Optimal Reactive Power Dispatch Formulated as Quadratic OPF and Solved via CS-SLP

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Abstract—*Increased penetration of inverter interfaced renewable energy resources creates challenges and opportunities for reactive power management in the modern electricity grid. Because of the multiplicity of new resources, new computational tools and optimization models are needed in formulating and solving the Optimal Reactive Power Dispatch Problem (ORPD). In this paper, we propose (1) an object-oriented ORPD formulation based on high-fidelity modeling of each device in the network, especially those with VAR/V control capability and (2) a two-step Convex Solution-Sequential Linear Programming algorithm. The proposed method introduces two innovations: (a) high fidelity quadratized models of each component of the power system with emphasis on those components that have VAR/V control capability; and (b) an object oriented convexification of the resulting quadratic OPF problem; the solution is obtained by first solving the convex problem using public solvers for convex problems and them removing the relaxation and solving the original OPF using SLP, starting from the solution of the relaxed (convex) problem.*

Index Terms-- Optimal Reactive Power Dispatch (ORPD), Convex relaxations, Sequential Linear Programming (SLP), Co-state Method

I. Introduction

Reactive Power Optimization (RPO) is of great importance for the operational security, reliability, and optimality of the electric energy system. Specifically, reactive power optimization results in near flat voltage profile for the power system, near minimal transmission losses, and almost any post mortem analysis of blackouts indicate that VAR/Volt control was a contributing factor of the blackout. This problem is typically formulated as an Optimal Power Flow (OPF). The objective of the OPF in a RPO problem is typically defined as a voltage profile optimization. A levelized voltage profile indicates good management and use or reactive power sources and that the transmission of power does not require excessive reactive power to support power flow.

The OPF was first introduced in the 1960s. Since then many formulations of the OPF problem have been developed to address issues such as efficiency, robustness, and specific objectives. The “DC” OPF, or DCOPF, is a formulation based on the “DC” approximation of an AC power system and it is considered to be the simplest and most approximate OPF. It simplifies the physical problem making the problem linear, which in turn results in significantly faster solution times. However, the solution to the DCOPF may not be close to a feasible solution of the actual power system as it ignores the reactive power constraints. It has been popularized in the operation of open markets to facilitate certain computations. The formulation of the OPF in which the power system is modeled as an AC network is known as the ACOPF. There are several variations of the ACOPF depending on (a) expressing voltages in rectangular or polar coordinates and (b) the AC network equations are written in terms of power [2] or in terms of electric current (or electric current injection) [3]. When the voltages are represented in rectangular coordinates with current injection, the formulation is referred to as the IV-ACOPF. Our formulation is of the IV-ACOPF type.

Several solution methods have been proposed and developed to solve the OPF problem. The state-of-the-art in OPF solvers is discussed in [4]. Two widely used solution approaches are linearization and convexification techniques. Successive Linear Programming has been widely used to solve the OPF since the 70's [5]. Recently, the authors of [6] have formulated the OPF using an IV formulation and solved it using SLP. The authors have also introduced a quadratization procedure by which all the nonlinearities in the model equations of the OPF are not higher than order 2 [7]. We refer to this formulation as the quadratic ACOPF, or QOPF for brevity. The QOPF formulation is object oriented and we have used the SLP for its solution.

Convex relaxation algorithms have attracted attention in recent years to solve the ACOPF problem. Relaxation techniques such as semidefinite relaxation (SDR) [8] and second order cone relaxation (SOCR) [9] have been used. The convexified problem approximates the actual problem; the attractiveness of this approach is that there are many efficient solution algorithms for convex optimization problems. In their trend setting work [10], J. Lavaei and S. H. Low derived a sufficient zero-duality gap that when met, global optimal solution is guaranteed for the convexified problem. However, in many cases the duality gap is not zero [11] indicating an approximate solution of the convexified problem. In all cases, the solution of the convexified problem must be translated to a physically realizable solution by going back to the unrelaxed model of the OPF. In this paper, we present the Quadratic Optimal Reactive Power Dispatch (QORPD), based on a

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physically-based object-oriented modeling approach, as well as a two-step solution method named Convex Solution - Sequential Linear Programming (CS-SLP) to achieve an optimal and physically realizable solution. The SLP part of the solution method is a slight modification of the SLP approach in [12]. The following features make the overall solution approach computationally efficient: (a) object oriented convexification via minimal relaxation, (b) efficient solution to the convex problem using mature solvers, and (c) computing a feasible and realizable solution by use of SLP starting from the convexified problem solution. The SLP algorithm has the following innovations: (1) gradual introduction of active constraints into the set of model constraints minimizing the size of the LP, and (2) dynamic limits on control movement (ensuring that the linearized model is valid in the constraint region).

The remainder of this paper is organized as follows. In Section II we present our quadratic modeling approach. In Section III we summarize our QORPD formulation and in Section IV we present our CS-SLP algorithm. In Section V we provide numerical examples. We conclude in Section VI with a brief discussion of our results.

II. QUADRATIC MODELING

The proposed ORPD formulation starts by modeling each device in the power system as a model object in a standard syntax named SCAQCF: **S**tate and **C**ontrol **A**lgebraic **Q**uadratic **C**ompanion **F**orm. Given a network with devices modeled as SCAQCF objects, the network model is formed in an object-oriented fashion by operating only on device objects.

A. *Device Model in SCAQCF syntax*

Each individual device is a mathematical object which expresses the physics of the device and the operational limits in terms of through, across, internal states and control variables. The general Quadratized Device Model syntax is as follows:

$$\mathbf{i} = Y_{ex1}\mathbf{x} + Y_{eu1}\mathbf{u} + C_{e1} \tag{1a}$$

$$0 = Y_{ex2}\mathbf{x} + Y_{eu2}\mathbf{u} + C_{e2} \tag{1b}$$

$$0 = Y_{ex3}\mathbf{x} + Y_{eu3}\mathbf{u} + \begin{Bmatrix} \vdots \\ \mathbf{x}^T \left\langle F_{ex3}^i \right\rangle \mathbf{x} \\ \vdots \end{Bmatrix} + \begin{Bmatrix} \vdots \\ \mathbf{u}^T \left\langle F_{eu3}^i \right\rangle \mathbf{u} \\ \vdots \end{Bmatrix} + C_{e3} \tag{1c}$$

$$\mathbf{h} = Y_{fx}\mathbf{x}(t) + Y_{fu}\mathbf{u}(t) + \begin{Bmatrix} \vdots \\ \mathbf{x}^T F_{fx}^i \mathbf{x} \\ \vdots \end{Bmatrix} + \begin{Bmatrix} \vdots \\ \mathbf{u}^T F_{fu}^i \mathbf{u} \\ \vdots \end{Bmatrix} + C_f \le 0 \tag{1d}$$

$$\mathbf{u}_{hmin} \le \mathbf{u}(t) \le \mathbf{u}_{hmax} \tag{1e}$$

where $\mathbf{i}$, $\mathbf{x}$ and $\mathbf{u}$ are vectors of the through, state, and control variables. Current (real and imaginary part), torque in rotating machinery, heat in thermal equipment, etc. are some of the variables that may appear in the through vector. Device terminal across variables (typically terminal voltages) as well as internal states which are additional states constitute the vector of state variables. The control variables may be generator reactive power generation, transformer taps, capacitor bank switching, etc. and they are represented with the vector u.

Equations 1a through 1c describe the physical laws that govern the operation of the device. Equation 1a is the set of terminal equation where there is one equation for each through variable, defined at the device terminals. Equation 1b and 1c are respectively linear and quadratic internal equations. Equation 1d represents the functional constraints of the device expressed as quadratic functions of the state and control. Equation 1e are the physical limits of the device controls. The equations are expressed in the metric system (MKSA). Two examples of power system devices expressed in the SCAQCF syntax are provided in Appendices A (model of a generator), and B (model of a regulating transformer). The models have been simplified to meet space constraints. Any device can be cast in the SCAQCF syntax, including lines, cables, inverters, etc. Regarding inverter models, we have developed a quadratized PWM inverter model consisting of three models: an averaging model of the voltage source inverter based on dynamic phasors, a controller and a Digital Signal Processor (DSP) to provide the proper feedback.

B. *Network Model in SCAQCF syntax*

Given a power system with N devices, each device model expressed in the SCAQCF syntax, the network model is computed. The network model is also in the SCAQCF syntax. This section describes the formation of the network model in SCAQCF syntax. The network model is obtained by applying the generalized KCL – Kirchhoff's Current Law (i.e. for electric circuits: sum of currents, for thermal circuits: sum of heat flow, for mechanical systems: sum of torques) at all nodes of the network. The through variables are substituted with the model equations that express the through variables as functions of the state. This way the through variables are eliminated from the network model. The internal equations of each model as well as the functional constraints and the limits of the controls are appended to the network model. The process is symbolically shown in the network formation box in Figure 1.

III. QORPD FORMULATION

The QORPD problem is formulated by first forming the network model with the network functional constraints; then the objective function of voltage profile levelization is added to the problem. The result is an OPF with the objective of optimizing the voltage profile subject to the network model (power flow equations), the functional constraints and limits on the control variables. The procedure is symbolically shown in Figure 1, block denoted "network formation."

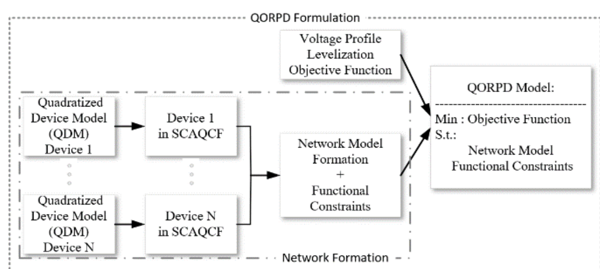

Figure 1. Complete QORPD formulation process.

The objective function is:

$$\min\ J = \sum_{i \in S_{nodes.}} \left(\frac{V_{i,mag} - V_{i,tar}}{\alpha_i V_{i,tar}} \right)^2 \qquad (2)$$

Where S_{nodes} is the set of selected nodes at which the voltage is being levelized. $V_{i,mag}$ and $V_{i,tar}$ are the voltage magnitude of the line to neutral voltage and targeted voltage value for selected node i. α_i is a user defined tolerance value.

The objective function is expanded into the quadratic form :

$$J = \sum_{i \in S_{nodes}} \frac{1}{\alpha_i^2 V_{i,tar}^2} \left(V_{i,real}^2 + V_{i,imag}^2 \right) + \sum_{i \in S_{nodes}} \frac{-2}{\alpha_i^2 V_{i,tar}} V_{i,mag} + \sum_{i \in S_{nodes}} \frac{1}{\alpha_i^2} \qquad (3)$$

The quadratic form is then cast into the AQCF syntax.

The mathematical model of the QORPD has the following form (not unlike other formulations, there is no sin and cos terms):

$$\begin{aligned}
\min: \quad & J = Y_{ox}\mathbf{x} + Y_{ou}\mathbf{u} + \mathbf{x}^T F_{ox3}\mathbf{x} + \mathbf{u}^T F_{ou3}\mathbf{u} + C_{e3} \\
\text{s.t.}: \quad & \mathbf{g} = Y_{ex}\mathbf{x} + \begin{Bmatrix} \vdots \\ \mathbf{x}^T F_{ex}^i \mathbf{x} \\ \vdots \end{Bmatrix} + Y_{eu}\mathbf{u} + \begin{Bmatrix} \vdots \\ \mathbf{u}^T F_{eu}^i \mathbf{u} \\ \vdots \end{Bmatrix} - B_e = 0 \\
& \mathbf{h} = Y_{fx}\mathbf{x} + Y_{fu}\mathbf{u} + \begin{Bmatrix} \vdots \\ \mathbf{x}^T F_{fx}^i \mathbf{x} \\ \vdots \end{Bmatrix} + \begin{Bmatrix} \vdots \\ \mathbf{u}^T F_{fu}^i \mathbf{u} \\ \vdots \end{Bmatrix} + C_f \le 0 \\
& \mathbf{u}_{\min} \le \mathbf{u} \le \mathbf{u}_{\max}
\end{aligned} \qquad (4)$$

IV. CS-SLP SOLUTION METHOD

The proposed QORPD in (4) is solved through the two-step procedure: step 1: convexification and convex solution, and step 2: sequential linear programming solution of the unrelaxed problem starting from the convex solution. The solution method has been named CS-SLP.

A. Algorithm Overview

The overall CS-SLP algorithm is shown in Figure 5. In the first step, the quadratized OPF model is formed, and the initial conditions are defined. Using the model and the initial conditions, the functional constraints are evaluated and the constraints that are active (violated) are identified. At each convex sequence (outer iteration), active constraints are added to the set of model constraints. The convexification method is described in subsection “B. Object-Oriented Convexification”. The convex problem is subsequently solved with a commercially available convex optimization solver (Gurobi). The solution is updated, the functional constraints are evaluated and the constraints that are active (violated) are identified. This first part terminates when there are no new violated constraints or when the maximum allowed number of outer iterations (a user-defined number) is reached.

Next, the original non-relaxed quadratic OPF is restored. Using the solution of the convex problem as the starting point, active constraints in the unrelaxed QORPD problem are identified and added to the model constraints. The objective function and the model constraints are linearized with respect to the control variables using the highly efficient co-state method. The result is the linearized problem for this iteration.

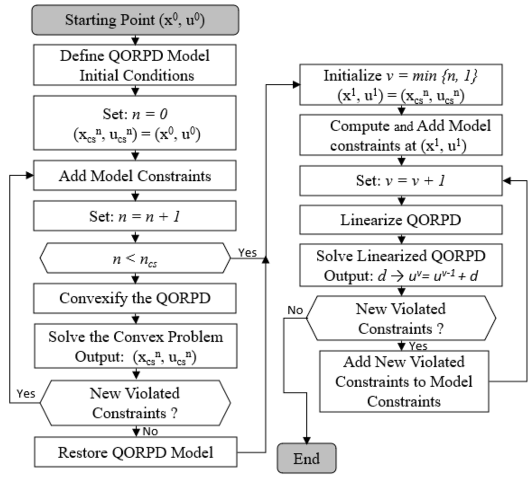

Figure 2. CS-SLP algorithm for solving QORPD.

Control excursion limits are defined at each iteration to ensure that the linearized model is valid in the constraint region. Subsequently, the linearized problem is solved, the controls are updated and the states for each network are computed (new iterate, x and u). The solution iterate is used to identify any new model constraints. The algorithm converges when no new model constraints have been identified. Otherwise, the algorithm proceeds to the next SLP iteration.

B. Object-Oriented Convexification

The convexification of the QOPF is performed by operating solely on the object of the QOPF. Here we present the underlying mechanics that support the convexification process. Without loss of generality, the nonlinearities in the QORPD model can be abstracted as $aw_1^2 + bw_1w_2 + cw_2^2$ where variables w_1, w_2, w_3 etc. can be either state or control variables. This expression would be convex if (a) its hessian is positive semidefinite (PSD) and (b) it’s an inequality.

1) Positive semidefiniteness:

The symmetrized hessian matrix for this expression is given as $H = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$. The LDLT factorization-based test or the determinant is used to assert the positive semi definiteness of the hessian matrix. If the hessian is found to not be PSD, terms are added according to the Table 1 to make it

The terms added for convexification in Table I are obtained by solving a small optimization problem that guarantees minimal additions, thus minimizing the approximation.

$$\min \quad \sum |d_i|, \qquad \textit{subject to}$$

$$\text{leading principal minor k of } \left(\mathbf{H} + \mathbf{d}^T \mathbf{I} \right) \ge 0, k = 1, \cdots, n$$

Where d_i are the elements of the diagonal vector $\mathbf{d}$ which is the vector of minimal terms to be added to obtain a PSD hessian.

TABLE I. ADDITIONS FOR CONVEXIFICATION

Case	Terms to be added
$a>0$, $c>0,\&\ ac\geq 0.25b^2$	NONE
$a\geq(b/2)$	$\left[\left(0.25b^2/a\right)-c\right]y_j^{\,2}$
$c\geq(b/2)$	$\left[\left(0.25b^2/c\right)-a\right]y_i^{\,2}$
$a\leq(b/2)\&\ c\leq(b/2)$	$\left(0.5\lvert b\rvert-a\right)y_i^{\,2}+\left(0.5\lvert b\rvert-c\right)y_j^{\,2}$

2) PSD Quadratic Equalities:

PDS hessian is not sufficient to make an expression convex. In addition to having a PSD hessian, the quadratic expression must also be an inequality. Each convexified quadratic equality is further decomposed into a linear equality and a quadratic inequality by introducing a positive variable z. The variable z is penalized in the objective function to ensure that the equality holds at the optimal solution. Mathematically this is expressed as follows:

Starting with a generalized convexified quadratic equality $0=a^Tx+b^Tu+\sum c_{ij}x_ix_j+\sum d_{k\ell}u_ku_\ell$, a positive variable z is introduced and expressions $0=a^Tx+b^Tu+z$ and $0=\sum c_{ij}x_ix_j+\sum d_{k\ell}u_ku_\ell-z$ are obtained. The quadratic equation is then converted into $\sum c_{ij}x_ix_j+\sum d_{k\ell}u_ku_\ell-z\leq 0$ and the variable z is penalized in the objective function.

The resulting convexified QOPF is:

$$
\begin{aligned}
\min: \quad & J=Y_{ox}\mathbf{x}+Y_{ou}\mathbf{u}+\mathbf{x}^TF_{ox3}\mathbf{x}+\mathbf{u}^TF_{ou3}\mathbf{u}+C_{e3}+\mathrm{p}\mathbf{z}^T\mathbf{z}\\
\text{s.t.}: \quad & \mathbf{g}_1=Y_{ex}\mathbf{x}+Y_{eu}\mathbf{u}+\mathbf{z}-B_e=0\\
& \mathbf{g}_2=\begin{Bmatrix}\vdots\\ \mathbf{x}^TF_{ex}^{i-c}\mathbf{x}\\ \vdots\end{Bmatrix}+\begin{Bmatrix}\vdots\\ \mathbf{u}^TF_{eu}^{i-c}\mathbf{u}\\ \vdots\end{Bmatrix}-\mathbf{z}\leq 0\\
& \mathbf{h}=Y_{fx}\mathbf{x}+Y_{fu}\mathbf{u}+\begin{Bmatrix}\vdots\\ \mathbf{x}^TF_{fx}^{i-c}\mathbf{x}\\ \vdots\end{Bmatrix}+\begin{Bmatrix}\vdots\\ \mathbf{u}^TF_{fu}^{i-c}\mathbf{u}\\ \vdots\end{Bmatrix}+C_f\leq 0\\
& \mathbf{u}_{\min}\leq\mathbf{u}\leq\mathbf{u}_{\max}
\end{aligned}
\tag{5}
$$

Where superscript c denotes the addition of entries to the matrix to make the hessian matrices PDS.

C. Linearization with Co-State Method

At the beginning of each SLP iteration ν, the algorithm substitutes the current operating point $\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)$ into the constraints of that network model. The violated constraints are added to the set of model constraints, which are the constraints considered in the linearized problem to be solved. The algorithm linearizes the model constraints as well as the objective function via the costate method. The linearized problem is solved until no violated constraints exist, indicting convergence. Starting from the QOPF in (5), and considering control excursion limits , $-\mathbf{u}_{\text{llim}}\leq\mathbf{d}\leq\mathbf{u}_{\text{llim}}$, the linearized problem is given as:

$$
\begin{aligned}
\min \quad & c^T\mathbf{d}+e\\
\text{s.t.} \quad & a\mathbf{d}+b\leq 0\\
& \mathbf{d}_{\min}\leq\mathbf{d}\leq\mathbf{d}_{\max}
\end{aligned}
\tag{6}
$$

where c, e, a and b are computed through the costate method as :

$$
\begin{aligned}
c^T&=\frac{\partial J\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)}{\partial\mathbf{u}}-\frac{\partial J\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)}{\partial\mathbf{x}}\left(\frac{\partial g\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)}{\partial\mathbf{x}}\right)^{-1}\frac{\partial g\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)}{\partial\mathbf{u}}\\
e&=J\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)\\
a&=\frac{\partial h_m\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)}{\partial\mathbf{u}}-\frac{\partial h_m\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)}{\partial\mathbf{x}}\left(\frac{\partial g\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)}{\partial\mathbf{x}}\right)^{-1}\frac{\partial g\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)}{\partial\mathbf{u}}\\
b&=h_m\left(\mathbf{x}^\nu,\mathbf{u}^\nu\right)
\end{aligned}
\tag{7}
$$

Entries of vectors $\mathbf{d}_{\min}$ and $\mathbf{d}_{\max}$ are given as $d_{\min,i}=\max\left(u_{\min,i}-u_i^\nu,-u_{\text{llim},i}\right)$ and $d_{\max,i}=\min\left(u_{\max,i}-u_i^\nu,u_{\text{llim},i}\right)$.

Further details on the SLP with co-state method can be found in [11].

V. Numerical Examples

This section provides performance results with two small and medium size systems. The first small size system is depicted in Figure 4. The remaining systems were obtained from the Grid Optimization Competition by ARPA-E. The basic parameters of the systems are given in Table II.

TABLE II. BASIC PARAMETERS OF SMALL AND MEDIUM SYSTEMS

System	Small 4-bus	Small Network_01R-10	Medium Network_02*-173
Buses	4	14	500
Generators	2	5	224
Transformers	1	3	193
Lines	3	17	540
Loads	1	11	281
Fixed Shunts	0	1	5
Switched Shunts	0	0	31
States	39	125	4093
Controls	4	9	338

In each of the test cases, the convexified problem provides a solution that has moved from the initial condition to an operating point which is close to the optimal solution for the original problem. SLP then starts from the solution of the convexified problem and searches for an optimal solution that is both feasible and optimal.

A characteristic of the method is that it keeps the model small by the gradual introdution of functional constraints. Only a portion of the total number of functional constraints is active and added to the set of model constraints. Table III provides the number of model contraints added to the optimization model as a percentage of total number of constrints. It also provides the number of active constraints in the final solution.

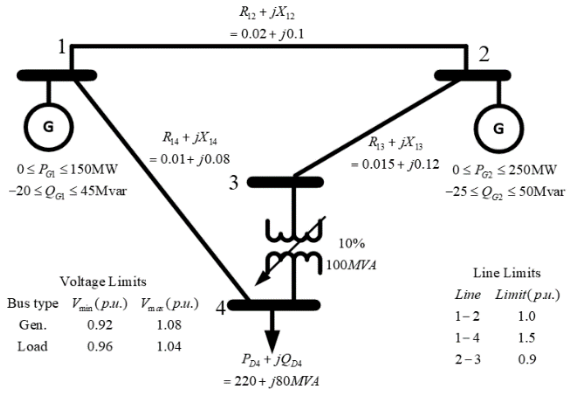

Figure 3. One-line diagram 4-Bus Sytem

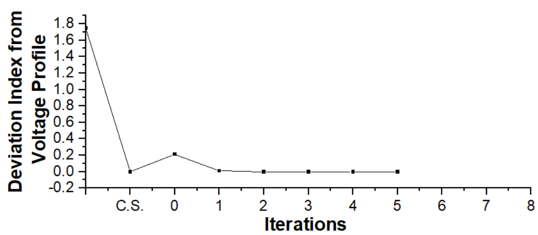

Figure 4. Deviation Index (objective function) from desired Voltage Profile vs iteration Count – 4-bus system, CS = Convexified Problem Solution

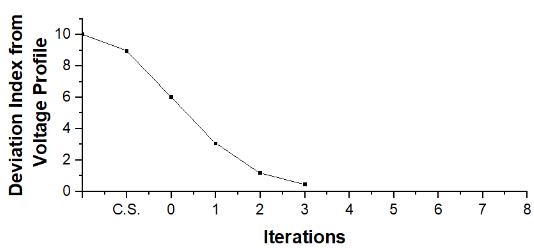

Figure 5. Deviation Index (objective function) from desired Voltage Profile vs iteration Count - Network_01R-10, CS = Convexified Problem Solution

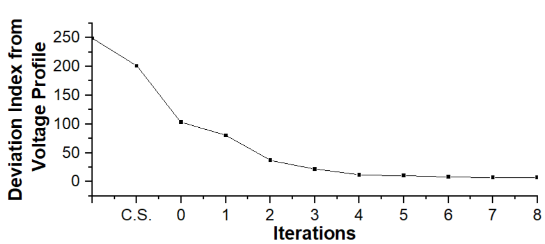

Figure 6. Deviation Index (objective function) from desired Voltage Profile vs iteration Count - Network_02*-173, CS = Convexified Problem Solution

VI. Conclusion

The ACOPF problem has been under intense research towards finding fast and reliable solution methods. Covexification of the problem to obtain a solution close to the optimal has been researched extensively. The present paper has made substantial contributions towards this goal. Specifically, an object oriented formulation has been introduced that results in a quadratic ACOPF formulation of the QORPD problem. This has been achieved by introducing the SCAQCF standard for modeling each component of the system with a set of equations that are linear and up to quadratic and cast into a specific syntax. Given the models of all components in this form, the network model is constructed by standard network modeling methods resulting in a network model expressed in the same format. Addition of an objective function and quadratization of the objective function results in an ACOPF formulation that is quadratic, the QACOPF. The second contribution is the fact that convexification of the quadratic ACOPF problem can be performed in an object oriented manner which minimizes the modifications of the original problem (minimal relaxations). Therefore, the distance between the convexified problem and the original problem is minimized. This is very important and it means that the solution to the convexified problem will be closer to the feasible optimal solution than any other convexification approach. The final contribution is the computation of the optimal solution to the original non-convex ACOPF problem by using a successive linear programing algorithm applied to the non-convex problem and starting from the solution of the convexified problem. This process results in an efficient solution methodology. The resulting methodology has been named CS-SLP (convexified solution – successive linear programming solution). The solution methodology for QORPD (formulated as QACOFP) the is a two-step process: (a) the QACOPF problem is first convexified by a series minimal relaxations and solved using a commercial convex solver; the solution is close to the optimal solution of the unrelaxed problem by virtue of the fact that the relaxation is minimal. (b) the original non-relaxed QACOPF is then solved via successive linear programming starting from the solution in (a), which allows the computation of the optimal feasible solution of the non-relaxed QACOPF.

TABLE III. BASIC PARAMETERS OF SMALL AND MEDIUM SYSTEMS

System	Small 4-bus	Small Network_01R-10	Medium Network_02*-173
# Model const.	7 (50%)	24 (37.5%)	672 (33.3%)
Max # Active const.	2 (14.3%)	2 (2.13%)	598 (29.17%)

The modeling approach allows for the seamless integration of different type of devices as all device models are mathematical objects of a standard syntax. The algorithm exploits the benefits of the quadratic nature of the QACOPF namely: (1) the object oriented convexification method operates on a quadratic model and simply makes minimal additions to the quadratic terms to make the hessian PSD, (2) the linearization of a quadratic problem using the co-state method keeps the model size small; dynamically adjusted control limits minimizes the linearization error in the solution of each linear program.

The following features make the overall solution approach computationally efficient: (a) object oriented convexification via minimal relaxations, (b) efficient solution to the convex problem using mature convex solvers, (c) gradual introduction of active constraints into the set of model constraints minimizing the size of the LP, and (d) dynamic limits on control movement.

APPENDIX A: GENERATOR MODEL

The generator model consists of three components: the machine model (two axes model), the exciter model and the governor model. For brevity, the model has been simplified. The model captures all the physics of the operation of the generator and provides high fidelity behavior of the generator VAR/V control capabilities.

1) The machine model: This machine model includes two damper wiundings. The single-phase equivalent in Figure A-1 is described by the following general QDM mathematical model:

$$I_{ar} = I_{1r} + I_{2r} + I_{0r} \quad \text{(A-1)}$$

$$I_{ai} = I_{1i} + I_{2i} + I_{0i} \quad \text{(A-2)}$$

$$I_{anr} = -I_{1r} - I_{2r} - I_{0r} \quad \text{(A-3)}$$

$$I_{ani} = -I_{1i} - I_{2i} - I_{0i} \quad \text{(A-4)}$$

$$I_{fr} = \frac{1}{r_f}\left(V_{fr} - V_{fnr}\right) \quad \text{(A-5)}$$

$$I_{fnr} = -\frac{1}{r_f}\left(V_{fr} - V_{fnr}\right) \quad \text{(A-6)}$$

$$I_{GC} = V_{GC} - C_{cost} \quad \text{(A-7)}$$

$$0 = \omega_{mr} - 2\pi \cdot 60/p \quad \text{(A-8)}$$

$$0 = \omega_{mi} - 0 \quad \text{(A-9)}$$

$$0 = V_{ar} - V_{anr} - V_{1r} \quad \text{(A-10)}$$

$$0 = V_{ai} - V_{ani} - V_{1i} \quad \text{(A-11)}$$

$$0 = I_{1r} - I_{dr} - I_{qr} \quad \text{(A-12)}$$

$$0 = I_{1i} - I_{di} - I_{qi} \quad \text{(A-13)}$$

$$0 = E_r - V_{1r} + rI_{dr} + rI_{qr} - x_d I_{di} - x_q I_{qi} \quad \text{(A-14)}$$

$$0 = E_i - V_{1i} + rI_{di} + rI_{qi} + x_d I_{dr} + x_q I_{qr} \quad \text{(A-15)}$$

$$0 = V_{ai} \quad \text{(A-16)}$$

$$0 = P_g + P_{specified} \quad \text{(A-17)}$$

$$0 = Q_g + Q_{specified} \quad \text{(A-18)}$$

$$0 = P_g - 3V_{1r}I_{1r} - 3V_{1i}I_{1i} \quad \text{(A-19)}$$

$$0 = Q_g + 3V_{1r}I_{1i} - 3V_{1i}I_{1r} \quad \text{(A-20)}$$

$$0 = E_r I_{dr} + E_i I_{di} \quad \text{(A-21)}$$

$$0 = E_i I_{qr} - E_r I_{qi} \quad \text{(A-22)}$$

$$0 = V_{ar}^2 + V_{ai}^2 - V_{specified}^2 \quad \text{(A-23)}$$

$$0 = cP_g^2 + bP_g + a - C_{cost} \quad \text{(A-24)}$$

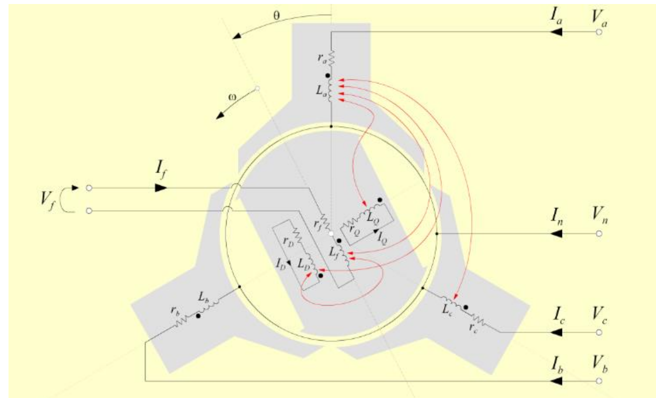

Figure A-1: Circuit diagram of two axis machine model

Depending on the control mode of the generator, the general machine model will adapt as follows: (a) in slack mode – equations A-17 and A-18 are excluded, (b) in PV control mode – equations A-16 and A-18 are excluded, and (c) in PQ control mode – equations A-16 and A-23 are excluded from the model

2) The exciter model: The exciter fulfils the function of providing direct current to the machine field winding for controlling the generating unit terminal voltage; this affects the generated reactive power. We are using the IEEE exciter model 1. For brevity the eqautions are omitted.

3) The governor model: The governor fulfils the function of controlling the generating unit output with feedback of frequency and set-point. The model is a PID controller described by the following equations:

$$T_m(t) = w(t) \quad \text{(A-25)}$$

$$0 = \omega_{m\_pseudo}(t) \quad \text{(A-26)}$$

$$f_{set}(t) = \omega_{set}(t)\frac{1}{2\pi}\frac{p}{2} \quad \text{(A-27)}$$

$$0 = \omega_{set\_pseudo}(t) \quad \text{(A-28)}$$

$$P_{set\_i}(t) = P_{set}(t) \quad \text{(A-29)}$$

$$0 = P_{set\_pseudo}(t) \quad \text{(A-30)}$$

$$0 = \Delta P_T(t) + \left(\frac{1}{2\pi R}\cdot\left(\omega_m(t) - \omega_{set}(t)\right) - \Delta P_C(t)\right) \quad \text{(A-31)}$$

$$0 = \frac{k}{2\pi}\left(\omega_m(t) - \omega_{set}(t)\right) \quad \text{(A-32)}$$

$$0 = \frac{k}{2\pi}\left(\omega_m(t) - \omega_{set}(t)\right) + B\left(P_m(t) - P_{set}(t)\right) \quad \text{(A-33)}$$

$$0 = P_m(t) - \Delta P_T(t) \quad \text{(A-34)}$$

$$0 = \omega_m(t)w(t) - P_m(t) \quad \text{(A-35)}$$

Depending on the control mode of the generator, governor model will adapt as follows: (a) in slack mode – equation A-32 is excluded, and (b) in non slack control modes – equation A-33 is excluded.

APPENDIX B: TCUL TRANSFORMER MODEL

The transformer with TCUL model is developed from first principles yielding a nonlinear model in which the tap appears in absolute value. The model is converted into an analytic quadratic model by introducing additional variables. The result is that the model of these transformer is an analytic quadratic model. It is important to note that commercially available programs use simplified models of TCULs supplemented with complex correction tables to account for the effect of the tap on the impedances of the transformer. The proposed quadratic model for TCULs is superior to the available models and translates to drastically increased computational efficiency of the overall problem. The positive sequence circuit model of a transformer with TCUL is shown in Figure B-1. Note that the figure shows the positive sequence of a wye-wye connected transformer. If the transformer is delta-wye, there will be a phase shift of 30 degrees. This model is not shown in this Appendix.

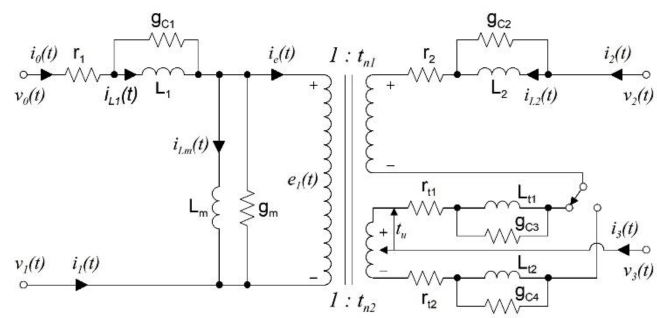

Figure B-1: Circuit diagram of transformer with TCUL – positive sequence

The positive sequence transformer model is presented here. The QDM of the transformer with TCUL is as follows,

$$I_{0r} = I_{L_{1r}} - \omega g_{c1} L_1 I_{L1i}, \; I_{0i} = I_{L_{1i}} + \omega g_{c1} L_1 I_{L1r}$$

$$I_{1r} = -I_{L_{1r}} + \omega g_{c1} L_1 I_{L1i}, \; I_{1i} = -I_{L_{1i}} - \omega g_{c1} L_1 I_{L1r}$$

$$I_{2r} = I_{L_{2r}} - \omega g_{c2} L_2 I_{L2i}, \; I_{2i} = I_{L_{2i}} + \omega g_{c2} L_2 I_{L2r}$$

$$I_{3r} = -I_{L_{2r}} + \omega g_{c2} L_2 I_{L2i}, \; I_{3i} = -I_{L_{2i}} - \omega g_{c2} L_2 I_{L2r}$$

$$0 = E_{1r} - \omega L_m I_{Lmi}, \; 0 = E_{1i} + \omega L_m I_{Lmr}$$

$$0 = -I_{L_{1r}} + \omega g_{c1} L_1 I_{L1i} + I_{cr} + I_{mr} + g_m E_{1r}$$

$$0 = -I_{L_{1i}} - \omega g_{c1} L_1 I_{L1r} + I_{ci} + I_{mi} + g_m E_{1i}$$

$$0 = V_{0r} - V_{1r} - E_{1r} - r_1 I_{L1r} + \left(1 + r_1 \omega g_{c1} L_1\right) I_{L1i}$$

$$0 = V_{0i} - V_{1i} - E_{1i} - r_1 I_{1i} - \left(1 + r_1 \omega g_{c1} L_1\right) I_{L1r}$$

$$0 = V_{2r} - V_{3r} - t_{n1}\left(1+y\right) E_{1r} - r_2 I_{L2r} + \omega\left(1 + r_2 g_{c2} L_2\right) I_{L2i}$$
$$-10^{-6} \frac{V_{rat.,\sec.}}{I_{rat.,\sec.}} I_{L_{tr}} - \frac{t_{n1}}{t_{n2}} r_t y_2 I_{L_{tr}} + \frac{t_{n1}}{t_{n2}} \omega L_t y_2 I_{L_{ti}} + \left(\frac{t_{n1}}{t_{n2}}\right)^2 \omega L_t g_{c4} y_3 I_{L_{ti}}$$

$$0 = V_{2i} - V_{3i} - t_{n1}\left(1+y\right) E_{1i} - r_2 I_{L_{2r}} - \omega\left(1 + r_2 g_{c2} L_2\right) I_{L_{2i}}$$
$$-10^{-6} \frac{V_{rat.,\sec.}}{I_{rat.,\sec.}} I_{L_{ti}} - \frac{t_{n1}}{t_{n2}} r_t y_2 I_{L_{ti}} - \frac{t_{n1}}{t_{n2}} \omega L_t y_2 I_{L_{tr}} - \left(\frac{t_{n1}}{t_{n2}}\right)^2 \omega L_t g_{c4} y_3 I_{L_{tr}}$$

$$0 = I_{L2r} - \omega g_{c2} L_2 I_{L2i} - I_{Ltr} + \omega g_{c4} \frac{t_{n1}}{t_{n2}} L_t y_2 I_{Lti}$$

$$0 = I_{L2i} + \omega g_{c2} L_2 I_{L2i} - I_{Ltr} - \omega g_{c4} \frac{t_{n1}}{t_{n2}} L_t y_2 I_{Ltr}$$

$$0 = I_{cr} t_{n1}\left(1+y\right) + I_{L2r} - \omega g_{c2} L_2 I_{L2r}$$

$$0 = I_{ci} t_{n1}\left(1+y\right) + I_{L2i} + \omega g_{c2} L_2 I_{L2r}$$

$$0 = y^2 - y_2^2, \; 0 = y_2 - y_1^2, \; 0 = y_3 - y_2^2$$

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