Advances in Discrete-State-Feedback Stabilization of Highly Nonlinear Hybrid Systems by Razumikhin Technique

Henglei Xu and Xuerong Mao

***Abstract*—In this paper, the authors apply the Razumikhin technique to investigate the stabilization of hybrid stochastic systems by feedback control based on discrete-time state observations, rather than the widely used comparison idea or Lyapunov functional method to this problem. Further, we extend the Razumikhin method to study the asymptotic boundedness of hybrid stochastic systems. The coefficients of these stochastic systems considered do not meet the usual linear growth condition, but are highly nonlinear. The control function designed can easily be implemented in reality. Meanwhile, a better bound for a class of stochastic systems could be obtained on the duration between two consecutive state observations comparing with the existing results. Two interesting examples, the application to stochastic volatility model and stochastic Cohen-Grossberg neural network, respectively, are provided to manifest the effectiveness of our new theory.**

***Index Terms*—Razumikhin technique, highly nonlinearity, Itô formula, discrete-state feedback control.**

I. Introduction

A lot of practical systems, such as economic systems [1], manufacturing systems [2] and neural networks [3], whose structure and parameters may change abruptly, can be modeled by stochastic differential equations (SDEs) driven by continuous-time Markov chain (also known as hybrid SDEs). Among many interesting topics in the study of hybrid SDEs, the automatic control has drawn intensive attention. There is enormous literature in this area. Here, we refer the reader to [4]–[9] and references therein.

In the case when a given hybrid SDE is unstable, it is a general practice to use feedback control to make the underlying system perform as desired, say stably. Up to 2013, the design of feedback control was based on continuous-time state observations. However, in the real world, it is extremely costly and impossible to have continuous observations of the state for all time. So it is more realistic and costs less if the state is only observed at discrete time $0, \tau, 2\tau, \cdots$. In 2013, Mao in [10] initiated the study of the discrete-state-feedback stabilization of hybrid SDEs. Since then, this problem has already been studied by several authors (see, e.g., [11]–[16]).

The principal procedure to investigate the discrete-state-feedback stabilization problem is: (i) design a feedback controller based on continuous-time state observations, which is able to stabilize the original unstable system, (ii) obtain a bound on the observation duration, say by τ^*, (iii) let this designed controller behave in discrete time and make sure $\tau < \tau^*$. This idea has been used very successfully in some branches of science and industry for many years (see, e.g., [10], [12], [13], [17]). As how to show the stability of the discrete-time controlled system, to the best of the authors' knowledge, there are currently two effective methods widely used.

One is to construct an auxiliary system, namely the continuous-time controlled system, which is proved to be stable in advance, and then make use of the state difference estimation between the discrete-time controlled system and the auxiliary system to indirectly derive the stability of the discrete-time controlled system. This indirect technique is often called the comparison theorem, through which we can take full advantage of the existing stabilization results by continuous-time feedback control. And there is very rich literature in this problem (see, e.g., [18]–[20]). However, the duration bound, τ^*, derived by using this method is usually not very sharp (e.g., [10], [13]). But what's worse, this method is not applicable for highly nonlinear hybrid SDEs (namely do not satisfy the linear growth condition). For example, in [21], Hu et al. showed that this method worked well if and only if the underlying system was globally Lipschitz continuous (at most times, satisfied the linear growth condition).

The other one is therefore becoming significant, which works directly to the discrete-time controlled system. This is easier to use because we do not need to first guarantee and verify the stability of an auxiliary system. Now, to implement this idea, the technique of Lyapunov functionals has received much attention. By making use of the Lyapunov functional method, You et al. in [11] also obtained a better τ^* than that in [10]. In the highly nonlinear area, [16] was the first paper to design feedback control based on discrete-time state observations to stabilize a given unstable highly nonlinear hybrid SDE by using the Lyapunov functional method. Nevertheless, it should also be pointed out that this approach depends closely on the construction of Lyapunov functionals. But as we all know, constructing Lyapunov functionals effectively is sometimes really a challenge work.

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H. Xu and X. Mao are with the Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, U.K. (e-mail: henglei.xu@strath.ac.uk, x.mao@strath.ac.uk).

On the other hand, we notice that the discrete-state-feedback controlled system is in fact a hybrid stochastic differential delay equation (SDDE) (see (18) below). In the study of the stability of delay systems, the Razumikhin technique has been proved as a very powerful tool. We cite [6], [22]–[26] to the reader for more details. So this begs a question naturally: can we use the Razumikhin method to investigate the stabilization problem of discrete-state feedback control? The answer is positive. In fact, Li et al. in [14] successfully applied the Razumikhin method to the discrete-state-feedback stabilization problem for a class of hybrid stochastic systems, and surprisingly, they even obtained a sharper τ^* than that in [10] and [11]. And to our knowledge, so far [14] has been the only paper to use the Razumikhin approach to investigate this kind of stabilization problem. But unfortunately, they still required system coefficients to meet the linear growth condition.

Consequently, motivated by [10], [14], [16], we will employ the Razumikhin technique to study the boundedness and exponential stability of highly nonlinear hybrid SDEs by feedback control based on discrete-time state observations in this paper. Certainly, we will work to the discrete-time controlled system directly. A number of main features which differ from those in [16] are highlighted below:

- The Razumikhin method is applied to study the discrete-state-feedback stabilization problem of highly nonlinear hybrid SDEs, which can avoid the difficulty of constructing appropriate Lyapunov functionals and much complicated analysis. Further, we extend the Razumikhin technique to investigate the asymptotic boundedness of hybrid stochastic systems.
- The conditions imposed on the control function and the original system are weaken, so we can include more general stochastic systems and deal with more complicated situations (see Example 1).
- Conditions imposed on the control function can be verified much more easily in practice, in particular comparing with Conditions 4.2 and 5.1 in [16]. Moreover, a better bound on the duration between two consecutive state observations will be obtained for a class of hybrid SDEs (see Example 2).

II. Razumikhin-type Theorem on Boundedness and Exponential Stability

Throughout this paper, unless otherwise specified, we use the following notations. If both a and b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $\mathbb{R}_+$ denote the collection of all non-negative real numbers. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If M is a vector or matrix, M^{T} denotes its transpose. If M is a matrix, denote its trace norm by $|A| = \sqrt{\text{trace}(A^{\mathrm{T}}A)}$. For any positive constant h, $C([-h, 0]; \mathbb{R}^n)$ represents the family of all continuous functions ϕ from $[-h, 0]$ to $\mathbb{R}^n$ with norm $||\phi|| = \sup_{-h\le\theta\le 0} |\phi(\theta)|$. Denote by $L^q_{\mathcal{F}_t}([-h, 0]; \mathbb{R}^n)$, $q > 0$, the family of all $\mathcal{F}_t$-measurable $C([-h, 0]; \mathbb{R}^n)$-valued random variables ϕ such that $E||\phi||^q < \infty$.

We let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (that is, it is increasing, right-continuous and $\mathcal{F}_0$ contains all P-null sets). If $\bar\Omega$ is a subset of Ω, denote by $I_{\bar\Omega}$ its indicator function, that is, $I_{\bar\Omega}(\omega) = 1$ if $\omega \in \bar\Omega$ and 0 otherwise. Let $B(t) = (B_1(t), B_2(t), \cdots, B_m(t))^{\mathrm{T}}$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with transition rate matrix $Q = (q_{ij})_{N\times N}$ given by

$$P(r(t+\Delta) = j | r(t) = i) = \begin{cases} 1 + q_{ij}\Delta + o(\Delta), & \text{if } i = j, \\ q_{ij}\Delta + o(\Delta), & \text{if } i \ne j, \end{cases}$$

as $\Delta \downarrow 0$. Here $q_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$, while $q_{ii} = -\sum_{j\ne i} q_{ij}$. We assume that the Markov chain $r(t)$ and the Brownian motion $B(t)$ are independent under the probability measure P.

Most of the existing results (e.g., [6], [22], [23]) mainly use the Razumikhin technique to derive the q-th moment exponential stability, but few focus on boundedness. Therefore, in this paper, we will extend the Razumikhin-type theorem to the asymptotic boundedness.

Let us first consider an n-dimensional stochastic hybrid functional differential equation (SFDE)

$$\mathrm{d}x(t) = F(x_t, t, r(t))\mathrm{d}t + G(x_t, t, r(t))\mathrm{d}B(t) \tag{1}$$

on $t \ge 0$ with the initial data

$$\{x(\theta) | -h \le \theta \le 0\} = \zeta \in C([-h, 0]; \mathbb{R}^n), \quad r(0) = r_0 \in S.$$

Here $x_t = \{x(t+\theta) | -h \le \theta \le 0\}$ is the past segment while

$$F : C([-h, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times S \to \mathbb{R}^n,$$
$$G : C([-h, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times m}$$

are Borel-measurable functions. For convenience, we extend $r(t)$ to $[-h, 0]$ by setting $r(\theta) = r_0$ for all $\theta \in [-h, 0]$.

To state our main results, we need a few more notations. Let $C(\mathbb{R}^n \times [-h, \infty); \mathbb{R}_+)$ stand for the collection of all continuous functions from $\mathbb{R}^n \times [-h, \infty)$ to $\mathbb{R}_+$. Denote by $C^{2,1}(\mathbb{R}^n \times [-h, \infty) \times S; \mathbb{R}_+)$ the family of all continuous non-negative functions $V(x, t, i)$ on $\mathbb{R}^n \times [-h, \infty) \times S$ such that for each $i \in S$, they are continuously twice differentiable in x and once in t. Given $V \in C^{2,1}(\mathbb{R}^n \times [-h, \infty) \times S; \mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $C([-h, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times S$ to $\mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(\phi, t, i) =& V_t(\phi(0), t, i) + V_x(\phi(0), t, i)F(\phi, t, i) \\ &+ \frac{1}{2}\text{trace}\left(G^{\mathrm{T}}(\phi, t, i)V_{xx}(\phi(0), t, i)G(\phi, t, i)\right) \\ &+ \sum_{j=1}^{N} q_{ij}V(\phi(0), t, j), \end{aligned} \tag{2}$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right)$ and $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{n\times n}$.

The following two assumptions are also required.

Assumption 2.1: Assume that for every initial data ζ and r_0, there exists a unique global solution $x(t)$ of SFDE (1). For some $q > 0$, the solution has the property that $E|x(t)|^q < \infty$

for any $t \geq 0$. Assume further that there exists a function $V(x,t,i) \in C^{2,1}(\mathbb{R}^n \times [-h,\infty) \times S; \mathbb{R}_+)$ such that

$$\sup_{-h\leq s\leq t} EV(x(s),s,r(s)) < \infty, \quad \forall t \geq -h,$$

and

$$\sup_{0\leq s\leq t} E|\mathcal{L}V(x_s,s,r(s))| < \infty, \quad \forall t \geq 0.$$

Moreover, assume that $EV(x(t),t,r(t))$ and $E\mathcal{L}V(x_t,t,r(t))$ are right-continuous functions on $t \geq 0$.

Assumption 2.2: For such $V(x,t,i)$ defined in Assumption 2.1, suppose that for any integer $k > 0$, there is a $L_k > 0$ such that

$$|V_x(\phi(0),t,i)G(\phi,t,i)| \leq L_k \tag{3}$$

for any $\phi \in C([-h,0];\mathbb{R}^n)$ with $||\phi|| \leq k$ and $(t,i) \in \mathbb{R}_+ \times S$.

Remark 1: Assumption 2.1 just provides a general setting of SFDE (1), which should have a global solution with some moment properties in terms of a proper function V. Thus in practice, we need to impose extra conditions that can easily be verified to guarantee this assumption. For example, in our subsequent control problem, we give Assumptions 3.1, 3.2, 4.1 and Rules 1, 2 to do this job. On the other hand, we highlight that Assumption 2.2 is a local requirement, which is used to estimate the Itô integral. But different from Assumption 2.1, it is given in terms of an arbitrary function ϕ, rather than the solution $x(s)$ or x_s. Therefore it can be guaranteed in practice, such as by imposing the local Lipschitz condition on G.

Now, we can give our new Razumikhin-type theorem.

Theorem 2.1: Let Assumptions 2.1 and 2.2 hold. For such $V(x,t,i)$ defined in Assumption 2.1, assume that there exist constants $p > 1$, $\lambda_1 \geq 0$ and $\lambda_2 > 0$ such that

$$E\mathcal{L}V(x_t,t,r(t)) \leq \lambda_1 - \lambda_2 EV(x_t(0),t,r(t)) \tag{4}$$

when x_t satisfies that

$$EV(x_t(\theta),t+\theta,r(t+\theta)) \leq pEV(x_t(0),t,r(t))$$

for any $\theta \in [-h,0]$. Then for any initial data ζ and r_0, the solution of SFDE (1) has the property that

$$\limsup_{t\to\infty} EV(x(t),t,r(t)) \leq \frac{\lambda_1}{\lambda},$$

where $\lambda = \min\left\{\lambda_2, \frac{\log p}{h}\right\}$. In particular, if $\lambda_1 = 0$,

$$\limsup_{t\to\infty} \frac{1}{t}\log EV(x(t),t,r(t)) \leq -\lambda.$$

Proof. Fix any initial data ζ and r_0. Let $\eta \in (0,\lambda)$ be arbitrary and set $\bar{\lambda} = \lambda - \eta$. For $t \geq 0$, define

$$U(t) = \sup_{-h\leq\theta\leq 0}\left(e^{\bar{\lambda}(t+\theta)}EV(x(t+\theta),t+\theta,r(t+\theta))\right).$$

From Assumption 2.1, we know that $U(t) < \infty$ for any $t \geq 0$, so $U(t)$ is well-defined. Letting $y_\eta(t) = \lambda_1\int_0^t e^{\bar{\lambda}s}\mathrm{d}s$, we then claim that

$$D^+(U(t) - y_\eta(t)) \leq 0, \qquad t \geq 0. \tag{5}$$

If assertion (5) is true, we have

$$U(t) - y_\eta(t) \leq U(0) - y_\eta(0) \leq M, \qquad t \geq 0,$$

where $M = \sup_{-h\leq\theta\leq 0} V(\zeta(\theta),\theta,r(\theta))$. It then follows that for any $t \geq 0$,

$$e^{\bar{\lambda}t}EV(x(t),t,r(t)) \leq M + \lambda_1\int_0^t e^{\bar{\lambda}s}\mathrm{d}s \leq M + \frac{\lambda_1}{\bar{\lambda}}e^{\bar{\lambda}t}.$$

Since η is arbitrary, we have

$$EV(x(t),t,r(t)) \leq Me^{-\lambda t} + \frac{\lambda_1}{\lambda}. \tag{6}$$

Finally, letting $t \to \infty$ gives

$$\limsup_{t\to\infty} EV(x(t),t,r(t)) \leq \frac{\lambda_1}{\lambda}.$$

If $\lambda_1 = 0$, we derive from (6) that

$$\limsup_{t\to\infty} \frac{1}{t}\log EV(x(t),t,r(t)) \leq -\lambda.$$

Now we show that assertion (5) is true. Fix $\hat{t} \geq 0$ arbitrarily. It is easy to observe that either

$$U(\hat{t}) > e^{\bar{\lambda}\hat{t}}EV(x(\hat{t}),\hat{t},r(\hat{t}))$$

or

$$U(\hat{t}) = e^{\bar{\lambda}\hat{t}}EV(x(\hat{t}),\hat{t},r(\hat{t})).$$

In the situation of the former, we derive from the right-continuity of $EV(x(\cdot),\cdot,r(\cdot))$ that for all sufficiently small $\Delta_1 \in (0,h)$,

$$U(\hat{t}) > e^{\bar{\lambda}t}EV(x(t),t,r(t)), \qquad \hat{t} \leq t \leq \hat{t} + \Delta_1. \tag{7}$$

For $\hat{t} + \Delta_1 - h \leq t < \hat{t}$, we naturally have

$$\begin{aligned}
&e^{\bar{\lambda}t}EV(x(t),t,r(t)) \\
\leq &\sup_{-h\leq\theta\leq 0}\left(e^{\bar{\lambda}(\hat{t}+\theta)}EV(x(\hat{t}+\theta),\hat{t}+\theta,r(\hat{t}+\theta))\right) = U(\hat{t}).
\end{aligned}$$

This together with (7) yields that $U(\hat{t}+\Delta_1) \leq U(\hat{t})$, and so

$$U(\hat{t}+\Delta_1) < U(\hat{t}) + \lambda_1\int_{\hat{t}}^{\hat{t}+\Delta_1} e^{\bar{\lambda}s}\mathrm{d}s.$$

Then we have

$$U(\hat{t}+\Delta_1) - y_\eta(\hat{t}+\Delta_1) < U(\hat{t}) - y_\eta(\hat{t}),$$

which indicates that

$$\begin{aligned}
&D^+(U(\hat{t}) - y_\eta(\hat{t})) \\
=&\limsup_{\Delta_1\to 0^+} \frac{\big(U(\hat{t}+\Delta_1) - y_\eta(\hat{t}+\Delta_1)\big) - \big(U(\hat{t}) - y_\eta(\hat{t})\big)}{\Delta_1} \leq 0.
\end{aligned}$$

On the other hand, if $U(\hat{t}) = e^{\bar{\lambda}\hat{t}}EV(x(\hat{t}),\hat{t},r(\hat{t}))$, we derive that for any $\theta \in [-h,0]$,

$$e^{\bar{\lambda}(\hat{t}+\theta)}EV(x(\hat{t}+\theta),\hat{t}+\theta,r(\hat{t}+\theta)) \leq e^{\bar{\lambda}\hat{t}}EV(x(\hat{t}),\hat{t},r(\hat{t})).$$

Consequently,

$$\begin{aligned}
EV(x(\hat{t}+\theta),\hat{t}+\theta,r(\hat{t}+\theta)) \leq& e^{-\bar{\lambda}\theta}EV(x(\hat{t}),\hat{t},r(\hat{t})) \\
\leq& e^{\bar{\lambda}h}EV(x(\hat{t}),\hat{t},r(\hat{t})) \\
\leq& pEV(x(\hat{t}),\hat{t},r(\hat{t})),
\end{aligned}$$

where we have used the fact that $p \geq e^{\lambda h}$. Then by condition (4), we have

$$\begin{aligned} &E\mathcal{L}V(x_{\hat{t}}, \hat{t}, r(\hat{t})) + \bar{\lambda} EV(x(\hat{t}), \hat{t}, r(\hat{t}) \\ \leq &E\mathcal{L}V(x_{\hat{t}}, \hat{t}, r(\hat{t})) + \lambda_2 EV(x(\hat{t}), \hat{t}, r(\hat{t})) \\ \leq &\lambda_1 < \lambda_1 + \varepsilon, \end{aligned}$$

where ε is an arbitrary positive constant. We therefore see from the right-continuity of $EV(x(t), t, r(t))$ and $E\mathcal{L}V(x_t, t, r(t))$ that for all $\Delta_2 \in (0, h)$ sufficiently small,

$$E\mathcal{L}V(x_t, t, r(t)) + \bar{\lambda} EV(x(t), t, r(t)) < \lambda_1 + \varepsilon \tag{8}$$

for any $\hat{t} \leq t \leq \hat{t} + \Delta_2$. For each integer $k \geq 1$, define the stopping time

$$\sigma_k(\omega) = \inf \left\{ t \geq \hat{t} \middle| |x(t, \omega)| \geq k \right\},$$

which represents the first exiting time of sample path $x(t, \omega)$ leaving from the area $\{x \in \mathbb{R}^n | |x| < k\}$ after time $\hat{t}$. But this could be infinity since it is possible that for some ω, $x(t, \omega)$ would never go beyond that area. In this situation, the time set $\left\{ t \geq \hat{t} \middle| |x(t, \omega)| \geq k \right\}$ is empty. Hence throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ is the empty set). For convenience, we denote $\sigma_k(\omega)$ by σ_k. Because SFDE (1) admits a unique global solution, we observe that σ_k is increasing to infinity almost surely as $k \to \infty$. For each $k \geq 1$, by the generalized Itô formula, we have

$$\begin{aligned} &e^{\bar{\lambda} \hat{t}_k} V(x(\hat{t}_k), \hat{t}_k, r(\hat{t}_k)) - e^{\bar{\lambda} \hat{t}} V(x(\hat{t}), \hat{t}, r(\hat{t})) \\ = &\int_{\hat{t}}^{\hat{t}_k} e^{\bar{\lambda} s} \left(\mathcal{L}V(x_s, s, r(s)) + \bar{\lambda} V(x(s), s, r(s)) \right) \mathrm{d}s + M_k, \end{aligned}$$

where $\hat{t}_k = (\hat{t} + \Delta_2) \wedge \sigma_k$ and

$$M_k = \int_{\hat{t}}^{\hat{t}_k} e^{\bar{\lambda} s} V_x(x(s), s, r(s)) G(x_s, s, r(s)) \mathrm{d}B(s).$$

Note that when $|x(\hat{t})| \geq k$ we have $\hat{t}_k = \hat{t}$; while $|x(\hat{t})| < k$, making use of (3), we see that for any $s \in [\hat{t}, \hat{t}_k]$

$$e^{\bar{\lambda} s} V_x(x(s), s, r(s)) G(x_s, s, r(s)) \leq e^{\bar{\lambda}(t + \Delta_2)} L_k < \infty.$$

Therefore, we have $EM_k = 0$ and hence

$$\begin{aligned} &E\left(e^{\bar{\lambda} \hat{t}_k} V(x(\hat{t}_k), \hat{t}_k, r(\hat{t}_k)) \right) - E\left(e^{\bar{\lambda} \hat{t}} V(x(\hat{t}), \hat{t}, r(\hat{t})) \right) \\ = &E \int_{\hat{t}}^{\hat{t}_k} e^{\bar{\lambda} s} \left(\mathcal{L}V(x_s, s, r(s)) + \bar{\lambda} V(x(s), s, r(s)) \right) \mathrm{d}s. \end{aligned}$$

It is easy to see that for each $k \geq 1$,

$$\begin{aligned} &\left| \int_{\hat{t}}^{\hat{t}_k} e^{\bar{\lambda} s} \left(\mathcal{L}V(x_s, s, r(s)) + \bar{\lambda} V(x(s), s, r(s)) \right) \mathrm{d}s \right| \\ \leq &\int_{\hat{t}}^{\hat{t} + \Delta_2} \left| e^{\bar{\lambda} s} \left(\mathcal{L}V(x_s, s, r(s)) + \bar{\lambda} V(x(s), s, r(s)) \right) \right| \mathrm{d}s. \end{aligned}$$

Since

$$\begin{aligned} &E \left| e^{\bar{\lambda} s} \left(\mathcal{L}V(x_s, s, r(s)) + \bar{\lambda} V(x(s), s, r(s)) \right) \right| \\ \leq &e^{\bar{\lambda} s} \left(E|\mathcal{L}V(x_s, s, r(s))| + \bar{\lambda} EV(x(s), s, r(s)) \right) < \infty \end{aligned}$$

holds for any $s \in [\hat{t}, \hat{t} + \Delta_2]$, by Fubini theorem, we have

$$\begin{aligned} &E \int_{\hat{t}}^{\hat{t} + \Delta_2} \left| e^{\bar{\lambda} s} \left(\mathcal{L}V(x_s, s, r(s)) + \bar{\lambda} V(x(s), s, r(s)) \right) \right| \mathrm{d}s \\ = &\int_{\hat{t}}^{\hat{t} + \Delta_2} E \left| e^{\bar{\lambda} s} \left(\mathcal{L}V(x_s, s, r(s)) + \bar{\lambda} V(x(s), s, r(s)) \right) \right| \mathrm{d}s \\ < &\infty. \end{aligned}$$

Letting $k \to \infty$ and using the Fatou lemma, the dominated convergence theorem, we obtain that

$$\begin{aligned} &e^{\bar{\lambda}(\hat{t} + \Delta_2)} EV(x(\hat{t} + \Delta_2), \hat{t} + \Delta_2, r(\hat{t} + \Delta_2)) \\ = &E\left(\liminf_{k \to \infty} e^{\bar{\lambda} t_k} V(x(t_k), t_k, r(t_k)) \right) \\ \leq &\liminf_{k \to \infty} E\left(e^{\bar{\lambda} t_k} V(x(t_k), t_k, r(t_k)) \right) \\ = &e^{\bar{\lambda} \hat{t}} EV(x(\hat{t}), \hat{t}, r(\hat{t})) + E \int_{\hat{t}}^{\hat{t} + \Delta_2} e^{\bar{\lambda} s} \Big(\mathcal{L}V(x_s, s, r(s)) \\ &+ \bar{\lambda} V(x(s), s, r(s)) \Big) \mathrm{d}s. \end{aligned} \tag{9}$$

Applying the Fubini theorem again as well as (8) yields that

$$\begin{aligned} &e^{\bar{\lambda}(\hat{t} + \Delta_2)} EV(x(\hat{t} + \Delta_2), \hat{t} + \Delta_2, r(\hat{t} + \Delta_2)) \\ < &e^{\bar{\lambda} \hat{t}} EV(x(\hat{t}), \hat{t}, r(\hat{t})) + \int_{\hat{t}}^{\hat{t} + \Delta_2} (\lambda_1 + \varepsilon) e^{\bar{\lambda} s} \mathrm{d}s. \end{aligned} \tag{10}$$

By analogy with (10), for any $\hat{t} \leq t \leq \hat{t} + \Delta_2$, we have

$$\begin{aligned} &e^{\bar{\lambda} t} EV(x(t), t, r(t)) \\ < &e^{\bar{\lambda} \hat{t}} EV(x(\hat{t}), \hat{t}, r(\hat{t})) + (\lambda_1 + \varepsilon) \int_{\hat{t}}^{t} e^{\bar{\lambda} s} \mathrm{d}s \\ \leq &U(\hat{t}) + (\lambda_1 + \varepsilon) \int_{\hat{t}}^{\hat{t} + \Delta_2} e^{\bar{\lambda} s} \mathrm{d}s. \end{aligned}$$

For $\hat{t} + \Delta_2 - h \leq t < \hat{t}$, it is also easy to see that

$$e^{\bar{\lambda} t} EV(x(t), t, r(t)) < U(\hat{t}) + (\lambda_1 + \varepsilon) \int_{\hat{t}}^{\hat{t} + \Delta_2} e^{\bar{\lambda} s} \mathrm{d}s$$

since $e^{\bar{\lambda} t} EV(x(t), t, r(t)) < U(\hat{t})$. Thus, we obtain that

$$U(\hat{t} + \Delta_2) \leq U(\hat{t}) + \lambda_1 \int_{\hat{t}}^{\hat{t} + \Delta_2} e^{\bar{\lambda} s} \mathrm{d}s + \varepsilon \int_{\hat{t}}^{\hat{t} + \Delta_2} e^{\bar{\lambda} s} \mathrm{d}s.$$

Letting $\Delta_2 \to 0$ implies that $D^+(U(\hat{t}) - y_\eta(\hat{t})) \leq \varepsilon e^{\bar{\lambda} \hat{t}}$. This holds for any $\varepsilon > 0$, so we must have $D^+(U(\hat{t}) - y_\eta(\hat{t})) \leq 0$.

Since $\hat{t}$ is chosen arbitrarily, claim (5) is true. This therefore completes the proof. □

It should be pointed out that condition (4) is given in terms of the SFDE solution. Although it can be checked easily in our later discrete-state-feedback stabilization problem, we give the following corollary to make our theory more applicable in reality, where condition (4) is replaced by a general one which does not involve the solution.

Corollary 2.1: Let Assumptions 2.1 and 2.2 hold. For such $V(x, t, i)$ defined in Assumption 2.1, assume that there are constants $p > 1$, $\lambda_1 \geq 0$ and $\lambda_2 > 0$ such that

$$E\left(\max_{1 \leq i \leq N} \mathcal{L}V(\phi, t, i) \right)$$

$$\leq \lambda_1 - \lambda_2 E\left(\max_{1\leq i\leq N} V(\phi(0),t,i)\right)$$

for those $\phi \in L^q_{\mathcal{F}_t}([-h,0];\mathbb{R}^n)$ satisfying

$$E\left(\min_{1\leq i\leq N} V(\phi(\theta),t+\theta,i)\right) \leq pE\left(\max_{1\leq i\leq N} V(\phi(0),t,i)\right)$$

for any $\theta \in [-h,0]$. Then for any initial data ζ and r_0, the solution of SFDE (1) has the property that

$$\limsup_{t\to\infty} EV(x(t),t,r(t)) \leq \frac{\lambda_1}{\lambda},$$

where $\lambda = \min\left\{\lambda_2, \frac{\log p}{h}\right\}$. In particular, if $\lambda_1 = 0$,

$$\limsup_{t\to\infty} \frac{1}{t}\log EV(x(t),t,r(t)) \leq -\lambda.$$

Finally, we make some comments about the right-continuity of EV and $E\mathcal{L}V$ required in Assumption 2.1.

Remark 2: From (8), we find that when using the Razumikhin technique, it is crucial to require both $EV(x(t),t,r(t))$ and $E\mathcal{L}V(x_t,t,r(t))$ to be right-continuous. But to guarantee the right-continuity of these two functions is not trivial, especially for highly nonlinear systems. We will introduce this in detail later in Lemmas 3.1, 4.1 and subsequent Remark 6. Moreover, even if we have the right-continuity of $V(x(t),t,r(t))$ and $\mathcal{L}V(x_t,t,r(t))$, we still cannot draw the conclusion that $EV(x(t),t,r(t))$ and $E\mathcal{L}V(x_t,t,r(t))$ are right-continuous. Because in general, only the right-continuity of a process cannot guarantee its expectation remains right-continuous. We will give an example in Appendix (Example 3) to show this.

III. Discrete-state-feedback stabilization problem

Consider an n-dimensional nonlinear hybrid SDE

$$\mathrm{d}y(t) = f(y(t),t,r(t))\mathrm{d}t + g(y(t),t,r(t))\mathrm{d}B(t) \qquad (11)$$

on $t \geq 0$, with the initial data $y(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in S$, where $f : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times m}$ are Borel measurable functions. In the classical theory of SDEs (see, e.g., [6]), the local Lipschitz condition and the linear growth condition would be imposed to system coefficients. The former makes sure SDE (11) has a unique maximal local solution, but may explode to infinity at finite time. The latter is then used to limit the growth of f and g, and force the local solution to become global. However, as we mentioned before, the linear growth condition is not of interest to us in this paper. As a consequence, we will keep the local Lipschitz condition, but replace the linear growth condition by two more general conditions, polynomial growth condition and Khasminskii-type condition, which are stated in the following two assumptions, respectively.

Assumption 3.1: Assume that there exist four constants $K_0 \geq 0$, $K_1 > 0$, $q_1 > 1$ and $q_2 \geq 1$ such that

$$\begin{aligned}|f(x,t,i)| &\leq K_0 + K_1(|x| + |x|^{q_1}),\\ |g(x,t,i)| &\leq K_0 + K_1(|x| + |x|^{q_2})\end{aligned} \qquad (12)$$

for all $(x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S$.

In this assumption, we call condition (12) the polynomial growth condition, which is required since we still do not want f and g to grow very rapidly. But it is not able to ensure the existence of global solution of SDE (11). The following assumption should be imposed.

Assumption 3.2: Assume that for any integer $b > 0$, there is a positive constants M_b such that

$$|f(x,t,i)-f(y,t,i)|\vee|g(x,t,i)-g(y,t,i)| \leq M_b|x-y| \quad (13)$$

for all $x,y \in \mathbb{R}^n$ with $|x|\vee|y| \leq b$ and $(t,i) \in \mathbb{R}_+\times S$. Assume also that there exist constants $\alpha > 0$ and $\bar{q} \geq 2q_1 + 2q_2 - 2$ such that

$$x^{\mathrm{T}} f(x,t,i) + \frac{\bar{q}-1}{2}|g(x,t,i)|^2 \leq \alpha(1+|x|^2) \qquad (14)$$

for all $(x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S$.

In Assumption 3.2, condition (13) is the local Lipscitz condition and condition (14) is known as the Khasminskii-type condition. Then by Theorem 3.19 in [6], under Assumption 3.2, for any initial data x_0 and r_0, SDE (11) admits a unique global solution $y(t)$ such that $\sup_{0\leq s\leq t} E|y(s)|^{\bar{q}} < \infty$ for any $t \geq 0$.

Remark 3: It should be pointed out that both Assumption 3.1 and Assumption 3.2 are needed in this paper. On the one hand, Assumption 3.2 is used to guarantee that SDE (11) has a unique global solution $y(t)$ with property that $E|y(s)|^{\bar{q}} < \infty$. On the other hand, Assumption 3.1 can imply that both $f(x(t),t,r(t))$ and $g(x(t),t,r(t))$ are in L^2, and these properties are very useful in the subsequent stability analysis. Meanwhile, the Khasminskii-type condition is given based on Assumption 3.1 since $\bar{q} \geq 2q_1 + 2q_2 - 2$. These two assumptions can also be found in [16] and [27]. But here we have a little stronger restriction on $\bar{q}$ that $\bar{q} \geq 2q_1 + 2q_2 - 2$, rather than $\bar{q} \geq 2q_1 \vee (q_1 + 2q_2 - 1)$. This is required to make sure Assumption 2.1 is true. More details can be seen in Lemmas 3.1, 4.1 and Remark 6. Additionally, if $q_1+1 > 2q_2$, $\bar{q}$ could sometimes be large arbitrarily. We will give an example in Appendix (Example 4) to illustrate this.

But the solution may not be stable. In the case when the given hybrid system (11) is unstable, we want to design a discrete-time feedback controller in the drift part so that the controlled system

$$\begin{aligned}\mathrm{d}x(t) =&(f(x(t),t,r(t)) + u(x([t/\tau]\tau),t,r(t)))\mathrm{d}t\\ &+ g(x(t),t,r(t))\mathrm{d}B(t)\end{aligned} \qquad (15)$$

becomes stable. Here, to avoid confusion, we denote by $x(t)$ the solution of the controlled system. The feedback control $u(x([t/\tau]\tau),t,r(t))$ is indeed based on discrete-time observations of the state $x(t)$ at time $0,\tau,2\tau,\cdots$. In this paper, we assume that the control function $u : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$ is Borel measurable and globally Lipschitz continuous.

Rule 1: Assume that there exists a positive constant K_2 such that

$$|u(x,t,i) - u(y,t,i)| \leq K_2|x-y| \qquad (16)$$

for all $(x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$. Moreover, for the stability purpose, assume that $u(0,t,i) \equiv 0$.

Note that when $K_0 = 0$, condition (12) forces $f(0,t,i) \equiv 0$ and $g(0,t,i) \equiv 0$. This means the controlled system (15) has the trivial solution when $K_0 = 0$, which is also required for the stability purpose. Moreover, Rule 1 implies the following linear growth condition

$$|u(x,t,i)| \le K_2|x|, \quad \forall (x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S. \tag{17}$$

In fact, the controlled system (15) is a hybrid SDDE with a bounded variable delay, if we define delay function $\xi : \mathbb{R}_+ \to [0,\tau]$ by $\xi(t) = t - k\tau$ for $k\tau \le t < (k+1)\tau$, $k = 0, 1, 2\cdots$. Thus the controlled system (15) can be rewritten as

$$\begin{aligned} \mathrm{d}x(t) =&(f(x(t),t,r(t)) + u(x(t-\xi(t)),t,r(t)))\mathrm{d}t \\ &+ g(x(t),t,r(t))\mathrm{d}B(t). \end{aligned} \tag{18}$$

As systems (15) and (18) are equivalent, we will mainly concentrate on the controlled system (18) in the rest of this paper. Then by Theorem 7.13 in [6], under Assumptions 3.1, 3.2 and Rule 1, for any initial data $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in S$, the controlled system (18) also has a unique global solution $x(t)$ such that $\sup_{0\le s\le t} E|x(s)|^{\bar q} < \infty$ for any $t \ge 0$. With a little more effort, we can have a better result.

Lemma 3.1: Let Assumptions 3.1, 3.2 and Rule 1 hold. For any initial data $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in S$, the solution of the controlled system (18) has the property that

$$E\left(\sup_{0\le s\le t}|x(s)|^{2q_1}\right) < \infty, \quad \forall t \ge 0. \tag{19}$$

Proof. Fix any time $t \ge 0$. Applying the Itô formula to $|x|^{2q_1}$, we see that for any $0 \le s \le t$

$$\begin{aligned} &|x(s)|^{2q_1} \\ =&|x_0|^{2q_1} + \int_0^s 2q_1|x(v)|^{2q_1-2}x^{\mathrm{T}}(v)(f(x(v),v,r(v)) \\ &+ u(x(v-\xi(v)),v,r(v)))\mathrm{d}v \\ &+ \int_0^s q_1|x(v)|^{2q_1-2}|g(x(v),v,r(v))|^2\mathrm{d}v \\ &+ \int_0^s q_1(2q_1-2)|x(v)|^{2q_1-4}\left|x^{\mathrm{T}}(v)g(x(v),v,r(v))\right|^2\mathrm{d}v \\ &+ \int_0^s 2q_1|x(v)|^{2q_1-2}x^{\mathrm{T}}(v)g(x(v),v,r(v))\mathrm{d}B(v). \end{aligned}$$

Since $\bar q \ge 2q_1 + 2q_2 - 2$, we derive from condition (14) that

$$x^{\mathrm{T}}f(x,t,i) + \frac{2q_1-1}{2}|g(x,t,i)|^2 \le \alpha(1+|x|^2)$$

for all $(x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S$. Using this and condition (17) as well as the Young inequality, and then taking expectations on both sides, we get

$$\begin{aligned} &E\left(\sup_{0\le s\le t}|x(s)|^{2q_1}\right) \\ \le&|x_0|^{2q_1} + J(t) + E\int_0^t 2q_1\alpha|x(v)|^{2q_1-2}\left(1+|x(v)|^2\right)\mathrm{d}v \\ &+ E\int_0^t \left((2q_1-1)K_2|x(v)|^{2q_1} + K_2|x(v-\xi(v))|^{2q_1}\right)\mathrm{d}v, \end{aligned} \tag{20}$$

where

$$\begin{aligned} J(t) =&E\left(\sup_{0\le s\le t}\left|\int_0^s 2q_1|x(v)|^{2q_1-2}x^{\mathrm{T}}(v)\right.\right. \\ &\left.\left.\times g(x(v),v,r(v))\mathrm{d}B(v)\right|\right). \end{aligned}$$

By the Burkholder-Davis-Gundy inequality, we compute

$$\begin{aligned} &J(t) \\ \le&3E\left(\int_0^t 4q_1^2|x(v)|^{4q_1-2}|g(x(v),v,r(v))|^2\mathrm{d}v\right)^{\frac12} \\ \le&3E\left(\int_0^t 8q_1^2K_1^2\left(|x(v)|^{4q_1} + |x(v)|^{4q_1+2q_2-2}\right)\mathrm{d}v\right)^{\frac12} \\ \le&3E\left(\int_0^t 8q_1^2K_1^2|x(v)|^{4q_1}\mathrm{d}v\right)^{\frac12} \\ &+ 3E\left(\int_0^t 8q_1^2K_1^2|x(v)|^{4q_1+2q_2-2}\mathrm{d}v\right)^{\frac12} \\ \le&E\left(\sup_{0\le s\le t}|x(s)|^{2q_1}\int_0^t 72q_1^2K_1^2|x(v)|^{2q_1}\mathrm{d}v\right)^{\frac12} \\ &+ E\left(\sup_{0\le s\le t}|x(s)|^{2q_1}\int_0^t 72q_1^2K_1^2|x(v)|^{2q_1+2q_2-2}\mathrm{d}v\right)^{\frac12} \\ \le&\frac14 E\left(\sup_{0\le s\le t}|x(s)|^{2q_1}\right) + 72q_1^2K_1^2E\int_0^t|x(v)|^{2q_1}\mathrm{d}v \\ &+ \frac14 E\left(\sup_{0\le s\le t}|x(s)|^{2q_1}\right) \\ &+ 72q_1^2K_1^2E\int_0^t|x(v)|^{2q_1+2q_2-2}\mathrm{d}v, \end{aligned}$$

where we have used the inequalities $\sqrt{ab} \le \frac14 a + b$ and $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$. Substituting this into (20) gives that

$$\begin{aligned} &E\left(\sup_{0\le s\le t}|x(s)|^{2q_1}\right) \\ \le&2|x_0|^{2q_1} + CE\int_0^t\Big(|x(v)|^{2q_1} + |x(v-\xi(v))|^{2q_1} \\ &+ |x(v)|^{2q_1-2} + |x(v)|^{2q_1+2q_2-2}\Big)\mathrm{d}v, \end{aligned}$$

where C is a positive constant depending on q_1, α, K_1 and K_2. Recalling that $\sup_{0\le s\le t} E|x(s)|^{\bar q} < \infty$, and using the Fubini theorem, we obtain that

$$\begin{aligned} &E\left(\sup_{0\le s\le t}|x(s)|^{2q_1}\right) \\ \le&2|x_0|^{2q_1} + 4C\left(1 + \sup_{0\le s\le t}E|x(s)|^{\bar q}\right)t < \infty. \end{aligned}$$

This completes the proof. □

Before ending this section, we assume further that for any fixed $(x,i) \in \mathbb{R}^n \times S$, $f(x,\cdot,i)$, $g(x,\cdot,i)$ and $u(x,\cdot,i)$ are right-continuous functions on $t \ge 0$. Then since f, g, u are all continuous in the first component x, we observe from the right continuity of $t - \xi(t)$, $r(t)$ and the continuity of solution $x(t)$ that the drift coefficient $f(x(t),t,r(t)) + u(x(t-\xi(t)),t,r(t))$ and the diffusion coefficient $g(x(t),t,r(t))$ are right-continuous on $t \ge 0$.

In fact, the requirements of f, g, u on the second component t is quite natural when studying the stability of hybrid SDEs because at most time (see, e.g., [6]), we always need the system coefficients to be right-continuous. While in general, we do not mention these conditions on t explicitly. But it should be pointed out that the Lipschitz conditions (global or local) in x are not enough.

IV. Stabilization results

We have only shown that there admits a global solution to the controlled system (18) with the property that $\sup_{0\le s\le t} E|x(s)|^{\bar q} < \infty$ and $E\left(\sup_{0\le s\le t}|x(s)|^{2q_1}\right) < \infty$ for any $t \ge 0$ under our standing Assumptions 3.1, 3.2 and Rule 1. In order for the controlled system (18) to be stable, we need to impose more conditions on the control function and the original system (11). In this section, we will show that with Assumption 4.1 and Rule 2, the controlled system (18) is (q_1+1)-th moment exponentially stable and almost surely exponentially stable.

Assumption 4.1: For each $i \in S$, assume that there are nonnegative constants c, $\bar c$ and positive constants ρ_i, $\bar\rho_i$, β_i, $\bar\beta_i$ such that

$$x^{\mathrm T} f(x,t,i) + \frac{1}{2}|g(x,t,i)|^2 \le c + \rho_i |x|^2 - \beta_i |x|^{q_1+1} \tag{21}$$

and

$$x^{\mathrm T} f(x,t,i) + \frac{q_1}{2}|g(x,t,i)|^2 \le \bar c + \bar\rho_i |x|^2 - \bar\beta_i |x|^{q_1+1} \tag{22}$$

for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Rule 2: For each $i \in S$, assume that we can find constant $\kappa_i \in \mathbb{R}$ for

$$x^{\mathrm T} u(x,t,i) \le \kappa_i |x|^2 \tag{23}$$

for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, while for

$$\mathcal{A} := -2\mathrm{diag}(\alpha_1,\cdots,\alpha_N) - Q \tag{24}$$

to be a non-singular M-matrix, where $\alpha_i = \rho_i + \kappa_i$.

Next, we make some comments about these two conditions that there are many control functions available and we can cover more general hybrid SDEs compared with [16].

Remark 4: Under Assumption 4.1, we can actually design lots of control functions to satisfy Rules 1 and 2. For example, design the control function in the linear form $u(x,t,i) = -2\rho_i x$. Then Rule 1 is satisfied with $K_2 = 2\max_{i\in S}|\rho_i|$, Rule 2 is true with $\kappa_i = -2\rho_i$ and $\alpha_i = -\rho_i$, while $\mathcal{A} = 2\mathrm{diag}(\rho_1,\cdots,\rho_N) - Q$ is a non-singular M-matrix. Combining with conditions (21), (22) and (23), we see that

$$\begin{aligned} &x^{\mathrm T}(f(x,t,i)+u(x,t,i)) + \frac{1}{2}|g(x,t,i)|^2 \\ \le& c + \alpha_i|x|^2 - \beta_i|x|^{q_1+1} \end{aligned} \tag{25}$$

and

$$\begin{aligned} &x^{\mathrm T}(f(x,t,i)+u(x,t,i)) + \frac{q_1}{2}|g(x,t,i)|^2 \\ \le& \bar c + \bar\alpha_i|x|^2 - \bar\beta_i|x|^{q_1+1}, \end{aligned} \tag{26}$$

where $\bar\alpha_i = \bar\rho_i + \kappa_i$. This is exactly Condition 4.1 in [16]. But differently, we now do not require the matrix

$$-(q_1+1)\mathrm{diag}(\bar\alpha_1,\cdots,\bar\alpha_N) - Q$$

also to be a non-singular M-matrix, since we will make use of the last term $-\beta_i|x|^{q_1+1}$ in (25) and set free parameters to balance the term $|x|^{q_1+1}$. In this case, there is no need to give such restriction on $\bar\alpha_i$, $i \in S$. This enables us to include more general hybrid SDEs and tackle more practical issues. We will give an example (Example 1) to illustrate this modification.

We set $(\eta_1,\cdots,\eta_N)^{\mathrm T} := \mathcal{A}^{-1}(\mu,\cdots,\mu)^{\mathrm T}$, where μ is a free parameter. As $\mathcal{A}$ is a non-singular M-matrix, all η_i are positive. Besides, once $\mathcal{A}$ is fixed, all η_i are propositional to the value of μ, namely $\eta_i = \sum_{j=1}^N (\mathcal{A}^{-1})_{ij}\mu$. This relationship is important to the determination of τ^*, so we set $\eta_m = \min_{i\in S}\eta_i = A_m\mu$ and $\eta_M = \max_{i\in S}\eta_i = A_M\mu$, where

$$A_m = \min_{i\in S}\left(\sum_{j=1}^N (\mathcal{A}^{-1})_{ij}\right), \quad A_M = \max_{i\in S}\left(\sum_{j=1}^N (\mathcal{A}^{-1})_{ij}\right),$$

which are two positive constants only depending on the matrix $\mathcal{A}$.

Define a function $V(x,t,i) \in C^{2,1}(\mathbb{R}^n \times [-\tau,\infty) \times S; \mathbb{R}_+)$ by

$$V(x,t,i) = \eta_i|x|^2 + |x|^{q_1+1} \tag{27}$$

while define a function $LV : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}$ by

$$\begin{aligned} LV(x,t,i) =& 2\eta_i\left(x^{\mathrm T}(f(x,t,i)+u(x,t,i)) + \frac{1}{2}|g(x,t,i)|^2\right) \\ &+ \sum_{j=1}^N q_{ij}\eta_j|x|^2 \\ &+ (q_1+1)|x|^{q_1-1}x^{\mathrm T}(f(x,t,i)+u(x,t,i)) \\ &+ \frac{q_1+1}{2}|x|^{q_1-1}|g(x,t,i)|^2 \\ &+ \frac{(q_1+1)(q_1-1)}{2}|x|^{q_1-3}|x^{\mathrm T}g(x,t,i)|^2. \end{aligned}$$

By (25) and (26), it is easy to derive that

$$\begin{aligned} LV(x,t,i) \le& 2c\eta_i + (q_1+1)\bar c - \mu|x|^2 - (q_1+1)\bar\beta_i|x|^{2q_1} \\ &- (2\eta_i\beta_i - (q_1+1)(\bar c + \alpha_i))|x|^{q_1+1}. \end{aligned} \tag{28}$$

Define, for $(\phi,t,i) \in C([-\tau,0];\mathbb{R}^n) \times \mathbb{R}_+ \times S$,

$$\begin{aligned} F(\phi,t,i) =& f(\phi(0),t,i) + u(\phi(-\xi(t)),t,i), \\ G(\phi,t,i) =& g(\phi(0),t,i). \end{aligned}$$

Then our controlled system (18) becomes SFDE (1) on $t \ge 0$ with initial data $x_0(\theta) = x_0$ for $\theta \in [-\tau,0]$. Here, $x_t = \{x(t+\theta)| -\tau \le \theta \le 0\}$.

In order to use the Razumikhin technique developed in Theorem 2.1, it is incredibly necessary to verify Assumptions 2.1 and 2.2 at first, which is stated as the following lemma.

Lemma 4.1: Assumptions 2.1 and 2.2 are true under Assumptions 3.1, 3.2, 4.1 and Rules 1, 2.

Proof. We divide the verification work step by step.

Step 1: We have already shown in Section III that controlled system (18) has a global solution satisfying that for any $t \ge 0$, $E|x(t)|^{\bar q} < \infty$.

Step 2: The function V has been given by $\eta_i|x|^2 + |x|^{q_1+1}$ in (27). Since $\bar q \ge 2q_1 + 2q_2 - 2$, it is easy to see that

$$\sup_{-\tau\le s\le t} EV(x(s),s,r(s))$$

$$\le \eta_M|x_0|^2 + |x_0|^{q_1+1} + \sup_{0\le s\le t}\left(\eta_M E|x(t)|^2 + E|x(t)|^{q_1+1}\right) < \infty.$$

Then by conditions (12) and (17), compute

$$\begin{aligned} &|\mathcal{L}V(x_t,t,r(t))| \\ \le& 2\eta_M K_1\left(|x(t)|^2 + |x(t)|^{q_1+1}\right) \\ &+ \eta_M K_2|x(t)|^2 + \eta_M K_2|x(t-\xi(t))|^2 \\ &+ 2\eta_M K_1^2\left(|x(t)|^2 + |x(t)|^{2q_2}\right) \\ &+ N\eta_M\left(\max_{1\le i,j\le N}|q_{ij}|\right)|x(t)|^2 \\ &+ (q_1+1)K_1\left(|x(t)|^{q_1+1} + |x(t)|^{2q_1}\right) \\ &+ q_1K_2|x(t)|^{q_1+1} + K_2|x(t-\xi(t))|^{q_1+1} \\ &+ q_1(q_1+1)K_1^2\left(|x(t)|^{q_1+1} + |x(t)|^{q_1+2q_2-1}\right) \\ \le& C\left(1 + \sup_{0\le s\le t}|x(s)|^{2q_1}\right), \end{aligned}$$

where C is a positive number independent from t. By Lemma 3.1, we see that $\sup_{0\le s\le t} E|\mathcal{L}V(x_s,s,r(s))| < \infty$.

Step 3: For any integer $k > 0$ and any $\phi \in C([-\tau,0];\mathbb{R}^n)$ with $||\phi|| \le k$ and $(t,i) \in \mathbb{R}_+ \times S$, since g is locally Lipschitz continuous in x and $g(0,t,i) \equiv 0$, it is easy to see that

$$\begin{aligned} &|V_x(\phi(0),t,i)G(\phi,t,i)| \\ =& \left|\left(2\eta_i\phi^{\mathrm{T}}(0) + (q_1+1)|\phi(0)|^{q_1-1}\phi^{\mathrm{T}}(0)\right)g(\phi(0),t,i)\right| \\ \le& 2M_k\eta_M k^2 + (q_1+1)M_k k^{q_1+1} := L_k, \end{aligned}$$

where we have set $G(\phi,t,i) = g(\phi(0),t,i)$ before. Thus, condition (3) is true.

Step 4: Recalling the discussions at the end of Section III, $V(x(t),t,r(t)$ and $\mathcal{L}V(x_t,t,r(t))$ are right-continuous on $t \ge 0$. Then for any sufficiently small $\Delta > 0$, we have

$$\sup_{t\le s\le t+\Delta}|V(x(s),s,r(s))| \le C\left(1 + \sup_{0\le s\le t+\Delta}|x(s)|^{q_1+1}\right)$$

and

$$\sup_{t\le s\le t+\Delta}|\mathcal{L}V(x_s,s,r(s))| \le C\left(1 + \sup_{0\le s\le t+\Delta}|x(s)|^{2q_1}\right),$$

where C is a positive number independent from t. Since in Lemma 3.1

$$E\left(\sup_{0\le s\le t+\Delta}|x(s)|^{2q_1}\right) < \infty,$$

using the Hölder inequality and the dominated convergence theorem shows that

$$\begin{aligned} \lim_{s\to t^+} EV(x(s),s,r(s)) =& E\left(\lim_{s\to t^+} V(x(s),s,r(s))\right) \\ =& EV(x(t),t,r(t)) \end{aligned}$$

and

$$\begin{aligned} \lim_{s\to t^+} E\mathcal{L}V(x_s,s,r(s)) =& E\left(\lim_{s\to t^+} \mathcal{L}V(x_s,s,r(s))\right) \\ =& E\mathcal{L}V(x_t,t,r(t)). \end{aligned}$$

As a result, $EV(x(t),t,r(t)$ and $E\mathcal{L}V(x_t,t,r(t))$ are right-continuous.

Up to now, all the conditions in Assumptions 2.1 and 2.2 are satisfied. □

In the study of discrete-state-feedback stabilization problem, no matter which method we pick up, it is very significant to estimate the difference between current state $x(t)$ and discrete-time state $x(t-\xi(t))$. We state this estimation as a lemma.

Lemma 4.2: Let all the conditions of Lemma 4.1 hold, Then for all $t \ge 0$, we have

$$\begin{aligned} &E|x_t(0) - x_t(-\xi(t))|^2 \\ \le& 2c\tau + K_0\tau + H_1(\mu)\tau \sup_{-\tau\le\theta\le 0} EV(x_t(\theta), t+\theta, r(t+\theta)), \end{aligned} \tag{29}$$

where

$$\begin{aligned} H_1(\mu) =& \left(\frac{2\alpha_M + K_1 + 3K_2}{A_m\mu} \vee \frac{2K_1q_1}{q_1+1}\right) \\ &+ \left(\frac{K_0 + K_1 + 3K_2}{A_m\mu} \vee \frac{2K_1}{q_1+1}\right), \end{aligned} \tag{30}$$

in which $\alpha_M = \max_{i\in S}\alpha_i$.

Proof. For any $t \ge 0$, there exists some integer $k \ge 0$ such that $k\tau \le t < (k+1)\tau$. Then it is easy to see that $v - \xi(v) = k\tau$ for $v \in [k\tau, t]$. Applying the Itô formula to (18) yields that

$$\begin{aligned} &|x(t) - x(t-\xi(t))|^2 \\ =& M(t) + \int_{k\tau}^t \Big(2(x(v) - x(k\tau))^{\mathrm{T}}(f(x(v),v,r(v)) \\ &+ u(x(k\tau),v,r(v))) + |g(x(v),v,r(v))|^2\Big)\mathrm{d}v \\ =& M(t) + \int_{k\tau}^t \Big(2x^{\mathrm{T}}(v)(f(x(v),v,r(v)) \\ &+ u(x(v),v,r(v))) + |g(x(v),v,r(v))|^2\Big)\mathrm{d}v \\ &- \int_{k\tau}^t 2x^{\mathrm{T}}(k\tau)f(x(v),v,r(v))\mathrm{d}v \\ &- \int_{k\tau}^t 2x^{\mathrm{T}}(v)u(x(v),v,r(v))\mathrm{d}v \\ &+ \int_{k\tau}^t 2x^{\mathrm{T}}(v)u(x(k\tau),v,r(v))\mathrm{d}v \\ &- \int_{k\tau}^t 2x^{\mathrm{T}}(k\tau)u(x(k\tau),v,r(v))\mathrm{d}v, \end{aligned} \tag{31}$$

where

$$M(t) = \int_{k\tau}^t (2(x(v) - x(k\tau))^{\mathrm{T}}g(x(v),v,r(v))\mathrm{d}B(v)$$

is a continuous martingale. From condition (25), we derive that

$$\begin{aligned} &\int_{k\tau}^t \Big(2x^{\mathrm{T}}(v)(f(x(v),v,r(v)) + u(x(v),v,r(v))) \\ &+ |g(x(v),v,r(v))|^2\Big)\mathrm{d}v \\ \le& 2c\tau + 2\alpha_M\int_{k\tau}^t |x(v)|^2\mathrm{d}v. \end{aligned}$$

Making use of (12) and (17) as well as the Young inequality, we obtain that

$$-\int_{k\tau}^t 2x^{\mathrm{T}}(k\tau)f(x(v),v,r(v))\mathrm{d}v$$

$$\begin{aligned}&\leq 2\int_{k\tau}^{t}|x(k\tau)|\left(K_0+K_1|x(v)|+K_1|x(v)|^{q_1}\right)\mathrm{d}v\\&\leq K_0\tau+K_1\int_{k\tau}^{t}|x(v)|^2\mathrm{d}v+(K_0+K_1)\int_{k\tau}^{t}|x(k\tau)|^2\mathrm{d}v\\&\quad+\frac{2K_1q_1}{q_1+1}\int_{k\tau}^{t}|x(v)|^{q_1+1}\mathrm{d}v+\frac{2K_1}{q_1+1}\int_{k\tau}^{t}|x(k\tau)|^{q_1+1}\mathrm{d}v\end{aligned}$$

and

$$\begin{aligned}&\int_{k\tau}^{t}2x^{\mathrm{T}}(v)u(x(k\tau),v,r(v))\mathrm{d}v\\&\leq 2K_2\int_{k\tau}^{t}|x(v)||x(k\tau)|\mathrm{d}v\\&\leq K_2\int_{k\tau}^{t}|x(v)|^2\mathrm{d}v+K_2\int_{k\tau}^{t}|x(k\tau)|^2\mathrm{d}v.\end{aligned}$$

Substituting these into (31) gives that

$$\begin{aligned}&|x(t)-x(t-\xi(t))|^2\\&\leq M(t)+2c\tau+K_0\tau+(2\alpha_M+K_1+3K_2)\int_{k\tau}^{t}|x(v)|^2\mathrm{d}v\\&\quad+\frac{2K_1q_1}{q_1+1}\int_{k\tau}^{t}|x(v)|^{q_1+1}\mathrm{d}v\\&\quad+(K_0+K_1+3K_2)\int_{k\tau}^{t}|x(k\tau)|^2\mathrm{d}v\\&\quad+\frac{2K_1}{q_1+1}\int_{k\tau}^{t}|x(k\tau)|^{q_1+1}\mathrm{d}v\\&\leq M(t)+2c\tau+K_0\tau+\left(\frac{2\alpha_M+K_1+3K_2}{\eta_m}\vee\frac{2K_1q_1}{q_1+1}\right)\\&\quad\times\int_{k\tau}^{t}\left(\eta_{r(v)}|x(v)|^2+|x(v)|^{q_1+1}\right)\mathrm{d}v\\&\quad+\left(\frac{K_0+K_1+3K_2}{\eta_m}\vee\frac{2K_1}{q_1+1}\right)\\&\quad\times\int_{k\tau}^{t}\left(\eta_{r(k\tau)}|x(k\tau)|^2+|x(k\tau)|^{q_1+1}\right)\mathrm{d}v.\end{aligned}$$

We can take expectations on both sides and then use the Fubini theorem to get that

$$\begin{aligned}&E|x(t)-x(t-\xi(t))|^2\\&\leq 2c\tau+K_0\tau+\left(\frac{2\alpha_M+K_1+3K_2}{A_m\mu}\vee\frac{2K_1q_1}{q_1+1}\right)\\&\quad\times\int_{k\tau}^{t}EV(x(v),v,r(v))\mathrm{d}v\\&\quad+\left(\frac{K_0+K_1+3K_2}{A_m\mu}\vee\frac{2K_1}{q_1+1}\right)\\&\quad\times\int_{k\tau}^{t}EV(x(k\tau),k\tau,r(k\tau))\mathrm{d}v.\end{aligned}\tag{32}$$

Noting the definition of x_t, we further have

$$\begin{aligned}&E|x_t(0)-x_t(-\xi(t))|^2\\&\leq 2c\tau+K_0\tau+H_1(\mu)\tau\sup_{-\tau\leq\theta\leq 0}EV(x_t(\theta),t+\theta,r(t+\theta)),\end{aligned}$$

which is the required assertion (29). □

In order to obtain the bound of τ, we need to guarantee the positivity of $H_1(\mu)$. But observing (30) in detail, we find that

$$H_1(\mu)\geq\frac{2K_1q_1}{q_1+1}+\frac{2K_1}{q_1+1}>0.$$

We now present our first stabilization result in this paper.

Theorem 4.1: Under the same conditions of Lemma 4.1, there is a positive number τ^* such that for any initial data x_0 and r_0, the solution of the controlled system (18) obeys that

$$\limsup_{t\to\infty}E|x(t)|^{q_1+1}<\infty$$

as long as $\tau<\tau^*$. In particular, if $K_0=0$, $c=0$ and $\bar{c}=0$, the solution satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log E|x(t)|^{q_1+1}<0.$$

In other words, the controlled system (18) is asymptotically bounded in (q_1+1)-th moment and exponentially stable in (q_1+1)-th moment when $K_0=0$, $c=0$ and $\bar{c}=0$ provided $\tau<\tau^*$.

Before giving the proof, to make this theorem can be implemented in practice, we make some comments on how to determine the value of τ^*.

Remark 5: Set

$$\begin{aligned}&\varphi(\gamma,\mu)\\&=\frac{1}{H_2(\gamma,\mu)}\left(\frac{1-\gamma}{A_M}\wedge\left(2A_m\mu\beta_m-(q_1+1)(\bar{c}+\bar{\alpha}_M)\right)\right),\end{aligned}$$

where $\bar{\alpha}_M=\max_{i\in S}\bar{\alpha}_i$, $\beta_m=\min_{i\in S}\beta_i$, $\bar{\beta}_m=\min_{i\in S}\bar{\beta}_i$,

$$H_2(\gamma,\mu)=H_1(\mu)\left(\frac{K_2^2A_M^2\mu}{\gamma}+\frac{(q_1+1)K_2^2}{4\bar{\beta}_m}\right)$$

and γ and μ are free parameters taking values in

$$\Lambda=(0,1)\times\left(\frac{(q_1+1)(\bar{c}+\bar{\alpha}_M)}{2A_m\beta_m}\vee 0,\infty\right).$$

Then τ^* is given by

$$\tau^*=\sup_{(\gamma,\mu)\in\Lambda}\varphi(\gamma,\mu).\tag{33}$$

With a little effort, we can show that there exists a $(\hat{\gamma},\hat{\mu})\in\Lambda$ such that $\varphi(\hat{\gamma},\hat{\mu})=\sup_{(\gamma,\mu)\in\Lambda}\varphi(\gamma,\mu)$. It is easy to see that $\varphi(\gamma,\mu)$ is a positive, continuous and bounded function in Λ. Moreover, we observe that $\bar{\varphi}(\gamma,\cdot)$ tends to 0 as γ tends to 0 or 1 and $\varphi(\cdot,\mu)$ tends to 0 as μ tends to $\left(\frac{(q_1+1)(\bar{c}+\bar{\alpha}_M)}{2A_m\beta_m}\vee 0\right)$ or ∞. As a result, there exists $(\hat{\gamma},\hat{\mu})\in\Lambda$ such that

$$\tau^*=\sup_{(\gamma,\mu)\in\Lambda}\varphi(\gamma,\mu)=\varphi(\hat{\gamma},\hat{\mu}).$$

From now on, the free parameters γ and μ are fixed as $\hat{\gamma}$ and $\hat{\mu}$, respectively. And also all η_i, $i\in S$ are fixed. Next, we show that Theorem 4.1 is true.

Proof. We have checked Assumptions 2.1 and 2.2 in Lemma 4.1. The rest work is to verify condition (4).

Recalling the definition of $\mathcal{L}V$ and LV, and making use of (16), (28), we see that

$$\begin{aligned}&\mathcal{L}V(x_t,t,r(t))\\&=LV(x_t(0),t,r(t))+V_x(x_t(0),t,r(t))\\&\quad\times\left(u(x_t(-\xi(t)),t,r(t))-u(x_t(0),t,r(t))\right)\\&\leq 2c\eta_{r(t)}+(q_1+1)\bar{c}-\hat{\mu}|x_t(0)|^2-(q_1+1)\bar{\beta}_{r(t)}|x_t(0)|^{2q_1}\\&\quad-(2\eta_{r(t)}\beta_{r(t)}-(q_1+1)(\bar{c}+\bar{\alpha}_{r(t)}))|x_t(0)|^{q_1+1}\end{aligned}$$

$$+ K_2\left(2\eta_{r(t)}|x_t(0)| + (q_1+1)|x_t(0)|^{q_1}\right) \times |x_t(0) - x_t(-\xi(t))|. \quad (34)$$

By the elementary inequality, we can get

$$2K_2\eta_{r(t)}|x_t(0)||x_t(0)-x_t(-\xi(t))| \le \hat{\gamma}\hat{\mu}|x_t(0)|^2 + \frac{K_2^2\eta_{r(t)}^2}{\hat{\gamma}\hat{\mu}}|x_t(0)-x_t(-\xi(t))|^2$$

and

$$(q_1+1)K_2|x_t(0)|^{q_1}|x_t(0)-x_t(-\xi(t))| \le (q_1+1)\bar{\beta}_{r(t)}|x_t(0)|^{2q_1} + \frac{(q_1+1)K_2^2}{4\bar{\beta}_{r(t)}}|x_t(0)-x_t(-\xi(t))|^2.$$

Substituting these into (34), we derive that

$$\begin{aligned}&\mathcal{L}V(x_t,t,r(t))\\ \le& 2c\eta_{r(t)} + (q_1+1)\bar{c} - (1-\hat{\gamma})\hat{\mu}|x_t(0)|^2\\ &- (2\eta_{r(t)}\beta_{r(t)} - (q_1+1)(\bar{c}+\bar{\alpha}_{r(t)}))|x_t(0)|^{q_1+1}\\ &+ \left(\frac{K_2^2\eta_{r(t)}^2}{\hat{\gamma}\hat{\mu}} + \frac{(q_1+1)K_2^2}{4\bar{\beta}_{r(t)}}\right)|x_t(0)-x_t(-\xi(t))|^2.\end{aligned}$$

Taking expectations on both sides and making use of (29), we derive that

$$\begin{aligned}&E\mathcal{L}\bar{V}(x_t,t,r(t))\\ \le&\lambda_1 - \frac{(1-\hat{\gamma})\hat{\mu}}{\eta_M}E\left(\eta_{r(t)}|x_t(0)|^2\right)\\ &- (2\eta_m\beta_m - (q_1+1)(\bar{c}+\bar{\alpha}_M))E\left(|x_t(0)|^{q_1+1}\right)\\ &+ H_2(\hat{\gamma},\hat{\mu})\tau \sup_{-\tau\le\theta\le 0} EV(x_t(\theta),t+\theta,r(t+\theta)),\end{aligned}$$

where

$$\lambda_1 = 2c\eta_M + (q_1+1)\bar{c} + 2c\tau\left(\frac{K_2^2\eta_M^2}{\hat{\gamma}\hat{\mu}} + \frac{(q_1+1)K_2^2}{4\bar{\beta}_m}\right)$$

is a constant. For any $\tau < \tau^*$, we observe that

$$\frac{1-\hat{\gamma}}{A_M} - H_2(\hat{\gamma},\hat{\mu})\tau > 0$$

and

$$(2A_m\hat{\mu}\beta_m - (q_1+1)(\bar{c}+\bar{\alpha}_M)) - H_2(\hat{\gamma},\hat{\mu})\tau > 0.$$

Recalling the fact that $\eta_m = A_m\hat{\mu}$ and $\eta_M = A_M\hat{\mu}$, there is some $p > 1$ so that

$$\frac{(1-\hat{\gamma})\hat{\mu}}{\eta_M} - pH_2(\hat{\gamma},\hat{\mu})\tau > 0$$

and

$$(2\eta_m\beta_m - (q_1+1)(\bar{c}+\bar{\alpha}_M)) - pH_2(\hat{\gamma},\hat{\mu})\tau > 0.$$

For all $t \ge 0$ and those x_t satisfying

$$EV(x_t(\theta),t+\theta,r(t+\theta)) \le pV(x_t(0),t,r(t)), \quad \forall\theta\in[-\tau,0],$$

we have

$$\begin{aligned}&E\mathcal{L}V(x_t,t,r(t))\\ \le&\lambda_1 - \left(\frac{(1-\hat{\gamma})\hat{\mu}}{\eta_M} - pH_2(\hat{\gamma},\hat{\mu})\tau\right)E\left(\eta_{r(t)}|x_t(0)|^2\right)\\ &- ((2\eta_m\beta_m - (q_1+1)(\bar{c}+\bar{\alpha}_M)) - pH_2(\hat{\gamma},\hat{\mu})\tau)\\ &\times E\left(|x_t(0)|^{q_1+1}\right)\\ \le&\lambda_1 - \lambda_2 EV(x_t(0),t,r(t)),\end{aligned}$$

where

$$\begin{aligned}\lambda_2 =& \left(\frac{(1-\hat{\gamma})\hat{\mu}}{\eta_M} - pH_2(\hat{\gamma},\hat{\mu})\tau\right)\\ &\wedge ((2\eta_m\beta_m - (q_1+1)(\bar{c}+\bar{\alpha}_M)) - pH_2(\hat{\gamma},\hat{\mu})\tau).\end{aligned}$$

Applying Theorem 2.1, we derive that

$$\limsup_{t\to\infty} E|x(t)|^{q_1+1} \le \frac{\lambda_1}{\lambda_2 \wedge \frac{\log p}{\tau}}.$$

In particular, if $K_0 = 0$, $c = 0$ and $\bar{c} = 0$, namely, $\lambda_1 = 0$

$$\limsup_{t\to\infty} \frac{1}{t}\log E|x(t)|^{q_1+1} \le -\left(\lambda_2 \wedge \frac{\log p}{\tau}\right).$$

This completes the proof. □

Remark 6: Compared with the Lyapunov functional method, the advantage of the Razumikhin method to study delay systems is that it can tackle the difficulty arisen from the nondifferentiability and fast change of the time delay. But we require EV and $E\mathcal{L}V$ to be right-continuous. For hybrid SDEs meeting the linear growth condition (e.g., [6]), it is very easy to prove the right-continuity of EV and $E\mathcal{L}V$ since we always have $E\left(\sup_{0\le s\le t}|x(s)|^r\right) < \infty$ for any $t \ge 0$ at any positive order r. However, in the highly nonlinear ones, we need to impose extra assumptions to guarantee this, see Lemmas 3.1 and 4.1. This is the main reason why we require $\bar{q}$ is no less than $2q_1 + 2q_2 - 2$.

Also for highly nonlinear SDEs, moment exponential stability in general cannot imply the almost sure exponential stability. However, this is possible in our case. We state this as our second theorem.

Theorem 4.2: Let all the conditions in Theorem 4.1 hold. If $K_0 = 0$, $c = 0$ and $\bar{c} = 0$, for any initial data x_0 and r_0, the solution of the controlled system (18) obeys that

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| < \infty \quad \text{a.s.} \quad (35)$$

as long as $\tau < \tau^*$.

We can use the same analysis as in the proof of Theorem 5.4 in [16] to show this theorem so we leave these proofs to the reader.

V. Examples

A couple of examples are given in this part to illustrate our theoretical results. In order to avoid complicated calculations, we let $B(t)$ be a scalar Brownian motion and $r(t)$ be a continuous Markov chain on the state space $S = \{1, 2\}$ with the transition rate matrix

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Certainly, $B(t)$ and $r(t)$ are independent.

Example 1: We first give an example to show that we can include more general stochastic systems compared with [16]. Consider a scalar hybrid SDE in financial mathematics, which can be regarded as a generalisation of the well-known Heston stochastic volatility 1.5-model (see, e.g., [1], [6], [28])

$$\mathrm{d}y(t) = y(t)\left(a_{r(t)} - b_{r(t)}|y(t)|\right)\mathrm{d}t + c_{r(t)}|y(t)|^{1.5}\mathrm{d}B(t), \tag{36}$$

on $t \geq 0$, where

$$a_1 = 2,\ a_2 = 0.2,\ b_1 = 2,\ b_2 = 2.4,\ c_1 = 1,\ c_2 = 0.5.$$

It is easy to verify that Assumptions 3.1 and 3.2 are satisfied with $K_0 = 0$, $K_1 = 2.4$, $q_1 = 2$, $q_2 = 1.5$ and $\alpha = 2$, $\bar{q} = 5$. Through computer simulation, we can find that hybrid SDE (36) is unstable (see Fig. 1).

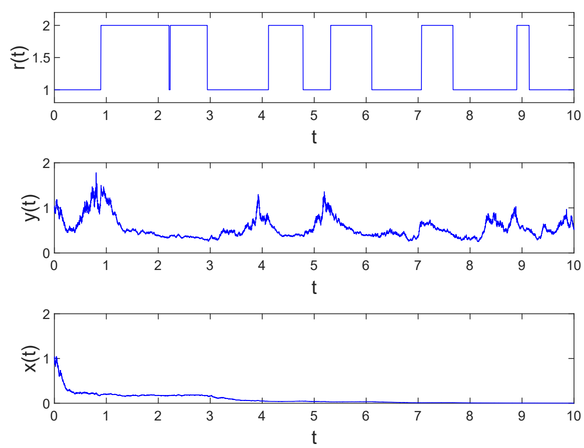

Fig. 1. Computer simulation of the sample paths of the Markov chain (the top one), and SDE (36) without control (the middle one) and the controlled system (37) with the control function (38) and $\tau = 0.00001$ (the bottom one) using the truncated Euler-Maruyama method [29] with time step size 10^{-6}.

Here we will discuss an interesting phenomena that the state can be observed fully in mode 1 but in mode 2, it is not observable. Therefore, we can only design feedback control in mode 1, based on discrete-time state observations of course, but we cannot have feedback control in mode 2. For example, the financial market can be roughly divided as "bullish" mode and "bearish" mode. Sometimes, only "bearish" mode can cause investors' much attention, where the market can be observed easily and needed extra control. In terms of mathematics, the controlled system is

$$\begin{aligned}\mathrm{d}x(t) = &\left(a_{r(t)}x(t) - b_{r(t)}|x(t)|x(t) + u(x([t/\tau]\tau), r(t))\right)\mathrm{d}t \\ &+ c_{r(t)}|x(t)|^{1.5}\mathrm{d}B(t),\end{aligned} \tag{37}$$

where

$$u(x,1) = -3x, \qquad u(x,2) = 0. \tag{38}$$

Here to avoid confusion, we denote by $x(t)$ the solution of the controlled system. It is easily observed that Rule 1 holds with $K_2 = 3$. For any initial data $x_0 \in \mathbb{R}$ and $r_0 \in S$, the controlled system (37) has a unique global solution on $t \in \mathbb{R}_+$ with the property that $E|x(t)|^5 < \infty$. But this is not enough for the stability purpose.

Let us now check Assumption 4.1. For $(x,i) \in \mathbb{R} \times S$, compute

$$x\left(a_i x - b_i|x|x\right) + \frac{c_i^2}{2}|x|^3 \leq \begin{cases} 2|x|^2 - 1.5|x|^3, & i = 1, \\ 0.2|x|^2 - 2.275|x|^3, & i = 2, \end{cases}$$

and

$$x\left(a_i x - b_i|x|x\right) + c_i^2|x|^3 \leq \begin{cases} 2|x|^2 - |x|^3, & i = 1, \\ 0.2|x|^2 - 2.15|x|^3, & i = 2. \end{cases}$$

It is easy to see that

$$\begin{aligned} &c = 0, \quad \rho_1 = 2, \quad \rho_2 = 0.2, \quad \beta_1 = 1.5, \quad \beta_2 = 2.275, \\ &\bar{c} = 0, \quad \bar{\rho}_1 = 2, \quad \bar{\rho}_2 = 0.2, \quad \bar{\beta}_1 = 1, \quad \bar{\beta}_2 = 2.15, \end{aligned}$$

and $\beta_m = 1.5$, $\bar{\beta}_m = 1$. Further, we have

$$x^{\mathrm{T}}u(x,i) = \begin{cases} -3|x|^2, & i = 1, \\ 0, & i = 2, \end{cases}$$

which implies that $\kappa_1 = -3$, $\kappa_2 = 0$. Consequently, $\alpha_1 = -1$, $\alpha_2 = 0.2$, $\bar{\alpha}_1 = -1$, $\bar{\alpha}_2 = 0.2$, $\alpha_M = 0.2$, $\bar{\alpha}_M = 0.2$ and $\mathcal{A} = \begin{pmatrix} 3 & -1 \\ -1 & 0.6 \end{pmatrix}$. We then obtain that $A_m = 2$, $A_M = 5$, which shows that $\mathcal{A}$ is a non-singular M-matrix. Hence, Rule 2 is fulfilled. We then derive that $\tau^* = 0.000019$ defined in (33). By Theorems 4.1 and 4.2, we can, therefore, conclude that the controlled system (37) is 3-th moment and almost surely exponentially stable provided $\tau < 0.000019$. We perform a computer simulation with $\tau = 0.00001$ and the initial data $x_0 = 1$ and $r_0 = 1$. The sample paths of the Markov chain, the solution of SDE (36) and the solution of the controlled system (37) are plotted in Fig. 1. The simulation supports our theoretical results clearly.

On the other hand, we see that

$$-(q_1+1)\mathrm{diag}(\bar{\alpha}_1, \bar{\alpha}_2) - Q = \begin{pmatrix} 5 & -1 \\ -1 & 0.2 \end{pmatrix},$$

which is obviously not a non-singular M-matrix. This indicates that the theory in [16] cannot be applied to this example, so we weaken the conditions in [16].

Example 2: In this example, we will show that we indeed improve the estimation of the bound of τ compared with [16]. Consider a stochastic Cohen-Grossberg neural network with Markovian switching consisting of 10 neurons, where the j-th neuron can be described by

$$\begin{aligned}\mathrm{d}y_j(t) = &-a(r(t))y_j(t)\Bigg(b(r(t))y_j^2(t) \\ &- \sum_{k=1}^{N} W_{jk}(r(t))\Upsilon_k(y_k(t), r(t))\Bigg)\mathrm{d}t \\ &+ d(i)y_j^2(t)\mathrm{d}B(t), \quad 1 \leq j \leq 10.\end{aligned} \tag{39}$$

Here, in mode i, $i = 1, 2$, $a(i)y_j$ represents the amplification function, $b(i)y_j^2$ is the behaved function, $W_{jk}(i)$ stands for the connection weight from neuron k to neuron j, $\Upsilon_j(y_j, i) = v(i)\frac{1-e^{-y_j}}{1+e^{-y_j}}$ is the neuron activation function, $d(i)y_j^2$ is the noise perturbation. For more information of stochastic Cohen-Grossberg neural network, we cite [30]–[33] as references.

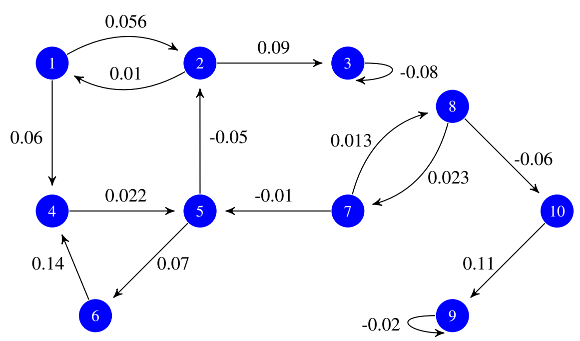

Fig. 2. The neuron network connection graph at mode 1.

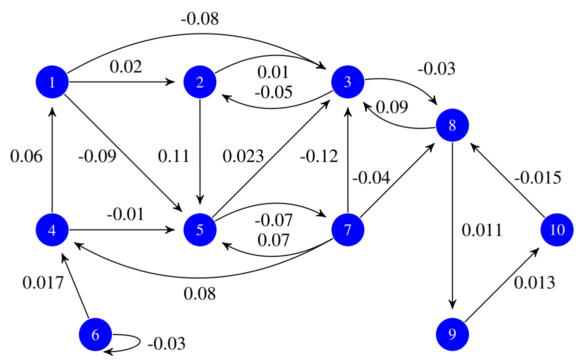

Fig. 3. The neuron network connection graph at mode 2.

The network parameters are given as $a(1) = 0.4$, $a(2) = 0.45$, $b(1) = 0.9$, $b(2) = 0.95$, $\upsilon(1) = 0.8$, $\upsilon(2) = 1$, $d(1) = 0.1$, $d(2) = 0.11$. The connection weight $W_{jk}(i)$ can be obtained from the network connection graphes with mode 1 in Fig. 2 and mode 2 in Fig. 3. Let us take Fig. 2 as an example to explain the network connection graph: node j stands for the j-th neuron, directed edge (j,k) means the output of the k-th neuron is connected with the input of the j-th neuron, the number on the edge (j,k) is the value of $W_{jk}(1)$, if there is no edge between two nodes, these two neurons do not have direct interaction and the value of $W_{jk}(1)$ is zero, such as $W_{32}(1) = 0.09$, $W_{31}(1) = 0$. Here, positive number represents the output-input connection is noninverting, negative is inverting.

Let $y = (y_1, \cdots, y_{10})^{\mathrm{T}}$, $y^2 = \left(y_1^2, \cdots, y_{10}^2\right)^{\mathrm{T}}$, $W(i) = (W_{jk}(i))_{10\times 10}$, $\Upsilon(y,i) = (\Upsilon_1(y_1,i), \cdots, \Upsilon_{10}(y_{10},i))^{\mathrm{T}}$. Then we can rewrite network (39) as a general type of SDE

$$\mathrm{d}y(t) = f(y(t), r(t))\mathrm{d}t + g(y(t), r(t))\mathrm{d}B(t), \tag{40}$$

where

$$f(y,i) = -a(i)\mathrm{diag}(y_1, \cdots, y_{10})\left(b(i)y^2 - W(i)\Gamma(y,i)\right)$$

and $g(y,i) = d(i)y^2$.

It is clear that both f and g are locally Lipschitz continuous. Noting that for $y \in \mathbb{R}^{10}$, $\left|y^2\right| \le |y|^2$ and $|\Upsilon(y,i)| \le \upsilon(i)$ since $\left|\frac{1-e^{-y_j}}{1+e^{-y_j}}\right| \le 1$, we derive that

$$\begin{aligned}|f(y,i)| \le& \left|-a(i)\mathrm{diag}(y_1, \cdots, y_{10})b(i)y^2\right| \\ &+ |-a(i)\mathrm{diag}(y_1, \cdots, y_{10})W(i)\Upsilon(y,i)| \\ \le& a(i)b(i)|y|\left|y^2\right| + a(i)|W(i)||y||\Upsilon(y,i)| \\ \le& a(i)\upsilon(i)|W(i)||y| + a(i)b(i)|y|^3\end{aligned}$$

and

$$|g(y,i)| = d(i)\left|y^2\right| \le d(i)|y|^2.$$

Then it is easy to verify that Assumption 3.1 holds with $K_0 = 0$, $K_1 = 0.4275$, $q_1 = 3$ and $q_2 = 2$. To verify Assumption 3.2, for $y \in \mathbb{R}^{10}$, compute

$$\begin{aligned}y^{\mathrm{T}} f(y,i) =& -a(i)\left(y^2\right)^{\mathrm{T}}\left(b(i)y^2 - W(i)\Upsilon(y,i)\right) \\ \le& -a(i)b(i)\left|y^2\right|^2 + a(i)\upsilon(i)|W(i)|\left|y^2\right| \\ \le& a(i)\upsilon(i)|W(i)||y|^2 - \frac{a(i)b(i)}{10}|y|^4\end{aligned}$$

because $|y|^4 \le 10\left|y^2\right|^2$. We hence see that

$$\begin{aligned}&y^{\mathrm{T}} f(y,i) + \frac{\bar{q}-1}{2}|g(y,i)|^2 \\ \le& a(i)\upsilon(i)|W(i)||y|^2 - \left(\frac{a(i)b(i)}{10} - \frac{\bar{q}-1}{2}(d(i))^2\right)|y|^4.\end{aligned}$$

As a result, Assumption 3.2 is satisfied with $\alpha = 0.084$ and $\bar{q} = 8$. Through computer simulation, we can find that SDE (40) is unstable (see Fig. 4). Then we want to design a discrete-state feedback control to stabilize SDE (40). To make it simple, our control function will have the form

$$u(y,i) = \pi(i)y, \tag{41}$$

where $\pi(1) = -3$, $\pi(2) = -3.1$. Then the controlled system becomes

$$\begin{aligned}\mathrm{d}x(t) =& (f(x(t), r(t)) + u(x([t/\tau]\tau), r(t)))\mathrm{d}t \\ &+ g(x(t), r(t))\mathrm{d}B(t).\end{aligned} \tag{42}$$

Here, to avoid confusion, we use $x(t)$ to denote the solution of the controlled system. It is easy to observe that Rule 1 holds with $K_2 = 3.1$. Then compute

$$\begin{aligned}&x^T f(x,i) + \frac{1}{2}|g(x,i)|^2 \\ \le& a(i)\upsilon(i)|W(i)||x|^2 - \left(\frac{a(i)b(i)}{10} - \frac{1}{2}(d(i))^2\right)|x|^4\end{aligned}$$

and

$$\begin{aligned}&x^T f(x,i) + \frac{q_1}{2}|g(x,i)|^2 \\ \le& a(i)\upsilon(i)|W(i)||x|^2 - \left(\frac{a(i)b(i)}{10} - \frac{3}{2}(d(i))^2\right)|x|^4\end{aligned}$$

for $(x,i) \in \mathbb{R}^{10} \times S$. Consequently, it is easy to see that Assumption 4.1 is satisfied with $c = 0$, $\bar{c} = 0$, $\rho_1 = 0.0474$, $\rho_2 = 0.016$, $\beta_1 = 0.031$, $\beta_2 = 0.0367$, $\bar{\rho}_1 = 0.0474$, $\bar{\rho}_2 = 0.016$, $\bar{\beta}_1 = 0.021$, $\bar{\beta}_2 = 0.0246$. Further, $\beta_m = 0.031$, $\bar{\beta}_m = 0.021$. We also have

$$x^{\mathrm{T}} u(x,i) = \pi(i)|x|^2,$$

which implies that $\kappa_1 = -3$, $\kappa_2 = -3.1$, $\alpha_1 = -2.9506$, $\alpha_2 = -3.016$, $\bar{\alpha}_1 = -2.9506$, $\bar{\alpha}_2 = -3.016$, $\alpha_M = -2.9506$, $\bar{\alpha}_M = -2.9506$, $\mathcal{A} = \begin{pmatrix} 6.9012 & -1 \\ -1 & 7.0319 \end{pmatrix}$. We then obtain that $A_m = 0.1662$ and $A_M = 0.169$, which shows that $\mathcal{A}$

is a non-singular M-matrix. Therefore, Rule 2 is fulfilled. Recalling the discussion in Remark 5 and making use of MATLAB, we derive that $\tau^* = 0.0071$. By Theorems 4.1 and 4.2, we conclude that the controlled system (42) is 4-th moment exponentially stable and almost surely exponentially stable provided $\tau < 0.0071$. We perform a computer simulation with $\tau = 0.001$ and the initial data $x_0 = (0.15, \cdots, 0.15)^{\mathrm{T}}$ and $r_0 = 1$. The sample paths of the Markov chain, the solution of SDE (39) and the solution of the controlled system (42) are plotted in Fig. 4. The simulation supports our theoretical results clearly.

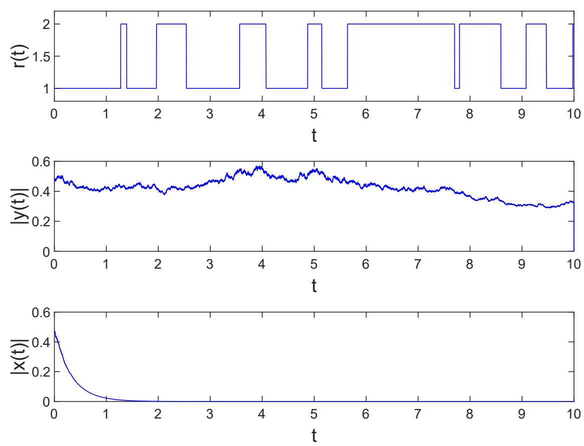

Fig. 4. Computer simulation of the sample paths of the Markov chain (the top one), and SDE (39) without control (the middle one) and the controlled system (42) with the control function (41) and $\tau = 0.001$ (the bottom one) using the truncated Euler-Maruyama method [29] with time step size 10^{-6}.

Next, we give a remark to demonstrate that we can improve the estimation of the bound of τ in this example.

Remark 7: We apply the theory developed in [16] to the unstable SDE (40). Here, for convenience, we use the same notations in [16]. It is easy to check that Assumptions 2.1, 2.2, 2.3 and Conditions 4.1, 4.2 in [16] are all satisfied. Hence, we only need to check Condition 5.1. But the reader may find, in practice, it is not easy to verify this condition, that is

$$LU(x,t,i) + \gamma_1 \left(2\eta_i|x| + (q_1+1)\bar{\eta}_i|x|^{q_1}\right)^2 + \gamma_2|f(x,t,i)|^2 + \gamma_3|g(x,t,i)|^2 \le -\gamma_4|x|^2 - \gamma_5|x|^{2q_1}, \tag{43}$$

since there are five positive free parameters γ_j, $j = 1,2,3,4,5$ to be chosen freely. These free parameters influence the bound of τ we obtain, namely Condition 5.2 in [16],

$$\tau < \frac{\sqrt{\gamma_4\gamma_1}}{2K_2^2} \wedge \frac{\sqrt{\gamma_1\gamma_2}}{\sqrt{2}K_2} \wedge \frac{\gamma_1\gamma_3}{K_2^2} \wedge \frac{1}{4\sqrt{2}K_2}. \tag{44}$$

And a bad choice may bring us a relatively small bound of τ. As a result, through this example, we will introduce a method to avoid lots of trails of γ_j, $j = 1,2,3,4,5$. Here,

$$(\bar{\eta}_1, \bar{\eta}_2)^{\mathrm{T}} := \bar{\mathcal{A}}^{-1}(1,1)^{\mathrm{T}} = (0.0846, 0.083)^{\mathrm{T}},$$

where $\bar{\mathcal{A}} = -(q_1+1)\mathrm{diag}(\bar{\alpha}_1, \bar{\alpha}_2) - Q$ is a non-singular M-matrix. The Lyapunov function U used in [16] now becomes

$$U(x,i) = \begin{cases} 0.338|x|^2 + 0.3384|x|^4, & i = 1, \\ 0.3325|x|^2 + 0.3321|x|^4, & i = 2, \end{cases}$$

and we have

$$LU(x,t,i) \le \begin{cases} -|x|^2 - 1.0105|x|^4 - 0.0071|x|^6, & i = 1, \\ -|x|^2 - 1.0122|x|^4 - 0.0082|x|^6, & i = 2. \end{cases}$$

Compute

$$\begin{aligned} &LU(x,t,1) + \gamma_1\left(2\eta_1|x| + (q_1+1)\bar{\eta}_1|x|^{q_1}\right)^2 \\ &\quad + \gamma_2|f(x,1)|^2 + \gamma_3|g(x,1)|^2 \\ &\le -(1 - 0.1142\gamma_1 - 0.0024\gamma_2)|x|^2 \\ &\quad - (1.0105 - 0.2287\gamma_1 - 0.0356\gamma_2 - 0.01\gamma_3)|x|^4 \\ &\quad - (0.0071 - 0.1145\gamma_1 - 0.1296\gamma_2)|x|^6 \end{aligned}$$

and

$$\begin{aligned} &LU(x,t,2) + \gamma_1\left(2\eta_2|x| + (q_1+1)\bar{\eta}_2|x|^{q_1}\right)^2 \\ &\quad + \gamma_2|f(x,2)|^2 + \gamma_3|g(x,2)|^2 \\ &\le -(1 - 0.1105\gamma_1 - 0.0071\gamma_2)|x|^2 \\ &\quad - (1.0122 - 0.2208\gamma_1 - 0.0719\gamma_2 - 0.0121\gamma_3)|x|^4 \\ &\quad - (0.0082 - 0.1103\gamma_1 - 0.1828\gamma_2)|x|^6. \end{aligned}$$

Then Condition 5.1 is satisfied with

$$\begin{aligned} \gamma_4 =& (1 - 0.1142\gamma_1 - 0.0024\gamma_2) \\ &\wedge (1 - 0.1105\gamma_1 - 0.0071\gamma_2), \\ \gamma_5 =& (0.0071 - 0.1145\gamma_1 - 0.1296\gamma_2) \\ &\wedge (0.0082 - 0.1103\gamma_1 - 0.1828\gamma_2) \end{aligned}$$

if we have

$$\begin{aligned} 1 - 0.1142\gamma_1 - 0.0024\gamma_2 &> 0, \\ 1 - 0.1105\gamma_1 - 0.0071\gamma_2 &> 0, \\ 1.0105 - 0.2287\gamma_1 - 0.0356\gamma_2 - 0.01\gamma_3 &> 0, \\ 1.0122 - 0.2208\gamma_1 - 0.0719\gamma_2 - 0.0121\gamma_3 &> 0, \\ 0.0071 - 0.1145\gamma_1 - 0.1296\gamma_2 &> 0, \\ 0.0082 - 0.1103\gamma_1 - 0.1828\gamma_2 &> 0. \end{aligned} \tag{45}$$

Through (45), we get a rough estimation of $\gamma_1, \gamma_2, \gamma_3$ that

$$(\gamma_1, \gamma_2, \gamma_3) \in \Gamma = (0, 0.0621) \times (0, 0.0447) \times (0, 83.6531).$$

In this situation, we can define a function $\tilde{\varphi}$ in Γ by

$$\begin{aligned} \tilde{\varphi}(\gamma_1,\gamma_2,\gamma_3) =& \frac{1}{19.22}\Big(\sqrt{(1 - 0.1142\gamma_1 - 0.0024\gamma_2)\gamma_1} \\ &\wedge \sqrt{(1 - 0.1105\gamma_1 - 0.0071\gamma_2)\gamma_1}\Big) \\ &\wedge \frac{\sqrt{\gamma_1\gamma_2}}{4.3841} \wedge \frac{\gamma_1\gamma_3}{9.61} \wedge \frac{1}{17.5362}. \end{aligned}$$

Then the bound of τ, say by τ_1^*, defined in (44) can be modified as

$$\tau_1^* = \sup\{\tilde{\varphi}(\gamma_1,\gamma_2,\gamma_3) | (\gamma_1,\gamma_2,\gamma_3) \in \Gamma, (45) \text{ is satisfied}\}.$$

Using MATLAB, we get $\tau_1^* = 0.0066$. By Theorem 5.3 in [16], the controlled system (42) is exponentially stable in 4-th moment and almost surely provided $\tau < 0.0066$. Through this, we see that we get a better bound of τ in this example. This shows the advantage of the Razumikhin technique in the estimation of the bound of τ.

VI. Conclusion

In this paper, we have discussed the stabilization of highly nonlinear hybrid SDEs by feedback control based on discrete-time state observations by the Razumikhin technique, rather than the comparison idea or the Lyapunov functional method in the most existing results. We firstly developed a Razumikhin-type theorem to study the asymptotic boundedness and moment exponential stability of hybrid SFDEs. Then we applied this generalized theory to our discrete-state-feedback stabilization problem and showed that the underlying unstable system could be stabilized in the sense of (q_1+1)-th moment exponential stability and almost sure exponential stability. Finally, two interesting examples and computer simulations were provided to demonstrate that we could include more general models and get a better estimation of τ^* compared with the existing results. We also observed that there were several advantages of the Razumikhin technique compared with the Lyapunov functional method in the discrete-state-feedback stabilization problem: the control function designed are easier to be implemented in practice; a better bound of τ could be obtained for a class of hybrid SDEs; much complicated analysis can be avoided.

Appendix

In this Appendix, we first give a counter example to show the right-continuity of a process sometimes cannot ensure its expectation keeps this property as discussed in Remark 2.

Example 3: Let $B(t)$ be a scalar Brownian motion. Define the stopping time $T = \inf\{t \ge 0 | B(t) = 1\}$. It is easy to see that $T < \infty$ a.s. from the recurrence of $B(t)$. Then for any $t \ge 0$, set $Y(t) = B(t \wedge T)$. By the well-known Doob martingale stopping theorem, $Y(t)$ is actually a continuous martingale vanishing at $t = 0$ with the property that

$$\lim_{t\to\infty} Y(t) = 1 \quad \text{a.s.}$$

Define a process $X(t)$ by

$$X(t) = \begin{cases} Y\left(\dfrac{1}{t-1}\right), & t > 1, \\ 1, & 0 \le t \le 1. \end{cases}$$

Since

$$\lim_{t\to 1^+} X(t) = \lim_{s\to\infty} Y(s) = 1 \quad \text{a.s.}$$

we observe that $X(t)$ is continuous (certainly right-continuous). However, we have for $0 \le t \le 1$, $EX(t) = 1$, and for $t > 1$

$$EX(t) = EY\left(\frac{1}{t-1}\right) = 0.$$

This means $EX(t)$ is not right-continuous at $t = 1$.

In Remark 3, we have discussed that when $q_1 + 1$ is strictly larger than $2q_2$, $\bar{q}$ sometimes could be arbitrarily large, which can be seen in the following example.

Example 4: Consider a scalar hybrid SDE discussed in [16], namely

$$\mathrm{d}y(t) = f(y(t), t, r(t))\mathrm{d}t + g(y(t), t, r(t))\mathrm{d}B(t) \tag{46}$$

on $t \ge 0$, where $B(t)$ is a one-dimensional Brownian motion, $r(t)$ takes values on $S = \{1, 2\}$, and f, g are defined by

$$f(y,t,1) = y - 3y^3, \qquad g(y,t,1) = |y|^{1.5},$$
$$f(y,t,2) = y - 2y^3, \qquad g(y,t,2) = 0.5|y|^{1.5}.$$

Let $\bar{q}$ be arbitrarily large. For $i = 1$,

$$\begin{aligned} & yf(y,t,1) + \frac{\bar{q}-1}{2}|g(y,t,1)|^2 \\ =& y^2 - 3y^4 + \frac{\bar{q}-1}{2}|y|^3 \le \left(1 + \frac{(\bar{q}-1)^2}{48}\right)|y|^2 \end{aligned}$$

since $(\bar{q}-1)|y|^3 \le 6|y|^4 + \frac{(\bar{q}-1)^2}{24}|y|^2$. For $i = 2$,

$$\begin{aligned} & yf(y,t,2) + \frac{\bar{q}-1}{2}|g(y,t,2)|^2 \\ =& y^2 - 2y^4 + \frac{\bar{q}-1}{8}|y|^3 \le \left(1 + \frac{(\bar{q}-1)^2}{512}\right)|y|^2 \end{aligned}$$

since $(\bar{q}-1)|y|^3 \le 16|y|^4 + \frac{(\bar{q}-1)^2}{64}|y|^2$. Then Assumption 3.2 is satisfied with any large $\bar{q}$ and $\alpha = 1 + \frac{(\bar{q}-1)^2}{48}$.

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Henglei Xu received the B.Sc. degree in applied mathematics from Harbin Institute of Technology (Weihai), Weihai, China, in 2020. He is currently working toward the Ph.D. degree in mathematics and statistics from Strathclyde University, Glasgow, U.K.. His Ph.D. programme is fully funded by Chinese Scholarship Council and Strathclyde University (PhD studentship). His current research interests include stabilization, stochastic control and differential equations and their applications.

Xuerong Mao received the PhD degree from Warwick University, Coventry, U.K., in 1989. He was SERC (Science and Engineering Research Council, U.K.) Post-Doctoral Research Fellow from 1989 to 1992. Moving to Scotland, he joined the University of Strathclyde, Glasgow, U.K., as a Lecturer in 1992, was promoted to Reader in 1995, and was made Professor in 1998 which post he still holds. He was elected as a Fellow of the Royal Society of Edinburgh (FRSE) in 2008. He has authored five books and over 300 research papers. His main research interests lie in the field of stochastic analysis including stochastic stability, stabilization, control, numerical solutions and stochastic modelling in finance, economic and population systems. He is a member of the editorial boards of several international journals.