators. In general S_j contains a local range-j piece, and products of lower-range terms, c.f. (6.2.56). Q_j is a purely local range-j operator.

In the case of the Heisenberg spin chain, there are an infinite number of non-zero local commuting operators Q_j. One can then query whether they are 'independent' enough to render the model integrable, and indeed this seems plausible. In the case of the eclectic spin chain, there are only a finite number of non-zero operators Q_j. This means that there exists an integer N, which depends on the length of the spin chain, such that

$$Q_j = 0 \qquad \forall j \geq N. \tag{6.2.60}$$

For example, consider the cyclic sector for $L = 5, M = 3, K = 1$. The charge $Q_2 = H_{\text{ec}}$ was given in (6.2.20). The next local charges are given explicitly by

$$Q_3 = \begin{pmatrix} 0 & 0 & -\frac{\xi_3^2}{2} & \frac{\xi_3^2}{2} & 0 & \frac{1}{2}\left(\xi_1^2 - \xi_2^2\right) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\xi_3^2}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{\xi_3^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{\xi_3^3}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\xi_3^3}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.2.61}$$

and $Q_j = 0$ for $j \geq 5$. Moreover, just like the Hamiltonian Q_2, all of the operators are nilpotent, i.e. there exists some integer M such that

$$(Q_j)^n = 0, \qquad \forall n \geq M, \qquad \forall j \geq 2. \tag{6.2.62}$$

For example, in the case described above we have $(Q_2)^5 = (Q_3)^3 = (Q_4)^2 = 0$. Therefore each Q_j is non-diagonalisable for $j \geq 2$.

Because of these facts, it is possible that the eclectic model does not deserve to be called an *integrable* model, as there may not be enough conserved quantities to constrain the system. In [6] the authors were unable to determine the Jordan block spectrum of $Q_2 = H_{\text{ec}}$ using techniques of integrability, which may not be a surprise if the model is not actually integrable. Despite this, there still exists a number of non-trivial algebraic relations in the Yang–Baxter equation (6.2.35). It is an interesting open question whether it is possible to use this algebra to determine the Jordan block spectra of Q_2 and the higher charges. It is possible that the model is integrable, but the integrability manifests itself in a novel way.

6.3 Combinatorial Solution of the Hypereclectic Spin Chain

In this section we demonstrate a combinatorial approach for the solution of the eclectic model. The combinatorial solution is for the Hamiltonian of the hypereclectic model H_{hec}, which is argued to also apply to the eclectic model H_{ec} by universality. The solution is elegant; it takes the form of a generating function $\mathcal{Z}(q)$ which enumerates the full spectrum of Jordan blocks for any particle sector labelled by L, M, K. The elegance of the solution raises the hope that it can be related to an integrability based approach in the future.

We first describe the solution for a single ϕ_3 field, also called a *wall*, because this field is non-interacting for the hypereclectic model. These insights then are generalised to an arbitrary number of walls, i.e. $K > 1$, to obtain the master generating function. We finally comment on the universality hypothesis, which asserts that the spectrum of the hypereclectic model coincides with the spectrum of the eclectic model for generic couplings, for special filling conditions $L - M \geq M - K \geq K$. We provide further evidence that this hypothesis is valid.

6.3.1 Solution for One Wall

We first describe a method to determine the full Jordan block spectrum for the hypereclectic spin chain in sectors where $K = 1$, i.e. there is a single, non-moving ϕ_3 field, which acts as a fixed wall. In these sectors the model is equivalent to a chiral XY spin chain with open boundary conditions. Throughout this section we denote the hypereclectic Hamiltonian defined in (6.2.7) $H_{\text{hec}} \equiv H$ and set $\xi_3 = 1$. We recall that the only non-trivial action of the Hamiltonian density $H_{i,i+1}$ is on $|21\rangle$:

$$\begin{aligned} &H|21\rangle = |12\rangle, \\ &H|12\rangle = H|23\rangle = H|32\rangle = H|31\rangle = H|13\rangle = H|11\rangle = H|22\rangle = H|33\rangle = 0. \end{aligned} \tag{6.3.1}$$

Since the ϕ_3 field does not move under the action of H, we can further restrict to sectors with a fixed position of ϕ_3. We will restrict to *static* states of the form $|j_1 j_2 \cdots j_{L-1} 3\rangle$, where $j_1, j_2, \ldots, j_{L-1} \in \{1, 2\}$. We will refer to the subspace of $V^{L,M,1}$ spanned by states of this form as $W^{L,M}$. We can access states where ϕ_3 is in a different position by acting with the translation operator U, so that the Hilbert space decomposes

$$V^{L,M,1} = \bigoplus_{j=0}^{L-1} U^j W^{L,M}. \tag{6.3.2}$$

We begin with a few simple examples, before describing the general solution for the Jordan block spectrum of H in $W^{L,M}$.

Example: L, $M = 2$, $K = 1$. The simplest situation is when $M = 2$ and $K = 1$. This means there is a single ϕ_3 field, a single ϕ_2 field, and $L - 2$ ϕ_1 fields. A natural basis for $W^{L,2}$ is given by $L - 1$ states

$$|211 \cdots 113\rangle, |121 \cdots 113\rangle, \ldots, |111 \cdots 123\rangle. \tag{6.3.3}$$

In this sector the states clearly form a single Jordan block of length $L - 1$, as can be seen by acting with H repeatedly on $|211 \cdots 113\rangle$

$$|211 \cdots 113\rangle \xrightarrow{H} |121 \cdots 113\rangle \xrightarrow{H} \cdots \xrightarrow{H} |111 \cdots 123\rangle \xrightarrow{H} 0. \tag{6.3.4}$$

We will refer to any state of the form $|2^{M-K} 1^{L-M} 3^K\rangle$ as *anti-locked*, and $|1^{L-M} 2^{M-K} 3^K\rangle$ as *locked*. Similarly, for the spaces $U^j W^{L,M}$, $j = 1, \ldots, L - 1$, there is a single Jordan block of length $L - 1$. Therefore for $M = 2$ and $K = 1$ we have

$$\text{JNF}_{L,2,1} = (L - 1)^L, \tag{6.3.5}$$

meaning there are L blocks of length $L - 1$.

Example: $\boldsymbol{L = 7, M = 3, K = 1}$. The situation becomes more intricate with increasing M, which we illustrate with the example $L = 7, M = 3, K = 1$. In this sector there are 4 ϕ_1 fields, 2 ϕ_2 fields, and a single ϕ_3 field. In $W^{7,3}$ there are 15 states. We use the important observation that the anti-locked state is always a top state for the longest Jordan block

$$\begin{aligned}
&|2211113\rangle && H^0 \\
\to&|2121113\rangle && H^1 \\
\to&|2112113\rangle + |1221113\rangle && H^2 \\
\to&|2111213\rangle + 2|1212113\rangle && H^3 \\
\to&|2111123\rangle + 3|1211213\rangle + 2|1122113\rangle && H^4 \\
\to&4|1211123\rangle + 5|1121213\rangle && H^5 \\
\to&5|1112213\rangle + 9|1121123\rangle && H^6 \\
\to&14|1112123\rangle && H^7 \\
\to&14|1111223\rangle && H^8 \\
\to&0, && H^9
\end{aligned} \tag{6.3.6}$$

so we have identified a Jordan block of length 9, whose eigenstate is proportional to the locked state $|1111223\rangle$. However, since there are 15 states in the sector there must be additional Jordan blocks.

We note that each of the 15 elementary states appear in the tower of states (6.3.6). We classify these 15 states by where in this state tower they appear, by defining the *level* S of an elementary state. We give the anti-locked state $|2211113\rangle$ $S = 8$ and the locked state $|1111223\rangle$ $S = 0$. In general, if an elementary state appears in the row H^k of (6.3.6), we give it $S = 8 - k$. Combinatorially, the S-value for a state is the total number of 1's to the right of each of the 2's. We definine $W_S^{7,3}$ to be the vector subspace of $W^{7,3}$ spanned by states with level S. Then we have

$$W^{7,3} = \bigoplus_{S=0}^{8} W_S^{7,3}, \tag{6.3.7}$$

and it is clear that

$$H : W_S^{7,3} \to W_{S-1}^{7,3}, \qquad HW_0^{7,3} = 0. \tag{6.3.8}$$

In light of this, the next natural place to look for a top state of a Jordan block is in $W_6^{7,3}$. This is because a single state from each $W_S^{7,3}$ is already contained in the largest Jordan block, and $W_6^{7,3}$ is the space with largest S with dimension larger than 1. We thus deduce that the top state for the next Jordan block must be of the form

$$\alpha|2112113\rangle + \beta|1221113\rangle \in W_6^{7,3}, \tag{6.3.9}$$

where $\alpha \neq \beta$ as we want the state to be linearly independent from the corresponding state in the length 9 block. We act repeatedly on this state with H until there is a possible choice for α and β which makes the state vanish

$$\begin{aligned}
&\alpha|1221113\rangle + \beta|2112113\rangle \\
&\to \beta|2111213\rangle + (\alpha+\beta)|1212113\rangle \\
&\to \beta|2111123\rangle + (\alpha+2\beta)|1211213\rangle + (\alpha+\beta)|1122113\rangle \\
&\to (\alpha+3\beta)|1211123\rangle + (2\alpha+3\beta)|1121213\rangle \\
&\to (2\alpha+3\beta)|1112213\rangle + (3\alpha+6\beta)|1121123\rangle \\
&\to (5\alpha+9\beta)|1112123\rangle.
\end{aligned}$$

We see that this yields the zero vector if $5\alpha + 9\beta = 0$, for example $\alpha = -9, \beta = 5$. Therefore this chain of states determines a Jordan block of length 5, with top state $5|2112113\rangle - 9|1221113\rangle \in W_6^{7,3}$ and eigenstate $-3|1112213\rangle + 3|1121123\rangle \in W_{8-6}^{7,3} = W_2^{7,3}$.

There must be a single Jordan block of length 1 remaining, and by state counting this must be contained in $W_4^{7,3}$, since this is the only space with dimension greater than 2. We make the ansatz for the top state

$$\alpha'|2111123\rangle + \beta'|1211213\rangle + \gamma'|1122113\rangle \in W_4^{7,3}. \tag{6.3.10}$$

This is easily checked to be an eigenstate of H for $\alpha' = -\beta' = \gamma' = 1$ and thus determines a Jordan block of length 1. The story is identical for the remaining spaces $U^j W^{7,3}$, $j = 1, \ldots, 6$, so the overall Jordan normal form for $L = 7, M = 3, K = 1$ is

$$\mathrm{JNF}_{7,3,1} = (9^7, 5^7, 1^7). \tag{6.3.11}$$

Let us step back and look at the state tower (6.3.6), from which we can see the dimensions

$$\dim W_S^{7,3}, \qquad S = 0, 1, \ldots, 8 \tag{6.3.12}$$

by counting the number of elementary states in each row. We note that these dimensions form a diamond, in that they start from 1 at $S = 8$, increase to a maximum of 3 at $S = 4$, and decrease symmetrically to 1 at $S = 0$. We encode these dimensions in a generating function

$$\bar{Z}_{7,3}(q) = \sum_{S=0}^{8} \dim W_S^{7,3} q^S = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8. \tag{6.3.13}$$

Because of this diamond structure it is actually possible to deduce the Jordan block structure in $W^{7,3}$ from the generating function, a purely combinatorial object, up to some possible subtleties described in the next section. Given the generating function (6.3.13) we identify the Jordan block of length 9 by the degree of the polynomial plus 1. We then subtract $1 + q + q^2 + \cdots + q^8$ to represent the fact that there is one state at each level in this largest block. We then normalise the resulting polynomial to have lowest power q^0, to arrive at the polynomial $1 + q + 2q^2 + q^3 + q^4$. This polynomial represents the states in the length 5 Jordan block. We subtract $1 + q + \cdots + q^4$ to exclude this Jordan block and normalise again. The resulting polynomial is simply 1, and represents the Jordan block of length 1. Overall the procedure is

$$\begin{aligned}
&1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8 \\
\to &1 + q + 2q^2 + q^3 + q^4 \\
\to &1,
\end{aligned} \tag{6.3.14}$$

from which we deduce the Jordan block spectrum $(9,5,1)$. We proceed to generalise these arguments and compute the generating function $\bar{Z}_{L,M}(q)$ for arbitrary L, M.

General L, M, and $K = 1$ - Generating Function. For general L, M we similarly grade the vector space in the static sector by the action of H

$$W^{L,M} = \bigoplus_{S=0}^{S_{\max}} W_S^{L,M}, \tag{6.3.15}$$

$$H : W_S^{L,M} \to W_{S-1}^{L,M}, \qquad HW_0^{L,M} = 0. \tag{6.3.16}$$

We have in general $S_{\max} = L_1 M_1$, where $L_1 \coloneqq L - M$ is the number of 1's in the sector and $M_1 \coloneqq M - 1$ is the number of 2's. The anti-locked state is $|2^{M_1} 1^{L_1} 3\rangle \in W_{S_{\max}}^{L,M}$ and the locked state is $|1^{L_1} 2^{M_1} 3\rangle \in W_0^{L,M}$. In general, an elementary state takes the form

$$|n_1, n_2, \ldots, n_{M_1}\rangle \coloneqq |\underbrace{1\cdots 1}_{n_0}\mathbf{2}\underbrace{1\cdots 1}_{n_1}\mathbf{2}\underbrace{1\cdots 1}_{n_2}\cdots\mathbf{2}\underbrace{1\cdots 1}_{n_{M_1}}\mathbf{3}\rangle, \tag{6.3.17}$$

where n_j is the number of 1's between the j^{th} and $(j+1)^{\text{th}}$ 2. Clearly they should satisfy

$$\sum_{j=0}^{M_1} n_j = L - M = L_1. \tag{6.3.18}$$

In this notation we can define the level S of a state, which counts the number of 1's on the right hand side of each of the 2's. Explicitly the state $|n_1, n_2, \ldots, n_{M_1}\rangle$ defined in (6.3.17) has

$$S = \sum_{j=1}^{M_1} j n_j. \tag{6.3.19}$$

As before we define $W_S^{L,M}$ to be spanned by elementary states with this level S. The Hamiltonian acts on (6.3.17) as

$$H : |n_1, n_2, \ldots, n_{M_1}\rangle \quad \to \quad \sum_{j=1}^{M_1} |n_1, n_2, \ldots, n_{j-1}+1, n_j - 1, \ldots, n_{M_1}\rangle. \tag{6.3.20}$$

(6.3.19) and (6.3.20) make it clear that H decreases S to $S-1$, i.e. if one acts on a level-S elementary state with H, a linear combination of level-$(S-1)$ elementary states is returned.

We now consider the problem of determining the dimensions of the spaces $W_S^{L,M}$. We would like to determine a generating function

$$\bar{Z}_{L,M}(q) = \sum_{S=0}^{S_{\max}} \dim W_S^{L,M} q^S. \tag{6.3.21}$$

These dimensions $\dim W_S^{L,M}$ are given by the number of partitions of the integer S into at most M_1 parts, each less than or equal to L_1. Expressing (6.3.19) as

$$S = (n_1 + n_2 + \cdots + n_{M_1}) + (n_2 + \cdots + n_{M_1}) + \cdots + n_{M_1} \tag{6.3.22}$$

one can notice that there is one-to-one correspondence between an elementary vector in (6.3.17) and such a restricted partition of S in (6.3.22). For example, consider the same example as above, $L = 7, M = 3, K = 1$. There are 3 elementary states in $W_4^{7,3}$:

$$\begin{aligned} &|2111123\rangle, \qquad (n_1 + n_2, n_2) = (4, 0), \\ &|1211213\rangle, \qquad (n_1 + n_2, n_2) = (3, 1), \\ &|1122113\rangle, \qquad (n_1 + n_2, n_2) = (2, 2). \end{aligned} \tag{6.3.23}$$

These correspond to the partitions of the integer 4 into at most $M_1 = 2$ parts, where each part is less than or equal to $L_1 = 4$. There are 3 such partitions $4 = 4 = 3 + 1 = 2 + 2$.

The restricted partitions described above can be generated in general by Gaussian (or q-) binomial coefficients [201]

$$\bar{Z}_{L,M}(q) = \sum_{S=0}^{S_{\max}} \dim W_S^{L,M} q^S = \binom{L-1}{M-1}_q = \prod_{k=1}^{M-1} \frac{1 - q^{L-k}}{1 - q^k}, \tag{6.3.24}$$

which is always a polynomial in q. Note that if we send $q \to 1$, the q-binomial reduces to the ordinary binomial coefficient and we have

$$\sum_{S=0}^{S_{\max}} \dim W_S^{L,M} = \binom{L-1}{M-1} = \dim W^{L,M}, \tag{6.3.25}$$

as expected because of (6.3.15). Properties of the q-binomial coefficient (6.3.24) can be used to understand the structure of graded spaces $W_S^{L,M}$. For example, the property

$$[q^k]\binom{L-1}{M-1}_q = [q^{(L-M)(M-1)-k}]\binom{L-1}{M-1}_q, \qquad k = 0, 1, \ldots, nm, \tag{6.3.26}$$

where $[q^k]f(q)$ is the coefficient of q^k in the polynomial $f(q)$, explains the symmetric structure of the state tower (6.3.6). We also have

$$[q^0]\binom{L-1}{M-1}_q = [q^{(L-M)(M-1)}]\binom{L-1}{M-1}_q = 1, \tag{6.3.27}$$

which reflects the fact that $W_{S_{\max}}^{L,M}$ and $W_0^{L,M}$ are one-dimensional, being spanned by the anti-locked and locked states respectively.

(6.3.24) generates a list of dimensions[3] $\mathbf{d}_S := \dim W_S^{L,M}$

$$(\mathbf{d}_{S_{\max}}, \mathbf{d}_{S_{\max}-1}, \ldots, \mathbf{d}_1, \mathbf{d}_0) \qquad \text{with} \quad \mathbf{d}_0 = \mathbf{d}_{S_{\max}} = 1. \tag{6.3.28}$$

By a further property of the q-binomial coefficient, the dimensions are increasing from the left to the right until the midpoint. After that they decrease symmetrically to 1, because of the symmetry

$$\mathbf{d}_S = \mathbf{d}_{\tilde{S}}, \qquad \tilde{S} := S_{\max} - S, \tag{6.3.29}$$

[3] $\mathbf{d}_S = \mathbf{d}_S(L_1, M_1)$, we suppress the L_1, M_1 dependence for now.

which reflects (6.3.26). For the space $W^{L,M}_{S_{\max}}$, there is only one elementary state $\psi_0 :=$ $|2^{M_1}1^{L_1}3\rangle$, the anti-locked state. By successive action of H, a Jordan string of states is generated

$$\psi_0 \xrightarrow{H} H\psi_0 \xrightarrow{H} H^2\psi_0 \xrightarrow{H} \cdots \xrightarrow{H} H^{S_{\max}}\psi_0 \xrightarrow{H} 0. \tag{6.3.30}$$

Therefore, this generates a Jordan block of size $S_{\max}+1$, which is the largest block in this sector.

The next dimension $\mathbf{d}_{S_{\max}-1}$ in (6.3.28) is also one, which can be computed from (6.3.24). This means $W^{L,M}_{S_{\max}-1}$ is spanned by $H\psi_0$, the first descendant of the anti-locked state in (6.3.30). Therefore there is no other independent vector in $W^{L,M}_{S_{\max}-1}$ which can generate a new Jordan string.

The top state of the second Jordan block arises at the first level $S = S_1$ below $S_{\max}$ whose dimension is bigger than 1. We can form $\mathbf{d}_{S_1} - 1$ linearly independent potential top states in $W^{L,M}_{S_1}$, which are linearly independent from the H-descendant of the anti-locked state. We denote these states by $\psi_j^{(S_1)}$ ($j = 1, \ldots, \mathbf{d}_{S_1} - 1$), and make the ansatz

$$\psi_j^{(S_1)} = \sum_{i=1}^{\mathbf{d}_{S_1}} \alpha_j^{(i)} e_i^{(S_1)}, \tag{6.3.31}$$

where $e_i^{(S_1)}$ are the elementary states in $W^{L,M}_{S_1}$. $\alpha_j^{(i)}$ are constants which are determined by the condition that each $\psi_j^{(S_1)}$ constitutes a top state for a new Jordan block. Each of these states generates a Jordan string

$$\psi_j^{(S_1)} \xrightarrow{H} H\psi_j^{(S_1)} \xrightarrow{H} H^2\psi_j^{(S_1)} \cdots \xrightarrow{H} H^{S_1-\tilde{S}_1}\psi_j^{(S_1)} \xrightarrow{H} 0, \quad j = 1, \ldots, \mathbf{d}_{S_1} - 1. \tag{6.3.32}$$

The condition $H^{S_1-\tilde{S}_1}\psi_j^{(S_1)} \xrightarrow{H} 0$ leads to a linear system of equations for the $\alpha_j^{(i)}$ which can be solved to determine the $\mathbf{d}_{S_1} - 1$ new top states. These new Jordan blocks each have size $S_1 - \tilde{S}_1 + 1$. The only possible subtlety is the potential for an 'unexpected shortening' of the Jordan block, that is the possibility for the equation $H^k\psi_j^{(S_1)} = 0$ to admit a solution in the $\alpha_j^{(i)}$ for some $k < S_1 - \tilde{S}_1 + 1$. While we did not rigorously disprove shortening in full generality, we verified for a large number values of L and M that it does not happen, see appendix C.1.

The third set of Jordan blocks occurs at a level S_2, which is the largest integer satisfying $\mathbf{d}_{S_2} > \mathbf{d}_{S_1}$. Then, as before, we can form $\mathbf{d}_{S_2} - \mathbf{d}_{S_1}$ linearly independent potential top states which are linearly independent from H-descendants of the previous vectors, ψ_0 and $\psi_j^{(S_1)}$. We make a similar ansatz for these potential top states

$$\psi_j^{(S_2)} = \sum_{i=1}^{\mathbf{d}_{S_2}} \beta_j^{(i)} e_i^{(S_2)}, \tag{6.3.33}$$

where $\beta_j^{(i)}$ are constants. These states create new Jordan strings

$$\psi_j^{(S_2)} \xrightarrow{H} H\psi_j^{(S_2)} \xrightarrow{H} H^2\psi_j^{(S_2)} \cdots \xrightarrow{H} H^{S_2-\tilde{S}_2}\psi_j^{(S_2)} \xrightarrow{H} 0, \quad j = 1, \ldots, \mathbf{d}_{S_2} - \mathbf{d}_{S_1}, \tag{6.3.34}$$

and the final condition $H^{S_2-\tilde{S}_2}\psi_j^{(S_2)} \xrightarrow{H} 0$ is solved to determine the constants $\beta_j^{(i)}$. This leads to $\mathbf{d}_{S_2} - \mathbf{d}_{S_1}$ Jordan blocks of size $S_2 - \tilde{S}_2 + 1$. This procedure can be continued until it reaches the maximum value of the dimension $\mathbf{d}_S$ which occurs at $S = [S_{\max}/2]$.

We note that for a given L, M, the dimensions $\mathbf{d}_S$ are sufficient to determine the sizes and multiplicities of the Jordan blocks. For example, for $L = 9, M = 5$ we compute using (6.3.24)

$$\begin{aligned}\bar{Z}_{9,5}(q) =& 1 + q + 2q^2 + 3q^3 + 5q^4 + 5q^5 + 7q^6 + 7q^7 + 8q^8 \\ &+ 7q^9 + 7q^{10} + 5q^{11} + 5q^{12} + 3q^{13} + 2q^{14} + q^{15} + q^{16},\end{aligned} \tag{6.3.35}$$

from which we can identify the Jordan normal form of H in $W^{9,5}$ to be $(17, 13, 11, 9^2, 5^2, 1)$ using the same procedure as (6.3.14). We can exhaust the Hilbert space by application of $U^j, j = 1, \ldots, 8$, so that overall we have

$$\mathrm{JNF}_{9,5,1} = (17^9, 13^9, 11^9, 9^{18}, 5^{18}, 1^9). \tag{6.3.36}$$

For higher K, see the next section 6.3.2, it is necessary to work with a slightly modified generating function for the dimensions of $W_S^{L,M}$. This modified generating function is symmetric under $q \to q^{-1}$:

$$Z_{L,M}(q) = q^{-S_{\max/2}} \bar{Z}_{L,M}(q) := \begin{bmatrix} L-1 \\ M-1 \end{bmatrix}_q = \prod_{k=1}^{M-1} \frac{q^{\frac{L-M+k}{2}} - q^{-\frac{L-M+k}{2}}}{q^{k/2} - q^{-k/2}}. \tag{6.3.37}$$

For example, (6.3.13) and (6.3.35) are modified to

$$Z_{7,3}(q) = q^{-4} + q^{-3} + 2q^{-2} + 2q^{-1} + 3 + 2q + 2q^2 + q^3 + q^4, \tag{6.3.38}$$

$$\begin{aligned}Z_{9,5}(q) =& q^{-8} + q^{-7} + 2q^{-6} + 3q^{-5} + 5q^{-4} + 5q^{-3} + 7q^{-2} + 7q^{-1} \\ &+ 8 + 7q + 7q^2 + 5q^3 + 5q^4 + 3q^5 + 2q^6 + q^7 + q^8.\end{aligned} \tag{6.3.39}$$

The modified function provides an elegant way to determine the sizes and multiplicities of the Jordan blocks in a sector uniquely. We have

$$Z_{L,M}(q) = \sum_j N_j [j]_q = N_1\, q^0 + N_2 \left(q^{-\frac{1}{2}} + q^{\frac{1}{2}}\right) + N_3 \left(q^{-1} + q^0 + q^1\right) + \ldots, \tag{6.3.40}$$

where N_j is the number of Jordan blocks of length j. $[j]_q$ is a modified q-number

$$[j]_q = \frac{q^{j/2} - q^{-j/2}}{q^{1/2} - q^{-1/2}} = \sum_{k=\frac{-j+1}{2}}^{\frac{j-1}{2}} q^k. \tag{6.3.41}$$

For example $Z_{7,3}(q)$ and $Z_{9,5}(q)$ can also be written

$$Z_{7,3}(q) = [1]_q + [5]_q + [9]_q, \tag{6.3.42}$$

$$Z_{9,5}(q) = [1]_q + 2[5]_q + 2[9]_q + [11]_q + [13]_q + [17]_q, \tag{6.3.43}$$

reflecting the Jordan block structures $(9, 5, 1)$ and $(17, 13, 11, 9^2, 5^2, 1)$ respectively.

Interestingly, a generating function which generates the Jordan block spectrum of all of $V^{L,M,1}$ can be obtained as a trace over the Hilbert space.

$$\mathcal{Z}_{L,M}(q) = \operatorname{tr} q^{\hat{S} - S_{\max}/2} = L \begin{bmatrix} L-1 \\ M-1 \end{bmatrix}_q. \tag{6.3.44}$$

This trace implements the counting of states explicitly, rather than using the q-binomial coefficient. $\hat{S}$ is a counting operator which acts on elementary states with well-defined values of S

$$\hat{S}|S\rangle = S|S\rangle, \tag{6.3.45}$$

and is extended by linearity.

Cyclicity classes. We briefly note that instead of considering states in $U^j W^{L,M}$ where the ϕ_3 field is in a fixed position, we could have considered states in any cyclicity class k. If we replaced the states $|j_1 j_2 \cdots j_{L-1} 3\rangle \to \mathcal{C}_k |j_1 j_2 \cdots j_{L-1} 3\rangle$ for any $k = 0, 1, \ldots, L-1$ the arguments of this section are unchanged because $[H, \mathcal{C}_k] = 0$, where $\mathcal{C}_k$ is the unnormalised projector defined in (6.2.15). Therefore the Jordan normal form of H is the same in $W^{L,M}$ and $V_k^{L,M,1}$ for any k. The distinction between static and cyclic bases is more intricate for $K > 1$.

6.3.2 Generalisation to Many Walls

Here we discuss the extension of the previous section to sectors with many walls, i.e. $K > 1$. The main observation is that $K > 1$ states behave essentially like a tensor product of K states with $K = 1$. Any elementary state $v \in V^{L,M,K}$ ending in a 3 can be written

$$v = v_1 \otimes v_2 \otimes \cdots \otimes v_K, \tag{6.3.46}$$

where $v_i \in W^{\ell_i + m_i + 1, m_i + 1}$ are elementary states themselves. We defined $W^{L,M}$ above (6.3.2). ℓ_i denotes the number of 1's in v_i and m_i denotes the number of 2's. The hypereclectic Hamiltonian H acts on states of the form (6.3.46) as

$$Hv = Hv_1 \otimes v_2 \otimes \cdots \otimes v_K + v_1 \otimes Hv_2 \otimes \cdots \otimes v_K + \cdots + v_1 \otimes v_2 \otimes \cdots \otimes Hv_K. \tag{6.3.47}$$

We define $\boldsymbol{\ell} \coloneqq (\ell_1, \ldots, \ell_K)$ and $\boldsymbol{m} \coloneqq (m_1, \ldots, m_K)$, which should satisfy

$$\sum_{i=1}^{K} \ell_i = L - M = L_1, \qquad \sum_{i=1}^{K} m_i = M - K = M_1. \tag{6.3.48}$$

We will denote the spaces $\bigotimes_{i=1}^{K} W^{\ell_i + m_i + 1, m_i + 1}$ as *subsectors*, and picking the vectors $\boldsymbol{\ell}, \boldsymbol{m}$ corresponds to a choice of subsector. We consider subsectors $(\boldsymbol{\ell}, \boldsymbol{m})$ satisfying (6.3.48) which are unique up to application of the translation operator U^j. In practise this means we identify $(\boldsymbol{\ell}, \boldsymbol{m}) \sim (\boldsymbol{\ell}', \boldsymbol{m}')$ if $\boldsymbol{\ell}, \boldsymbol{\ell}'$ and $\boldsymbol{m}, \boldsymbol{m}'$ are related by the same cyclic permutation σ^n

$$(\boldsymbol{\ell}, \boldsymbol{m}) \sim (\boldsymbol{\ell}', \boldsymbol{m}') \quad \longleftrightarrow \quad (\boldsymbol{\ell}', \boldsymbol{m}') = (\sigma^n \boldsymbol{\ell}, \sigma^n \boldsymbol{m}), \tag{6.3.49}$$

$$\sigma(\ell_1, \ell_2, \ldots, \ell_K) \equiv (\ell_2, \ldots, \ell_K, \ell_1). \tag{6.3.50}$$

In this way we can describe all the states in $V^{L,M,K}$ using the translation operator U. Overall we have

$$V^{L,M,K} = \bigoplus_{(\boldsymbol{\ell}, \boldsymbol{m})/\sim} \bigoplus_{j=1}^{L/S_{\boldsymbol{l},\boldsymbol{m}}} U^j \bigotimes_{i=1}^{K} W^{\ell_i + m_i + 1, m_i + 1}, \tag{6.3.51}$$

where we introduced the *symmetry factor* for a subsector $S_{\boldsymbol{l},\boldsymbol{m}}$. The symmetry factor reflects the fact that some subsectors are especially symmetric with respect to cyclicity. This occurs when there is an $n < K$ such that

$$(\sigma^n \boldsymbol{\ell}, \sigma^n \boldsymbol{m}) = (\boldsymbol{\ell}, \boldsymbol{m}), \tag{6.3.52}$$

where σ is the cyclic permutation defined in (6.3.50). In this case we give the subsector a symmetry factor $S_{\boldsymbol{\ell},\boldsymbol{m}} = K/n$. For example, let $L = 14, M = 8, K = 4$ and take the subsector $\boldsymbol{\ell} = (2,1,2,1), \boldsymbol{m} = (1,1,1,1)$. We have $\sigma^2 \boldsymbol{\ell} = \boldsymbol{\ell}$ and $\sigma^2 \boldsymbol{m} = \boldsymbol{m}$ and so $S_{\boldsymbol{\ell},\boldsymbol{m}} = 4/2 = 2$ in this case.

Example: $\boldsymbol{L = 7, M = 4, K = 2}$. We begin with the simple example $L = 7, M = 4, K = 2$. In this sector there are three ϕ_1 fields, two ϕ_2 fields, two ϕ_3 fields and $\frac{7!}{3!2!2!} = 210$ total states. In table 6.1 we show the 6 inequivalent choices of $(\boldsymbol{\ell}, \boldsymbol{m})$, which corresponds to the 6 ways to decompose the states into $K = 1$ states, on which H acts block diagonally.

	Form of state	Number of states	$\boldsymbol{\ell}, \boldsymbol{m}$	JNF
1	$\|111223\rangle \otimes \|3\rangle$	$10 \times 1 = 10$	$(3,0),(2,0)$	$7 \oplus 3$
2	$\|11123\rangle \otimes \|23\rangle$	$4 \times 1 = 4$	$(3,0),(1,1)$	4
3	$\|11223\rangle \otimes \|13\rangle$	$6 \times 1 = 6$	$(2,1),(2,0)$	$5 \oplus 1$
4	$\|1123\rangle \otimes \|123\rangle$	$3 \times 2 = 6$	$(2,1),(1,1)$	$3 \otimes 2$
5	$\|1223\rangle \otimes \|113\rangle$	$3 \times 1 = 3$	$(1,2),(2,0)$	3
6	$\|1113\rangle \otimes \|223\rangle$	$1 \times 1 = 1$	$(3,0),(0,2)$	1

Table 6.1: Decomposition of $L = 7, M = 4, K = 2$ states into $K = 1$ states. The 3's should be regarded as fixed, whereas the 1's and 2's can be permuted within their ket.

All subsectors except for **4** behave trivially as a single $K = 1$ sector under the action of H. Their Jordan normal forms were determined in the previous section and are listed in the table. We look at states of the form **4** in a bit more detail. These states have the form of an $L = 4, M = 2, K = 1$ state and an $L = 3, M = 2, K = 1$ state glued together, which consist of a single Jordan block of size 3 and 2 respectively. The natural 'anti-locked' state comes from gluing together the anti-locked states of the respective $K = 1$ parts $|2113213\rangle$. We act successively on this state with H

$$\begin{aligned} &|2113213\rangle \to |1213213\rangle + |2113123\rangle \\ &\to |1123213\rangle + 2|1213123\rangle \to 3|1123123\rangle \to 0, \end{aligned} \tag{6.3.53}$$

which is a Jordan block of length 4. There is a further Jordan block of length 2 obtained by making the ansatz for a new top state

$$\gamma_1|1213213\rangle + \gamma_2|2113123\rangle, \tag{6.3.54}$$

and similarly to the last section this gives a Jordan block of length 2 for $\gamma_1 = -1, \gamma_2 = 2$. Thus the Jordan decomposition of the subsector **4** is $(4,2)$. Since the Jordan decompositions of the $K = 1$ sectors are (3) and (2) respectively, we denote this as $3 \otimes 2 = 4 \oplus 2$.

At the level of generating functions, we can deduce the Jordan normal form of the 'tensor product' sectors by multiplying the generating functions of the corresponding $K = 1$ sectors. For example, for the subsector **4** we have

$$Z^{\boldsymbol{4}}_{7,4,2}(q) = Z_{4,2}(q)Z_{3,2}(q) = (q^{-1} + 1 + q)(q^{-1/2} + q^{1/2}) \\ = q^{-3/2} + 2q^{-1/2} + 2q^{1/2} + q^{3/2}, \tag{6.3.55}$$

from which the Jordan normal form $(4, 2)$ can be easily deduced using (6.3.40). To obtain the full generating function for each of the subsectors in $L = 7, M = 4, K = 2$ we can simply add the generating functions for each of the subsectors $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{6}$

$$Z_{7,4,2}(q) = \sum_{\boldsymbol{i}=\mathbf{1}}^{\mathbf{6}} Z^{\boldsymbol{i}}_{7,4,2}(q) \\ = q^{-3} + 2q^{-2} + 2q^{-3/2} + 4q^{-1} + 3q^{-1/2} + 6 + 3q^{1/2} + 4q + 2q^{3/2} + 2q^2 + q^3. \tag{6.3.56}$$

Using (6.3.40) leads to the following Jordan normal form:

$$\mathrm{JNF}_{7,4,2} = (7, 5, 4^2, 3^2, 2, 1^2). \tag{6.3.57}$$

In this sector there are no subtleties with cyclicity and the rest of the Hilbert space can be exhausted by application of the translation operator U^j, $j = 1, \ldots, 6$. For each j we have the same argument as before, so the full Jordan block structure can be obtained as seven copies of (6.3.57)

$$\mathrm{JNF}^{\mathrm{tot}}_{7,4,2} = (7^7, 5^7, 4^{14}, 3^{14}, 2^7, 1^{14}). \tag{6.3.58}$$

At the level of the generating function this can be obtained by multiplying (6.3.56) by $L = 7$. However, there are cases where cyclic symmetry leads to some subtleties, as we discuss next.

Example: $\boldsymbol{L = 8, M = 4, K = 2}$. Let us consider the case of $L = 8, M = 4, K = 2$. There are $\frac{8!}{4!4!2!} = 420$ states in this sector. Therein one finds an $(\boldsymbol{\ell}, \boldsymbol{m})$ subsector that is symmetric with respect to cyclicity. In table 6.2 we break the states into $K = 1$ states as in the previous section, where we replaced $4 \otimes 2 = 5 \oplus 3$ and $3 \otimes 3 = 5 \oplus 3 \oplus 1$

	Form of state	Number of states	$\boldsymbol{\ell}, \boldsymbol{m}$	Jordan decomposition
1	$\|1111223\rangle \otimes \|3\rangle$	$15 \times 1 = 15$	$(4,0), (2,0)$	$9 \oplus 5 \oplus 1$
2	$\|111223\rangle \otimes \|13\rangle$	$10 \times 1 = 10$	$(3,1), (2,0)$	$7 \oplus 3$
3	$\|111123\rangle \otimes \|23\rangle$	$5 \times 1 = 5$	$(4,0), (1,1)$	5
4	$\|11123\rangle \otimes \|123\rangle$	$4 \times 2 = 8$	$(3,1), (1,1)$	$4 \otimes 2 = 5 \oplus 3$
5	$\|11223\rangle \otimes \|113\rangle$	$6 \times 1 = 6$	$(2,2), (2,0)$	$5 \oplus 1$
6	$\|11113\rangle \otimes \|223\rangle$	$1 \times 1 = 1$	$(4,0), (0,2)$	1
7	$\|1123\rangle \otimes \|1123\rangle$	$3 \times 3 = 9$	$(2,2), (1,1)$	$3 \otimes 3 = 5 \oplus 3 \oplus 1$
8	$\|1223\rangle \otimes \|1113\rangle$	$3 \times 1 = 3$	$(1,3), (2,0)$	3

Table 6.2: Decomposition of $L = 8, M = 4, K = 2$ states into $K = 1$ states.

by multiplying the appropriate $K = 1$ generating functions and naively extracting the resulting Jordan block structures using (6.3.40). We see that $\mathbf{7}$ is the subsector where the issues with cyclicity emerge. For the other subsectors we can exhaust the rest of the state space by acting with $U^j, j = 1, \dots, 7$. However for subsector $\mathbf{7}$ applying the translation U^4 maps the states to a state in the same subsector, which reflects the fact this subsector has a symmetry factor $S_{\boldsymbol{\ell},\boldsymbol{m}} = 2$. Therefore acting with $U^j, j = 0, 1, \dots, 7$ leads to a double counting by a factor of 2. We can realise this at the level of an overall generating function for the $L = 8, M = 4, K = 2$ sector by multiplying by $1/S_{\boldsymbol{\ell},\boldsymbol{m}} = 1/2$ for the subsector $\mathbf{7}$

$$\mathcal{Z}_{8,4,2}(q) = 8(Z^{\mathbf{1}}_{8,4,2} + Z^{\mathbf{2}}_{8,4,2} + Z^{\mathbf{3}}_{8,4,2} + Z^{\mathbf{4}}_{8,4,2} + Z^{\mathbf{5}}_{8,4,2} + Z^{\mathbf{6}}_{8,4,2} + \frac{1}{2}Z^{\mathbf{7}}_{8,4,2} + Z^{\mathbf{8}}_{8,4,2}). \quad (6.3.59)$$

We compute (6.3.59) to be

$$\mathcal{Z}_{8,4,2}(q) = 8q^{-4} + 16q^{-3} + 52q^{-2} + 80q^{-1} + 108 + 80q + 52q^2 + 16q^3 + 8q^4. \quad (6.3.60)$$

Using (6.3.40) we identify the Jordan normal form to be

$$\mathrm{JNF}^{\mathrm{tot}}_{8,4,2} = (9^8, 7^8, 5^{36}, 3^{28}, 1^{28}). \quad (6.3.61)$$

General $\boldsymbol{L, M, K}$. Here we generalise the above observations to arbitrary L, M, K sectors. Given an L, M, K sector we consider a subsector $\bigotimes_{i=1}^K W^{\ell_i+m_i+1, m_i+1}$ defined by the vectors $\boldsymbol{\ell}, \boldsymbol{m}$. The anti-locked state takes the form

$$\Omega = |(2\cdots21\cdots1)_1\mathbf{3}(2\cdots21\cdots1)_2\mathbf{3}\cdots(2\cdots21\cdots1)_K\mathbf{3}\rangle, \quad (6.3.62)$$

where (ℓ_j, m_j) are the numbers of 1's and 2's in the j^{th} bracket. Recall that we have

$$\sum_{j=1}^K \ell_j = L_1 = L - M, \quad \sum_{j=1}^K m_j = M_1 = M - K. \quad (6.3.63)$$

As for $K = 1$, we can grade the vector space by the action of H

$$\bigotimes_{i=1}^K W^{\ell_i+m_i+1, m_i+1} = \bigoplus_{S=0}^{S_{\max}} W_S^{\boldsymbol{\ell},\boldsymbol{m}}, \quad (6.3.64)$$

where $W^{\boldsymbol{\ell},\boldsymbol{m}}_{S_{\max}}$ is spanned by the anti-locked state and H lowers the level $S \to S-1$. By acting successively with H on Ω, we will arrive at the locked state

$$|(1\cdots12\cdots2)_1\mathbf{3}(1\cdots12\cdots2)_2\mathbf{3}\cdots(1\cdots12\cdots2)_K\mathbf{3}\rangle. \quad (6.3.65)$$

There will be many different configurations in the middle with lower values of S. For the anti-locked state we have

$$S = S_{\max} = \boldsymbol{\ell}\cdot\boldsymbol{m} = \sum_{j=1}^K \ell_j m_j, \quad (6.3.66)$$

and so the size of the largest Jordan block in each subsector is $S_{\max} + 1$. If we define the number of actions of H on the j^{th} bracket as n_j, a general state has a level

$$S = \sum_{j=1}^K s_j = S_{\max} - N, \qquad s_j = \ell_j m_j - n_j, \quad N = \sum_{j=1}^K n_j, \quad \text{with} \quad 0 \le n_j \le \ell_j m_j. \quad (6.3.67)$$

The anti-locked state has $S = S_{\max}$ (or $N = 0$) and the locked state has $S = 0$ (or $N = S_{\max}$).

Now consider states obtained by acting with H N times on the anti-locked state,

$$H^N \Omega = \sum_{n_1=0}^{\ell_1 m_1} \cdots \sum_{n_K=0}^{\ell_K m_K} |H^{n_1}(2\cdots 21\cdots 1)\mathbf{3}H^{n_2}(2\cdots 21\cdots 1)\mathbf{3}\cdots H^{n_K}(2\cdots 21\cdots 1)\mathbf{3}\rangle. \tag{6.3.68}$$

The number of elementary states generated by each $H^{n_j}(2\cdots 21\cdots 1)$ was found in section 6.3.1 to be $\mathbf{d}_{l_j m_j - n_j}(\ell_j, m_j)$, which appeared as a coefficient of the q-binomial $\binom{\ell_j+m_j}{m_j}_q$ as defined in (6.3.24). Therefore we can compute the number of elementary states at each level S to be

$$\mathbf{D}^{\boldsymbol{\ell},\boldsymbol{m}}_{S_{\max}-N} \equiv \dim W^{\boldsymbol{\ell},\boldsymbol{m}}_{S_{\max}-N} = \sum_{n_1=0}^{\ell_1 m_1} \cdots \sum_{n_K=0}^{\ell_K m_K} \prod_{j=1}^{K} \mathbf{d}_{l_j m_j - n_j}(\ell_j, m_j), \qquad \text{with} \quad \sum_{j=1}^{K} n_j = N. \tag{6.3.69}$$

This can be recast into a generating function

$$\begin{aligned} \bar{Z}^{\boldsymbol{\ell},\boldsymbol{m}}(q) &= \sum_{N=0}^{S_{\max}} \mathbf{D}^{\boldsymbol{\ell},\boldsymbol{m}}_{S_{\max}-N}\, q^N = \sum_{N=0}^{S_{\max}} \left[\sum_{n_1=0}^{\ell_1 m_1} \cdots \sum_{n_K=0}^{\ell_K m_K} \delta_{N, \sum_{i=1}^K n_i} \prod_{j=1}^{K} \mathbf{d}_{l_j m_j - n_j}(\ell_j, m_j) \right] q^N \\ &= \sum_{n_1=0}^{\ell_1 m_1} \cdots \sum_{n_K=0}^{\ell_K m_K} \prod_{j=1}^{K} \left[\mathbf{d}_{l_j m_j - n_j}(\ell_j, m_j) q^{n_j} \right] = \prod_{j=1}^{K} \left[\prod_{k=1}^{m_j} \frac{1 - q^{\ell_j + m_j + 1 - k}}{1 - q^k} \right] \end{aligned} \tag{6.3.70}$$

using the expression for $K = 1$ in (6.3.24). This may be expressed through q-binomials as

$$\bar{Z}^{\boldsymbol{\ell},\boldsymbol{m}}(q) = \prod_{j=1}^{K} \binom{l_j + m_j}{m_j}_q. \tag{6.3.71}$$

This proves that the generating function for an $\boldsymbol{\ell}, \boldsymbol{m}$ subsector is simply a product of the corresponding $K = 1$ generating functions. For example, if we take $L = 13, M = 7, K = 3$ and consider the subsector $\boldsymbol{\ell} = (3, 2, 1), \boldsymbol{m} = (2, 1, 1)$ we find

$$\begin{aligned} \bar{Z}^{\boldsymbol{\ell},\boldsymbol{m}}(q) &= \tbinom{5}{2}_q \tbinom{3}{1}_q \tbinom{2}{1}_q = \textstyle\sum_{N=0}^{9} \mathbf{D}^{\boldsymbol{l},\boldsymbol{m}}_{9-N} q^N \\ &= 1 + 3q + 6q^2 + 9q^3 + 11q^4 + 11q^5 + 9q^6 + 6q^7 + 3q^8 + q^9. \end{aligned} \tag{6.3.72}$$

Analagously to the $K = 1$ case, we can use (6.3.14) to determine the Jordan block spectrum in this subsector

$$\mathrm{JNF}^{\boldsymbol{\ell},\boldsymbol{m}}_{13,7,3} = (2^2, 4^3, 6^3, 8^2, 10). \tag{6.3.73}$$

Since the states belonging to a given partition $\boldsymbol{\ell}, \boldsymbol{m}$ of (L_1, M_1) are not mixed with those in a different partition, the total Jordan block spectrum is just direct sum of all the spectrum sets.

One can sum over all inequivalent partitions formally. For this purpose, it is necessary to use the modified q-binomial coefficients defined in (6.3.37)

$$Z^{\boldsymbol{\ell},\boldsymbol{m}}(q) = \prod_{j=1}^{K} \begin{bmatrix} \ell_j + m_j \\ m_j \end{bmatrix}_q = \prod_{j=1}^{K} q^{-\ell_j m_j/2} \binom{l_j + m_j}{m_j}_q. \tag{6.3.74}$$

For each $\boldsymbol{\ell}, \boldsymbol{m}$ subsector we can exhaust the rest of the state space by acting with the translation operator $U^j, j = 1, \ldots, L-1$. The arguments of this section do not change in these cases, and so the overall generating function for a subsector can be obtained by simply multiplying it by L. The only exception is $\boldsymbol{\ell}, \boldsymbol{m}$ subsectors which have a symmetry factor $S_{\boldsymbol{\ell},\boldsymbol{m}} \neq 1$. Adjusting for this possibility, we can define the generating function for a whole L, M, K sector as a sum over inequivalent partitions

$$\mathcal{Z}_{L,M,K}(q) = \sum_{(\boldsymbol{\ell},\boldsymbol{m})/\sim} \frac{L}{S_{\boldsymbol{\ell},\boldsymbol{m}}} Z^{\boldsymbol{\ell},\boldsymbol{m}}(q). \tag{6.3.75}$$

This total generating function gives the complete Jordan block spectrum, as in (6.3.40):

$$\mathcal{Z}_{L,M,K}(q) = \sum_j N_j [j]_q = N_1\, q^0 + N_2 \left(q^{-\frac{1}{2}} + q^{\frac{1}{2}}\right) + N_3 \left(q^{-1} + q^0 + q^1\right) + \ldots. \tag{6.3.76}$$

As for the $K = 1$ case, $\mathcal{Z}_{L,M,K}(q)$ can alternatively be computed as a trace over the entire Hilbert space

$$\mathcal{Z}_{L,M,K}(q) = \operatorname{tr} q^{\hat{S} - \hat{S}_{\max}/2}, \tag{6.3.77}$$

where $\hat{S}$ measures the level S of an elementary state and $\hat{S}_{\max}$ measures $S_{\max} = \boldsymbol{\ell} \cdot \boldsymbol{m}$ of a state in an $\boldsymbol{\ell}, \boldsymbol{m}$ subsector. Both operators are extended to the full Hilbert space by linearity. We can define $\hat{S}' \equiv \hat{S} - \hat{S}_{\max}/2$ for brevity.

Cyclicity classes. The expression (6.3.77), which can also be expressed as (6.3.75), gives a generating function that describes the Jordan block spectrum of the hypereclectic model in an arbitrary sector of operators defined by L, M, K. However, in certain circumstances it might be useful to compute the Jordan block spectrum in a specific cyclicity class k, for example the cyclic sector $k = 0$ relevant to quantum field theory. In this case, the formula (6.3.77) still applies

$$\mathcal{Z}^k_{L,M,K}(q) = \operatorname{tr}_k q^{\hat{S}'}, \tag{6.3.78}$$

where we take care to trace only over states of a fixed cyclicity k.

6.3.3 Universality

In the previous sections we described a method to find the full Jordan block spectrum of the hypereclectic model, as opposed to the more interesting eclectic model. However, we claim a *universality hypothesis*: The Jordan block spectrum of the eclectic model for generic couplings ξ_1, ξ_2, ξ_3 is identical to that of the hypereclectic model, provided L, M, K satisfy

$$L_1 = L - M \geq K, \qquad M_1 = M - K \geq K. \tag{6.3.79}$$

(6.3.79) implies that the number of ϕ_3 fields in the sector does not exceed the number of ϕ_1's or ϕ_2's. Without loss of generality we can further take

$$L - M \geq M - K \geq K. \tag{6.3.80}$$

In this section we provide evidence that this conjecture is true, and in appendix C.2 we prove it for $K = 1$. Throughout this section we will consider (6.3.80) to be satisfied, otherwise we can simply relabel the fields so that it is. It is possible to fine-tune the couplings to break down the Jordan block structure in certain cyclicity classes, as discussed in appendix C.3. Since the ϕ_3 fields no longer act as walls it is useful to work with states of a fixed cyclicity k, see section 6.2.2. For definiteness in the following examples we will restrict to the cyclic sector $k = 0$, the case relevant to single trace operators in quantum field theory.

Eclectic Spin Chain and Level S. Recall that for elementary states in $K = 1$ sectors we defined a level S, which corresponds to the total number of 1's to the right of each of the 2's in a state. Here we work with cyclic states

$$|j_1 j_2 \cdots j_{L-1} 3\rangle_0 = \mathcal{C}_0 |j_1 j_2 \cdots j_{L-1} 3\rangle = \sum_{j=0}^{L-1} U^j |j_1 j_2 \cdots j_{L-1} 3\rangle. \tag{6.3.81}$$

We define S in an analogous manner for states of the form (6.3.81). For example the state $|1211213\rangle_0$ has $S = 4$. Let us define V_S to be the vector subspace of $V^{L,M,1}$ spanned by cyclic states with level S.[4] We saw previously that the hypereclectic Hamiltonian maps states in V_S to V_{S-1}

$$H_3 : V_S \to V_{S-1}, \qquad H_3 V_0 = 0. \tag{6.3.82}$$

Let us investigate the action of the full eclectic Hamiltonian $H_{\text{ec}} = H_1 + H_2 + H_3$ on the vector spaces V_S. We find that

$$H_1 : V_S \to V_{S-L_1}, \qquad H_2 : V_S \to V_{S-M_1}. \tag{6.3.83}$$

Since $L_1 \geq M_1 \geq 1$ (6.3.83) implies that H_1 and H_2 decrease S for a state by a greater than or equal amount to H_3. This already makes plausible that they will not affect the Jordan normal form of H_3, since H_2 and H_1 will annihilate states faster than H_3. For example, consider the anti-locked state $|221113\rangle_0 \in V_6$ for $L = 6, M = 3, K = 1$, so that $L_1 = 3, M_1 = 2$. Then

$$\begin{aligned} H_1 |221113\rangle_0 &= |211123\rangle_0 \in V_3, \\ H_2 |221113\rangle_0 &= |122113\rangle_0 \in V_4, \\ H_3 |221113\rangle_0 &= |212113\rangle_0 \in V_5. \end{aligned} \tag{6.3.84}$$

Example: $L = 7, M = 3, K = 1$. Let us consider the eclectic model for $L = 7, M = 3, K = 1$. In the hypereclectic model this sector has the Jordan block spectrum $(9, 5, 1)$ in $W^{9,5}$. Here we show that the eclectic model has the same Jordan block spectrum in the cyclic sector.

The anti-locked state in the cyclic sector $|2211113\rangle_0 \in V_8$ again determines a Jordan block of length 9. The first descendant of the anti-locked state is

$$H_{\text{ec}} |2211113\rangle_0 = \xi_1 |2111123\rangle_0 + \xi_2 |1221113\rangle_0 + \xi_3 |2121113\rangle_0. \tag{6.3.85}$$

[4] We suppress the L, M dependence of V_S.

Note that the coefficients of ξ_1, ξ_2, and ξ_3 are states with $S = 4, 6$, and 7 respectively, which reflects equations (6.3.82) and (6.3.83). In general acting with a power of H_{ec} on $|2211113\rangle_0$ gives

$$H_{\text{ec}}^n|2211113\rangle_0 = H_3^n|2211113\rangle_0 \ + \ \text{lower } S \text{ states.} \tag{6.3.86}$$

It is then easy to see that $H^9|2211113\rangle_0 = 0$ and thus $|2211113\rangle_0$ is the top state for a Jordan block of length 9, as before.

In the hypereclectic case the top state of the next Jordan block is

$$\psi^{(6)} = -9|1221113\rangle_0 + 5|2112113\rangle_0 \in V_6, \tag{6.3.87}$$

which satisfies $H_3^5\psi^{(6)} = 0$. Thus $\psi^{(6)}$ determines a Jordan block of length 5 for H_3. However, in this case things are a bit trickier in the eclectic model. We have

$$H_{\text{ec}}^5\psi^{(6)} = 15\xi_2\xi_3^4|1111223\rangle_0 \neq 0. \tag{6.3.88}$$

It is however possible to modify the top state (6.3.87) by adding states of lower S, such that the residual term (6.3.88) vanishes. In this case it is sufficient to add states with $S = 5$ to $\psi^{(6)}$. Since $\dim V_5 = 2$ we can add 2 states, to arrive at a new top state

$$\chi^{(6)} = \psi^{(6)} + \gamma_1|1212113\rangle_0 + \gamma_2|2111213\rangle. \tag{6.3.89}$$

This state satisfies

$$H_{\text{ec}}^5\chi^{(6)} = (-15\xi_2 + (5\gamma_1 + 4\gamma_2)\xi_3)\xi_3^4|1111223\rangle_0, \tag{6.3.90}$$

which vanishes for $5\gamma_1 + 4\gamma_2 = 15\xi_2/\xi_3$. Note that this defines a one-parameter family of top states. Therefore the eclectic model also has a Jordan block of length 5 in this sector, with a slightly modified top state (6.3.89) which contains lower $S = 5$ states.

In the hypereclectic model the top state for the final Jordan block is

$$\psi^{(4)} = |2111123\rangle_0 - |1211213\rangle_0 + |1122113\rangle_0 \in V_4, \tag{6.3.91}$$

which satisfies $H_3\psi^{(4)} = 0$ and thus determines a Jordan block of length 1. The action of the eclectic Hamiltonian on this state gives a residual

$$H_{\text{ec}}\psi^{(4)} = -\xi_1|1111223\rangle_0 - \xi_2|1112213\rangle_0 - \xi_2|1121123\rangle_0 \neq 0, \tag{6.3.92}$$

which consists of states with $S = 0$ and $S = 2$. As before we can eliminate this residual by adding states of lower S to the top state (6.3.91). We first try to add states with $S = 3$, and since $\dim V_3 = 2$ we add 2 states

$$\chi^{(4)} = \psi^{(4)} + \alpha_1|1121213\rangle_0 + \alpha_2|1211123\rangle_0. \tag{6.3.93}$$

We check that for $\alpha_1 = \xi_2/\xi_3, \alpha_2 = -2\xi_2/\xi_3$ the $S = 2$ states in the residual (6.3.92) vanish

$$H_{\text{ec}}\chi^{(4)} = -\xi_1|1111223\rangle_0 + \frac{\xi_2^2}{\xi_3}|1112123\rangle_0, \tag{6.3.94}$$

however we find a new residual consisting of an $S = 1$ and an $S = 0$ state. These can be removed by adding $S = 2$ states into the top state ansatz

$$\bar{\chi}^{(4)} = \chi^{(4)} + \beta_1|1112213\rangle_0 + \beta_2|1121123\rangle_0, \tag{6.3.95}$$

and setting $\beta_1 = \xi_1/\xi_2, \beta_2 = -\xi_1/\xi_2 - \xi_2^2/\xi_3^2$. With these choices for α_i and β_i we have

$$H_{\text{ec}}\bar{\chi}^{(4)} = 0, \tag{6.3.96}$$

and so we have identified the Jordan block of length one in this sector of the eclectic model. In summary, by taking a top state for the hypereclectic model at a level S, we can manufacture a top state (of a Jordan block of the same length) for the eclectic model by adding appropriate combinations of states with lower values of S. We will argue that it is always possible to add these states of lower S, thus rendering the Jordan block spectra of the hypereclectic and eclectic models equivalent.

General Argument for $K = 1$. Here we sketch a proof of universality for $K = 1$, where the filling condition (6.3.79) is trivially satisfied, if all three particles are present. We find it useful to first introduce the notion of *supereclectic* models. These are intermediate models between the eclectic model H_{ec} and the hypereclectic model H_3, defined by setting only a single coupling ξ_1 or ξ_2 equal to zero

$$H_{\text{super},i} = H_i + H_3, \qquad i = 1, 2. \tag{6.3.97}$$

For both of these cases it is possible to prove rigorously that $H_{\text{super},i}$ has the same Jordan normal form as H_3 for generic couplings. The general strategy of the proof is reminiscent of the example given in the warmup example above. For the hypereclectic model, at a level S satisfying $\mathbf{d}_S > \mathbf{d}_{S+1}$ we can construct $\mathbf{d}_S - \mathbf{d}_{S+1}$ top states

$$\psi^{(S)} = \sum_{j=1}^{\mathbf{d}_S} \alpha_j^{(S)} e_j^{(S)}, \tag{6.3.98}$$

where $\alpha_j^{(S)}$ are known coefficients and $e_j^{(S)}$ are the elementary states at level S. $\psi^{(S)}$ is the top state for a Jordan block of length $S - \tilde{S} + 1$

$$H_3^{S-\tilde{S}+1}\psi^{(S)} = 0, \tag{6.3.99}$$

where we recall $\tilde{S} = S_{\max} - S = (L - M)(M - 1) - S$. We show that it is always possible to modify the state by adding a linear combination of states with lower S

$$\psi^{(S)} \to \psi_i^{(S)} = \psi^{(S)} + \sum_{n=0}^{S-1} \varphi^{(n)} \tag{6.3.100}$$

where $\varphi^{(n)} \in V_n$. The modified state is a top state for a Jordan block of the same length in the supereclectic model $H_{\text{super},i}$

$$H_{\text{super},i}^{S-\tilde{S}+1}\psi_i^{(S)} = 0, \tag{6.3.101}$$

which renders the Jordan normal forms of $H_{\text{super},i}$ and H_3 equivalent for generic couplings. The technical details of this proof are given in appendix C.2. This argument can then be slightly modified to motivate that the Jordan normal forms of H_{ec} and H_3 are also equivalent.

Universality for $\boldsymbol{K > 1}$. It is more complicated to show universality for $K > 1$. One main difference from the $K = 1$ case of the supereclectic models, as explained in appendix C.2, is that the action of h_j on $\varphi^{(S)}$ in general generates several states with differing S-values. If we interpret $\hat{S}(h_j\varphi^{(S)})$ in (C.2.4) as the largest among these and replace L_1 with the associated ℓ_j, the same logic should be valid, so that one can construct for the supereclectic models all subleading states in (6.3.69).

For the eclectic model, however, the critical simplification used in (C.2.22) is not valid. While we obtained numerical evidence for universality in the general case, we were unable to provide a proof.

This concludes the portion of the thesis which describes our main research results. We conclude with a summary of our work and some outlook.

Chapter 7

Conclusions and Outlook

In this thesis we studied several aspects of Feynman integrals and correlation functions, all of which have some connection to the (dynamical) fishnet theory, which represents a strong-twist limit of the celebrated $\mathcal{N} = 4$ super Yang–Mills theory.

After a detailed review of the core concepts, we studied the action of the conformal group on Minkowski space, in the explicit setting of the conformal box integral. We gave a geometric explanation of the breaking of conformal invariance, and explicitly quantified it by classifying the set of conformally equivalent configurations of four points. We showed explicitly that the functional form of the box integral in kinematic regions different from the Euclidean region can differ from the Bloch–Wigner function by discontinuities thereof. In view of the Osterwalder-Schrader theorem this is not a surprise, and indeed the functional form of any locally conformally invariant correlation function should differ from its form in the Euclidean region by some discontinuity thereof. However, in which variable the discontinuity should be taken probably needs to be studied on a case by case basis. The conformal box integral is a very simple setting to see this mechanism in detail.

We also investigated the extent to which the box can be constrained using its Yangian invariance. The space of Yangian invariants in this case turned out to be spanned by the Bloch–Wigner function and its discontinuities. Using discrete symmetries we were able to get surprisingly far in constraining the box integral in each kinematic region, and indeed we could fix it up to twelve undetermined constants. It would be interesting to find a way to fix these constants without resorting to an explicit analytic continuation from the Euclidean region, as we did. Can it be done in a way more in line with integrability, for example by studying the star-triangle relation in each kinematic region?

Although in this chapter we focused on the case of the one-loop box integral, it would be interesting to look at higher-point/higher-loop conformal integrals directly in Minkowski space, study their analytic properties in different kinematic regions, and analyse the extent to which conformal symmetry is broken. Is there a systematic way to determine which discontinuities in the cross-ratios one needs to take to recover the functional expression away from the Euclidean region? Can higher-point analogues of the double-infinity configurations we introduced be useful in this regard? Natural examples to look at would be the higher loop Basso–Dixon graphs and the conformal n-gons. The conformal pentagon would be an interesting place to start, since its analytic form is already known, see (2.3.59). Since there are five independent cross ratios in this case, the analytic structure is certainly more intricate.

We proceeded to study the Basso–Dixon integrals, which represent exact correlators in the fishnet theory, and contribute to several observables in $\mathcal{N}=4$ SYM. Although they have a very elegant functional form in terms of polylogs, this has not yet been derived purely using integrability. If the fishnet theory is integrable in the planar limit (and there is much evidence for this) then a derivation of the Basso–Dixon formula based on integrability would be very natural. We initiated the study of the action of the Yangian algebra on the Basso–Dixon graphs. While the higher-point fishnet graphs are exactly Yangian invariant, there are issues in taking the four-point limit to the Basso–Dixon graphs, which lead to an inhomogeneity on the right hand side of the invariance equation. We analysed this inhomogeneity in detail, and derived the Yangian Ward identities for the Basso–Dixon integrals, which we numerically verified up to four loops.

We generalised these identities to the integrable two-parameter family of fishnet models proposed by Olivucci and Kazakov. We found that these equations separate neatly when specified to two dimensions, which lead to one of the simplest incarnations of the Yangian bootstrap, in the case of the two-dimensional conformal box integral. In particular, we solved the resulting ordinary differential equations by separation of variables in terms of Legendre functions. In fact, the Basso–Dixon graphs were calculated in 2D using Sklyanin's separation of variables in [190]. It would be very interesting to investigate whether there is a link between these facts, and how to make this explicit.

At the moment it is unclear how to *solve* our Yangian Ward identities for the conformal functions $\phi_{\alpha\beta}$, due to the presence of the inhomogeneity. It would be very interesting to devise a method to solve these identities, for example using a hypergeometric ansatz for a family of Basso–Dixon graphs with general propagator powers. Another direction to explore is the action of the higher-level Yangian generators on the Basso–Dixon graphs. Although level-one invariance is destroyed beyond the box integral, it is still possible that the higher-loop graphs are annihilated by the higher-level generators. If this is the case, this would lead to higher-order PDEs in z and $\bar{z}$ which $\phi_{\alpha\beta}$ satisfy. A natural conjecture would be that $\phi_{\alpha\beta}$ is annihilated by $\widehat{\mathrm{J}}^a_{(n)}$, where $n=\max(\alpha,\beta)$.

Notably, our derivation of the Ward identities relies on the Yangian invariance of correlation functions in the fishnet theory. Since we verified the resulting equations on many Feynman integrals, there is strong evidence for their validity. Still it would be important to derive the initial invariance of correlators from first principles, e.g. from a Yangian invariance of the action. This would require to extend the methods of [120] developed for $\mathcal{N}=4$ SYM, such that they can also be applied to the class of fishnet theories.

Interestingly, our results for the two-dimensional box integral are single-valued in the complex variable z. Although a theory of single-valued harmonic polylogs is by now well-developed, there is no such theory for functions of an elliptic nature. It would be interesting to construct a class of elliptic functions which are single-valued in the sense of Bloch–Wigner, as these would likely play an important role in the blossoming subject of elliptic Feynman integrals.

We then changed direction and looked at a different aspect of correlation functions in strongly-twisted $\mathcal{N}=4$ SYM: namely we studied the one-loop dilatation operator in a three scalar sector of the *dynamical* fishnet theory. This operator, although integrable in the Yang–Baxter sense, is non-diagonalisable and has so far resisted all attempts at a solution from a Bethe ansatz approach. Using combinatorial arguments, we introduced an elegant generating function which fully classifies the Jordan block spectrum of the related hypereclectic model. For a single 'wall', this function can be expressed as a q-deformation of the binomial coefficient $\binom{L-1}{M-1}$, where L is the length of the spin chain and $L-M$ is the number of excitations of type 2. For higher numbers of walls the generating function can be written as a sum of products of q-binomials. This sum involves an unwieldy sum over subsectors with a symmetry factor $S_{\ell,m}$. The formula was already improved in [202] to remove this factor, and which used Pólya counting explicitly to describe the cyclic states.

The elegance of these formulas suggests that there is some connection to integrability, which was not used in their derivation. Is the q of the generating function related to the spectral parameter u of the transfer matrix? The q parameter of a quantum deformed group? Can the non-trivial algebraic relations generated by the Yang–Baxter equation be used in any way?

It is important to note that our derivation for the Jordan block structure of the eclectic model relies on the universality hypothesis and the lack of so-called unexpected shortening. While we have good evidence that both of these points are valid, it would be valuable to provide a rigorous proof.

Since the sizes and multiplicities of the Jordan blocks determine the form of the two-point functions in logarithmic conformal field theory, it would be interesting to explicitly calculate, using Feynman diagrams, the two-point functions in a logarithmic multiplet, and verify that logarithms appear in the precise manner predicted by [33]. One could start by looking at the rank-two multiplet which appears in the cyclic sector for $L=3, M=2, K=1$. One could then study if wrapping has any effect on the Jordan block structures for lower lengths.

It has been suggested in [203] that logarithmic multiplets in fishnet theory are protected i.e. they do not receive any corrections beyond one loop. It would be interesting to prove this explicitly in the case of the eclectic model. It seems plausible, given that the dilatation operator at higher loops appears to decrease the S value of states faster than the one-loop dilatation operator, and so doesn't affect the Jordan normal form.

A final interesting conceptual question is whether the Jordan block spectrum of other non-diagonalisable spin chains, integrable or not, can be described by similar generating functions. Or is this particular to the (hyper)eclectic spin chain? There would be a few natural ways to test this. For example, one could study the dilatation operator in other non-diagonalisable sectors of (dynamical) fishnet theory. These sectors could contain derivative fields/fermions, and would be more intricate to analyse. There are also different strong twist limits of $\mathcal{N}=4$ SYM available, which should contain new non-diagonalisable models, see [6]. One could also consider the dilatation operator in the strong twist limit of ABJM theory [194]. In this case the first quantum correction to the dilatation operator appears at two loops, and this would probably be a chiral version of the alternating spin chain given in [145].

Finally, it would be important to leverage all the insights obtained about the integrability of the bi-scalar fishnet theory, and apply this to understand the origin of integrability in $\mathcal{N}=4$ SYM. It seems plausible that conformal symmetry is the driving force behind the integrability of this theory, given that this is the only symmetry retained in the strong-twist limit to the fishnet theory. A starting point to understanding how to transport this information is to understand it for the three-parameter dynamical fishnet model. A study of the correlation functions of this theory was initiated in [167], although an interpretation in terms of integrable conformal spin chains is missing.

Appendix A

Conformal Feynman Parametrisations

In this appendix we give explicit Feynman parametrisations for scalar ladder integrals and the vector ladder integrals appearing in this thesis, particularly in chapter 5. We show some details of the calculation for the vector double ladder integral but otherwise we just state the results.

A.1 Scalar Ladders

We give conformal Feynman parametrisations for the scalar ladders $\phi_{\ell 1}$, obtained as specialisations of (5.2.1) and (5.2.3), and cases with more general propagator powers.

Box. The Feynman parametrisation for the conformal box function reads

$$\phi_{11}(u,v) = \int_0^\infty \mathrm{d}^2\alpha \frac{1}{(1+\alpha_1 u + \alpha_2 v) D_{\alpha_1\alpha_2}}, \tag{A.1.1}$$

where $D_{\alpha_1\alpha_2} \coloneqq \alpha_1 + \alpha_2 + \alpha_1\alpha_2$.

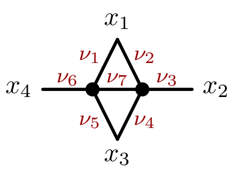
Double Ladder. For the double ladder, the Feynman parametrisation of the conformal function is

$$\phi_{21}(u,v) = \int_0^\infty \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{1}{(1+\beta_1 u + \beta_2 v) D_{\beta_1\beta_2} D_{\alpha_1\alpha_2}}, \tag{A.1.2}$$

where $\int_0^\beta \mathrm{d}^2\alpha \coloneqq \int_0^{\beta_1} \mathrm{d}\alpha_1 \int_0^{\beta_2} \mathrm{d}\alpha_2$. For the modified double ladder defined in (5.3.50) we have

$$\phi_{21}^{\omega D}(u,v) = \frac{1}{\Gamma_\omega^2 \Gamma_{D/2-\omega}^2} \int_0^\infty \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{[(\beta_1-\alpha_1)(\beta_2-\alpha_2)\alpha_1\alpha_2]^{\omega-1}}{(1+\beta_1 u + \beta_2 v)^{D/2-\omega}(D_{\beta_1\beta_2} D_{\alpha_1\alpha_2})^{\omega}}. \tag{A.1.3}$$

The double ladder for fully generic conformal propagator powers (5.3.4) can be written as

$I_{21}^{\nu,D} =$ 

$= V_{2,1}^{\nu,D} \phi_{21}^{\nu,D}(u,v)$, (A.1.4) where $V_{2,1}^{\nu,D} = x_{12}^{-\nu_1-\nu_2-\nu_3+\nu_4+\nu_5+\nu_6} x_{14}^{-\nu_1-\nu_2-\nu_6+\nu_3+\nu_4+\nu_5} x_{13}^{-\nu_4-\nu_5} x_{24}^{-\nu_3-\nu_4-\nu_5-\nu_6+\nu_1+\nu_2}$. The conformal function can be Feynman parametrised as

$$\phi_{21}^{\nu,D}(u,v) = c_{21}^{\nu,D} \int_0^\infty \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{(\beta_1-\alpha_1)^{\nu_5-1}(\beta_2-\alpha_2)^{\nu_1-1}\alpha_1^{\nu_4-1}\alpha_2^{\nu_2-1}}{(1+\beta_1 u+\beta_2 v)^{\nu_6}(D_{\beta_1\beta_2})^{D/2-\nu_6}(D_{\alpha_1\alpha_2})^{D/2-\nu_7}}, \tag{A.1.5}$$

where the prefactor is written as

$$c_{21}^{\nu,D} = v^{\nu_2-\nu_5} \frac{\Gamma_{D/2-\nu_7}\Gamma_{D/2-\nu_6}}{\prod_{i\neq 6}\Gamma_{\nu_i}}. \tag{A.1.6}$$

Triple Ladder. The triple ladder Feynman parametrisation is given by

$$\phi_{31}(u,v) = \int_0^\infty \mathrm{d}^2\gamma \int_0^\gamma \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{1}{(1+\gamma_1 u+\gamma_2 v)D_{\gamma_1\gamma_2}D_{\beta_1\beta_2}D_{\alpha_1\alpha_2}}. \tag{A.1.7}$$

For the modified triple ladder which appears in the theory (5.3.48) we have

$$\phi_{31}^{\omega D}(u,v) = \frac{1}{\Gamma_\omega^3\Gamma_{\bar\omega}^3} \int_0^\infty \mathrm{d}^2\gamma \int_0^\gamma \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{[(\gamma_1-\beta_1)(\gamma_2-\beta_2)(\beta_1-\alpha_1)(\beta_2-\alpha_2)\alpha_1\alpha_2]^{\omega-1}}{(1+\gamma_1 u+\gamma_2 v)^{\bar\omega}(D_{\gamma_1\gamma_2}D_{\beta_1\beta_2}D_{\alpha_1\alpha_2})^\omega} \tag{A.1.8}$$

where we remind $\bar\omega = D/2-\omega$.

***ℓ*-ladder.** The pattern persists for general ℓ-ladders whose conformal functions are Feynman parametrised by

$$\phi_{\ell 1}(u,v) = \int_0^\infty \mathrm{d}^2\alpha_n \left(\prod_{i=1}^{n-1}\int_0^{\alpha_{i+1}} \mathrm{d}^2\alpha_i\right) \frac{1}{(1+u\alpha_{n,1}+v\alpha_{n,2})(\prod_{j=1}^n D_{\alpha_{j,1}\alpha_{j,2}})}, \tag{A.1.9}$$

and in the generalised case we have

$$\phi_{\ell 1}^{\omega D} = \frac{1}{\Gamma_\omega^\ell\Gamma_{\bar\omega}^\ell} \int_0^\infty \mathrm{d}^2\alpha_n \left(\prod_{i=1}^{n-1}\int_0^{\alpha_{i+1}} \mathrm{d}^2\alpha_i\right) \frac{\left[\prod_{k=2}^{\ell}(\alpha_{k,1}-\alpha_{k-1,1})(\alpha_{k,2}-\alpha_{k-1,2})\alpha_{1,1}\alpha_{1,2}\right]^{\omega-1}}{(1+u\alpha_{n,1}+v\alpha_{n,2})^{\bar\omega}(\prod_{j=1}^n D_{\alpha_{j,1}\alpha_{j,2}})^\omega}. \tag{A.1.10}$$

A.2 Vector Ladders

Here we provide Feynman parametrisations for the vector ladder integrals.

Double Ladder. We derive the conformal Feynman parametrisations of the vector integral coefficients of $I_{21}^{\mu,2}$, defined in (5.2.19). We start with

$$\begin{aligned} x_{13}^4 x_{24}^2 I_{21}^{\mu,2} &= \int \frac{\mathrm{d}^4x_a}{\pi^2}\frac{\mathrm{d}^4x_b}{\pi^2} \frac{x_{13}^4x_{24}^2x_{b1}^\mu}{x_{a1}^2x_{a3}^2x_{a4}^2x_{ab}^2x_{b1}^4x_{b2}^2x_{b3}^2} \\ &= \int \frac{\mathrm{d}^4x_b}{\pi^2}\frac{x_{13}^4x_{24}^2x_{b1}^\mu}{x_{b1}^4x_{b2}^2x_{b3}^2}\int\frac{\mathrm{d}^4x_a}{\pi^2}\frac{1}{x_{ab}^2x_{a1}^2x_{a3}^2x_{a4}^2}. \end{aligned} \tag{A.2.1}$$

The second integral is just a scalar box. With Feynman parameters (and subsequently evaluating one of the parameter integrals) it evaluates to [67]

$$\begin{aligned}&\int \frac{\mathrm{d}^4 x_a}{\pi^2} \frac{1}{x_{ab}^2 x_{a1}^2 x_{a3}^2 x_{a4}^2}\\ &= \int_0^\infty [\mathrm{d}^2\alpha] \frac{1}{(\alpha_1 x_{b1}^2 + \alpha_3 x_{b3}^2 + \alpha_4 x_{b4}^2)(\alpha_1\alpha_3 x_{13}^2 + \alpha_1\alpha_4 x_{14}^2 + \alpha_3\alpha_4 x_{34}^2)},\end{aligned} \tag{A.2.2}$$

where $[\mathrm{d}^2\alpha]$ denotes a projective integral over $\alpha_1, \alpha_3, \alpha_4$. For example we could take $[\mathrm{d}^2\alpha] = \mathrm{d}\alpha_1 \mathrm{d}\alpha_3 \delta(\alpha_4 - 1)$. The second factor of the integrand does not depend on x_b so we omit it for now. We need to compute

$$\int \frac{\mathrm{d}^4 x_b}{\pi^2} \frac{x_{b1}^\mu}{x_{b1}^4 x_{b2}^2 x_{b3}^2 (\alpha_1 x_{b1}^2 + \alpha_3 x_{b3}^2 + \alpha_4 x_{b4}^2)}, \tag{A.2.3}$$

where we freely exchange loop and parametric integrals. Using Feynman parameters this is

$$\frac{\Gamma_5}{\pi^2} \int [\mathrm{d}^3\gamma] \int \mathrm{d}^4 x_b \frac{\gamma_1 x_{b1}^\mu}{(\gamma_1 x_{b1}^2 + \gamma_2 x_{b2}^2 + \gamma_3 x_{b3}^2 + \gamma_4(\alpha_1 x_{b1}^2 + \alpha_3 x_{b3}^2 + \alpha_4 x_{b4}^2))^5}, \tag{A.2.4}$$

The denominator of (A.2.4) can be written as

$$\sigma^5 \left(l^2 + \frac{\Delta}{\sigma^2} \right)^5, \tag{A.2.5}$$

where

$$\begin{aligned}\sigma =&\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4(\alpha_1 + \alpha_3 + \alpha_4),\\ l^\mu =&x_b^\mu - \frac{x_1^\mu(\gamma_1 + \gamma_4\alpha_1) + x_2^\mu \gamma_2 + x_3^\mu(\gamma_3 + \alpha_3\gamma_4) + x_4^\mu \gamma_4\alpha_4}{\sigma},\\ \Delta =&(\gamma_1 + \gamma_4\alpha_1)\gamma_2 x_{12}^2 + (\gamma_1 + \gamma_4\alpha_1)(\gamma_3 + \gamma_4\alpha_3)x_{13}^2 + (\gamma_1 + \gamma_4\alpha_1)\gamma_4\alpha_4 x_{14}^2\\ &+ \gamma_2(\gamma_3 + \gamma_4\alpha_3)x_{23}^2 + \gamma_2\gamma_4\alpha_4 x_{24}^2 + (\gamma_3 + \gamma_4\alpha_3)\gamma_4\alpha_4 x_{34}^2.\end{aligned} \tag{A.2.6}$$

The numerator x_{b1}^μ can be written

$$x_{b1}^\mu = l^\mu - \frac{1}{\sigma} \sum_{i=2}^4 B_i x_{1i}^\mu, \tag{A.2.7}$$

where

$$B_2 = \gamma_2, \qquad B_3 = \gamma_3 + \gamma_4\alpha_3, \qquad B_4 = \gamma_4\alpha_4. \tag{A.2.8}$$

The integral over the l^μ piece vanishes because it is an odd integrand, so (A.2.3) reduces to

$$-\sum_{i=2}^4 x_{1i}^\mu \int_0^\infty [\mathrm{d}^3\gamma] \int_0^\infty \mathrm{d}l \frac{24\gamma_1 l B_i}{\sigma^6(l^2 + \frac{\Delta}{\sigma^2})}, \tag{A.2.9}$$

where we integrate over $l := |l^\mu|$. Performing the integral over l leads to

$$\frac{1}{2} x_{13}^4 x_{24}^2 I_{21}^{\mu,2} = -\sum_{i=2}^4 x_{1i}^\mu \int [\mathrm{d}^2\alpha][\mathrm{d}^3\gamma] \frac{x_{13}^4 x_{24}^2 \gamma_1 B_i}{(\alpha_1\alpha_3 x_{13}^2 + \alpha_1\alpha_4 x_{14}^2 + \alpha_3\alpha_4 x_{34}^2)\Delta^3}. \tag{A.2.10}$$

We choose to perform the γ_2 integral, and make the rescaling of Feynman parameters

$$\begin{aligned} &\alpha_1 \to x_{34}^2\alpha_1, && \alpha_3 \to x_{14}^2\alpha_3, && \alpha_4 \to x_{13}^2\alpha_4, \\ &\gamma_1 \to \frac{x_{34}^2}{x_{13}^2x_{24}^2}\gamma_1, && \gamma_3 \to \frac{x_{14}^2}{x_{13}^2x_{24}^2}\gamma_3, && \gamma_4 \to \frac{1}{x_{13}^2x_{24}^2}\gamma_4. \end{aligned} \tag{A.2.11}$$

This rescaling ensures that the resulting integrands are manifestly conformal invariant [66]. We finally de-project $\alpha_4 = \gamma_4 = 1$ and make the change of variables $\gamma_i = \beta_i - \alpha_i$ for $i = 1, 3$. This leads to the required form

$$x_{13}^4 x_{24}^2 I_{21}^{\mu,2} = -\frac{x_{12}^\mu}{x_{12}^2}F_2(u,v) - \frac{x_{13}^\mu}{x_{13}^2}F_3(u,v) - \frac{x_{14}^\mu}{x_{14}^2}F_4(u,v), \tag{A.2.12}$$

with (we further relabel $\alpha_3, \beta_3 \to \alpha_2, \beta_2$)

$$F_2(u,v) = u\int_0^\infty \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{\beta_1 - \alpha_1}{(1+\beta_1 u + \beta_2 v)^2 D_{\beta_1\beta_2} D_{\alpha_1\alpha_2}}, \tag{A.2.13}$$

$$F_3(u,v) = \int_0^\infty \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{\beta_2(\beta_1 - \alpha_1)}{(1+\beta_1 u + \beta_2 v)(D_{\beta_1\beta_2})^2 D_{\alpha_1\alpha_2}}, \tag{A.2.14}$$

$$F_4(u,v) = \int_0^\infty \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{\beta_1 - \alpha_1}{(1+\beta_1 u + \beta_2 v)(D_{\beta_1\beta_2})^2 D_{\alpha_1\alpha_2}}, \tag{A.2.15}$$

where we remind that $\int_0^\beta \mathrm{d}^2\alpha$ is shorthand for $\int_0^{\beta_1} \mathrm{d}\alpha_1 \int_0^{\beta_2} \mathrm{d}\alpha_2$ and $D_{\alpha_1\alpha_2} = \alpha_1 + \alpha_2 + \alpha_1\alpha_2$.

Triple Ladder. Here we have

$$\begin{aligned} x_{13}^6 x_{24}^2 I_{31}^{\mu,3} &= -\frac{x_{12}^\mu}{x_{12}^2}G_2(u,v) - \frac{x_{13}^\mu}{x_{13}^2}G_3(u,v) - \frac{x_{14}^\mu}{x_{14}^2}G_4(u,v), \\ x_{13}^6 x_{24}^2 I_{31}^{\mu,2} &= -\frac{x_{12}^\mu}{x_{12}^2}\bar{G}_2(u,v) - \frac{x_{13}^\mu}{x_{13}^2}\bar{G}_3(u,v) - \frac{x_{14}^\mu}{x_{14}^2}\bar{G}_4(u,v), \end{aligned} \tag{A.2.16}$$

where

$$\begin{aligned} G_2(u,v) &= \int_0^\infty \mathrm{d}^2\gamma \int_0^\gamma \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{u(\gamma_1 - \beta_1)}{(1+u\gamma_1 + v\gamma_2)^2 D_{\gamma_1\gamma_2} D_{\beta_1\beta_2} D_{\alpha_1\alpha_2}}, \\ G_3(u,v) &= \int_0^\infty \mathrm{d}^2\gamma \int_0^\gamma \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{\gamma_2(\gamma_1 - \beta_1)}{(1+u\gamma_1 + v\gamma_2) D_{\gamma_1\gamma_2}^2 D_{\beta_1\beta_2} D_{\alpha_1\alpha_2}}, \\ G_4(u,v) &= \int_0^\infty \mathrm{d}^2\gamma \int_0^\gamma \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{\gamma_1 - \beta_1}{(1+u\gamma_1 + v\gamma_2) D_{\gamma_1\gamma_2}^2 D_{\beta_1\beta_2} D_{\alpha_1\alpha_2}}, \end{aligned} \tag{A.2.17}$$

For $\bar{G}_i$ we have

$$\bar{G}_i = \int_0^\infty [\mathrm{d}^2\alpha][\mathrm{d}^2\beta][\mathrm{d}^2\gamma] \frac{E_i}{\Lambda_1\Lambda_2\Lambda_3^2\Lambda_4}, \tag{A.2.18}$$

where

$$E_2 = u\gamma\beta_4, \qquad E_3 = \gamma_3 + u\beta_3\gamma + v\alpha_3\bar{\gamma}, \qquad E_4 = v\bar{\gamma}\alpha_2, \tag{A.2.19}$$

and

$$
\begin{aligned}
\Lambda_1 &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \\
\Lambda_2 &= \beta_1\beta_3 + \beta_1\beta_4 + \beta_3\beta_4, \\
\Lambda_3 &= \gamma_3 + v\bar{\gamma}(\alpha_2 + \alpha_3) + u\gamma(\beta_3 + \beta_4), \\
\Lambda_4 &= (\gamma\beta_1 + \bar{\gamma}\alpha_1)F_3 + (\gamma_3 + u\gamma\beta_3 + v\bar{\gamma}\alpha_3)(\bar{\gamma}\alpha_2 + \gamma\beta_4) + \gamma\bar{\gamma}\alpha_2\beta_4.
\end{aligned} \tag{A.2.20}
$$

Quadruple Ladder. Here we can write

$$
\begin{aligned}
x_{13}^8 x_{24}^2 I_{41}^{\mu,4} &= -\frac{x_{12}^\mu}{x_{12}^2} V_2(u,v) - \frac{x_{13}^\mu}{x_{13}^2} V_3(u,v) - \frac{x_{14}^\mu}{x_{14}^2} V_4(u,v), \\
x_{13}^8 x_{24}^2 I_{41}^{\mu,3} &= -\frac{x_{12}^\mu}{x_{12}^2} \bar{V}_2(u,v) - \frac{x_{13}^\mu}{x_{13}^2} \bar{V}_3(u,v) - \frac{x_{14}^\mu}{x_{14}^2} \bar{V}_4(u,v),
\end{aligned} \tag{A.2.21}
$$

where

$$
\begin{aligned}
V_2(u,v) &= \int_0^\infty \mathrm{d}^2\delta \int_0^\delta \mathrm{d}^2\gamma \int_0^\gamma \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{u(\delta_1 - \gamma_1)}{(1 + u\delta_1 + v\delta_2)^2 D_{\delta_1\delta_2} D_{\gamma_1\gamma_2} D_{\beta_1\beta_2} D_{\alpha_1\alpha_2}}, \\
V_3(u,v) &= \int_0^\infty \mathrm{d}^2\delta \int_0^\delta \mathrm{d}^2\gamma \int_0^\gamma \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{\delta_2(\delta_1 - \gamma_1)}{(1 + u\delta_1 + v\delta_2) D_{\delta_1\delta_2}^2 D_{\gamma_1\gamma_2} D_{\beta_1\beta_2} D_{\alpha_1\alpha_2}}, \\
V_4(u,v) &= \int_0^\infty \mathrm{d}^2\delta \int_0^\delta \mathrm{d}^2\gamma \int_0^\gamma \mathrm{d}^2\beta \int_0^\beta \mathrm{d}^2\alpha \frac{(\delta_1 - \gamma_1)}{(1 + u\delta_1 + v\delta_2) D_{\delta_1\delta_2}^2 D_{\gamma_1\gamma_2} D_{\beta_1\beta_2} D_{\alpha_1\alpha_2}}.
\end{aligned} \tag{A.2.22}
$$

For $\bar{V}_i$ we have

$$
\bar{V}_i = \int_0^\infty [\mathrm{d}^2\alpha][\mathrm{d}^2\beta][\mathrm{d}^2\gamma][\mathrm{d}^2\delta] \frac{\tilde{E}_i}{\tilde{\Lambda}_1\tilde{\Lambda}_2\tilde{\Lambda}_3^2\tilde{\Lambda}_4\tilde{\Lambda}_5}, \tag{A.2.23}
$$

where here

$$
\tilde{E}_2 = uv\bar{\delta}\gamma\beta_2, \qquad \tilde{E}_3 = v(\delta_3 + v(\alpha_3\delta + \bar{\delta}\gamma_3) + u\bar{\delta}\gamma\beta_3), \qquad \tilde{E}_4 = v^2\delta\alpha_4, \tag{A.2.24}
$$

and morevover

$$
\begin{aligned}
\tilde{\Lambda}_1 &= \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_3\alpha_4, \\
\tilde{\Lambda}_2 &= \beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3, \\
\tilde{\Lambda}_3 &= u\bar{\delta}\gamma(\beta_2 + \beta_3) + v(\alpha_3\delta + \bar{\delta}\gamma_3 + \alpha_4\delta) + \delta_3, \\
\tilde{\Lambda}_4 &= \alpha_4\beta_2\gamma\delta\bar{\delta} + \beta_2\gamma\bar{\delta}u\left(\alpha_1\delta + \bar{\delta}\left(\beta_1\gamma + \gamma_1\right)\right) + \beta_2\gamma\bar{\delta}\left(\delta_3 + \beta_3\gamma\bar{\delta}u + \alpha_3\delta v + \gamma_3\bar{\delta}v\right) \\
&\quad + \alpha_4\delta\left(\delta_3 + \beta_3\gamma\bar{\delta}u + \alpha_3\delta v + \gamma_3\bar{\delta}v\right) + \alpha_4\delta v\left(\alpha_1\delta + \bar{\delta}\left(\beta_1\gamma + \gamma_1\right)\right) \\
&\quad + \left(\alpha_1\delta + \bar{\delta}\left(\beta_1\gamma + \gamma_1\right)\right)\left(\delta_3 + \beta_3\gamma\bar{\delta}u + \alpha_3\delta v + \gamma_3\bar{\delta}v\right), \\
\tilde{\Lambda}_5 &= u\gamma((\gamma_1 + \gamma\beta_1)(\beta_2 + \beta_3) + \gamma\beta_2\beta_3) + v\gamma_3(\gamma_1 + \gamma\beta_1 + \gamma\beta_2).
\end{aligned} \tag{A.2.25}
$$

Appendix B

Box Integral Extras

B.1 Kinematic Regions

We list explicitly in table B.1 the elements of K_i for $i = 1, 2, 3, 4$. The elements of $\bar{K}_i$ can be obtained by flipping all the signs of K_i.

K_1	K_2
$\begin{bmatrix}++\\++\\++\end{bmatrix}, \begin{bmatrix}-+\\+-\\-+\end{bmatrix}, \begin{bmatrix}-+\\-+\\+-\end{bmatrix}, \begin{bmatrix}+-\\-+\\-+\end{bmatrix}$ $\begin{bmatrix}+-\\+-\\+-\end{bmatrix}, \begin{bmatrix}++\\--\\--\end{bmatrix}, \begin{bmatrix}--\\--\\-+\end{bmatrix}, \begin{bmatrix}--\\++\\--\end{bmatrix}$	$\begin{bmatrix}-+\\++\\++\end{bmatrix}, \begin{bmatrix}++\\+-\\-+\end{bmatrix}, \begin{bmatrix}++\\-+\\+-\end{bmatrix}, \begin{bmatrix}--\\-+\\-+\end{bmatrix}$ $\begin{bmatrix}--\\+-\\+-\end{bmatrix}, \begin{bmatrix}-+\\--\\--\end{bmatrix}, \begin{bmatrix}+-\\--\\++\end{bmatrix}, \begin{bmatrix}+-\\++\\--\end{bmatrix}$
$K3$	$K4$
$\begin{bmatrix}++\\-+\\++\end{bmatrix}, \begin{bmatrix}-+\\--\\-+\end{bmatrix}, \begin{bmatrix}-+\\++\\+-\end{bmatrix}, \begin{bmatrix}+-\\++\\-+\end{bmatrix}$ $\begin{bmatrix}+-\\--\\+-\end{bmatrix}, \begin{bmatrix}++\\+-\\--\end{bmatrix}, \begin{bmatrix}--\\+-\\++\end{bmatrix}, \begin{bmatrix}--\\-+\\--\end{bmatrix}$	$\begin{bmatrix}++\\++\\-+\end{bmatrix}, \begin{bmatrix}-+\\+-\\++\end{bmatrix}, \begin{bmatrix}-+\\-+\\--\end{bmatrix}, \begin{bmatrix}+-\\-+\\++\end{bmatrix}$ $\begin{bmatrix}+-\\+-\\--\end{bmatrix}, \begin{bmatrix}++\\--\\+-\end{bmatrix}, \begin{bmatrix}--\\--\\-+\end{bmatrix}, \begin{bmatrix}--\\++\\+-\end{bmatrix}$

Table B.1: Explicit signs of kinematics in K_i.

B.2 $V_{\mathbb{C}} \subset \bar{V}_{1,z,\bar{z}}$ proof.

We prove that configurations $w \in V_{\mathbb{C}}$ are necessarily contained within $\bar{V}_{1,z,\bar{z}}$. By following the arguments leading to (4.2.29) we can find $A_1 \in \mathrm{Conf}(\mathbb{R}^{1,3})$ such that

$$A_1 w = \{0, y_2, y_3, \iota\}. \tag{B.2.1}$$

There are eight possibilities for $(\mathrm{sgn}(y_2^2), \mathrm{sgn}(y_3^2), \mathrm{sgn}(y_{23}^2))$, corresponding to the eight possibilities for the signs of kinematics K_i or $\bar{K}_i$. The only sign assignment which can possibly lead to a conformal plane configuration with $z, \bar{z} \in \mathbb{C} \setminus \mathbb{R}$ is $(- - -)$, which occurs when $w \in \bar{V}_1$. In this case we can rotate/rescale y_3 using $A_2 \in \mathrm{Conf}(\mathbb{R}^{1,3})$ to be the spacelike unit vector $e_3 = (0, 0, 0, 1)$

$$A_2 A_1 w = \{0, z_2, e_3, \iota\}, \tag{B.2.2}$$

where $z_2 = (t, p, q, r) := A_2 y_2$. The stabiliser of 0, e_3, ι is $SO^+(1, 2)$ acting on the first three coordinates of $\mathbb{R}^{1,3}$. The final transformation $A_3 \in \mathrm{Conf}(\mathbb{R}^{1,3})$ mapping w to a Minkowskian conformal plane depends on the sign of $c := t^2 - p^2 - q^2$, which is as yet undetermined because $z_2^2 < 0$. If $\mathrm{sgn}(c) > 0$ then w can be mapped to $w_{--}(a, \eta)$ (see table 4.2), which has $z, \bar{z} \in \mathbb{R}$. If $\mathrm{sgn}(c) < 0$ then w can be mapped to $w_{\mathbb{C}}(r, \phi)$, which has $z, \bar{z} \in \mathbb{C} \setminus \mathbb{R}$. If $(\mathrm{sgn}(y_2^2), \mathrm{sgn}(y_3^2), \mathrm{sgn}(y_{23}^2))$ is not $(- - -)$ it is always possible to find a permutation $\sigma \in S_4$ and a conformal transformation A to map $(0, y_2, y_3, \iota)$ to a Minkowskian conformal plane with real $z, \bar{z}$. Therefore w can only have $z, \bar{z} \in \mathbb{C} \setminus \mathbb{R}$ if $w \in \bar{V}_1$, and so $V_{\mathbb{C}} \subset \bar{V}_{1,z,\bar{z}}$.

B.3 Missing Kinematic Region

We show our numerical procedure for excluding the possibility of configurations $w \in V_1$ with $\mathrm{sgn}(k(w)) = k^* = \left[\begin{smallmatrix}--\\++\end{smallmatrix}\right]$ and $z, \bar{z} \in (0, 1)$. We can bring such an $w = \{x_1, x_2, x_3, x_4\} \in V_1$ to a simpler form using conformal transformations which do not change the signs of the kinematics. We translate x_1 to the origin and rotate/rescale x_2 to e_3. We can then use an $SO^+(1, 2)$ transformation to rotate x_3 into the e_0, e_3 plane. We can finally use an element of $SO(2)$ to eliminate one of the spatial coordinates from x_4. The resulting configuration is

$$w_a = \{y_1, y_2, y_3, y_4\} = \left\{0, e_3, \frac{1}{2}(v_3 + u_3, 0, 0, v_3 - u_3), \frac{1}{2}(v_4 + u_4, 0, h, v_4 - u_4)\right\}, \tag{B.3.1}$$

where u_i, v_i are light cone coordinates and h is the residual spatial coordinate of y_4. The kinematics of w_a are

$$k(w_a) = \begin{bmatrix} -1, (u_3 - u_4)(v_3 - v_4) - \frac{h^2}{4} \\ (1 + u_3)(-1 + v_3), u_4 v_4 - \frac{h^2}{4} \\ u_3 v_3, (1 + u_4)(-1 + v_4) - \frac{h^2}{4} \end{bmatrix}. \tag{B.3.2}$$

w_a is subject to the constraint $\mathrm{sgn}(k(w_a)) = \left[\begin{smallmatrix}--\\++\end{smallmatrix}\right]$. For $h = 0$ w_a is a two-dimensional configuration and it is easy to prove that $z, \bar{z} \in (-\infty, 0)$ or $(1, \infty)$. In this case we have

$$z, \bar{z} = 1 - \frac{u_4(1 + u_3)}{u_3(1 + u_4)}, \; 1 - \frac{v_4(1 - v_3)}{v_3(1 - v_4)}. \tag{B.3.3}$$

Imposing the constraints $\mathrm{sgn}(k(w_a)) = \left[\begin{smallmatrix}--\\++\end{smallmatrix}\right]$ for $h = 0$ one can see that $z, \bar{z} \in (-\infty, 0)$ or $(1, \infty)$, in particular $z, \bar{z} \notin (0, 1)$. For $h \neq 0$ this fact is checked numerically (figure B.1). It was also checked numerically by taking random configurations $w \in V_1$ with $z, \bar{z} \in (0, 1)$, making on the order of 10^7 SCTs to these, and checking $\mathrm{sgn}(k(w))$ after each iteration. Indeed $\mathrm{sgn}(k(w)) = \left[\begin{smallmatrix}--\\++\end{smallmatrix}\right]$ was never observed, as expected.

It is also worth commenting about the 'kinematics reversed' case, where $w \in \bar{V}_1$ and $\mathrm{sgn}(k(w)) = \left[\begin{smallmatrix}++\\--\end{smallmatrix}\right] = Pk^*$. In this case $z, \bar{z} \in (-\infty, 0), z, \bar{z} \in (0, 1), z, \bar{z} \in (1, \infty)$ or $z, \bar{z} \in \mathbb{C}$ are all possible. Similarly to above, any such configuration can be conformally mapped to a simpler configuration w_b without changing the signs of the kinematics

$$w_b = \{z_1, z_2, z_3, z_4\} = \left\{0, \frac{1}{2}(v_2 + u_2, 0, 0, v_2 - u_2), e_3, \frac{1}{2}(v_4 + u_4, 0, h, v_4 - u_4)\right\}. \tag{B.3.4}$$

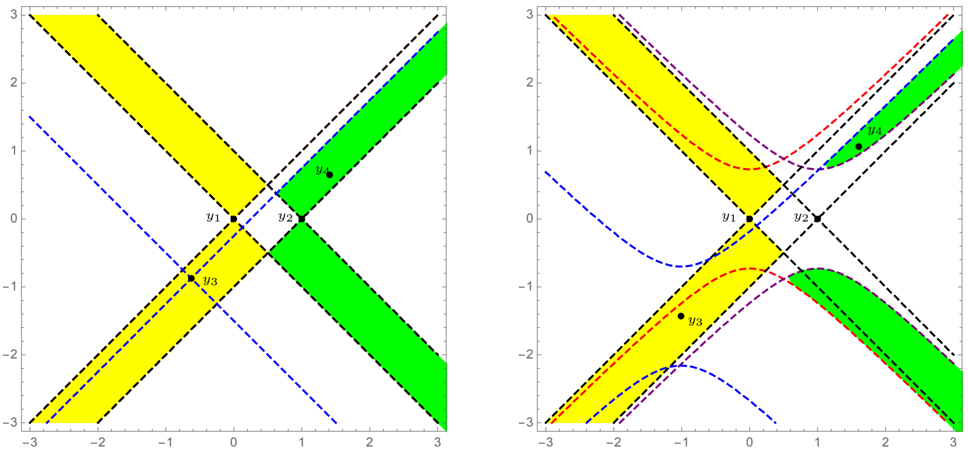


Figure B.1: Configuration w_a. y_3 can be placed anywhere in yellow region, after which y_4 must be placed in green region to enforce $\text{sgn}(k(w_a)) = k^*$. The left picture corresponds to $h = 0$ and the right picture to $h \neq 0$. In both cases for this choice of y_3 placing y_4 in the upper green region gives $z, \bar{z} \in (-\infty, 0)$, and placing y_4 in lower green region gives $z, \bar{z} \in (1, \infty)$. In the right picture the red, purple, and blue dashed lines correspond to the curves $y_{14}^2 = 0, y_{24}^2 = 0$, and $y_{34}^2 = 0$ respectively and reduce to light cones at $h = 0$.

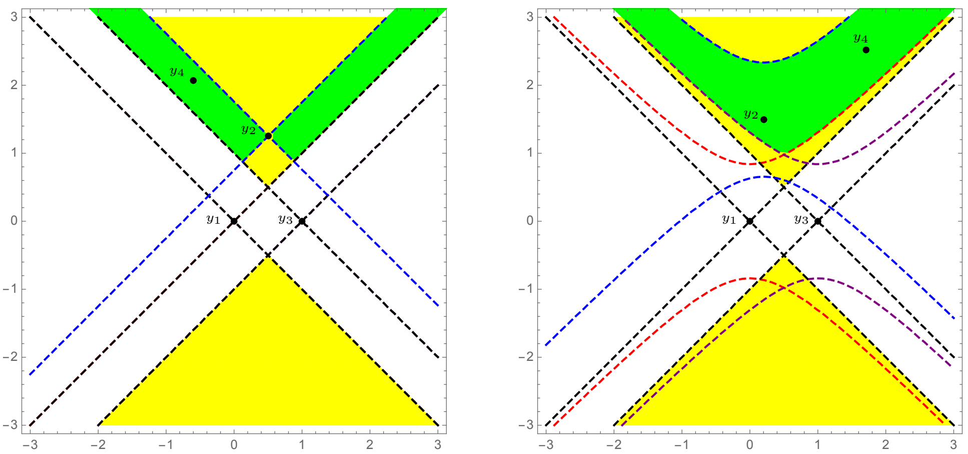


Figure B.2: Configuration w_b. y_2 can be placed anywhere in yellow region (which overlaps with green region), after which y_4 must be placed in green region to enforce $\text{sgn}(k(w_b)) = Pk^*$. The left picture corresponds to $h = 0$ and the right picture to $h \neq 0$. For $h = 0$ for this choice of y_2 placing y_4 in the left green region gives $z, \bar{z} \in (-\infty, 0)$, and placing y_4 in right green region gives $z, \bar{z} \in (1, \infty)$. For $h \neq 0$ the green regions merge and $z, \bar{z} \in (-\infty, 0), z, \bar{z} \in (0, 1), z, \bar{z} \in (1, \infty)$ and $z, \bar{z} \in \mathbb{C}$ are all possible. The red, purple, and blue dashed lines correspond to the curves $y_{14}^2 = 0, y_{34}^2 = 0$, and $y_{24}^2 = 0$ and reduce to light cones at $h = 0$. $z, \bar{z} \in (0, 1)$ is found to be 'rare' and occurs only when y_4 is placed near the intersection of the red and purple curves.

When $h = 0$ in this case it is again easily proved that only $z, \bar{z} \in (-\infty, 0)$ and $z, \bar{z} \in (1, \infty)$ are possible (figure B.2 left). For $h \neq 0$ each of $z, \bar{z} \in (-\infty, 0), z, \bar{z} \in (0, 1), z, \bar{z} \in (1, \infty)$ or $z, \bar{z} \in \mathbb{C}$ are all possible (figure B.2 right). Configurations with $z, \bar{z} \in (0, 1)$ are observed to be much rarer than the other cases.

B.4 Double Infinity Calculation

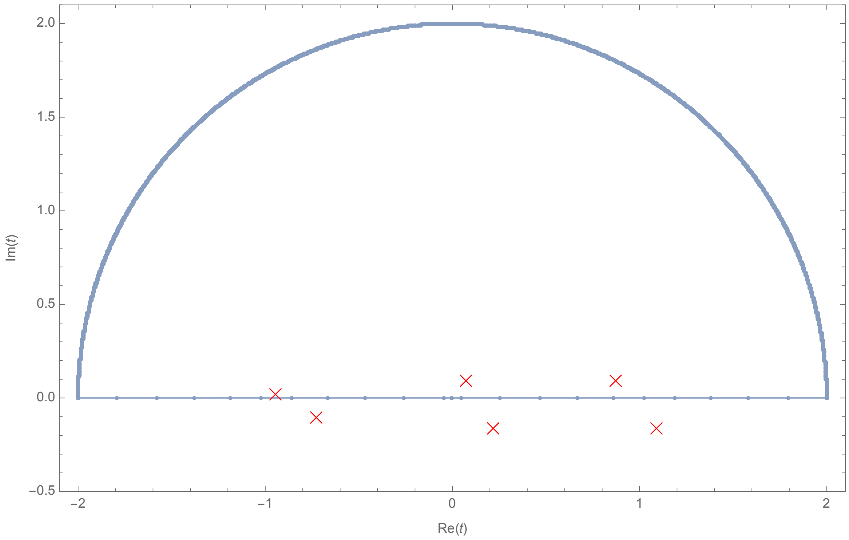


Figure B.3: Semicircular contour in the complex t plane, the radius is taken to infinity. The poles are indicated by red crosses, and the parametric poles are indicated schematically.

After a simple factorisation the box integral (4.3.22) becomes $\phi_{\boldsymbol{X}}^{+-}(\xi_+,\xi_-)=$

$$\int_{x,r,t}\frac{\frac{2i}{\pi}r^2}{(t-r+i\epsilon)(t+r-i\epsilon)(t-1-r+i\epsilon)(t-1+r-i\epsilon)(t+rx-\xi_++i\epsilon')(t-rx-\xi_--i\epsilon')}. \tag{B.4.1}$$

There are six poles of the integrand in the t plane, three in the upper half-plane and three in the lower half-plane. We take a positively oriented semicircular contour, closed at infinity in the upper half-plane (see figure B.3), so that only the poles $t_1=-r+i\epsilon$, $t_2=-r+1+i\epsilon$, $t_3=rx+\xi_-+i\epsilon'$ contribute to the integral. The integrand decreases sufficiently fast as $|t|\to\infty$, so we can use the residue theorem to recover

$$\begin{aligned}\phi_{\boldsymbol{X}}^{+-}(\xi_+,\xi_-)=\int_{x,r}\Bigg(&\frac{r}{(r-(-\frac{1}{2}+i\epsilon))(r-(\frac{-\xi_++i\epsilon}{1-x}))(r-(\frac{-\xi_-+i\epsilon}{1+x}))}\\ &-\frac{r}{(r-(\frac{1}{2}+i\epsilon))(r-(\frac{1-\xi_++i\epsilon}{1-x}))(r-(\frac{1-\xi_-+i\epsilon}{1+x}))}\\ &-\frac{2r^2}{(r-(\frac{-\xi_-+i\epsilon}{1+x}))(r-(\frac{\xi_-+i\epsilon}{1-x}))(r-(\frac{1-\xi_-+i\epsilon}{1+x}))(r-(\frac{\xi_--1+i\epsilon}{1-x}))(r-(\frac{\Delta\xi-i\epsilon'}{2x}))}\Bigg),\end{aligned} \tag{B.4.2}$$

where $\int_{x,r}:=\int_{-1}^{1}dx\int_0^\infty dr$ and $\Delta\xi:=\xi_+-\xi_-$. To perform the r integral we use a key-hole contour C, which comes in from $-\infty$ to 0 below the real axis, travels anticlockwise in an infinitesimal circle around 0, and leaves from 0 to $-\infty$ above the real axis, see figure B.4. If $f(r)$ is a function such that $\log(r)f(r)$ vanishes sufficiently quickly near the origin and at infinity, and $\log(z)$ is the complex logarithm with the branch cut chosen on the negative real axis, then

$$\int_C f(z)\log(z)dz=\int_{-\infty}^{0}(\log(r)+i\pi)f(r)dr+\int_0^{-\infty}(\log(r)-i\pi)f(r)dr=2\pi i\int_{-\infty}^{0}dr\,f(r). \tag{B.4.3}$$

Furthermore, using the residue theorem we can deduce a formula for the integral to be

$$\int_{-\infty}^{0} f(r)dr = \sum_k \mathrm{Res}_{r=r_k} f(r) \log r, \tag{B.4.4}$$

where the sum is over all poles in the complex r plane. There are a number of

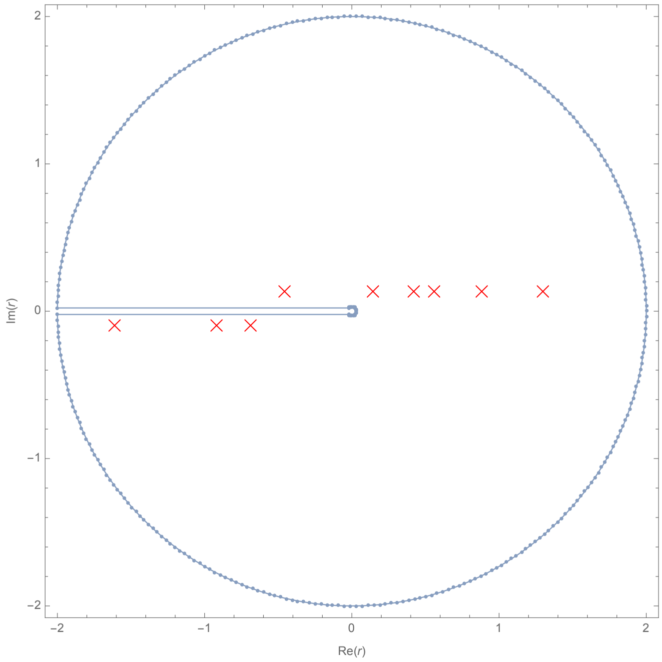


Figure B.4: The Hankel contour in the complex r plane is recovered by sending the radius of the large circle to infinity. The poles are indicated by red crosses, and the parametric poles are indicated schematically.

cancellations upon computing the residues. Indeed making the substitution $r' = -r$ the contributions from the two poles at $r' = \frac{\xi_- - i\epsilon}{1+x}$ cancel, as well as those from the poles at $r' = \frac{\xi_- -1 - i\epsilon}{1+x}$. Therefore after performing the r' integral the expression becomes reasonably compact

$$\begin{aligned}
\phi^{+-}_{\boldsymbol{X}}(\xi_+, \xi_-) = 2\int_{-1}^{1} dx \Bigg(&- \frac{\log(\frac{1}{2})}{(x - (1 - 2\xi_+ + i\epsilon x))(x - (-1 + 2\xi_- + i\epsilon x))} \\
&- \frac{\log(-\frac{1}{2} - i\epsilon)}{(x - (-1 + 2\xi_+ - i\epsilon x))(x - (1 - 2\xi_- - i\epsilon x))} \\
&+ \frac{(\Delta\xi)^2}{(\Box\xi)^2(\Box\xi - 2)^2} \frac{4x \log(\frac{-\Delta\xi + i\epsilon'}{2x})}{(x^2 - (\frac{-\Delta\xi + i\epsilon x}{\Box\xi})^2)(x^2 - (\frac{-\Delta\xi + i\epsilon x}{\Box\xi - 2})^2)} \\
&+ \Bigg(\frac{\xi_+}{\Box\xi} \frac{\log(\frac{\xi_+ - i\epsilon}{1-x})}{(x - (1 - 2\xi_+ + i\epsilon x))(x - (\frac{-\Delta\xi + i\epsilon x}{\Box\xi}))} \\
&+ \frac{\xi_-}{\Box\xi} \frac{\log(\frac{-\xi_- - i\epsilon}{1-x})}{(x - (1 - 2\xi_- - i\epsilon x))(x - (\frac{\Delta\xi - i\epsilon x}{\Box\xi}))} + (\xi_\pm \to 1 - \xi_\mp) \Bigg) \Bigg),
\end{aligned} \tag{B.4.5}$$

where we defined $\Box\xi := \xi_+ + \xi_-$. Note that this expression is checked to be invariant under $\xi_\pm \to 1 - \xi_\mp$ after noting that under this replacement $\Delta\xi \to \Delta\xi$ and $\Box\xi \to 2 - \Box\xi$.

It would be nice to directly integrate the above expression into logs/dilogs, however there is still a small subtlety. The terms $i\epsilon x$ are not of definite sign over the integration range, which is not desirable as the $i\epsilon$ should specify the branch of our final expression. Therefore we split the x integral from -1 to 0 and from 0 to 1. As a toy example we replace

$$\begin{aligned}\int_{-1}^{1}\frac{dx}{x-(a+i\epsilon x)} &= \int_{-1}^{0}\frac{dx}{x-(a-i\epsilon)}+\int_{0}^{1}\frac{dx}{x-(a+i\epsilon)} \\ &= \int_{0}^{1}dx\left(\frac{-1}{x-(-a+i\epsilon)}+\frac{1}{x-(a+i\epsilon)}\right).\end{aligned} \tag{B.4.6}$$

We perform this procedure to each of the five terms in the integral above. It is useful to introduce a shorthand for the combinations that appear in the denominators, so we define

$$r_{1\pm} \coloneqq 1-2\xi_+ \pm i\epsilon, \qquad r_{2\pm} \coloneqq -1+2\xi_- \pm i\epsilon, \tag{B.4.7}$$

$$s_{1\pm} \coloneqq \frac{-\Delta\xi \pm i\epsilon}{\Box\xi}, \qquad s_{2\pm} \coloneqq \frac{-\Delta\xi \pm i\epsilon}{\Box\xi - 2}. \tag{B.4.8}$$

Furthermore for the third term we change variables $x^2 = y$ and split up the logarithm. The resulting expression for the box integral is

$$\begin{aligned}\phi^{+-}_{\boldsymbol{X}}(\xi_+,\xi_-) = 2\int_0^1 dx\Bigg(&\frac{-\log(\frac{1}{2})}{(x-r_{1+})(x-r_{2+})}+\frac{-\log(\frac{1}{2})}{(x+r_{1-})(x+r_{2-})}+\frac{-\log(-\frac{1}{2}-i\epsilon)}{(x+r_{1+})(x+r_{2+})} \\ &+\frac{(\Delta\xi)^2}{(\Box\xi)^2(\Box\xi-2)^2}\left(\frac{2\log(-\frac{\Delta\xi}{2}+i\epsilon')-\log(x)}{(x-s_{1+}^2)(x-s_{2+}^2)}+\frac{-2\log(\frac{\Delta\xi}{2}-i\epsilon')+\log(x)}{(x-s_{1-}^2)(x-s_{2-}^2)}\right) \\ &-\frac{\log(-\frac{1}{2}-i\epsilon)}{(x-r_{1-})(x-r_{2-})}+\left(\frac{\xi_+}{\Box\xi}\left\{\frac{\log(\xi_+-i\epsilon)-\log(1-x)}{(x-r_{1+})(x-s_{1+})}+\frac{\log(\xi_+-i\epsilon)-\log(1+x)}{(x+r_{1-})(x+s_{1-})}\right\}\right. \\ &\left.+\frac{\xi_-}{\Box\xi}\left\{\frac{\log(-\xi_--i\epsilon)-\log(1-x)}{(x+r_{2+})(x+s_{1+})}+\frac{\log(-\xi_--i\epsilon)-\log(1+x)}{(x-r_{2-})(x-s_{1-})}\right\}+(\xi_\pm\to 1-\xi_\mp)\right)\Bigg).\end{aligned} \tag{B.4.9}$$

The remaining integrals may now be safely computed in terms of logs and dilogs, and the final answer is given in (4.3.24). The calculation for the configuration $\boldsymbol{X}^{--}$ is easier, because if we perform the t integral by closing the contour in the upper half-plane, there are only two poles to consider. The final answer is given in (4.3.27).

B.5 Properties of Regularised Yangian Invariants

In the main text, we regularised the functions g_i that span the space of Yangian invariants by adding infinitesimal imaginary shifts to their arguments. The purpose of these shifts is to clarify on which side of a branch cut the function is supposed to be evaluated. Of course, different choices of the sign of the shifts do not change the value of the basis element but merely its functional representation. We therefore summarise the relations between the basis elements in different regularisations as

$$\begin{pmatrix} g_1^+(z,\bar{z}) \\ g_2^+(z,\bar{z}) \\ g_3^+(z,\bar{z}) \\ g_4^+(z,\bar{z}) \end{pmatrix} = \begin{pmatrix} 1 & \theta_1\bar{\eta}_0 - \bar{\theta}_1\eta_0 & \theta_0\bar{\eta}_1 - \bar{\theta}_0\eta_1 & \bar{\theta}_1\theta_0 - \theta_1\bar{\theta}_0 \\ 0 & 1-\theta_0-\bar{\theta}_0 & \theta_0+\bar{\theta}_0 & -\theta_0-\bar{\theta}_0 \\ 0 & \theta_1+\bar{\theta}_1 & 1-\theta_1-\bar{\theta}_1 & \theta_1+\bar{\theta}_1 \\ 0 & -\eta_0\eta_1-\bar{\eta}_0\bar{\eta}_1 & \eta_0\eta_1+\bar{\eta}_0\bar{\eta}_1 & -1+\theta_0+\bar{\theta}_0+\theta_1+\bar{\theta}_1 \end{pmatrix} \begin{pmatrix} g_1^-(z,\bar{z}) \\ g_2^-(z,\bar{z}) \\ g_3^-(z,\bar{z}) \\ g_4^-(z,\bar{z}) \end{pmatrix}, \tag{B.5.1}$$

where for compactness we use the notation

$$\eta_a = 1-\theta_a, \qquad \theta_0 = \theta(-z), \qquad \theta_1 = \theta(z-1), \tag{B.5.2}$$

$$\bar{\eta}_a = 1-\bar{\theta}_a, \qquad \bar{\theta}_0 = \theta(-\bar{z}), \qquad \bar{\theta}_1 = \theta(\bar{z}-1). \tag{B.5.3}$$

The relation (B.5.1) is for $z,\bar{z} \in \mathbb{R}\setminus\{0,1\}$; of course g_i^+ and g_i^- are indistinguishable when $z \in \mathbb{C}\setminus\mathbb{R}, \bar{z} = z^*$. We also note the relation between the function basis f_i and g_i

$$g_4^\pm = f_3^\pm - f_2^\pm \pm 2\pi i f_1, \tag{B.5.4}$$

which is valid for all $z,\bar{z} \in \mathbb{R}\setminus\{0,1\}$. We also list some functional relations satisfied by the functions g_i^+ and g_i^-, for all possible $z,\bar{z}$. Here g_1 transforms very nicely:

$$\begin{aligned} g_1^\pm(z,\bar{z}) =& g_1^\mp(1-z,1-\bar{z}) = \frac{1}{u} g_1^\mp(\tfrac{1}{z},\tfrac{1}{\bar{z}}) \\ =& \frac{1}{v} g_1^\pm\left(\tfrac{1}{1-z},\tfrac{1}{1-\bar{z}}\right) = \frac{1}{u} g_1^\pm(1-\tfrac{1}{z},1-\tfrac{1}{\bar{z}}) = \frac{1}{v} g_1^\mp\left(\tfrac{z}{z-1},\tfrac{\bar{z}}{\bar{z}-1}\right). \end{aligned} \tag{B.5.5}$$

On the other hand g_2, g_3, and g_4 have a reduced symmetry under permutations:

$$g_2^\pm(z,\bar{z}) = \frac{1}{u} g_2^\mp(\tfrac{1}{z},\tfrac{1}{\bar{z}}), \tag{B.5.6}$$

$$g_3^\pm(z,\bar{z}) = \frac{1}{v} g_3^\mp\left(\tfrac{z}{z-1},\tfrac{\bar{z}}{\bar{z}-1}\right), \tag{B.5.7}$$

$$g_4^\pm(z,\bar{z}) = g_4^\mp(1-z,1-\bar{z}). \tag{B.5.8}$$

In general $g_i^\pm$ are mapped into each other under permutations for $i=1,2,3$. For example

$$g_3^\pm(1-z,1-\bar{z}) = -g_2^\mp(z,\bar{z}). \tag{B.5.9}$$

Appendix C

Eclectic Details

C.1 Unexpected Shortening

Here we reformulate the conditions for the unwanted 'unexpected shortening' described in section 6.3.1. In a sector with general L, M, $K = 1$, we argued for the existence of a top state in $W_S^{L,M}$, where S was such that $\mathbf{d}_S > \mathbf{d}_{S+1}$:

$$\psi^{(S)} = \sum_{i=1}^{\mathbf{d}_S} \alpha_i e_i^{(S)}, \tag{C.1.1}$$

where α_i are constants and $e_i^{(S)}$ are the elementary states in $W_S^{L,M}$. Acting with a power of H on this state gives

$$H^k \psi^{(S)} = \sum_{i=1}^{\mathbf{d}_{S-k}} \sum_{j=1}^{\mathbf{d}_S} A_{ij}^{(k)} \alpha_j e_i^{(S-k)} = \sum_{i=1}^{\mathbf{d}_{S-k}} (A^{(k)}\alpha)_i e_i^{(S-k)}, \tag{C.1.2}$$

where $A^{(k)}$ is a $\mathbf{d}_{S-k} \times \mathbf{d}_S$ matrix, and α is a vector of length $\mathbf{d}_S$ with entries α_i. The top state $\psi^{(S)}$ defines a Jordan block of length k if $H^k\psi^{(S)} = 0$, or equivalently the homogeneous linear system

$$A^{(k)}\alpha = 0 \tag{C.1.3}$$

admits at least one non-trivial solution in α. We conjecture that the rank of $A^{(k)}$ is always maximal:

$$\mathrm{rank}(A^{(k)}) = \min(\mathbf{d}_{S-k}, \mathbf{d}_S). \tag{C.1.4}$$

In this case, it is well-known that (C.1.3) can only admit a non-trivial solution in α if and only if $\mathrm{rank}(A^{(k)}) < \mathbf{d}_S$. Moreover, the number of independent non-trivial solutions is $\mathbf{d}_S - \mathrm{rank}(A^{(k)})$. Therefore a non-trivial solution only exists when $\mathbf{d}_{S-k} < \mathbf{d}_S$. This occurs precisely when $k = S - \tilde{S} + 1$, as can be deduced from (6.3.24). Therefore, if the rank of $A^{(k)}$ is always maximal, the top state $\psi^{(S)}$ determines $\mathbf{d}_S - \mathbf{d}_{S+1}$ Jordan blocks, each of length $S - \tilde{S} + 1$. We checked the rank of $A^{(k)}$ for all top states and for all values of k, up to $L = 30, M = 6$, and always found it to be maximal, in line with our conjecture.

C.2 Universality Details

In this section we prove that $H_{\text{super},i}$, defined in (6.3.97), has the same Jordan block structure as the hypereclectic model H_3 for $K = 1$, under the assumption discussed in appendix C.1. Then we describe how to modify these arguments to include the full eclectic Hamiltonian, and sketch a universality proof for $K = 1$.

Universality for $\boldsymbol{H}_{\textbf{super,1}}$. We start with the first supereclectic model defined in (6.3.97), $H_{\text{super},1}$. Consider a top vector $\psi^{(S)}$ for the hypereclectic model at a level S. This vector determines a Jordan block of length $n_S := S - \tilde{S} + 1$

$$H_3^{n_S} \psi^{(S)} = 0. \tag{C.2.1}$$

We can expand $H_{\text{super},1}^{n_S}$ as

$$H_{\text{super},1}^{n_S} = \sum_{k=0}^{n_S} \binom{n_S}{k} H_1^k H_3^{n_S-k} = H_3^{n_S} + n_S H_1 H_3^{n_S-1} + \frac{n_S(n_S-1)}{2} H_1^2 H_3^{n_S-2} + \cdots , \tag{C.2.2}$$

where we have used $[H_1, H_3] = 0$. We introduce a shorthand notation

$$H_{\text{super},1}^{n_S} = \sum_{j=0}^{n_S} h_j, \quad h_0 = H_3^{n_S}, \quad h_j := \binom{n_S}{k} H_1^j H_3^{n_S-j}, \quad j = 1, \ldots, n_S. \tag{C.2.3}$$

Because of (6.3.82) and (6.3.83) each h_j lowers the S-value of a state by $j(L_1-1)+n_S$. In other words, given a vector $\varphi^{(S)} \in V_S$ we have

$$\hat{S}(h_j \varphi^{(S)}) = S - j(L_1-1) - n_S = (\tilde{S}-1) - j(L_1-1). \tag{C.2.4}$$

In particular, $h_j \varphi^{(S)} = 0$ if this value is negative. Now let us consider the $\tilde{S}$ value of a top vector to be in an interval

$$\ell(L_1-1) \leq \tilde{S} - 1 < (\ell+1)(L_1-1). \tag{C.2.5}$$

In this case, all operators h_j with $j > \ell$ will annihilate the top vector and its descendants. Therefore, we may consider only operators $h_0, h_1, \ldots, h_\ell$ and disregard others in (C.2.3).

For this S value of the top vector of the hypereclectic model $\psi^{(S)}$, we claim that we can construct a corresponding top vector $\psi_1^{(S)}$ of the supereclectic model $H_{\text{super},1}$, defined by

$$H_{\text{super},1}^{n_S} \psi_1^{(S)} = 0, \tag{C.2.6}$$

via the ansatz

$$\psi_1^{(S)} = \varphi_0 + \varphi_1 + \cdots + \varphi_\ell, \qquad \varphi_0 = \psi^{(S)}, \tag{C.2.7}$$

if the top vector has $\tilde{S}$ which satisfies (C.2.5). The condition (C.2.6) can be written as

$$\begin{aligned} & (h_0 + h_1 + h_2 + \cdots + h_\ell)(\varphi_0 + \varphi_1 + \cdots + \varphi_\ell) \\ = \; & (h_0\varphi_0) + (h_0\varphi_1 + h_1\varphi_0) + \cdots + (h_0\varphi_\ell + h_1\varphi_{\ell-1} + \cdots + h_\ell\varphi_0) + \cdots = 0, \end{aligned} \tag{C.2.8}$$

where we have grouped terms in a very particular way. The first term $h_0\varphi_0$ in (C.2.8) vanishes due to (C.2.1). Now we want to find φ_1 in the second bracket from the restriction that it vanishes

$$h_0\varphi_1 + h_1\varphi_0 = 0. \tag{C.2.9}$$

Since $\hat{S}(h_1\varphi_0) = (\tilde{S}-1) - (L_1-1)$ from (C.2.4), this equation should be expressed by elementary vectors with this S value. There are $\mathbf{d}_{(\tilde{S}-1)-(L_1-1)}$ of them, which becomes the number of constraints.[1] This equation also determines $\hat{S}(\varphi_1) = \hat{S}(h_1\varphi_0) + n_S = S - (L_1-1)$. Therefore, φ_1 can be expressed as a linear combination of $\mathbf{d}_{S-(L_1-1)}$ elementary states. Since $\mathbf{d}_{S-(L_1-1)} = \mathbf{d}_{\tilde{S}+(L_1-1)} > \mathbf{d}_{(\tilde{S}-1)-(L_1-1)}$, one can solve coefficients of the linear combination from (C.2.9) (not always unique). This proves that we can always find the solution φ_1.

We require the next bracket in (C.2.8) to vanish:

$$h_0\varphi_2 + h_1\varphi_1 + h_2\varphi_0 = 0. \tag{C.2.10}$$

Again, one can find that $\hat{S}(h_1\varphi_1) = \hat{S}(h_2\varphi_0) = (\tilde{S}-1) - 2(L_1-1)$, from which we determine $\hat{S}(\varphi_2) = S - 2(L_1-1)$. Since the maximum number of constraints is smaller than that of the coefficients due to $\mathbf{d}_{S-2(L_1-1)} > \mathbf{d}_{(\tilde{S}-1)-2(L_1-1)}$, one can find φ_2 from the known vectors φ_1 and φ_0 using (C.2.10).

One can easily generalise this argument up to the ℓ-th bracket in (C.2.8):

$$h_0\varphi_\ell + h_1\varphi_{\ell-1} + \cdots + h_\ell\varphi_0 = 0, \tag{C.2.11}$$

where the vectors $\varphi_j,\ j = 0, \ldots, \ell-1$ have already been found in previous steps. Since $\hat{S}(\varphi_j) = S - j(L_1-1)$ we have $\hat{S}(h_j\varphi_{\ell-j}) = (\tilde{S}-1) - \ell(L_1-1)$ for $j = 1, \ldots, \ell$. This determines S-value of the unknown vector φ_ℓ to be $\hat{S}(\varphi_\ell) = S - \ell(L_1-1)$. Again, the maximum number of constraints in (C.2.11) is smaller than the number of coefficients in the expansion of φ_ℓ in terms of elementary states, which guarantees that we can always find its solution.

There are more terms which we did not include in the second line of (C.2.8), but it is easy to show they all vanish. For example, the $(\ell+1)$-th bracket would be

$$h_1\varphi_\ell + \cdots + h_\ell\varphi_1. \tag{C.2.12}$$

Their S-values should be $(\tilde{S}-1) - (\ell+1)(L_1-1)$, which is negative due to (C.2.5). This means that all these vectors vanish.

This proves our universality conjecture for the supereclectic model $H_{\mathrm{super},1}$ by constructing the top vector explicitly as

$$\psi_1^{(S)} = \psi^{(S)} + \varphi_1 + \cdots + \varphi_\ell, \tag{C.2.13}$$

for $\tilde{S}$ in (C.2.5).

Because $\tilde{S} \leq S_{\max}/2$ ($\tilde{S} \leq S$ by definition), the interval (C.2.5) is limited by the maximum value of ℓ which is

$$\ell_{\max} = \left[\frac{L_1 M_1}{2(L_1-1)}\right], \tag{C.2.14}$$

where $[x]$ is the largest integer not exceeding x.

[1] In fact, this is the maximum number of constraints since some of the elementary vectors may not appear.

Universality for $H_{\mathrm{super},2}$. The second supereclectic model $H_{\mathrm{super},2}$ defined in (6.3.97) can be analysed in exactly the same way. Again, one can express

$$H^{n_S}_{\mathrm{super},2} = \sum_{m=0}^{n_S} g_m, \quad g_m := \binom{n_S}{m} H_2^m H_3^{n_S-m}, \quad m = 0, \ldots, n_S, \quad g_0 = h_0 = H_3^{n_S}. \tag{C.2.15}$$

Each g_m lowers S-values as follows:

$$\hat{S}(g_m \phi^{(S)}) = S - m(M_1 - 1) - n_S. \tag{C.2.16}$$

In the same way as before, a top vector with level S (and corresponding $\tilde{S}$) with

$$m(M_1 - 1) \le \tilde{S} - 1 < (m+1)(M_1 - 1), \tag{C.2.17}$$

we only need to consider terms in (C.2.15) up to g_m.

The remaining procedure is identical to the previous case. One can always find $\tilde{\varphi}_k$ from $\tilde{\varphi}_0, \ldots, \tilde{\varphi}_{k-1}$ using

$$g_0 \tilde{\varphi}_k + g_1 \tilde{\varphi}_{k-1} + \cdots + g_k \tilde{\varphi}_0 = 0, \quad k = 1, \ldots, m. \tag{C.2.18}$$

This proves the universality conjecture for $H_{\mathrm{super},2}$ by constructing the top vector explicitly as

$$\psi_2^{(S)} = \psi^{(S)} + \tilde{\varphi}_1 + \cdots + \tilde{\varphi}_m, \tag{C.2.19}$$

for $\tilde{S}$ in (C.2.17), where m should be limited by the maximum value

$$m_{\max} = \left[\frac{L_1 M_1}{2(M_1 - 1)} \right]. \tag{C.2.20}$$

Universality for General Eclectic Model. Powers of H_{ec} can be written as

$$H^{n_S}_{\mathrm{ec}} = \sum_{k=0}^{n_S} \binom{n_S}{k} (H_1 + H_2)^k H_3^{n_S-k}. \tag{C.2.21}$$

This expression can be simplified greatly by observing that $H_1 H_2 = H_2 H_1 = 0$ in sectors where $K = 1$. This can be seen by acting with H_1 on any state

$$H_1 |\mathbf{2}1 \cdots 1\mathbf{2}1 \cdots 1 \cdots \mathbf{2}1 \cdots 1\mathbf{3}\rangle = |1 \cdots 1\mathbf{2}1 \cdots 1 \cdots \mathbf{2}1 \cdots 1\mathbf{2}\mathbf{3}\rangle. \tag{C.2.22}$$

Then, H_2 will annihilate the resulting state since it cannot contain $\mathbf{13}$. Therefore we can remove any terms with both H_1 and H_2 in the expansion (C.2.21), which leads to

$$H^{n_S}_{\mathrm{ec}} = h_0 + (g_1 + g_2 + \cdots + g_{n_S}) + (h_1 + h_2 + \cdots + h_{n_S}). \tag{C.2.23}$$

We can restrict the interval for $\tilde{S}$ by the two relations (C.2.5) and (C.2.17). Since $L_1 \ge M_1$, for a given ℓ we can find m such that

$$m(M_1 - 1) \le \ell(L_1 - 1) < (m+1)(M_1 - 1). \tag{C.2.24}$$

In this case, the intersection of the two intervals is given by

$$m(M_1 - 1) \le \ell(L_1 - 1) \le \tilde{S} - 1 < (m+1)(M_1 - 1). \tag{C.2.25}$$

For these values of S, the expansion of power of the eclectic Hamiltonian is truncated to

$$H_{\text{ec}}^{n_S} = h_0 + (g_1 + g_2 + \cdots + g_m) + (h_1 + h_2 + \cdots + h_\ell). \tag{C.2.26}$$

We now claim that the top vector of the eclectic model can be always constructed from the hypereclectic top state $\psi^{(S)} = \varphi_0$ as follows:

$$\psi_{\text{ec}}^{(S)} = \varphi_0 + \sum_{i=1}^{m} \tilde{\varphi}_i + \sum_{j=1}^{\ell} \varphi_j. \tag{C.2.27}$$

Let us provide the detailed proof for the simplest case $m = 2, \ell = 1$, with

$$2(M_1 - 1) \leq (L_1 - 1) \leq \tilde{S} - 1 < 3(M_1 - 1). \tag{C.2.28}$$

We will show that the top vector for the eclectic model can be constructed as

$$\psi_{\text{ec}}^{(S)} = \varphi_0 + \tilde{\varphi}_1 + \tilde{\varphi}_2 + \varphi_1. \tag{C.2.29}$$

One can expand $H_{\text{ec}}^{n_S} \psi_{\text{ec}}^{(S)} = 0$ as

$$\begin{aligned} & (h_0 + g_1 + g_2 + h_1)(\varphi_0 + \tilde{\varphi}_1 + \tilde{\varphi}_2 + \varphi_1) = (h_0\varphi_0) + (g_0\tilde{\varphi}_1 + g_1\varphi_0) + \\ + \quad & (g_0\tilde{\varphi}_2 + g_1\tilde{\varphi}_1 + g_2\varphi_0) + (g_0\varphi_1 + g_1\tilde{\varphi}_2 + g_2\tilde{\varphi}_1 + h_1\varphi_0) + \cdots = 0. \end{aligned} \tag{C.2.30}$$

The first three brackets in (C.2.30) have already been solved for $H_{\text{super},2}$, therefore we only need to consider the fourth term and ellipsis. The S-values of each term have already been computed as

$$\hat{S}(g_1\tilde{\varphi}_2) = \hat{S}(g_2\tilde{\varphi}_1) = (\tilde{S} - 1) - 3(M_1 - 1) < \hat{S}(h_1\varphi_0) = (\tilde{S} - 1) - (L_1 - 1). \tag{C.2.31}$$

Therefore, φ_1 can be determined from φ_0 in the same way as for $H_{\text{super},1}$ with additional subleading terms in S from the known $\tilde{\varphi}_1, \tilde{\varphi}_2$. The terms in the ellipsis in (C.2.30) are

$$\cdots = g_1\varphi_1 + g_2\tilde{\varphi}_2 + h_1\tilde{\varphi}_1 + g_2\varphi_1 + h_1\tilde{\varphi}_2 + h_1\varphi_1. \tag{C.2.32}$$

The S-values for these vectors are given by

$$\begin{aligned} & \hat{S}(g_i\varphi_j) = \hat{S}(h_j\tilde{\varphi}_i) = (\tilde{S} - 1) - j(L_1 - 1) - i(M_1 - 1), \\ & \hat{S}(h_i\varphi_j) = (\tilde{S} - 1) - (i + j)(L_1 - 1), \quad \hat{S}(g_i\tilde{\varphi}_j) = (\tilde{S} - 1) - (i + j)(M_1 - 1). \end{aligned} \tag{C.2.33}$$

It is not difficult to see from (C.2.28) that all these vectors should vanish since their S-values are all negative.

This procedure can now be generalised in principle to any value of (ℓ, m), although it is hard to give general, explicit expressions, since the mixed interval depends closely on explicit vaues of L_1, M_1.

C.3 Fine Tuning and Cyclicity Classes

Although we have proven the universality hypothesis for generic values of the couplings ξ_i for $K = 1$, it is possible to fine-tune the couplings to destroy the Jordan block structures in a particular cyclicity class. We give a simple example of this occurring, for the sector $L = 5, M = 3, K = 1$. There are 30 states in this sector:

$$\begin{aligned} &\mathcal{C}_k|22113\rangle, \quad \mathcal{C}_k|21213\rangle, \quad \mathcal{C}_k|12213\rangle, \\ &\mathcal{C}_k|21123\rangle, \quad \mathcal{C}_k|12123\rangle, \quad \mathcal{C}_k|11223\rangle, \end{aligned} \tag{C.3.1}$$

where $\mathcal{C}_k$ is the unnormalised projector defined in (6.2.15) and $k = 0, 1, 2, 3, 4$ labels the cyclicity class. In each cyclicity class k the hypereclectic model H_3 has Jordan decomposition $(5, 1)$, so that the overall Jordan decomposition is $(5^5, 1^5)$. The other models related to H_3 by permutations of the fields H_1 and H_2 have Jordan decomposition (3,2,1) in each cyclicity class. For generic ξ_i we have argued that the eclectic Hamiltonian $H_{\text{ec}} = H_1 + H_2 + H_3$ also has the Jordan decomposition $(5^5, 1^5)$, since this sector satisfies the filling conditions (6.3.80). Setting $\xi_3 = 0$ leads to a Jordan decomposition $(3^5, 2^5, 1^5)$. Interestingly, this decomposition can be further refined by tuning ξ_1 and ξ_2. Let us act with $H_{\text{ec}}|_{\xi_3=0}$ on the top state $\mathcal{C}_k|22113\rangle$:

$$\begin{aligned} \mathcal{C}_k|22113\rangle &\to \omega^k \xi_1 \mathcal{C}_k|21123\rangle + \omega^{-k} \xi_2 \mathcal{C}_k|12213\rangle \\ &\to (\omega^{2k}\xi_1^2 + \omega^{-2k}\xi_2^2)\mathcal{C}_k|11223\rangle \to 0, \end{aligned} \tag{C.3.2}$$

where $\omega = e^{2\pi i/5}$ and we used $\mathcal{C}_k U^{\pm 1}\psi = \omega^{\pm k}\mathcal{C}_k\psi, [H_i, \mathcal{C}_k] = 0$. For generic couplings this gives a length 3 block in each cyclicity class. However, if we tune the couplings such that $\xi_2^2 = -\omega^{4k}\xi_1^2$ the block splits into a 2-block and a 1-block in this cyclicity class k. There are two further top states in this sector:

$$\begin{aligned} &\mathcal{C}_k|21213\rangle \to (\xi_1\omega^k + \xi_2\omega^{-k})\mathcal{C}_k|12123\rangle \to 0, \\ &\xi_2\omega^{-k}\mathcal{C}_k|21123\rangle - \xi_1\omega^k\mathcal{C}_k|12213\rangle \to 0. \end{aligned} \tag{C.3.3}$$

The first of these is a 2-block, which can be broken into two 1-blocks in a single cyclicity class if $\xi_2 = -\omega^{2k}\xi_1$. The next of these is always a 1-block. From this example we see explicitly that finer Jordan block decompositions can be obtained in specific cyclicity classes by tuning the couplings appropriately.

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