A Granular-Based Approach to Address Multiplicative Consistency of Reciprocal Preference Relations in Decision-Making

F.J. Cabrerizo
University of Granada
cabrerizo@decsai.ugr.es

A. Kaklauskas
Vilnius Tech
arturas.kaklauskas@vgtu.lt

I.J. Pérez
University of Granada
ijperez@decsai.ugr.es

E. Herrera-Viedma
University of Granada
viedma@decsai.ugr.es

Abstract

Decisions, having the possibility to have important consequences on people's lives, are made every day. For this reason, there exists a great need for making good decisions in today's world. Because consistency has been assumed to be a rationality measure, inconsistent judgments are considered to lead to bad decisions. This study aims to introduce a new granular-based approach to deal with consistency, concretely multiplicative consistency, of reciprocal preference relations in decision-making. Firstly, we present a process of an optimal distribution of information granularity maximizing the consistency of the reciprocal preference relation. Secondly, based on it, we develop an interactive procedure for multiplicative consistency improvement with the implication of the decision maker. Several numerical examples are conducted to validate the effectiveness of this granular-based approach.

Keywords: Consistency, information granularity, decision-making, reciprocal preference relations.

1. Introduction

Decision-making is the act of selecting between two or more possible solutions to a problem. Both for business and life, decision-making and problem-solving are essential skills. We will not even become aware of it, but we make decisions on a daily basis. It consists in a succession of steps taken by a decision maker to decide the best alternative meeting the needs of the problem to solve. In one of these steps, the decision maker must evaluate the alternatives and then choose the best one. We can find a number of preference elicitation methods to model the decision maker's judgments about the alternatives (Millet, 1997). For instance, the decision maker can provide a vector of alternatives ordered from best to worst, give an utility value to each alternative representing the fulfilment degree on a specified criterion from her or his viewpoint, or express her or his preference degree of one alternative over another one. The latter gives rise to a preference relation by repeatedly applying these pairwise comparisons.

Preference relations have been chosen in decision-making to model decision maker's evaluations for two main reasons. Firstly, they facilitate the aggregation of individual decision maker evaluations into group ones (Fodor & Roubens, 1994). Secondly, they allow the decision maker to give more precise evaluations (Millet, 1997). Despite that, a preference relation requires more information that the one needed, which could lead to contradictions because of the complex nature of the decision-making process itself and the insufficiency of the current knowledge (García-Lapresta & Montero, 2006). Consistency has been comprehended as a measure of rationality that allows performance degrees. Therefore, a consistency degree related to a preference relation acts as a sound measure of its reliability and quality. A low value means that the preference relation contains many inconsistencies leading to wrong decisions. As a result, it is very important to get preference relations with consistency degrees as high as possible before continuing with their usage in the further decision analysis. It ensures the validity of the resulting decisions (Herrera-Viedma et al., 2004).

Many consistency improvement approaches dealing with preference relations have been conceived in the literature (Li et al., 2019). They seek to increase the consistency of the preference relation, but most approaches achieved this goal in a way that gives rise to high divergences between the evaluations provided

URI: https://hdl.handle.net/10125/102823
978-0-9981331-6-4

by the decision maker and the modified ones, which produces a notable information loss. To address this concern, the consistency improvement approaches can incorporate a number of conditions into the modification process of the evaluations. For instance, to control the difference between the modified and original evaluations given by the decision maker in the AHP (analytic hierarchy process), Pedrycz and Song (2011) made use of a central concept of Granular Computing, namely information granularity allocation (Pedrycz, 2014). The underlying idea consists in treating the evaluations as information granules that provide the flexibility degree that is necessary to improve the consistency. Based on this idea, several consistency improvement methods have been developed (Cabrerizo et al., 2021; Cabrerizo et al., 2017; Pedrycz, 2014), which deal with different types of preference relations. Their common characteristic is that they are based on an information granularity whose allocation is uniform. Even though these methods have the ability of improving the consistency of the preference relation at the same time that they limit the range in which the evaluations are modified according to the granules of information, their performance could actually be improved by allowing an optimal distribution of a distinct information granularity to the elements of the preference relation (Zhang et al., 2022). This distribution, by being more flexible, could achieve a better consistency than the approaches assuming that the information granularity is distributed in a uniform way.

Based on the idea introduced by Zhang et al. (2022), this study aims to introduce a new granular-based approach to deal with consistency of reciprocal preference relations. First, based on an established average level of information granularity, a consistency improvement process with an optimal distribution of information granularity is presented. Due to the limitation offered by the average level of information granularity, it can improve the consistency degree of the reciprocal preference relation whereas preserving the original evaluations to the greatest extent. On the one hand, to characterize the consistency, the property of multiplicative transitivity is assumed (to model the cardinal consistency of a reciprocal preference relation this property is the most appropriate as it was shown by Chiclana et al. (2009)). On the other hand, the underlying optimization problem is solved by means of the differential evolution (DE) algorithm (Storn & Price, 1997). Second, an interactive consistency improvement procedure involving an active implication of the decision maker is developed. It is based on the optimal consistency improvement process and offers a complete application framework for practical situations.

To proceed with this approach and its realization, we organize this study in the following way. Section 2 recalls the needed knowledge in relation to the reciprocal preference relation, the related consistency degree, and the DE algorithm. Section 3 further explains both the process of multiplicative consistency improvement with an optimal distribution of information granularity and the interactive procedure for multiplicative consistency improvement. Section 4 conducts a number of numerical experiments and analyzes the performance of the proposed approach. Lastly, Section 5 concludes the research conducted and points out new directions for future investigations.

2. Preliminaries

2.1. Reciprocal preference relations and consistency

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of alternatives solving a given decision problem. In decision-making, the objective is to order the alternatives as a solution to the problem, according to the evaluations given by a sole decision maker or a group (Carlsson et al., 2012).

It has already been mentioned that pairwise comparisons in the form of preference relations have widely been used in decision-making. However, to model the decision maker's evaluations, in addition to the preference elicitation method, a representation domain must be established (Herrera-Viedma et al., 2021). Concretely, the fuzzy sets and their extensions have generally been used as they can manage efficiently the vagueness of the not well-defined evaluations given by decision makers (human beings) (Bustince et al., 2016). Among them, the reciprocal preference relations (Świtalski, 1999), which have been utilized to characterize preference degrees in the fuzzy set theory, are the subject of this research.

Definition 2.1 *(Kacprzyk, 1986) "A reciprocal preference relation R defined over a set of alternatives A is given by its membership function $\mu_R : A \times A \to [0, 1]$, $\mu_R(a_i, a_j) = r_{ij}$, that verifies $r_{ij} + r_{ji} = 1$, $\forall i, j = 1, \ldots, n$, $i \neq j$."*

Generally, R is represented by a matrix $R = [r_{ij}]$ of size $n \times n$. For a decision maker, $r_{ij} = \mu_R(a_i, a_j)$, being $i \neq j$, designates her or his preference degree of a_i over a_j. Notably, a value ≤ 0.5 is given to r_{ij} if the decision maker has a preference for a_j over a_i; 0.5 is given to r_{ij} if the decision maker has an equal preference for a_i and a_j; and a value ≥ 0.5 is given to r_{ij} if the decision maker has a preference for a_i over a_j. As the entries of the leading diagonal are not used, they are commonly denoted as "–" (Kacprzyk, 1986).

There exist three hierarchic and strictly necessary levels of rationality that have to be considered when dealing with preference relations: (i) Indifference is required between an alternative and itself, (ii) is a_i is preferred to a_j, then a_j cannot be preferred to a_i, and (iii) if a_i is preferred to a_j, and a_j is preferred to a_k, then practical thinking suggests that a_i should be preferred to a_k (Chiclana et al., 2009).

The definition of a reciprocal preference relation guarantees the first and the second levels of rationality, but not the third one, which is related to the transitivity in the pairwise comparisons between any three alternatives. However, to consider a preference relation as consistent, it must satisfy the third level of rationality (Chiclana et al., 2009). In addition, any property satisfying transitivity is named as consistency property. Because of that, the consistency of a reciprocal preference relation is based on the transitivity notion (De Baets et al., 2006; Świtalski, 1999), which has been characterized in several forms as, for instance, minimum transitivity, moderate stochastic transitivity, maximum transitivity, strong stochastic transitivity, additive transitivity and multiplicative transitivity, to cite some of them (Chiclana et al., 2009; Herrera-Viedma et al., 2004). Among them, we focus on the property of multiplicative transitivity as Chiclana et al. (2009) proved it is the most convenient to model cardinal consistency of reciprocal preference relations. This property is formulated in the following way:

$$r_{ij} \cdot r_{jk} \cdot r_{ki} = r_{ik} \cdot r_{kj} \cdot r_{ji} \tag{1}$$

being $r_{ij} > 0 \; \forall i, j$.

Assuming reciprocity, the multiplicative transitivity property may be formulated as follows by simple algebraic manipulation (Chiclana et al., 2009):

$$r_{ik} = \frac{r_{ij} \cdot r_{jk}}{r_{ij} \cdot r_{jk} + (1 - r_{ij}) \cdot (1 - r_{jk})} \tag{2}$$

R is referred to as "multiplicative consistent" whether for each three alternatives, i.e., a_i, a_j, a_k, belonging to a set of alternatives A, then (2) is satisfied by r_{ij}, r_{jk}, and r_{ik}.

To estimate r_{ik}, being $i \neq k$, an intermediate alternative a_j and (2) can be applied:

$$mr_{ik}^{j} = \frac{r_{ij} \cdot r_{jk}}{r_{ij} \cdot r_{jk} + (1 - r_{ij}) \cdot (1 - r_{jk})} \tag{3}$$

The average of all mr_{ik}^{j} estimates the value of r_{ik}, which is denoted as mr_{ik}:

$$mr_{ik} = \frac{1}{n-2} \sum_{j=1; j \neq i,k}^{n} mr_{ik}^{j} \tag{4}$$

Then, the error, denoted as ϵr_{ik}, between the preference degree and its estimated value is measured as $|r_{ik} - mr_{ik}|$. These values can be utilized to measure the consistency degree of R, denoted as $cd(R)$, as:

$$cd(R) = \frac{2}{n^2 - n} \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} (1 - \epsilon r_{ik}) \tag{5}$$

If $cd(R) = 1$, the consistency reached is the highest. The lower $cd(R)$ is, the lower the consistency has been reached.

2.2. DE algorithm

Storn and Price (1997) proposed it as a population-based metaheuristic search technique that aims at evolving a population of $NP \geq 4$ d-dimensional vectors encoding the candidate solutions, $\mathbf{x}_i^g = (x_{i,1}^g, \ldots, x_{i,d}^g)$, $i = 1, \ldots, NP$, in the direction of the global optimum. To cover the search space to the greatest extent, the population should be initialized by randomizing uniformly the candidate solutions within the search space constrained by the established maximum and minimum parameter boundaries $\mathbf{x}_{max} = (x_{max,1}, \ldots, x_{max,d})$ and $\mathbf{x}_{min} = (x_{min,1}, \ldots, x_{min,d})$. At generation the $g = 0$, the jth parameter in the ith candidate solution can be initialized as:

$$x_{i,j}^0 = x_{min,j} + U(0,1) \cdot (x_{max,j} - x_{min,j}) \tag{6}$$

where $U(0,1)$ is a uniform distribution over $[0,1]$.

After the initialization, three operations (mutation, crossover, selection) are repeatedly applied generation after generation until a given criterion is satisfied. First, in the current population, the mutation operation produces a mutant vector $\mathbf{v}_i^g$ as regards to each $\mathbf{x}_i^g$. At the generation g, for each $\mathbf{x}_i^g$, its corresponding $\mathbf{v}_i^g$ is obtained via a given mutation strategy. The basic DE uses the "DE/rand/1" strategy, which is defined as:

$$\mathbf{v}_i^g = \mathbf{x}_a^g + F \cdot (\mathbf{x}_b^g - \mathbf{x}_c^g) \tag{7}$$

being a, b, and c, integers produced randomly in the interval $[1, NP]$ that are mutually exclusive and also distinct from i. They are produced randomly once for every mutant vector. The scaling factor $F \in [0,2]$ is a control parameter for scaling the difference vector.

Second, the crossover operation is applied to every $\mathbf{x}_i^g$ and its $\mathbf{v}_i^g$ to produce a trial vector $\mathbf{u}_i^g = (u_{i,1}^g, \ldots, u_{i,d}^g)$. In the basic version of the DE, the crossover operation is defined as:

$$u_{i,j}^g = \begin{cases} v_{i,j}^g, & \text{if } k \leq CR \text{ or } k = j \\ x_{i,j}^g, & \text{otherwise} \end{cases} \tag{8}$$

being k an integer produced randomly in the interval $[1, d]$ and $CR \in [0, 1]$ a crossover rate controlling the portion of parameter values copied from the mutant vector. Whether the value of any component of the obtained trial vector exceeds the corresponding lower and upper bounds, it must be uniformly and randomly reinitialized within the specified interval.

Third, the selection operator is applied, which is as follows:

$$\mathbf{x}_i^{g+1} = \begin{cases} \mathbf{u}_i^g, & \text{if } f(\mathbf{u}_i^g) \geq f(\mathbf{x}_i^g) \\ \mathbf{x}_i^g, & \text{otherwise} \end{cases} \quad (9)$$

being $f : \mathbb{R}^d \rightarrow \mathbb{R}$ the fitness function that must be optimized (maximized, in this case).

The DE algorithm has as a principal advantage that only three parameters must be adjusted, namely NP, F, and CR. Based on previous works (Das & Suganthan, 2011), a good value for NP is between $3d$ and $8d$, the value of CR is between 0.3 and 0.9, and $F = 0.8$.

3. Granular-based approach to address multiplicative consistency

Based on an optimal distribution of information granularity, first, this section describes a multiplicative consistency improvement process for reciprocal preference relations. Second, an interactive procedure for multiplicative consistency improvement, which guides the decision maker to change the pairwise comparisons, is developed.

3.1. Multiplicative consistency improvement process

From a practical viewpoint, the decision maker usually provides a reciprocal preference relation R that is inconsistent to some extent. For this reason, to improve its consistency, the decision maker must allow a certain adaptability and gives up her or his first pairwise comparisons. These modifications result in an information loss in the decision-making process. Based on the ideas presented by Pedrycz and Song (2011), the $[0, 1]$-values of the entries of the reciprocal preference relation can be replaced by granules of information, which gives rise to a granular reciprocal preference relation (Cabrerizo et al., 2017). Particularly, interval-valued entries are constructed if the granules of information are in the form of intervals. In such a case, if ε_{ij} is the information granularity level injected to r_{ij}, then an excessive information loss can be prevented during the consistency improvement process because of the limitation imposed by ε_{ij}.

Definition 3.1 *Let $R = [r_{ij}]$ be a reciprocal preference relation of size $n \times n$. $G(R) = [g_{ij}]$ of size $n \times n$ is the granular (interval-valued) reciprocal preference relation connected to R, being $G(\cdot)$ a family of interval-valued reciprocal preference relations and $g_{ij} = [r_{ij} - 0.5\varepsilon_{ij}, r_{ij} + 0.5\varepsilon_{ij}]$.*

Here, for the alternatives a_i and a_j, taking into account the reciprocity property, we principally focus on modifying the value of $r_{ij} \in [0, 1]$ within the following interval:

$$g_{ij} = [\max(0, r_{ij} - 0.5\varepsilon_{ij}), \min(1, r_{ij} + 0.5\varepsilon_{ij})] \quad (10)$$

Let $\Psi = \{(i, j) \mid i > j\}$ be the set of pairs of alternatives whose preference degree will be changed at first. Let $\overline{\Psi} = \{(i, j) \mid (i, j) \notin \Psi \wedge i \neq j\}$. Clearly, the cardinality, m, of Ψ is equal to $0.5n(n - 1)$. On the one hand, the needed adaptability may be injected to the decision-making process by the information granularity level. On the other hand, it could lead to the problem of loss of information. Consequently, we must limit and evaluate the average level of information granularity ε to guarantee a certain information accuracy. This is performed as:

$$m\varepsilon = \sum_{(i,j) \in \Psi} \varepsilon_{ij} \quad (11)$$

The value of ε is a parameter of the model that is given by the decision maker. The higher the value assigned to ε, the higher the decision maker's compromise and adaptability degree.

Let $\overline{R} \in G(R)$ be a component of the family of interval-valued reciprocal preference relations $G(\cdot)$, i.e.:

$$\begin{cases} \overline{r}_{ij} \in g_{ij} & (i, j) \in \Psi \\ \overline{r}_{ij} = 1 - \overline{r}_{ji} & (i, j) \in \overline{\Psi} \\ \overline{r}_{ij} = \text{“–”} & i = j \end{cases} \quad (12)$$

Considering an information granularity level distributed uniformly, i.e., $\varepsilon_{ij} = \varepsilon$, we carry out a random experiment with the purpose of analyzing the impact of the distributed granularity level. For a reciprocal preference relation R and average level of information granularity ε, a reciprocal preference relation $\overline{R}$ is randomly obtained coming from $G(R)$, i.e., the interval-valued representative of R. Next, using (5), we calculate the consistency degree of $\overline{R}$. To perform this experiment, the following R is considered:

$$R = \begin{bmatrix} - & 0.70 & 0.60 & 0.40 & 0.20 \\ 0.30 & - & 0.30 & 0.10 & 0.40 \\ 0.40 & 0.70 & - & 0.90 & 0.60 \\ 0.60 & 0.90 & 0.10 & - & 0.70 \\ 0.80 & 0.60 & 0.40 & 0.30 & - \end{bmatrix}$$

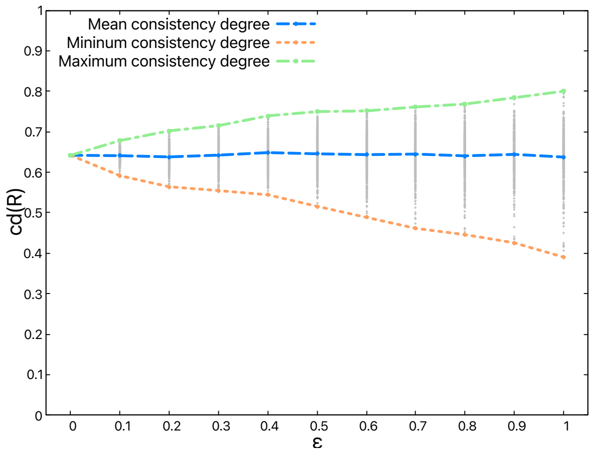

Figure 1. Consistency degree versus ε.

Taking into consideration several levels of information granularity ε, we run the random experiment 600 times. The related plots of ε versus the consistency degrees of the randomly obtained reciprocal preference relations $\overline{R}$ coming from $G(R)$ are displayed in Fig. 1. We can observe that, on the one hand, the minimum consistency degree decreases when the value assigned to ε is incremented. On the other hand, the maximum consistency degree increases and the average value stays essentially equal. As a result, by allowing an established level of information granularity, a modified reciprocal preference relation with a lower or a higher consistency degree may be generated. In particular, it is determined by the distribution of the information granularity, i.e., ε_{ij}, and the formation of the modified reciprocal preference relation, i.e., $\overline{r}_{ij}$. Therefore, a pertinent point is how to compute $\overline{r}_{ij}$ and ε_{ij} to make sure the achievement of a high consistency degree.

When we generate $\overline{R}$ from $G(R)$, we aim to obtain a reciprocal preference relation having a consistency degree as high as possible, i.e.:

$$\max cd(\overline{R}) \tag{13}$$

On the other hand, the modified reciprocal preference relation comes from an interval-valued (granular) representation of R, i.e.:

$$\overline{R} \in G(R) \tag{14}$$

Based on (11)–(13), we construct the model for improving the multiplicative consistency with an optimal information granularity distribution as:

$$\begin{cases} \max\limits_{\overline{r}_{ij},\varepsilon_{ij}} cd(\overline{R}) \\ \text{s.t.} \begin{cases} \overline{R} \in G(R) \\ m\varepsilon = \sum_{(i,j)\in\Psi} \varepsilon_{ij} \end{cases} \end{cases} \tag{15}$$

being $\overline{r}_{ij}$ and ε_{ij} the decision variables. Essentially, we can contemplate the model as a decision-making process composed of two sequential steps as ε_{ij} determines $\overline{r}_{ij}$.

In model (15), there are $n(n-1)$ decision variables. Nevertheless, considering that $\overline{R} \in G(R)$, the interval to which the value of $\overline{r}_{ij}$ belongs is determined by (10). On account of that, one has:

$$\begin{cases} 0 \leq 2|r_{ij} - \overline{r}_{ij}| \leq \varepsilon_{ij} \\ \overline{r}_{ij} \in [0,1] \end{cases} \tag{16}$$

In view of this, we may convert the model (15) into the following one:

$$\begin{cases} \max\limits_{\overline{r}_{ij}} cd(\overline{R}) \\ \text{s.t.} \begin{cases} \sum_{(i,j)\in\Psi} 2|r_{ij} - \overline{r}_{ij}| \leq m\varepsilon \\ \overline{r}_{ij} \in [0,1] & (i,j) \in \Psi \\ \overline{r}_{ij} = 1 - \overline{r}_{ji} & (i,j) \in \overline{\Psi} \\ \overline{r}_{ij} = \text{“–”} & i = j \end{cases} \end{cases} \tag{17}$$

being $\overline{r}_{ij}$ the decision variables. Its number now is $0.5n(n-1)$.

In model (17), the mean of the absolute distance between $\overline{r}_{ij}$ and r_{ij} in l_1-norm is limited due to ε. Consequently, a delimited information difference between the adjusted and the original reciprocal preference relation is guaranteed.

To solve the process of multiplicative consistency improvement, two steps are carried out. Firstly, based on (17), the value assigned to every $\overline{r}_{ij}$ is calculated. Secondly, the value assigned to every ε_{ij}, $(i,j) \in \Psi$, is computed. In the latter, let $\Delta = m\varepsilon - \sum_{(i,j)\in\Psi} 2|r_{ij} - \overline{r}_{ij}|$. Given (16), it is clear that countless values can be assigned to ε as solution whether $\Delta > 0$. Without loss of generality, in the process of multiplicative consistency improvement, ε is distributed as follows:

$$\varepsilon_{ij} = \begin{cases} 2|r_{ij} - \overline{r}_{ij}| + \frac{1}{\#\Upsilon}\Delta & (i,j) \in \Upsilon \\ 0 & (i,j) \in \Psi - \Upsilon \end{cases} \tag{18}$$

being $\Upsilon = \{(i,j) \mid (i,j) \in \Psi \wedge r_{ij} \neq \overline{r}_{ij}\}$ and $\#\Upsilon$ its cardinality. Notably, whether $\Delta = 0$, then $\varepsilon_{ij} = 2|r_{ij} - \overline{r}_{ij}|$ is the only one solution.

To solve this optimization problem, viz., model (17), the DE algorithm described in 2.2 is used for two main reasons. First, its simplicity (Das & Suganthan, 2011). Second, it has already demonstrated good performance in similar optimization problems (Zhang et al., 2022). Anyway, any other optimization method such as the particle swarm optimization could be applied as it has also been applied to this kind of problems (Cabrerizo et al., 2021).

3.2. Interactive procedure for multiplicative consistency improvement

There exist two kinds of methodologies managing consistency in preference relations (Li et al., 2019), particularly, (i) methods based on automatic adjustment and (ii) methods based on interactive adjustment. The process of multiplicative consistency improvement presented in Section 3.1 belongs to the former, i.e., the multiplicative consistency is improved by means of an optimization model without the participation of the decision maker. However, in practical decision-making processes, the decision-maker's implication is essential in the alteration of the preference relation. As a result, considering the proposed process of multiplicative consistency improvement, we further construct an interactive procedure for multiplicative consistency improvement involving the decision maker. Here, the optimal adjusted reciprocal preference relation, $\overline{R}$, returned by the process described in Section 3.1 is only taken into account as a decision support that the decision maker can use as an aid to change the values assigned to the pairwise comparisons.

The main point of the interactive procedure is to return the optimal adjusted reciprocal preference relation, $\overline{R}$, to the decision maker, who should reconstruct a new adjusted reciprocal preference relation R' according to it. To construct $R' = [r'_{ij}]$, the following interval-valued components are suggested:

$$r'_{ij} \in [\min(r_{ij}\overline{r}_{ij}), \max(r_{ij}, \overline{r}_{ij})] \quad (19)$$

Therefore, we have:

$$r'_{ij} = \min(r_{ij}, \overline{r}_{ij})\beta_{ij} + \max(r_{ij}, \overline{r}_{ij})(1 - \beta_{ij}) \quad (20)$$

being $\beta_{ij} \in [0, 1]$ a value decided by the decision maker.

In the interactive procedure for multiplicative consistency improvement there exist two stop conditions. On the one hand, it stops whether a maximum number of iterations, max_iter, is reached by the current iteration. On the other hand, it stops whether the adjustment of the reciprocal preference relation is less than a threshold δ, which establishes a tolerance degree. In particular, based on the original/adjusted reciprocal preference relation given by the decision maker, the number of iterations is recorded and the deviation, DEV, between the original and the adjusted reciprocal preference relation is obtained by calculating the distance between both reciprocal preference relations:

$$DEV(R, R') = \frac{2}{n^2 - n} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |r_{ij} - r'_{ij}| \quad (21)$$

Whether the deviation is greater or equal than δ and the number of the current iteration is less than max_iter, the feedback is given to the decision maker based on the optimal adjusted reciprocal preference relation returned by the process described in Section 3.1. According to it, the decision maker may change her or his reciprocal preference relation. Concretely, the decision maker may refuse, partially accept, or entirely accept the recommendation received. Let k be the iteration number and let R^k be the reciprocal preference relation given by the decision maker at the k iteration. Algorithm 1 describes the steps of this interactive procedure.

Algorithm 1:

Input: $R, \varepsilon, max\_iter, \delta$
Output: R'

```
k = 0, R^0 = R, DEV^0 = δ
while k < max_iter and DEV^k ≥ δ do
    Calculate R̄^k according to (17)
    Construct R' as suggested by (19)
    R^{k+1} = R'
    Calculate DEV^{k+1}(R^k, R^{k+1})
    k = k + 1
end while
return R'
```

4. Illustrative examples

First, three numerical examples are performed to show the applicability of the proposed process of multiplicative consistency improvement to increase the consistency degree of the reciprocal preference relations by means of the optimization of the level of information granularity and the adjusted reciprocal preference relations. Second, its effectiveness is illustrated by offering some comparisons. Third, the application of the interactive procedure for multiplicative consistency improvement is shown by using the same numerical examples.

4.1. Numerical examples with the process of multiplicative consistency improvement

We consider three reciprocal preference relations, R_1, R_2 and R_3, of dimensions 5×5, 7×7, and 8×8, respectively, to show the applicability of the optimal process of multiplicative consistency improvement:

$$R_1 = \begin{bmatrix} - & 0.70 & 0.60 & 0.40 & 0.20 \\ 0.30 & - & 0.30 & 0.10 & 0.40 \\ 0.40 & 0.70 & - & 0.90 & 0.60 \\ 0.60 & 0.90 & 0.10 & - & 0.70 \\ 0.80 & 0.60 & 0.40 & 0.30 & - \end{bmatrix}$$

$$R_2 = \begin{bmatrix} - & 0.60 & 0.70 & 0.20 & 0.40 & 0.30 & 0.10 \\ 0.40 & - & 0.40 & 0.10 & 0.60 & 0.20 & 0.30 \\ 0.30 & 0.60 & - & 0.70 & 0.90 & 0.90 & 0.20 \\ 0.80 & 0.90 & 0.30 & - & 0.70 & 0.30 & 0.70 \\ 0.60 & 0.40 & 0.10 & 0.30 & - & 0.80 & 0.80 \\ 0.70 & 0.80 & 0.10 & 0.70 & 0.20 & - & 0.90 \\ 0.90 & 0.70 & 0.80 & 0.30 & 0.20 & 0.10 & - \end{bmatrix}$$

$$R_3 = \begin{bmatrix} - & 0.30 & 0.80 & 0.80 & 0.30 & 0.20 & 0.40 & 0.60 \\ 0.70 & - & 0.60 & 0.60 & 0.60 & 0.70 & 0.20 & 0.30 \\ 0.20 & 0.40 & - & 0.70 & 0.30 & 0.60 & 0.80 & 0.80 \\ 0.20 & 0.40 & 0.30 & - & 0.70 & 0.80 & 0.20 & 0.60 \\ 0.70 & 0.40 & 0.70 & 0.30 & - & 0.80 & 0.60 & 0.70 \\ 0.80 & 0.30 & 0.40 & 0.20 & 0.20 & - & 0.20 & 0.70 \\ 0.60 & 0.80 & 0.20 & 0.80 & 0.40 & 0.80 & - & 0.70 \\ 0.40 & 0.70 & 0.20 & 0.40 & 0.30 & 0.30 & 0.30 & - \end{bmatrix}$$

Using (5), the consistency degrees of R_1, R_2 and R_3, are $cd(R_1) = 0.642$, $cd(R_2) = 0.578$ and $cr(R_3) = 0.688$, respectively. Before applying the proposed process of multiplicative consistency improvement, we establish two average levels of information granularity: $\varepsilon = 0.1$ and $\varepsilon = 0.2$. Subsequently, we apply the process of multiplicative consistency improvement to optimize the modified reciprocal preference relations, $\overline{R_1}$, $\overline{R_2}$ and $\overline{R_3}$, related to R_1, R_2 and R_3, and the distribution of information granularity ε.

To optimize the entries of the adjusted reciprocal preference relations, $\overline{R_1}$, $\overline{R_2}$ and $\overline{R_3}$, we run the DE algorithm. The search parameters of DE are set to be $F = 0.8$, $NP = 8(0.5n(n-1))$ and $CR = 0.9$. Then, using (18), we calculate the information granularity distribution according to the adjusted reciprocal preference relations that have been obtained. The matrix $D = [\varepsilon_{ij}]$, being ε_{ij} the distributed information granularity, conveys the information granularity distribution. Taking into account R_1, R_2 and R_3, the adjusted reciprocal preference relations and the matrices containing the information granularity distribution are the following.

Given $\varepsilon = 0.1$, the values of $\overline{R_1}$ and D_1 for R_1 are:

$$\overline{R}_1 = \begin{bmatrix} - & 0.70 & 0.47 & 0.40 & 0.38 \\ 0.30 & - & 0.30 & 0.10 & 0.40 \\ 0.53 & 0.70 & - & 0.72 & 0.60 \\ 0.60 & 0.90 & 0.28 & - & 0.69 \\ 0.62 & 0.60 & 0.40 & 0.31 & - \end{bmatrix}$$

$$D_1 = \begin{bmatrix} - & 0.00 & 0.26 & 0.00 & 0.36 \\ - & - & 0.00 & 0.00 & 0.00 \\ - & - & - & 0.36 & 0.00 \\ - & - & - & - & 0.02 \\ - & - & - & - & - \end{bmatrix}$$

Given $\varepsilon = 0.2$, the values of $\overline{R_1}$ and D_1 for R_1 are:

$$\overline{R}_1 = \begin{bmatrix} - & 0.63 & 0.46 & 0.38 & 0.41 \\ 0.37 & - & 0.27 & 0.24 & 0.44 \\ 0.54 & 0.73 & - & 0.59 & 0.61 \\ 0.62 & 0.76 & 0.41 & - & 0.68 \\ 0.59 & 0.56 & 0.39 & 0.32 & - \end{bmatrix}$$

$$D_1 = \begin{bmatrix} - & 0.142 & 0.282 & 0.042 & 0.422 \\ - & - & 0.062 & 0.282 & 0.082 \\ - & - & - & 0.622 & 0.022 \\ - & - & - & - & 0.042 \\ - & - & - & - & - \end{bmatrix}$$

Given $\varepsilon = 0.1$, the values of $\overline{R_2}$ and D_2 for R_2 are:

$$\overline{R}_2 = \begin{bmatrix} - & 0.59 & 0.58 & 0.10 & 0.38 & 0.29 & 0.22 \\ 0.41 & - & 0.42 & 0.11 & 0.53 & 0.21 & 0.30 \\ 0.42 & 0.58 & - & 0.71 & 0.82 & 0.81 & 0.23 \\ 0.90 & 0.89 & 0.29 & - & 0.66 & 0.35 & 0.69 \\ 0.62 & 0.47 & 0.18 & 0.34 & - & 0.58 & 0.78 \\ 0.71 & 0.79 & 0.19 & 0.65 & 0.42 & - & 0.89 \\ 0.78 & 0.70 & 0.77 & 0.31 & 0.22 & 0.11 & - \end{bmatrix}$$

$$D_2 = \begin{bmatrix} - & 0.02 & 0.24 & 0.20 & 0.04 & 0.02 & 0.24 \\ - & - & 0.04 & 0.02 & 0.14 & 0.02 & 0.00 \\ - & - & - & 0.02 & 0.16 & 0.18 & 0.06 \\ - & - & - & - & 0.08 & 0.10 & 0.02 \\ - & - & - & - & - & 0.44 & 0.04 \\ - & - & - & - & - & - & 0.02 \\ - & - & - & - & - & - & - \end{bmatrix}$$

Given $\varepsilon = 0.2$, the values of $\overline{R_2}$ and D_2 for R_2 are:

$$\overline{R}_2 = \begin{bmatrix} - & 0.58 & 0.50 & 0.21 & 0.35 & 0.39 & 0.26 \\ 0.42 & - & 0.42 & 0.19 & 0.47 & 0.19 & 0.26 \\ 0.50 & 0.58 & - & 0.65 & 0.63 & 0.72 & 0.26 \\ 0.79 & 0.81 & 0.35 & - & 0.69 & 0.32 & 0.58 \\ 0.65 & 0.53 & 0.37 & 0.31 & - & 0.61 & 0.69 \\ 0.61 & 0.81 & 0.28 & 0.68 & 0.39 & - & 0.77 \\ 0.74 & 0.74 & 0.74 & 0.42 & 0.31 & 0.23 & - \end{bmatrix}$$

$$D_2 = \begin{bmatrix} - & 0.046 & 0.406 & 0.026 & 0.106 & 0.186 & 0.166 \\ - & - & 0.046 & 0.186 & 0.266 & 0.026 & 0.086 \\ - & - & - & 0.106 & 0.546 & 0.366 & 0.126 \\ - & - & - & - & 0.026 & 0.046 & 0.246 \\ - & - & - & - & - & 0.386 & 0.226 \\ - & - & - & - & - & - & 0.266 \\ - & - & - & - & - & - & - \end{bmatrix}$$

Given $\varepsilon = 0.1$, the values of $\overline{R_3}$ and D_3 for R_3 are:

$$\overline{R}_3 = \begin{bmatrix} - & 0.27 & 0.64 & 0.79 & 0.33 & 0.37 & 0.39 & 0.59 \\ 0.73 & - & 0.60 & 0.64 & 0.60 & 0.53 & 0.26 & 0.34 \\ 0.36 & 0.40 & - & 0.70 & 0.31 & 0.59 & 0.65 & 0.81 \\ 0.21 & 0.36 & 0.30 & - & 0.66 & 0.77 & 0.28 & 0.57 \\ 0.67 & 0.40 & 0.69 & 0.34 & - & 0.78 & 0.58 & 0.62 \\ 0.63 & 0.47 & 0.42 & 0.23 & 0.22 & - & 0.14 & 0.63 \\ 0.61 & 0.74 & 0.35 & 0.72 & 0.42 & 0.86 & - & 0.64 \\ 0.41 & 0.66 & 0.19 & 0.43 & 0.38 & 0.37 & 0.36 & - \end{bmatrix}$$

$$D_3 = \begin{bmatrix} - & 0.06 & 0.32 & 0.02 & 0.06 & 0.34 & 0.02 & 0.02 \\ - & - & 0.00 & 0.08 & 0.00 & 0.34 & 0.12 & 0.08 \\ - & - & - & 0.00 & 0.02 & 0.02 & 0.30 & 0.02 \\ - & - & - & - & 0.08 & 0.06 & 0.16 & 0.06 \\ - & - & - & - & - & 0.04 & 0.04 & 0.16 \\ - & - & - & - & - & - & 0.12 & 0.14 \\ - & - & - & - & - & - & - & 0.12 \\ - & - & - & - & - & - & - & - \end{bmatrix}$$

Given $\varepsilon = 0.2$, the values of $\overline{R_3}$ and D_3 for R_3 are:

$$\overline{R}_3 = \begin{bmatrix} - & 0.40 & 0.39 & 0.65 & 0.31 & 0.58 & 0.41 & 0.64 \\ 0.60 & - & 0.60 & 0.59 & 0.61 & 0.70 & 0.33 & 0.57 \\ 0.61 & 0.40 & - & 0.59 & 0.37 & 0.64 & 0.53 & 0.81 \\ 0.35 & 0.41 & 0.41 & - & 0.50 & 0.82 & 0.25 & 0.57 \\ 0.69 & 0.39 & 0.63 & 0.50 & - & 0.77 & 0.46 & 0.67 \\ 0.42 & 0.30 & 0.36 & 0.18 & 0.23 & - & 0.22 & 0.53 \\ 0.59 & 0.67 & 0.47 & 0.75 & 0.54 & 0.78 & - & 0.73 \\ 0.36 & 0.43 & 0.19 & 0.43 & 0.33 & 0.47 & 0.27 & - \end{bmatrix}$$

$$D_3 = \begin{bmatrix} - & 0.202 & 0.822 & 0.302 & 0.022 & 0.762 & 0.022 & 0.082 \\ - & - & 0.000 & 0.022 & 0.022 & 0.000 & 0.260 & 0.542 \\ - & - & - & 0.222 & 0.142 & 0.082 & 0.542 & 0.022 \\ - & - & - & - & 0.402 & 0.042 & 0.102 & 0.062 \\ - & - & - & - & - & 0.062 & 0.282 & 0.062 \\ - & - & - & - & - & - & 0.042 & 0.342 \\ - & - & - & - & - & - & - & 0.062 \\ - & - & - & - & - & - & - & - \end{bmatrix}$$

Considering the optimal adjusted reciprocal preference relations, we determine their consistency degrees, which are shown in Table 1. These three numerical examples have demonstrated that the consistency degrees of the adjusted reciprocal preference relations can be improved by allowing a given ε. In addition, for $\varepsilon = 0.2$, the consistency degree reached in the three numerical examples has been greater than the one achieved for $\varepsilon = 0.1$. It means that a higher value of ε allows to achieve a greater improvement of the multiplicative consistency.

Table 1. Values of $cd(R_l)$ for chosen values of ε.

	$\varepsilon = 0.1$	$\varepsilon = 0.2$
$cd(R_1)$	0.729	0.750
$cd(R_2)$	0.630	0.660
$cd(R_3)$	0.721	0.749

4.2. Comparative study

As a means to validate the effectiveness of the proposed process, some simulated experiments are conducted in which it is compared with an approach based on a uniform distribution of information granularity (Cabrerizo et al., 2017), i.e., the level of information granularity ε is uniformly distributed among all the entries of the reciprocal preference relation. For comparative purposes, we refer to this approach as UD-process.

We randomly produce 150 reciprocal preference relations of distinct dimensions. The dimensions of the reciprocal preference relations R_l are $5(l = 1, \dots, 30)$, $6(l = 31, \dots, 60)$, $7(l = 61, \dots, 90)$, $8(l = 91, \dots, 120)$, and $9(l = 121, \dots, 150)$.

Let ε be 0.1 and 0.2, respectively. Based on the maximization of the consistency degree, both the UD-process and the proposed process are applied to obtain the adjusted reciprocal preference relations. The relation between the optimized and the original consistency degree, i.e., $cd(\overline{R})$ and $cd(R)$, is illustrated in Figs. 2 and 3. It can be observed that the values of $cd(\overline{R})$ in both procedures are higher than the values of $cd(R)$ for all the reciprocal preference relations. Furthermore, the values of $cd(\overline{R})$ are evidently higher for the proposed process than those reached by the UD-process. It illustrates the proposed process achieves better multiplicative consistency degrees.

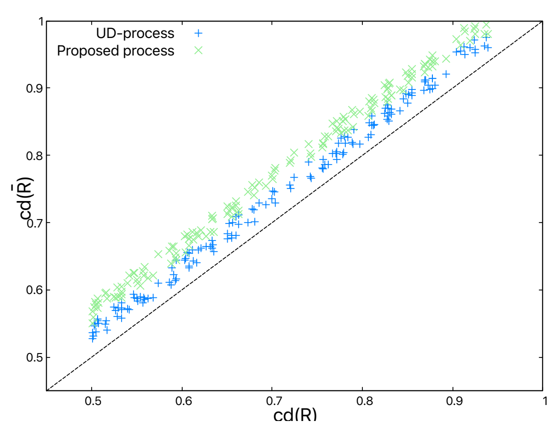

Figure 2. Optimized cd ($\varepsilon = 0.1$). Proposed process versus UD-process.

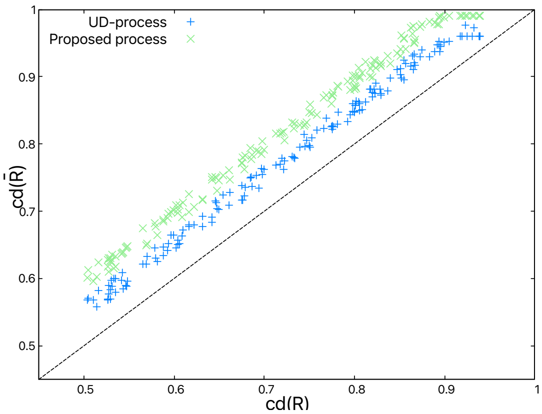

Figure 3. Optimized cd ($\varepsilon = 0.2$). Proposed process versus UD-process.

4.3. Numerical examples with the interactive procedure

To show the performance of the interactive multiplicative consistency improvement procedure, the reciprocal preference relations, R_1, R_2 and R_3, are also considered. In these examples, the decision maker's modification behavior in the interactive procedure is modeled by assuming that she or he agrees with the optimal adjusted reciprocal preference relation produced by the process of multiplicative consistency improvement described in Section 3.1, i.e., we suppose that the decision maker entirely accepts the recommendation received.

Let $max\_iter = 4$, $\delta = 0$, and $\varepsilon = 0.1$. Let k be the iteration number and let $cd(R_l)$ be the multiplicative consistency degree of the reciprocal preference R_l. Table 2 shows the values of the multiplicative consistency degrees of R_1, R_2 and R_3, in successive iterations of the interactive procedure (when $k = 0$, the consistency degree is the one related to the original reciprocal preference relation). As shown in Table 2, the values of the consistency degrees improve with the increasing numbers of iteration. It demonstrates the effectiveness of the interactive procedure.

Table 2. Values of $cd(R_l)$ in successive iterations of the interactive procedure.

	$k=0$	$k=1$	$k=2$	$k=3$
$cd(R_1)$	0.642	0.729	0.750	0.774
$cd(R_2)$	0.578	0.630	0.660	0.680
$cd(R_3)$	0.688	0.721	0.749	0.771

5. Conclusions

This study has presented an optimal granular-based approach addressing multiplicative consistency of reciprocal preference relations. First, based on a given average level of information granularity, a process of multiplicative consistency improvement with an optimal distribution of information granularity has been introduced. Although the increasing of consistency comes to detriment of improving the loss of information, and vice versa, i.e., the consistency degree and the information loss are in conflict, due to the bounded average information granularity, this process can improve the multiplicative consistency with a limited loss of information between the original and the adjusted reciprocal preference relation. Second, using this process, an interactive procedure for multiplicative consistency improvement has been constructed. The fundamental distinction between the former (automatic adjustment method) and the latter (interactive adjustment approach) is the participation of the decision maker. In many real-world decision-making processes, automatic adjustment methods cannot manage the question of preference degrees with low reliability without the participation of the decision maker, which is required during the adjustment step to increase the acceptability and validity of the solution adopted. In such a way, the interactive procedure is appropriate for real-world applications, whereas the optimal process of multiplicative consistency improvement can be used to offer an optimal a rational adjustment advice to the decision maker. Several numerical examples have been conducted to validate the effectiveness of both approaches.

The multiplicative transitivity property has been used in this study to characterize the cardinal consistency of the reciprocal preference relations due to Chiclana et al. (2009) proved it is the most appropriate one. However, as a first step at attempting to extend this study, any other consistency measure could also be used (Li et al., 2019). In addition, even though the computation time of the multiplicative consistency improvement process takes only a few seconds, it would be interesting to study how this time is increased with reciprocal preference relations having a greater number of components.

Acknowledgements

This work was supported by FEDER/Junta de Andalucía-Consejería de Transformación Económica, Industria, Conocimiento y Universidades / Proyecto

B-TIC-590-UGR20, by the Andalusian government through project P20_00673, by the program “Ayudas para la recualificación del sistema universitario español para 2021–2023” funded by the Ministerio de Universidades and Unión Europea - NextGenerationEU, and by the project PID2019-103880RB-I00 funded by MCIN / AEI / 10.13039/501100011033.

References

Bustince, H., Berrenechea, E., Pagola, M., Fernández, J., Xu, Z. S., Bedregal, B., Montero, J., Hagras, H., Herrera, F., & B. De Baets. (2016). A historial account of types of fuzzy sets and their relationships. *IEEE Transactions on Fuzzy Systems*, *24*(1), 179–194.

Cabrerizo, F. J., Pérez, I. J., Morente-Molinera, J. A., & Herrera-Viedma, E. (2021). A distribution of information granularity to deal with inconsistency in multi-criteria and heterogeneous group decision making. *Proceedings of the 54th Hawaii International Conference on System Sciences (HICSS 54)*, 1908–1917.

Cabrerizo, F. J., Pérez, I. J., Pedrycz, W., & Herrera-Viedma, E. (2017). An improvement of multiplicative consistency of reciprocal preference relations: A framework of Granular Computing. *Proceedings of the 2017 IEEE International Conference on Systems, Man, and Cybernetics (SMC)*, 1262–1267.

Carlsson, C., Brunelli, M., & Mezei, J. (2012). Decision making with a fuzzy ontology. *Soft Computing*, *16*(7), 1143–1152.

Chiclana, F., Herrera-Viedma, E., Alonso, S., & Herrera, F. (2009). Cardinal consistency of reciprocal preference relations: A characterization of multiplicative consistency. *IEEE Transactions on Fuzzy Systems*, *17*(1), 14–23.

Das, S., & Suganthan, P. N. (2011). Differential evolution: A survey of the state-of-the-art. *IEEE Transactions on Evolutionary Computation*, *15*(1), 4–31.

De Baets, B., Meyer, H. D., & De Schuymer, B. (2006). Cyclic evaluation of transitivity of reciprocal relations. *Social Choice and Welfare*, *26*, 217–238.

Fodor, J., & Roubens, M. (1994). *Fuzzy preference modelling and multicriteria decision support*. Kluwer Academic Publishers.

García-Lapresta, J. L., & Montero, J. (2006). Consistency in preference modeling. In B. Bouchon-Meunier, G. Coletti, & R. R. Yager (Eds.), *Modern information processing: From theory to applications* (pp. 87–97). Elsevier.

Herrera-Viedma, E., Herrera, F., Chiclana, F., & Luque, M. (2004). Some issues on consistency of fuzzy preference relations. *European Journal of Operational Research*, *154*(1), 98–109.

Herrera-Viedma, E., Palomares, I., Li, C.-C., Cabrerizo, F. J., Dong, Y. C., Chiclana, F., & Herrera, F. (2021). Revisiting fuzzy and linguistic decision making: Scenarios and challenges for making wiser decisions in a better way. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, *51*(1), 191–208.

Kacprzyk, J. (1986). Group decision making with a fuzzy linguistic majority. *Fuzzy Sets and Systems*, *18*(2), 105–118.

Li, C.-C., Dong, Y. C., Xu, Y., Chiclana, F., Herrera-Viedma, E., & Herrera, F. (2019). An overview on managing additive consistency of reciprocal preference relations for consistency-driven decision making and fusion: Taxonomy and future directions. *Information Fusion*, *52*, 143–156.

Millet, I. (1997). The effectiveness of alternative preference elicitation methods in the analytic hierarchy process. *Journal of Multi-Criteria Decision Analysis*, *6*(1), 41–51.

Pedrycz, W. (2014). Allocation of information granularity in optimization and decision-making models: Towards building the foundations of Granular Computing. *European Journal of Operational Research*, *232*(1), 137–145.

Pedrycz, W., & Song, M. (2011). Analytic hierarchy process (AHP) in group decision making and its optimization with an allocation of information granularity. *IEEE Transactions on Fuzzy Systems*, *19*(3), 527–539.

Storn, R., & Price, K. (1997). Differential evolution – A simple and efficient heuristic for global optimization over continuous spaces. *Journal of Global Optimization*, *11*(4), 341–359.

Świtalski, Z. (1999). Rationality of fuzzy reciprocal preference relations. *Fuzzy Sets and Systems*, *107*(2), 187–190.

Zhang, B., Pedrycz, W., Fayek, A. R., & Dong, Y. C. (2022). A differential evolution-based consistency improvement method in AHP with an optimal allocation of information granularity. *IEEE Transactions on Cybernetics*, *52*(7), 6733–6744.