Speculative trading, prospect theory and transaction costs

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Abstract

A speculative agent with prospect theory preference chooses the optimal time to purchase and then to sell an indivisible risky asset to maximise the expected utility of the round-trip profit net of transaction costs. The optimisation problem is formulated as a sequential optimal stopping problem, and we provide a complete characterisation of the solution. Depending on the preference and market parameters, the optimal strategy can be "buy and hold", "buy low, sell high", "buy high, sell higher" or "no trading". Behavioural preference and market friction interact in a subtle way which yields surprising implications on the agent's trading patterns. For example, increasing the market entry fee does not necessarily curb speculative trading, but instead may induce a higher reference point under which the agent becomes more risk-seeking and in turn is more likely to trade.

Keywords Sequential optimal stopping · S-shaped utility · Transaction costs · Entry-and-exit strategies

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1 Introduction

When it comes to modelling trading behaviour, the standard economic paradigm is the maximisation of risk-averse agents' expected utility in a frictionless market. This criterion, however, has been criticised on many levels. In terms of trading environment, financial friction is omnipresent in reality where transactions are subject to various

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costs. In terms of agents' preferences, the behavioural economics literature suggests that many individuals do not make decisions in accordance to expected utility theory. First, utilities are not necessarily derived from final wealth, but typically what matters is the change in wealth relative to some reference point. Second, individuals are usually risk-averse over the domain of gains, but risk-seeking over the domain of losses—this can be captured by an S-shaped utility function. Finally, individuals may fail to take portfolio effects into account when making investment decisions, and this phenomenon is known as narrow framing. These psychological ideas are explored for example in the seminal work of Kahneman and Tversky [16], Tversky and Kahneman [26, 27] and Kahneman and Lovallo [15].

In this paper, we develop a tractable dynamic trading model which captures a number of stylised behavioural biases of individuals as well as market frictions. In our setup, trading is costly due to proportional transaction costs as well as a fixed market entry fee. The goal of an agent is to find the optimal time to buy and then to sell an indivisible risky asset to maximise the expected utility of the round-trip profit under the prospect theory preference of Tversky and Kahneman [27]. While a realistic economy can consist of multiple assets, we can interpret the assumption of a single indivisible asset as a manifestation of narrow framing such that the trading decisions associated with one particular unit of indivisible asset can be completely isolated from the other investment opportunities. We believe the model is the best suitable to describe the trading behaviour of speculative agents. These "less-than-fully rational" agents purchase and sell an asset with a narrow objective of making a one-off round-trip profit rather than supporting consumption or stipulating a long-term portfolio growth.

A sequential optimal stopping problem featuring an S-shaped utility function is solved to identify the entry and exit time of the market by the agent. The solution approach is based on a backward induction idea. In the first stage, we focus on the exit strategy of the agent: Conditionally on the ownership of the asset purchased at a given price level (which determines the agent's reference point), the optimal liquidation problem is solved. Then the value function of this exit problem reflects the utility value of purchasing the asset at a different price level. Upon comparison against the utility value of inaction, we obtain the payoff function of the real option to purchase the asset which is then used in the second stage problem concerning the entry decision of the agent: The agent picks the optimal time to enter the trade so as to maximise the expected payoff of this real option to purchase the asset.

The traditional route to analyse an optimal stopping problem is to first conjecture a candidate optimal stopping rule, and then the dynamic programming principle is invoked to derive a free boundary value problem that the value function should satisfy. Then one can attempt to solve for the free boundaries via value matching and smooth pasting. For this approach to work, we need to correctly identify the form of the optimal stopping rule, but this exercise may be not trivial. As it turns out, the optimal continuation region of our entry problem can either be connected or disconnected, depending on the model parameters. It is thus difficult to adopt such a guess-and-verify approach since we do not know the correct form of the optimal stopping rule upfront. In our analysis, a martingale method is employed to solve the underlying optimal stopping problems, which has the important advantage that no a priori conjecture on the optimal strategy is required. The optimal continuation/stopping set can

be deduced directly by studying the smallest concave majorant of a suitably scaled payoff function.

Despite its relatively simple nature, our model is capable of generating a rich variety of trading behaviours such as "buy and hold", "buy low, sell high", "buy high, sell higher" and "no trading". The risk-seeking preference of a behavioural agent over the loss domain will typically encourage him to enter the trade, but his precise trading behaviour depends crucially on the level of transaction costs relative to his preference parameters. Generally speaking, a high proportional (fixed) transaction cost discourages trading at a high (low) nominal price. When proportional costs are high and the asset is expensive, the agent prefers waiting until the price level declines, and hence he is more inclined to consider a "buy low, sell high" strategy. But if instead the fixed entry fee is high and the asset is cheap, the agent might prefer delaying the purchase decision until the asset reaches a higher price level, and this leads to a trading pattern of "buy high, sell higher".

Both behavioural preferences and market frictions are studied extensively as separate topics in the mathematical finance literature. To the best of our knowledge, however, their interaction has not been explored to date. Under prospect theory, the risk attitude of the agent is heavily influenced by the reference point. In our model, the reference point is endogenised so that it depends on the cost of purchase including the transaction cost paid. The level of transaction cost therefore has a direct impact on the agent's risk preference. This subtle interaction between risk preference and transaction cost leads to interesting policy implications on how speculative trading can be curbed effectively. For example, a surprising result is that imposing a fixed market entry fee might indeed accelerate rather than cool down trading participation.

Our paper is closely related to the literature on optimal stopping under an S-shaped utility function. Kyle et al. [18] and Henderson [9] consider a one-off optimal liquidation problem in which the agent solves for the optimal time to liquidate an endowed risky asset to maximise an expected prospect theory utility. They do not consider the purchase decision, and the reference point is taken as some exogenously given status quo. A main contribution of our paper is that we further endogenise the reference point which depends on the purchase price of the asset, and the optimal purchase price must be determined as a part of the optimisation problem. The recent work of Henderson and Muscat [11] extends the model of Henderson [9] by considering partial liquidation of multiple indivisible assets. Both the present paper and [11] consider a sequential optimal stopping problem as the underlying mathematical framework. However, the economic natures of the problems are completely different—we study the sequential decision of purchase and sale, while they exclusively focus on sales.

Another relevant class of works is the realisation utility model which further incorporates a reinvestment possibility within a behavioural optimal stopping model, such as Barberis and Xiong [2], Ingersoll and Jin [14], He and Yang [8], Kong et al. [17] and Dai et al. [3]. In such models, the agent repeatedly purchases and sells an asset to maximise the sum of utility bursts realised from the gain and loss associated with each round-trip transaction. In a certain sense, these models consider an endogenised reference point which is continuously updated based on the historical prices within each trading episode. However, the purchase decision is exogenously given in many of the models where the agent is simply assumed to buy the asset again immediately

after a sale. These cited papers on realisation utility models all feature transaction costs which are required to make the problems well posed. As a result, the purchase pattern is not entirely realistic: If the agent is willing to sell an asset and then instantaneously repurchase an identical (or a different, but statistically identical) asset, then the agent is essentially throwing away money in the form of transaction costs without altering his own financial position. Only He and Yang [8] carefully analyse the purchase decision of the agent, but in any case they find that the purchase strategy is trivial—the agent either buys the asset immediately after a sale or never enters the trade again. Our model differs from the realisation utility model in that we do not consider perpetual reinvestment opportunities (which can be understood as narrow framing that the agent only focuses on a single episode of the trading experience when evaluating the entry and exit strategies). Nonetheless, the optimal purchase region of our model is non-trivial under typical parameters and encapsulates many realistic trading strategies.

Beyond the context of behavioural economics, there are a few works attempting to model the sequential purchase and sale decisions under an optimal switching framework. However, identification of a modelling setup which can generate reasonable trading patterns proves to be much more difficult than expected. On the one hand, Zervos et al. [29] report that "... the prime example of an asset price process, namely, the geometric Brownian motion, does not allow optimal buying and selling strategies that have a sequential nature". Indeed, existing literature which gives "buy low, sell high" as an optimal trading strategy often relies on extra statistical features of the asset price process such as mean reversion. See for example Zhang and Zhang [30], Song et al. [25], Leung et al. [20] and Leung and Li [19]. On the other hand, momentum-based trading strategies are also rarely studied in the mathematical finance literature. The scarce examples include the work of Dai et al. [5] and Dai et al. [4] who find that a trend-following strategy is optimal under a regime-switching model for the asset price. We contribute to this strand of literature by showing that a trading model based on a simple geometric Brownian motion can also generate many realistic trading patterns including both a reversal strategy (buy low, sell high) and a momentum strategy (buy high, sell higher). This is achieved via incorporating standard elements of behavioural preferences and market frictions.

The optimal investment rule in the classical Merton [23, 24] portfolio selection problem can also be viewed as a buy low, sell high strategy: Since the agent keeps a constant fraction of wealth invested in the risky asset, extra units of risky asset are sold (purchased) when the price increases (falls), ceteris paribus. In our paper, we focus on a single indivisible asset and do not consider portfolio effects.

The rest of the paper is organised as follows. Section 2 provides a description of the model and the underlying optimisation problem. In Sect. 3, we outline the solution methods to a standard optimal stopping problem and discuss heuristically how the solution to our sequential optimal stopping problem can be characterised via the idea of backward induction. The main results are collected in Sect. 4. Some comparative statics results and their policy implications are discussed in Sect. 5. Several extensions of the baseline model are discussed in Sect. 6. Section 7 concludes. A few technical proofs are deferred to the Appendix.

2 Problem description

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual conditions and supporting a one-dimensional Brownian motion $B = (B_t)_{t \geq 0}$. There is a single indivisible risky asset in the economy. Its price process $P = (P_t)_{t \geq 0}$ is modelled by a one-dimensional diffusion with state space $\mathcal{J} \subseteq \mathbb{R}_+$ and dynamics of

$$dP_t = \mu(P_t)dt + \sigma(P_t)dB_t,$$

where $\mu : \mathcal{J} \to \mathbb{R}$ and $\sigma : \mathcal{J} \to (0, \infty)$ are Borel functions. We assume that $\mathcal{J}$ is an interval with endpoints $0 \leq a_{\mathcal{J}} < b_{\mathcal{J}} \leq \infty$ and that P is regular in $(a_{\mathcal{J}}, b_{\mathcal{J}})$, i.e., for any $p, y \in (a_{\mathcal{J}}, b_{\mathcal{J}})$, we have $\mathbb{P}[\tau_y < \infty | P_0 = p] > 0$, where $\tau_y := \inf\{t \geq 0 : P_t = y\}$.

We assume that the interest rate is zero in our exposition. For the non-zero interest rate case, one can interpret the process P as the numéraire-adjusted price of the asset. Then the drift term $\mu(\cdot)$ can be viewed as the instantaneous excess return of the risky asset.

Trading in the asset is costly. If the agent wants to purchase the asset at its current price p, he will need to pay $\lambda p + \Psi$ to initiate the trade, where $\lambda \in [1, \infty)$ so that $\lambda - 1$ represents the proportional transaction cost on purchases and $\Psi \geq 0$ represents a fixed market entry fee. When the agent sells the asset at price p, he will only receive γp, where $\gamma \in (0, 1]$ so that $1 - \gamma$ represents the proportional transaction cost on sales.

The preference of the agent is described by the prospect theory of Tversky and Kahneman [27]. Under this framework, utility is derived from gains and losses relative to some reference point rather than from the total wealth. Individuals are typically risk-averse over the domain of gains and risk-seeking over the domain of losses. This can be captured by an S-shaped utility function $U : \mathbb{R} \to \mathbb{R}$ with $U(0) = 0$ and such that U is concave (resp. convex) over $\mathbb{R}_+$ (resp. $\mathbb{R}_-$). Finally, individuals also exhibit loss-aversion so that the negative utility increment brought by a unit of loss is much larger in magnitude than the positive utility increment from a unit of gain.

In the behavioural optimal liquidation literature such as Kyle et al. [18] and Henderson [9], the liquidation payoff is always compared against some exogenously given constant reference point. In our setup, we assume the reference point depends on both an exogenous constant R as well as on the amount paid by the agent to purchase the asset. Suppose the agent has executed a speculative round-trip trade where he has bought and then sold the asset at stopping times τ and ν (with $\tau \leq \nu$), respectively. The liquidation payoff γP_ν is evaluated against $\lambda P_\tau + \Psi + R$ as the reference point, where $\lambda P_\tau + \Psi$ is the capital spent on purchasing the asset and R is a constant outside the model specification. The parameter R can be interpreted as a preference parameter of the agent which reflects his "aspiration level" in the sense of Lopes and Oden [21], where a more motivated agent will set a higher economic benchmark as a profit target to beat. The realised utility of this round-trip trade is $U(\gamma P_\nu - \lambda P_\tau - \Psi - R)$.

A caveat, however, is that the agent is not obliged to enter or exit the trade at all if it is undesirable to do so. A realisation of $\tau = \infty$ refers to the case that the purchase decision is deferred indefinitely, which is economically equivalent to not entering the trade at all. The liquidation value is zero because there is nothing to be sold,

and the reference point becomes R since the required cash outflow $\lambda P_\tau + \Psi$ for the purchase has never materialised. Thus the prospect theory value under this strategy is simply $U(-R)$. Similarly, the agent may enter the trade at some time point but never liquidate the asset. This corresponds to a realisation of $\tau < \infty$ and $\nu = \infty$. In this case, the liquidation value is again zero and evaluated against the reference point $\lambda P_\tau + \Psi + R$. To summarise all the possibilities, the realised prospect theory utility associated with a trading strategy (τ, ν) is written as

$$
\begin{cases}
U(\gamma P_\nu - \lambda P_\tau - \Psi - R) & \text{if } \tau < \infty, \nu < \infty, \\
U(-R) & \text{if } \tau = \infty, \\
U(-\lambda P_\tau - \Psi - R) & \text{if } \tau < \infty, \nu = \infty.
\end{cases}
\tag{2.1}
$$

The objective of the agent is to find the optimal purchase time τ and sale time ν to maximise the expected value of (2.1). Define the objective function as

$$
J(p; \tau, \nu) := \mathbb{E}\big[U\big(\gamma P_\nu \mathbb{1}_{\{\tau < \infty, \nu < \infty\}} - (\lambda P_\tau + \Psi)\mathbb{1}_{\{\tau < \infty\}} - R\big) \big| P_0 = p \big]. \tag{2.2}
$$

Formally, the agent is solving the sequential optimal stopping problem

$$
\mathcal{V}(p) := \sup_{\tau, \nu \in \mathcal{T}: \tau \leq \nu} J(p; \tau, \nu), \tag{2.3}
$$

where $\mathcal{T}$ is the set of $(\mathcal{F}_t)$-stopping times valued in $\mathbb{R}_+ \cup \{+\infty\}$. Problem (2.3) has two features which make it non-standard relative to a typical optimal stopping problem. First, the decision space is two-dimensional. Second, the objective function has an explicit dependence on the stopping times τ, ν via the indicator functions, which further complicates the analysis.

Remark 2.1 Similarly to Henderson [9], Xu and Zhou [28] and Henderson et al. [10], we do not explicitly consider subjective discounting in our baseline model. On the one hand, our model features cash flows at different time points and it is not entirely clear what is the most appropriate way to apply subjective discounting, because the standard prospect theory framework is not directly applicable to intertemporal choices. On the other hand, under discounting, an impatient agent is much more inclined to delay losses and to realise profits earlier, which will lead to an extreme *disposition effect* which is not consistent with the empirical trading pattern of retail investors; see the discussion in Henderson [9]. At a mathematical level, introducing subjective discounting also makes our problem harder to be analysed in full generality. We briefly discuss in Sect. 6.2 how subjective discounting might be incorporated and explore (numerically in some cases) how it affects the optimal trading behaviour.

3 The solution methods

In this section, we give an overview of how problem (2.3) can be solved. We begin by offering a short summary about the solution approach to solve a standard optimal stopping problem for a one-dimensional diffusion.

3.1 The martingale method for optimal stopping problems

We review the martingale method to solve an undiscounted optimal stopping problem; this is based on Dayanik and Karatzas [6].

Consider a general problem in the form of

$$V(p) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[G\left(P_\tau\right)|P_0 = p\right]$$

for some payoff function G. In the standard theory of optimal stopping, the optimal stopping time can be characterised as the first exit time of the process from some open set $\mathcal{C}$, so that $\tau = \inf\{t \geq 0 : P_t \notin C\}$. In a one-dimensional diffusion setting, it is sufficient to consider stopping times which have the form $\tau_{a,b} := \tau_a \wedge \tau_b$, where $\tau_a := \inf\{t \geq 0 : P_t = a\}$ and $\tau_b := \inf\{t \geq 0 : P_t = b\}$ with $a \leq p \leq b$. Here $[a, b] \subseteq \mathcal{J}$ is an unknown interval to be identified (and it depends on p in general).

Let $s(\cdot)$ be the scale function of the process P (it is unique up to an affine transformation), defined as a strictly increasing function such that $\Theta := s(P)$ is a local martingale. A simple application of Itô's lemma shows that $s(\cdot)$ should solve the second order differential equation

$$\frac{\sigma^2(p)}{2} s''(p) + \mu(p)s'(p) = 0. \tag{3.1}$$

Let $\theta := s(p)$. Then

$$
\begin{aligned}
J(p; \tau_{a,b}) &:= \mathbb{E}[G(P_{\tau_{a,b}})|P_0 = p] \\
&= \mathbb{E}\left[G\left(s^{-1}(\Theta_{\tau_{a,b}})\right)\big|\Theta_0 = \theta\right] \\
&= \mathbb{E}[\phi(\Theta_{\tau_{a,b}})|\Theta_0 = \theta] \\
&= \mathbb{P}[\tau_a < \tau_b|\Theta_0 = \theta]\phi\left(s(a)\right) + \mathbb{P}[\tau_b < \tau_a|\Theta_0 = \theta]\phi\left(s(b)\right) \\
&= \frac{s(b) - \theta}{s(b) - s(a)}\phi\left(s(a)\right) + \frac{\theta - s(a)}{s(b) - s(a)}\phi\left(s(b)\right),
\end{aligned}
$$

where $\phi := G \circ s^{-1}$. The above can be maximised with respect to a and b. Moreover, the dummy variables a and b can be replaced by $a' = s(a)$ and $b' = s(b)$. Hence

$$
\begin{aligned}
V(p) &= \sup_{a,b : a \leq p \leq b} J(p; \tau_{a,b}) \\
&= \sup_{a',b' : a' \leq \theta \leq b'} \left(\frac{b' - \theta}{b' - a'}\phi(a') + \frac{\theta - a'}{b' - a'}\phi(b')\right) =: v(\theta),
\end{aligned}
$$

and thus $V(p) = v(s(p))$. The scaled value function $v(\theta)$ can be characterised as the smallest concave majorant of $\phi(\theta) = G(s^{-1}(\theta))$, the scaled payoff function over $s(\mathcal{J})$, which is defined as an interval with endpoints $s(a_\mathcal{J})$ and $s(b_\mathcal{J})$. The continuation and stopping set associated with the optimal stopping rule are given by $\mathcal{C} = \{p \in \mathcal{J} : v(s(p)) > \phi(s(p))\}$ and $\mathcal{S} = \{p \in \mathcal{J} : v(s(p)) = \phi(s(p))\}$, respectively.

3.2 Decomposition of the sequential optimal stopping problem

In the following two subsections, we discuss heuristically how the value function of (2.3) can be constructed by considering two sub-problems based on the idea of backward induction. The wellposedness conditions as well as a formal verification of optimality are explored in Sect. 4 when we specialise the modelling setup.

3.2.1 Exit problem

Suppose for the moment that the agent has already purchased the asset at some known price level q via paying $\lambda q + \Psi$ at some time in the past. Conditionally on this information, the reference point has been fixed at the known constant level $H := \lambda q + \Psi + R$. Suppose the current time is labelled as $t = 0$ and the current price of the asset is $P_0 = p$. The goal of the agent in the exit problem is to find the optimal time to sell this owned asset to maximise the expected prospect theory value of the sale proceeds relative to the reference point H. If the asset is (ever) sold at time ν, the utility of gain and loss relative to the reference point is $U(\gamma P_\nu \mathbb{1}_{\{\nu<\infty\}} - H)$ after taking the transaction cost on a sale into account. Since the realised utility is increasing in P_ν and the process P is nonnegative, there is in general no incentive for the agent to forgo the sale opportunity. Hence heuristically, one can drop the indicator function $\mathbb{1}_{\{\nu<\infty\}}$ and it is sufficient to consider the objective function $G_1(P_\nu; H) := U(\gamma P_\nu - H)$. The agent then solves the optimal stopping problem

$$V_1(p; H) := \sup_{\nu \in \mathcal{T}} \mathbb{E}[G_1(P_\nu; H)|P_0 = p] = \sup_{\nu \in \mathcal{T}} \mathbb{E}[U(\gamma P_\nu - H)|P_0 = p] \quad (3.2)$$

to determine the optimal time of the asset sale. The value function of the exit problem is then given by $V_1(p; H) = \bar{g}_1(s(p); H)$, where $\bar{g}_1 = \bar{g}_1(\theta; H)$ is the smallest concave majorant of

$$g_1(\theta; H) := G_1(s^{-1}(\theta); H) = U(\gamma s^{-1}(\theta) - H).$$

We write the optimiser to problem (3.2) as $\nu^*(p; H)$; it depends on the initial price level p and the given reference point H.

3.2.2 Entry problem

Now we assume the agent does not own any asset to begin with. His objective is to determine the optimal time to purchase (and then to sell) the asset to maximise the expected utility of the liquidation proceeds relative to the endogenised reference point.

At a given current asset price level p, there are two possible actions for the agent. First, he can opt to initiate the speculative trade by buying the asset now, which fixes the reference point as $\lambda p + \Psi + R$, and then sell it later in the future. When the asset is liquidated at his choice of the sale time ν, the realised utility is $U(\gamma P_\nu - \lambda p - \Psi - R)$. Conditionally on the decision to purchase the asset today at price p, the agent can find the best time of sale to maximise his expected utility by solving problem (3.2)

for $H = \lambda p + \Psi + R$. Then the best possible expected utility he can attain is

$$\sup_{v \in \mathcal{T}} \mathbb{E}[U(\gamma P_v - \lambda p - \Psi - R)|P_0 = p] = V_1(p; \lambda p + \Psi + R),$$

provided that he decides to enter the trade at the given price of p.

Alternatively, the agent can forgo the opportunity to enter the trade and stay away from the market forever. In this case, the payoff is zero and the reference point is simply equal to R. The utility he will receive is just the constant $U(-R)$.

Therefore, the opportunity to enter the speculative trade can be viewed as a real option. At a given price level p, the agent is willing to enter the trade only if the maximal expected utility of trading is not less than that of inaction, i.e., if

$$V_1(p; \lambda p + \Psi + R) \geq U(-R).$$

This is similar to a financial option being in the money. The payoff of this real option to the agent in utility terms as a function of the price level p is given by

$$G_2(p) := \max\{V_1(p; \lambda p + \Psi + R), U(-R)\}. \tag{3.3}$$

The entry problem for the agent is to find the optimal time to initiate the trade so as to maximise the expected value of (3.3). This is equivalent to solving

$$V_2(p) := \sup_{\tau \in \mathcal{T}} \mathbb{E}[G_2(P_\tau)|P_0 = p]$$

$$= \sup_{\tau \in \mathcal{T}} \mathbb{E}[\max\{V_1(P_\tau; \lambda P_\tau + \Psi + R), U(-R)\}|P_0 = p], \tag{3.4}$$

if the exit problem value function V_1 is well defined. We identify $\bar{g}_2 = \bar{g}_2(\theta)$ as the smallest concave majorant of

$$g_2(\theta) := G_2(s^{-1}(\theta)) = \max\{V_1(s^{-1}(\theta); \lambda s^{-1}(\theta) + \Psi + R), U(-R)\}$$

$$= \max\{\bar{g}_1(\theta; \lambda s^{-1}(\theta) + \Psi + R), U(-R)\}.$$

Then the value function of the entry problem is $V_2(p) = \bar{g}_2(s(p))$.

Let the optimiser to (3.4) be $\tau^*(p)$. With p being the initial price of the asset at $t = 0$, the agent will purchase the asset at the time $t = \tau^*(p)$. Then conditionally on the realisation of the entry price level $P_{\tau^*(p)}$, the agent solves the exit problem (3.2) with initial value $P_{\tau^*(p)}$ and reference point $H = \lambda P_{\tau^*(p)} + \Psi + R$. The corresponding optimiser is given by $v^*(P_{\tau^*(p)}; \lambda P_{\tau^*(p)} + \Psi + R)$ and reflects the time lapse between the initiation and closure of the trade. In particular, the agent will sell the asset at the time $t = \tau^*(p) + v^*(P_{\tau^*(p)}; \lambda P_{\tau^*(p)} + \Psi + R)$. This gives a complete characterisation of the optimal entry and exit strategy of the agent. Note that it is possible to have $\mathbb{P}[\tau^*(p) < \infty] < 1$. In other words, there is a possibility that the entry strategy is not executed in finite time, and hence there is no decision to sell (see the discussion in Sect. 4).

Intuitively, we expect $\mathcal{V}(p) = V_2(p)$, where $\mathcal{V}$ is the value function of the original sequential optimal stopping problem (2.3). This must be verified formally. Without any further specifications of the utility function U and the underlying price process P, however, it is hard to make further progress. For example, it is not even clear upfront whether (2.3) is a well-posed problem with a finite value function.

4 Main results

The procedures described in Sect. 3 are very generic and can guide us to write down a candidate for the value function of the sequential optimal stopping problem under a range of model specifications. To derive stronger analytical results, we specialise **in the rest of this paper** to the piecewise power utility function of Tversky and Kahneman [27] in the form

$$U(x) = \begin{cases} x^\alpha & \text{for } x > 0, \\ -k|x|^\alpha & \text{for } x \leq 0. \end{cases}$$

Here $\alpha \in (0, 1)$ so that $1 - \alpha$ is the level of risk-aversion and risk-seeking on the domains of gains and losses, and $k > 1$ controls the degree of loss-aversion. Experimental results in [27] give estimates of $\alpha = 0.88$ and $k = 2.25$.

The price process of the risky asset $P = (P_t)_{t \geq 0}$ is assumed to be a geometric Brownian motion,

$$dP_t = P_t(\mu dt + \sigma dB_t),$$

with $\mu \geq 0$ and $\sigma > 0$ being the constant drift and volatility of the asset. Define $\beta := 1 - \frac{2\mu}{\sigma^2} \leq 1$. Then by substituting $\mu(p) = \mu p$ and $\sigma(p) = \sigma p$ in (3.1), the scale function of P can be found as

$$s(x) = \begin{cases} x^\beta & \text{if } \beta > 0, \\ \ln x & \text{if } \beta = 0, \\ x^{-\beta} & \text{if } \beta < 0. \end{cases}$$

Finally, we assume $R > 0$ so that the aspiration level of the agent is always positive. This is not unreasonable since this parameter can be understood as some performance benchmark that an agent wants to outperform, and such a goal is typically positive.

We begin with a necessary condition under which (2.3) is well posed.

Lemma 4.1 *If $\beta \leq 0$ or $0 < \beta < \alpha$, then there exists a sequence of stopping times $(\tau_n, \nu_n)_{n=1,2,\ldots}$ such that $J(p; \tau_n, \nu_n) \to +\infty$ as $n \to \infty$, where $J(p; \cdot, \cdot)$ is defined in (2.2).*

Proof Consider the sequence of stopping times defined by

$$\tau_n := 0, \qquad \nu_n := \inf\{t \geq 0 : P_t \geq n\}. \tag{4.1}$$

If $\beta \leq 0$, then

$$P_t = P_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right) = P_0 \exp\left(\sigma\left(-\frac{\sigma}{2}\beta t + B_t\right)\right)$$

so that the Brownian motion in the exponent has nonnegative drift. Hence P can reach any positive level in finite time, and so $\nu_n < \infty$ and $P_{\nu_n} = n$, almost surely.

Hence

$$J(p; \tau_n, v_n) = \mathbb{E}\left[U\left(\gamma P_{v_n} \mathbb{1}_{\{\tau_n < \infty, v_n < \infty\}} - (\lambda P_{\tau_n} + \Psi)\mathbb{1}_{\{\tau_n < \infty\}} - R\right)\middle| P_0 = p\right]$$
$$= U(\gamma n - \lambda p - \Psi - R) \longrightarrow +\infty$$

as $n \to \infty$.

If $\beta > 0$, then instead P cannot reach any arbitrary level above its starting value in finite time and we have $\lim_{t\to\infty} P_t = 0$ almost surely. Then $\{v_n < \infty\} = \{P_{v_n} = n\}$, and for sufficiently large $n > p$, one can compute that

$$\mathbb{P}[v_n = \infty | P_0 = p] = \frac{n^\beta - p^\beta}{n^\beta}, \qquad \mathbb{P}[v_n < \infty | P_0 = p] = \frac{p^\beta}{n^\beta}.$$

Then

$$J(p; \tau_n, v_n) = \mathbb{P}[v_n = \infty | P_0 = p] U(-\lambda p - \Psi - R)$$
$$+ \mathbb{P}[v_n < \infty | P_0 = p] U(\gamma n - \lambda p - \Psi - R)$$
$$= -\frac{n^\beta - p^\beta}{n^\beta} k(\lambda p + \Psi + R)^\alpha + \frac{p^\beta}{n^\beta}(\gamma n - \lambda p - \Psi - R)^\alpha \longrightarrow +\infty$$

as $n \to \infty$ if $\beta < \alpha$. $\qquad\square$

Mathematically speaking, the sequential optimal stopping problem (2.3) is ill posed under the parameter combination in Lemma 4.1, and its value function diverges to infinity. This arises when the performance of the asset is too good relative to the agent's risk-aversion level over gains. Equation (4.1) corresponds to a "buy and hold" trading rule as a possible optimal strategy: the agent purchases the asset immediately from the outset, and the profit-target level of sale can be set arbitrarily high.

Remark 4.2 Empirically, historical returns on equities are excessively high relative to their risk level. For example, the annualised mean and standard deviation of the equity premium (i.e., excess return above the risk-free rate) of the U.S. market over the time period 1889–1978 are 6.18% and 16.67%, respectively (see Mehra and Prescott [22]), so that $\beta = 1 - 2\mu/\sigma \approx -3.45$, while the estimates based on the more recent time period 1950–2015 are 7.15% and 16.83% (see Bai and Lu [1]) so that $\beta \approx -4.05$. Although this may cast doubt on the practical relevance of the condition $0 < \alpha \leq \beta$ (where buy and hold is an optimal strategy if the condition does not hold), we should like to emphasise that β in general is a noisy statistical quantity which is hard to forecast. In our model, β should be interpreted as the agent's subjective assessment of the asset performance which can be much more conservative than the historical estimates. The parameter μ may also encapsulate subjective discounting which further lowers its value; see Sect. 6.2.

From here onwards, we focus on the case $0 < \alpha \leq \beta$ which is a necessary condition for (2.3) to be well posed. The form of the solutions to the exit problem (3.2) and

entry problem (3.4) are first provided, and then we discuss the economic intuition behind the associated trading strategies. Towards the end of this section, the optimality of the value function of the entry problem is formally verified to show that it indeed corresponds to the solution of the sequential optimal stopping problem (2.3).

We first state the solution to the exit problem (3.2); a similar result can be found in Henderson [9].

Lemma 4.3 *For the exit problem* (3.2), *if* $0 < \alpha \leq \beta$, *the agent will sell the asset when its price level first reaches* $\frac{cH}{\gamma}$ *or above, where* $c > 1$ *is a constant given by the solution to the equation*

$$\frac{\alpha}{\beta} c(c-1)^{\alpha-1} - (c-1)^{\alpha} - k = 0. \tag{4.2}$$

The value function is given by

$$V_1(p; H) = \begin{cases} \frac{\alpha}{\beta} c^{1-\beta}(c-1)^{\alpha-1} H^{\alpha-\beta}(\gamma p)^{\beta} - kH^{\alpha} & \text{for } p < \frac{cH}{\gamma}, \\ (\gamma p - H)^{\alpha} & \text{for } p \geq \frac{cH}{\gamma}. \end{cases} \tag{4.3}$$

Proof Recall the notation introduced in Sect. 3.2.1. For $\beta > 0$, the scaled payoff function of the exit problem is given by

$$g_1(\theta) = g_1(\theta; H) = G_1\big(s^{-1}(\theta); H\big) = U(\gamma \theta^{\frac{1}{\beta}} - H)$$

$$= \begin{cases} -k(H - \gamma \theta^{\frac{1}{\beta}})^{\alpha} & \text{for } 0 \leq \theta < (\frac{H}{\gamma})^{\beta}, \\ (\gamma \theta^{\frac{1}{\beta}} - H)^{\alpha} & \text{for } \theta \geq (\frac{H}{\gamma})^{\beta}. \end{cases}$$

It is straightforward to work out the derivatives of g_1 as

$$g_1'(\theta) = \begin{cases} \frac{k\alpha\gamma}{\beta} \theta^{\frac{1}{\beta}-1}(H - \gamma \theta^{\frac{1}{\beta}})^{\alpha-1} & \text{for } 0 \leq \theta < (\frac{H}{\gamma})^{\beta}, \\ \frac{\alpha\gamma}{\beta} \theta^{\frac{1}{\beta}-1}(\gamma \theta^{\frac{1}{\beta}} - H)^{\alpha-1} & \text{for } \theta \geq (\frac{H}{\gamma})^{\beta}, \end{cases}$$

and

$$g_1''(\theta) = \begin{cases} \frac{k\alpha\gamma}{\beta} \theta^{\frac{1}{\beta}-2}(H - \gamma \theta^{\frac{1}{\beta}})^{\alpha-2}(\frac{\gamma(\beta-\alpha)}{\beta} \theta^{\frac{1}{\beta}} + \frac{1-\beta}{\beta} H) & \text{for } 0 \leq \theta < (\frac{H}{\gamma})^{\beta}, \\ \frac{\alpha\gamma}{\beta} \theta^{\frac{1}{\beta}-2}(\gamma \theta^{\frac{1}{\beta}} - H)^{\alpha-2}(\frac{\gamma(\alpha-\beta)}{\beta} \theta^{\frac{1}{\beta}} - \frac{1-\beta}{\beta} H) & \text{for } \theta \geq (\frac{H}{\gamma})^{\beta}. \end{cases}$$

Given the standing assumption $\beta \leq 1$ and the condition $0 < \alpha \leq \beta$, g_1 is increasing concave for $\theta > (\frac{H}{\gamma})^{\beta}$ and increasing convex for $0 \leq \theta < (\frac{H}{\gamma})^{\beta}$. The smallest concave majorant of g_1 can be formed by drawing a straight line from $(0, g_1(0))$ which touches g_1 at some $\theta^* > (\frac{H}{\gamma})^{\beta}$. In particular, θ^* is a solution to $\frac{g_1(\theta)-g_1(0)}{\theta} = g_1'(\theta)$ for $\theta > (\frac{H}{\gamma})^{\beta}$, i.e.,

$$\frac{\alpha\gamma}{\beta} \theta^{\frac{1}{\beta}-1}(\gamma \theta^{\frac{1}{\beta}} - H)^{\alpha-1} = \frac{(\gamma \theta^{\frac{1}{\beta}} - H)^{\alpha} + kH^{\alpha}}{\theta}.$$

We conjecture that the solution has the form $\theta^* = c^\beta (\frac{H}{\gamma})^\beta$ for some constant $c > 1$. Then direct substitution shows that the constant c should solve (4.2). The smallest concave majorant of g_1 is then

$$\bar{g}_1(\theta) = \begin{cases} g_1(0) + \theta g_1'(\theta^*) & \text{for } 0 \le \theta < \theta^*, \\ g_1(\theta) & \text{for } \theta > \theta^*, \end{cases}$$

$$= \begin{cases} -kH^\alpha + \frac{\alpha}{\beta} H^{\alpha-\beta} c^{1-\beta} (c-1)^{\alpha-1} \gamma^\beta \theta & \text{for } 0 \le \theta < c^\beta (\frac{H}{\gamma})^\beta, \\ (\gamma \theta^{\frac{1}{\beta}} - H)^\alpha & \text{for } \theta > c^\beta (\frac{H}{\gamma})^\beta. \end{cases}$$

The value function is given by $V_1(p; H) = \bar{g}_1(s(p)) = \bar{g}_1(p^\beta)$, leading to (4.3). The corresponding optimal stopping time is

$$\tau = \inf\{t \ge 0 : \bar{g}_1(\Theta_t) = g_1(\Theta_t)\}$$

$$= \inf\{t \ge 0 : \Theta_t \ge \theta^*\} = \inf\left\{t \ge 0 : P_t \ge c\left(\frac{H}{\gamma}\right)\right\}. \qquad \square$$

The optimal sale strategy is a gain-exit rule where the agent is looking to sell the asset when its price is sufficiently high without considering stop-loss. Note that the gain-exit target $\frac{cH}{\gamma}$ is increasing in transaction costs (i.e., decreasing in γ). This means that the agent tends to delay the sale decision in a more costly trading environment.

Remark 4.4 Inspired by the literature, we expect that more sophisticated exit strategies can be observed under alternative model setups. For example, the agent will consider stop-loss when the asset has negative drift (see Sect. 6.1 and Henderson [9]) or when there are multiple trading opportunities as in the realisation utility models (see Sect. 6.3.2 as well as Ingersoll and Jin [14] and He and Yang [8]). Introducing jump-diffusion price dynamics will further result in disconnected waiting regions where an agent who has recently suffered from a huge jump loss may refuse to stop-loss (Dai et al. [3] and Kong et al. [17]). We opt to work with a simpler baseline model to make the whole entry-and-exit problem as tractable as possible.

Remark 4.5 Since we require $\beta > 0$, if the initial price of the asset is below the gain-exit target, there is a strictly positive probability that the asset is never sold. Moreover, an agent who fails to sell the asset at his target gain-exit level will suffer a total loss in the long run.

Remark 4.6 The expression of the target gain-exit threshold and in turn the value function of the exit problem are available in closed form, thanks to the specialisation that the degree of risk-aversion over gains is the same as that of risk-seeking over losses. This allows us to make a lot of analytical progress when studying the entry problem. We also lose the closed-form expressions in Lemma 4.3 if a fixed transaction cost on sales is introduced; in this case, the agent will only sell the asset when the utility proceed $U(\gamma p - H - \Gamma)$ from the sale is larger than that of inaction $U(-H)$,

where $\Gamma \geq 0$ represents a fixed market exit fee. Then the payoff function of the exit problem becomes

$$G_1(p; H) := \max\{U(\gamma p - H - \Gamma), U(-H)\}.$$

We now proceed to describe the optimal entry strategy of the agent. The proofs of the two propositions below are given in the Appendix.

Proposition 4.7 *Suppose $\alpha \leq \beta < 1$. For the entry problem* (3.4)*, the value function is given by $V_2(p) = \bar{g}_2(p^\beta)$, where $\bar{g}_2 = \bar{g}_2(\theta)$ is the smallest concave majorant of*

$$g_2(\theta) := \max\{v_1(\theta), -kR^\alpha\} := \max\left\{(R + \Psi)^\alpha f\left(\left(\frac{\gamma}{R + \Psi}\right)^\beta \theta\right), -kR^\alpha\right\} \quad (4.4)$$

with

$$f(x) := \frac{\frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1}x - k(\frac{\lambda}{\gamma}x^{1/\beta} + 1)^\beta}{(\frac{\lambda}{\gamma}x^{1/\beta} + 1)^{\beta-\alpha}}. \quad (4.5)$$

There are three different cases:

(1) *If $\frac{\lambda}{\gamma} \leq (\frac{\alpha}{\beta k}c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$, there exists $p_1^* \in [0, \infty)$ such that the agent will enter the trade when the asset price is at or above p_1^*. The value function is*

$$V_2(p) = \begin{cases} \frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1}(\lambda p + \Psi + R)^{\alpha-\beta}(\gamma p)^\beta \\ \quad -k(\lambda p + \Psi + R)^\alpha & \text{for } p \geq p_1^*, \\ (\frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1}(\lambda p_1^* + \Psi + R)^{\alpha-\beta}(\gamma p_1^*)^\beta \\ \quad -k(\lambda p_1^* + \Psi + R)^\alpha + kR^\alpha)\frac{p^\beta}{(p_1^*)^\beta} - kR^\alpha & \text{for } p < p_1^*. \end{cases} \quad (4.6)$$

(2) *If $\frac{\lambda}{\gamma} > (\frac{\alpha}{\beta k}c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$, there exists a constant $C \in (0, \infty)$ independent of Ψ and R such that*

(a) *if $\Psi < CR$, there exist $0 \leq p_1^* < p_2^* < \infty$ such that the agent will enter the trade when the asset price is within the interval $[p_1^*, p_2^*]$. The value function is*

$$V_2(p) = \begin{cases} \frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1}(\lambda p_2^* + \Psi + R)^{\alpha-\beta}(\gamma p_2^*)^\beta \\ \quad -k(\lambda p_2^* + \Psi + R)^\alpha & \text{for } p > p_2^*, \\ \frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1}(\lambda p + \Psi + R)^{\alpha-\beta}(\gamma p)^\beta \\ \quad -k(\lambda p + \Psi + R)^\alpha & \text{for } p_1^* \leq p \leq p_2^*, \\ (\frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1}(\lambda p_1^* + \Psi + R)^{\alpha-\beta}(\gamma p_1^*)^\beta \\ \quad -k(\lambda p_1^* + \Psi + R)^\alpha + kR^\alpha)\frac{p^\beta}{(p_1^*)^\beta} - kR^\alpha & \text{for } p < p_1^*. \end{cases} \quad (4.7)$$

(b) *if $\Psi \geq CR$, the agent will never enter the trade. The value function is $V_2(p) = -kR^\alpha$. Moreover, in cases* (1) *and* (2)(a)*, $p_1^* = \frac{R+\Psi}{\gamma}(x_1^*)^{1/\beta}$, where x_1^* is*

the smaller solution to the equation

$$k = \left(1 + \frac{\Psi}{R}\right)^{\alpha}$$

$$\times \frac{k(\frac{\lambda}{\gamma}x^{\frac{1}{\beta}} + 1)^{\beta}(\frac{\lambda}{\gamma}(1 - \frac{\alpha}{\beta})x^{\frac{1}{\beta}} + 1) - \frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1}\frac{\lambda}{\gamma}(1 - \frac{\alpha}{\beta})x^{\frac{1}{\beta}+1}}{(\frac{\lambda}{\gamma}x^{\frac{1}{\beta}} + 1)^{\beta-\alpha+1}}. \quad (4.8)$$

In case (2)(a), $p_2^* = \frac{R+\Psi}{\gamma}(x_2^*)^{1/\beta}$, where x_2^* is the unique solution to the equation

$$c^{1-\beta}(c-1)^{\alpha-1}\left(x^{-\frac{1}{\beta}} + \frac{\lambda\alpha}{\gamma\beta}\right) - \frac{k\lambda}{\gamma}\left(x^{-\frac{1}{\beta}} + \frac{\lambda}{\gamma}\right)^{\beta} = 0. \quad (4.9)$$

In the special case $\beta = 1$ where the asset has zero drift, the results are slightly different from those in Proposition 4.7.

Proposition 4.8 *Suppose $\alpha < \beta = 1$. For the entry problem (3.4), we have*:

(1) *If $\frac{\lambda}{\gamma} \leq \frac{\alpha}{k}(c-1)^{\alpha-1}$, there exists $p_1^* \in [0, \infty)$ such that the agent will enter the trade when the asset price is at or above p_1^*. The value function is given by (4.6) on setting $\beta = 1$.*

(2) *If $\frac{\alpha}{k}(c-1)^{\alpha-1} < \frac{\lambda}{\gamma} < \frac{(c-1)^{\alpha-1}}{k}$, there exists a constant $C \in (0, \infty)$ independent of Ψ and R such that*

(a) *if $\Psi < CR$, there exist $0 \leq p_1^* < p_2^* < \infty$ such that the agent will enter the trade when the asset price is within the interval $[p_1^*, p_2^*]$. The value function is given by (4.7) on setting $\beta = 1$.*

(b) *if $\Psi \geq CR$, the agent will never enter the trade. The value function is $V_2(p) = -kR^{\alpha}$.*

(3) *If $\frac{\lambda}{\gamma} \geq \frac{(c-1)^{\alpha-1}}{k}$, the agent will never enter the trade. The value function is $V_2(p) = -kR^{\alpha}$.*

Moreover, p_1^* has the same form as in Proposition 4.7 on setting $\beta = 1$, and the expression of p_2^* is available in closed form as
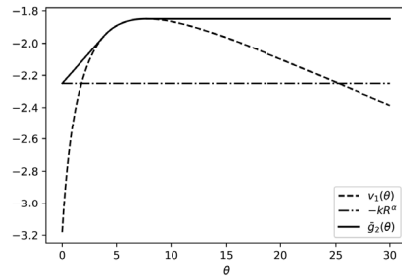
$$p_2^* = \frac{(R + \Psi)((c-1)^{\alpha-1} - k\lambda/\gamma)}{\lambda(k\lambda/\gamma - \alpha(c-1)^{\alpha-1})}.$$

Remark 4.9 For certain types of agents such as retail investors, fixed transaction costs are typically insignificant so that their trading behaviour might be better described by the specialisation $\Psi = 0$. In this case, Propositions 4.7 and 4.8 can be simplified; see Corollary 4.11 and the discussion thereafter.
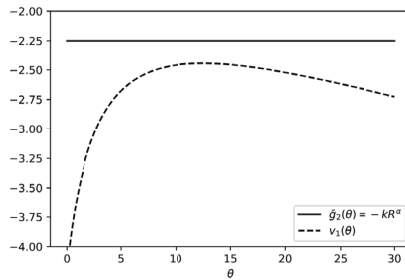
The value function of the entry problem is characterised as the smallest concave majorant of the payoff function in (4.4). Indeed, v_1 defined in (4.4) is simply the scaled value function of the exit problem so that $v_1(\theta) = V_1(\theta^{1/\beta}; \lambda\theta^{1/\beta} + \Psi + R)$,
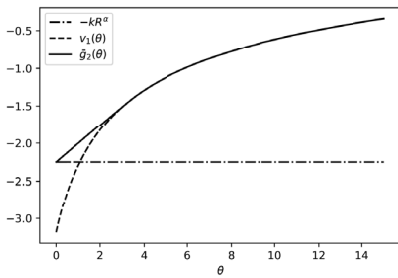
(a) Case (1) with $\lambda = 1.01$, $\gamma = 0.99$, $\Psi = 1$.   (b) Case (2)(a) with $\lambda = 1.1$, $\gamma = 0.9$, $\Psi = 1$.

(c) Case (2)(b) with $\lambda = 1.1$, $\gamma = 0.9$, $\Psi = 2.5$.

Fig. 1 The scaled value functions of the entry problem for the different cases of Proposition 4.7. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $R = 1$, $\beta = 0.85$

as discussed in Sect. 3.2.2. At a mathematical level, the various cases arising in Propositions 4.7 and 4.8 are due to the different possible shapes of v_1 under different combinations of parameters. Some illustrations are given in Fig. 1.

Economically, the optimal entry strategy crucially depends on the level of transaction costs relative to the market and preference parameters. A fixed market entry fee in general discourages trading when the asset price is low. Paying a flat fee of $10 to purchase an asset at $20 is much less appealing compared to the case that the asset is trading at $1000, because in the former case the asset has to increase in value by 50% just to break even against the fixed transaction fee paid. Meanwhile, proportional transaction costs are the most significant for an asset trading at a high nominal price. A 10% transaction fee charged on a million worth of property is much more expensive in monetary terms relative to the same percentage fee charged on a penny stock.

In case (1) of both Propositions 4.7 and 4.8, the proportional transaction costs are relatively low. Hence the agent does not mind purchasing the asset at a high nominal price. He will just avoid purchasing the asset when its price is very low due to the consideration of fixed transaction costs, and therefore the purchase region has the form $[p_1^*, \infty)$.

In case (2)(a), proportional transaction costs start becoming significant. On the one hand, the agent avoids initiating the trade when the asset price is too low since the fixed entry cost will be too large relative to the size of the trade. On the other hand,

the agent does not want to trade an expensive asset when the proportional costs are large. Upon balancing these two factors, the agent will wait when the asset price is either too low or too high, and will only purchase the asset when the price first enters an interval $[p_1^*, p_2^*]$. A very interesting feature of the optimal entry strategy is that the waiting region here is disconnected.

In case (2)(b) of Proposition 4.7, or cases (2)(b) and (3) of Proposition 4.8, the overall level of transaction costs is too high and hence the agent is discouraged from entering the trade in the first place. The key difference between Propositions 4.7 and 4.8 is that when the asset has a strictly positive drift ($\beta < 1$, i.e., $\mu > 0$), one must impose a strictly positive fixed entry cost in order to stop the agent from trading at all price levels (if $\Psi = 0$, then either case (1) or (2)(a) in Proposition 4.7 applies, in which case the agent is willing to enter the trade at a certain price level). When the asset is a statistically fair gamble ($\beta = 1$, i.e., $\mu = 0$), a high proportional transaction cost alone is sufficient to discourage the agent from trading. It is interesting to note that the trading decision also depends on the agent's aspiration level R. Comparing cases (2)(a) and (2)(b) in Propositions 4.7 and 4.8, a low value of R will more often lead to the "no trading" case. The economic interpretation is that an agent with low aspiration level (i.e., a low target benchmark) is less likely to participate in trading, especially when the (proportional) costs of trading are high. In Sect. 6.4, we briefly discuss how the aspiration level R may be endogenised.

When viewed in conjunction with the results of wellposedness (Lemma 4.1) and the optimal exit strategy (Lemma 4.3), our model can encapsulate many styles of trading behaviour. First, if $\beta \leq 0$ or $\beta < \alpha < 1$ so that the problem is ill posed, we have already shown that a simple "buy and hold" strategy of the form (4.1) is optimal in the sense that the attained utility level can be arbitrarily high.

In cases (1) or (2)(a) of Propositions 4.7 and 4.8, if the asset price starts below p_1^* at time zero, the agent will purchase the asset when its price level increases to p_1^*. Note that similarly to Remark 4.5, the price process P may fail to reach a fixed level $p_1^* > P_0$ in finite time. In this case, the entry strategy will not be executed and the payoff to the agent is zero. But otherwise, if a purchase is realised, then at the time of purchase, the reference point is set as $H = \lambda p_1^* + \Psi + R$. Then due to Lemma 4.3, the agent is looking to sell this asset later when its price level further increases to $\frac{cH}{\gamma} = \frac{c(\lambda p_1^* + \Psi + R)}{\gamma}$. Since $c > 1$, $\gamma \leq 1$ and $\lambda \geq 1$, it is clear that the target sale level $\frac{c(\lambda p_1^* + \Psi + R)}{\gamma}$ is strictly larger than p_1^*. This trading rule is thus a momentum strategy of the form "buy high, sell higher".

If the asset price starts above p_2^* at time zero in case (2)(a), the agent will buy the asset when its price level drops to p_2^* and later sell when the price increases to $\frac{c(\lambda p_2^* + \Psi + R)}{\gamma} > p_2^*$. This is a counter-trend trading strategy of the form "buy low, sell high".

Finally, in the high transaction cost cases (2)(b) of Proposition 4.7 and (2)(b) or (3) of Proposition 4.8, the agent will never participate in trading at any asset price level.

The various cases above are generated by different levels of transaction costs relative to the other model parameters. The following two corollaries further highlight the role of transaction costs in relationship to the optimal trading strategies.

Corollary 4.10 *If $\lambda = \gamma = 1$, the agent will purchase the asset when its price level is at or above p_1^* for some $p_1^* \in [0, \infty)$.*

Proof The result follows if we can show that $(\frac{\alpha}{\beta k} c^{1-\beta} (c-1)^{\alpha-1})^{\frac{1}{\beta}} > 1$ so that case (1) of Propositions 4.7 and 4.8 always applies when $\lambda = \gamma = 1$. Rewriting the required inequality in equivalent ways gives

$$
\left(\frac{\alpha}{\beta k} c^{1-\beta} (c-1)^{\alpha-1} \right)^{\frac{1}{\beta}} > 1 \iff \frac{\alpha}{\beta} c^{1-\beta} (c-1)^{\alpha-1} > k
$$

$$
\iff \frac{\alpha}{\beta} c^{1-\beta} (c-1)^{\alpha-1} > \frac{\alpha}{\beta} c(c-1)^{\alpha-1} - (c-1)^{\alpha}
$$

$$
\iff (c-1)^{\alpha} - \frac{\alpha}{\beta} c(c-1)^{\alpha-1} (1 - c^{-\beta}) > 0,
$$

where we have used (4.2). Using simple calculus, we can show that

$$
F(x) := (x-1)^{\alpha} - \frac{\alpha}{\beta} x(x-1)^{\alpha-1} (1 - x^{-\beta}) > 0
$$

for all $x > 1$. This concludes the proof. $\qquad\square$

Corollary 4.11 *If the model parameters satisfy the conditions in cases (1) or (2)(a) in Propositions 4.7 and 4.8 and if $\Psi = 0$, then we have $p_1^* = 0$.*

Proof This follows immediately from Propositions 4.7 and 4.8 by observing that $x = 0$ is the solution to (4.8) when $\Psi = 0$. $\qquad\square$

From Corollary 4.10, if there is no proportional transaction cost, then the agent does not care about entering the trade at a high nominal price level because he no longer needs to worry about the large magnitude of the trading fee arising from the proportional nature of the transaction costs. Hence "buy low, sell high" will not be observed as an optimal strategy. Similarly, Corollary 4.11 suggests that in the absence of a fixed market entry fee, the agent is happy to purchase an asset at any arbitrarily low price (in the non-degenerate case) since now he does not need to take the size of the trade into account against any fixed cost for break-even considerations. Thus "buy high, sell higher" will not be an optimal strategy in this special case.

Remark 4.12 If we further assume $R = \Psi = 0$ (where the main results in Propositions 4.7 and 4.8 still hold except that (4.4) will have a different and simpler expression), then only case (1) or case (2)(b) of Proposition 4.7 can arise for $\beta < 1$. Alternatively, only cases (1), (2)(b) or (3) of Proposition 4.8 can arise for $\beta = 1$. But $p_1^* = 0$ when $\Psi = 0$ as in Corollary 4.11. The entry behaviour is thus trivial—the agent either is willing to purchase the asset at any price level if $\frac{\lambda}{\gamma} \leq (\frac{\alpha}{\beta k} c^{1-\beta} (c-1)^{\alpha-1})^{\frac{1}{\beta}}$, or never enters the trade if $\frac{\lambda}{\gamma} > (\frac{\alpha}{\beta k} c^{1-\beta} (c-1)^{\alpha-1})^{\frac{1}{\beta}}$. From the modelling perspective, the constant $R + \Psi$ is a crucial component which enables the model to produce non-trivial purchase behaviour; see Sect. 6.3.2 as well.

The decomposition of the original purchase-and-sale problem (2.3) into two sub-problems of optimal exit and optimal entry is based on some economic heuristics described in Sects. 3.2.1 and 3.2.2. To close this section, we formally show that the value function of the entry problem in Propositions 4.7 and 4.8 indeed corresponds to the value function of the sequential optimal stopping problem (2.3).

Theorem 4.13 *Recall the definition of V_2 in Propositions 4.7 and 4.8 as the value function of the entry problem (3.4). We have $\mathcal{V}(p) = V_2(p)$, where $\mathcal{V}$ is the value function of the sequential optimal stopping problem (2.3). The optimal purchase and sale rules are given by*

$$\tau^* := \inf\{t \geq 0 : V_2(P_t) = G_2(P_t)\},$$

$$\nu^* := \inf\left\{t \geq \tau^* : P_t \geq \frac{c}{\gamma}(\lambda P_{\tau^*} + \Psi + R)\right\}. \tag{4.10}$$

Proof Starting from (2.2), we have for any $\tau, \nu \in \mathcal{T}$ with $\tau \leq \nu$ that

$$J(p; \tau, \nu)$$
$$= \mathbb{E}\big[U\big(\gamma P_\nu \mathbb{1}_{\{\tau < \infty, \nu < \infty\}} - (\lambda P_\tau + \Psi)\mathbb{1}_{\{\tau < \infty\}} - R\big)\big|P_0 = p\big]$$
$$\leq \mathbb{E}\big[U\big((\gamma P_\nu - \lambda P_\tau - \Psi)\mathbb{1}_{\{\tau < \infty\}} - R\big)\big|P_0 = p\big]$$
$$= \mathbb{P}[\tau < \infty|P_0 = p]\mathbb{E}[U(\gamma P_\nu - \lambda P_\tau - \Psi - R)|P_0 = p, \tau < \infty]$$
$$\quad + \mathbb{P}[\tau = \infty|P_0 = p]U(-R)$$
$$= \mathbb{E}\big[\mathbb{E}[U(\gamma P_\nu - \lambda P_\tau - \Psi - R)|P_\tau, \tau < \infty]\big|P_0 = p, \tau < \infty\big]\mathbb{P}[\tau < \infty|P_0 = p]$$
$$\quad + \mathbb{P}[\tau = \infty|P_0 = p]U(-R)$$
$$= \mathbb{E}\big[\mathbb{E}[G_1(\gamma P_\nu; \lambda P_\tau + \Psi + R)|P_\tau, \tau < \infty]\big|P_0 = p, \tau < \infty\big]\mathbb{P}[\tau < \infty|P_0 = p]$$
$$\quad + \mathbb{P}[\tau = \infty|P_0 = p]U(-R)$$
$$\leq \mathbb{E}\bigg[\sup_{\nu \in \mathcal{T}:\nu \geq \tau} \mathbb{E}[G_1(\gamma P_\nu; \lambda P_\tau + \Psi + R)|P_\tau, \tau < \infty]\bigg|P_0 = p, \tau < \infty\bigg]$$
$$\quad \times \mathbb{P}[\tau < \infty|P_0 = p] + \mathbb{P}[\tau = \infty|P_0 = p]U(-R).$$

But using the Markovian structure of the problem,

$$\sup_{\nu \in \mathcal{T}:\nu \geq \tau} \mathbb{E}[G_1(\gamma P_\nu; \lambda P_\tau + \Psi + R)|P_\tau = s, \tau < \infty]$$
$$= \sup_{\nu \in \mathcal{T}:\nu \geq 0} \mathbb{E}[G_1(\gamma P_\nu; \lambda P_0 + \Psi + R)|P_0 = s]$$
$$= V_1(s; \lambda s + \Psi + R).$$

Then we have

$$
\begin{aligned}
J(p; \tau, \nu) &\leq \mathbb{E}[V_1(P_\tau; \lambda P_\tau + \Psi + R)|P_0 = p, \tau < \infty]\,\mathbb{P}[\tau < \infty|P_0 = p] \\
&\quad + \mathbb{P}[\tau = \infty|P_0 = p]U(-R) \\
&\leq \mathbb{E}[\max\{V_1(P_\tau; \lambda P_\tau + \Psi + R), U(-R)\}|P_0 = p] \\
&\leq \sup_{\tau \in \mathcal{T}} \mathbb{E}[\max\{V_1(P_\tau; \lambda P_\tau + \Psi + R), U(-R)\}|P_0 = p] \\
&= \sup_{\tau \in \mathcal{T}} \mathbb{E}[G_2(P_\tau)|P_0 = p] \\
&= V_2(p).
\end{aligned}
$$

Taking the supremum on both sides leads to $\mathcal{V}(p) \leq V_2(p)$.

To complete the proof, it is sufficient to show $J(p; \tau^*, \nu^*) = V_2(p)$, where τ^*, ν^* are defined in (4.10). This can be directly verified for the various cases covered in Propositions 4.7 and 4.8 with different initial price level p.

As an example, we cover case (2)(a) in Proposition 4.7 and a level $p < p_1^*$. We can deduce from the shape of V_2 in this case that $\tau^* = \inf\{t \geq 0 : P_t \notin (0, p_1^*)\}$. Since $\beta > 0$, there are two possibilities: the asset price reaches the purchase target level p_1^* in finite time, which happens with probability $\mathbb{P}[\tau^* < \infty|P_0 = p] = \frac{p^\beta}{(p_1^*)^\beta}$, or it never reaches p_1^* in finite time so that the agent never enters a trade and faces a realised utility of $U(-R)$, which happens with probability $\mathbb{P}[\tau^* = \infty|P_0 = p] = 1 - \frac{p^\beta}{(p_1^*)^\beta}$. In the first scenario, after the agent purchases the asset at price p_1^*, he will sell the asset when its price further increases to $x^* := \frac{c}{\gamma}(\lambda p_1^* + \Psi + R)$. The conditional probability of a successful sale is $\mathbb{P}[\nu^* < \infty|P_{\tau^*} = p_1^*] = \frac{(p_1^*)^\beta}{(x^*)^\beta}$, where the realised utility is $U(\gamma x^* - \lambda p_1^* - \Psi - R)$. Otherwise, the target sale level is never reached with conditional probability $\mathbb{P}[\nu^* = \infty|P_{\tau^*} = p_1^*] = 1 - \frac{(p_1^*)^\beta}{(x^*)^\beta}$, and the realised utility becomes $U(-\lambda p_1^* - \Psi - R)$. Then we can directly compute

$$
\begin{aligned}
&J(p; \tau^*, \nu^*) \\
&= \mathbb{E}\big[U\big(\gamma P_{\nu^*}\mathbb{1}_{\{\tau^* < \infty, \nu^* < \infty\}} - (\lambda P_{\tau^*} + \Psi)\mathbb{1}_{\{\tau^* < \infty\}} - R\big)\big|P_0 = p\big] \\
&= \mathbb{P}[\tau^* = \infty|P_0 = p]U(-R) \\
&\quad + \mathbb{P}[\tau^* < \infty, \nu^* = \infty|P_0 = p]U(-\lambda p_1^* - \Psi - R) \\
&\quad + \mathbb{P}[\tau^* < \infty, \nu^* < \infty|P_0 = p]U(\gamma x^* - \lambda p_1^* - \Psi - R) \\
&= -\frac{(p_1^*)^\beta - p^\beta}{(p_1^*)^\beta}kR^\alpha - k\frac{p^\beta}{(p_1^*)^\beta}\frac{(x^*)^\beta - (p_1^*)^\beta}{(x^*)^\beta}(\lambda p_1^* + \Psi + R)^\alpha \\
&\quad + \frac{p^\beta}{(p_1^*)^\beta}\frac{(p_1^*)^\beta}{(x^*)^\beta}(\gamma x^* - \lambda p_1^* - \Psi - R)^\alpha
\end{aligned}
$$

$$= \frac{p^\beta}{(p_1^*)^\beta} \left(\frac{(p_1^*)^\beta}{(x^*)^\beta} (\gamma x^* - \lambda p_1^* - \Psi - R)^\alpha - k \frac{(x^*)^\beta - (p_1^*)^\beta}{(x^*)^\beta} (\lambda p_1^* + \Psi + R)^\alpha \right)$$

$$- \frac{(p_1^*)^\beta - p^\beta}{(p_1^*)^\beta} k R^\alpha$$

$$= \frac{p^\beta}{(p_1^*)^\beta} \left((\gamma p_1^*)^\beta c^{-\beta} ((c-1)^\alpha + k)(\lambda p_1^* + \Psi + R)^{\alpha-\beta} - k(\lambda p_1^* + \Psi + R)^\alpha \right)$$

$$- \frac{(p_1^*)^\beta - p^\beta}{(p_1^*)^\beta} k R^\alpha$$

$$= \frac{p^\beta}{(p_1^*)^\beta} \left((\gamma p_1^*)^\beta \frac{\alpha}{\beta} c^{1-\beta} (c-1)^{\alpha-1} (\lambda p_1^* + \Psi + R)^{\alpha-\beta} - k(\lambda p_1^* + \Psi + R)^\alpha \right)$$

$$- \frac{(p_1^*)^\beta - p^\beta}{(p_1^*)^\beta} k R^\alpha$$

$$= \frac{p^\beta}{(p_1^*)^\beta} \left((\gamma p_1^*)^\beta \frac{\alpha}{\beta} c^{1-\beta} (c-1)^{\alpha-1} (\lambda p_1^* + \Psi + R)^{\alpha-\beta} \right.$$

$$\left. - k(\lambda p_1^* + \Psi + R)^\alpha + k R^\alpha \right) - k R^\alpha$$

$$= V_2(p),$$

where we have used the definition $x^* := \frac{c}{\gamma} (\lambda p_1^* + \Psi + R)$ and the fact that c is the solution to (4.2). The other cases can be handled similarly. $\qquad \square$

5 Comparative statics of the optimal trading strategies

The critical trading boundaries in Propositions 4.7 and 4.8, although not available in closed form in general, can be analysed to shed light on the comparative statics of the optimal trading strategies with respect to a few underlying model parameters. The proof of the following proposition is given in the Appendix.

Proposition 5.1 *Away from case* (2)(b) *in Proposition* 4.7 *and cases* (2)(b) *and* (3) *in Proposition* 4.8, *we have that*

1) p_1^* *is decreasing in γ and increasing in Ψ;*
2) p_2^* *is decreasing in λ, increasing in γ and increasing in Ψ.*

Figure 2(b) shows the critical purchase boundaries p_1^* and p_2^* as γ varies. For very large values of γ such that the condition in case (1) of Proposition 4.7 holds, the optimal strategy is to buy the asset when its price exceeds p_1^*, and the agent is willing to enter the trade no matter how high the price is. Once γ is smaller than a certain critical value (labelled by the vertical dotted line in the figure), the parameter condition in case (2)(a) of Proposition 4.7 applies. The optimal strategy now becomes
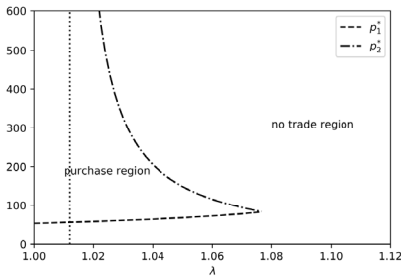
(a) Comparative statics with respect to λ. The dotted vertical line marks the critical value of λ above which p_2^* exists.
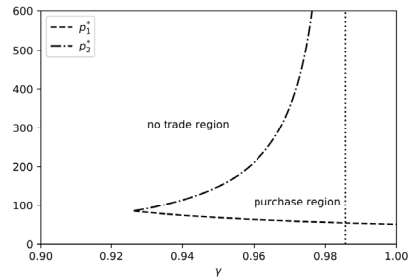
(b) Comparative statics with respect to γ. The dotted vertical line marks the critical value of γ below which p_2^* exists.
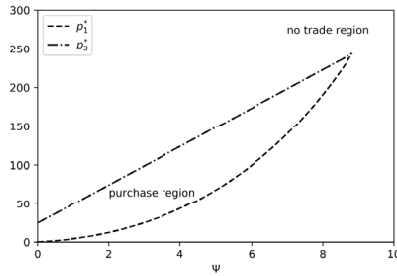
(c) Comparative statics with respect to Ψ.

Fig. 2 Comparative statics of the optimal purchase boundaries p_1^* and p_2^*. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $R = 1$, $\beta = 0.85$, $\lambda = 1.05$, $\gamma = 0.95$, $\Psi = 5$

Fig. 3 An illustration that p_1^*, the lower bound of the purchase region, need not be monotonic with respect to λ. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $\beta = 0.85$, $\lambda = 1.05$, $\gamma = 0.95$, $\Psi = 1$
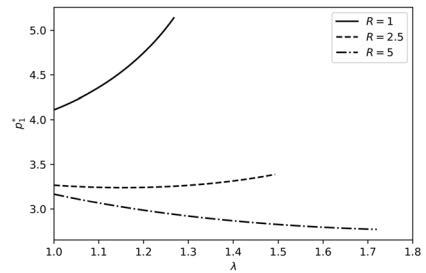
to purchase the asset only when its price is within a bounded range $[p_1^*, p_2^*]$. As γ further decreases, p_1^* increases while p_2^* decreases so that the purchase region $[p_1^*, p_2^*]$ shrinks. Once γ reaches another critical value, p_1^* and p_2^* converge and the purchase region vanishes entirely. This corresponds to case (2)(b) of Proposition 4.7 that the agent will not enter the trade at any price level. As a reminder, the constant C in case (2) of Propositions 4.7 and 4.8 depends on λ and γ. Increasing $\frac{\lambda}{\gamma}$ will result in a switch from case (2)(a) to case (2)(b).

We do not mention in Proposition 5.1 the effect of λ on p_1^*. Indeed, p_1^* is not monotonic in λ in general; see Fig. 3. Hence when viewed in conjunction with p_2^*, the

purchase region $[p_1^*, p_2^*]$ does not necessarily shrink uniformly when the proportional cost on purchases increases, i.e., the agent need not delay the purchase decision. Similar observations regarding potential non-monotonicity of trading decisions with respect to (proportional) transaction costs are made by Hobson et al. [12, 13] in the context of portfolio optimisation.

Similarly, we can also examine the impact of the fixed market entry cost on the purchase decision. As shown in Fig. 2(c), p_1^* and p_2^* are both increasing in Ψ. The fact that p_2^* is increasing in Ψ is indeed somewhat counter-intuitive and has a few interesting policy implications. Suppose there is a market regulator who wants to discourage the agent from purchasing the asset (for example, as a means to cool down a highly speculative real estate market). A natural measure to curb trading participation is to increase transaction costs. However, Fig. 2 reveals that there is a subtle difference between the impact of proportional and fixed transaction costs on the agent's trading behaviour.

From Fig. 2(b), the effect of increasing proportional transaction costs on sales (i.e., decreasing γ) is "monotonic" in terms of changing the trading decision of the agent. At any given current asset price level, decreasing γ (while all other parameters are held fixed) can only take the price from the purchase region to the no-trade region. Increasing proportional transaction costs on sales can therefore unambiguously suppress the trading activities in the market.

In contrast, the impact of the fixed market entry cost is somewhat unclear. Take Fig. 2(c) as an example and suppose the current price of the asset is \$100. If there is no fixed market entry fee initially (i.e., $\Psi = 0$), the agent will not participate in trading as the price is in the no-trade region. However, increasing Ψ from zero to \$4 will now put the price in the purchase region so that the agent is willing to purchase the asset immediately (given that the current asset price stays the same at \$100). This is exactly opposite to the intended outcome of the market regulator, because the increase in Ψ actually encourages trading participation.

The rationale behind this phenomenon is as follows. The speculative agent is evaluating the trading opportunity by comparing the potential sale proceeds against the reference point which is partly determined by the initial capital $\lambda p + \Psi$ required to enter the trade if the purchase price is p. Increasing the fixed market entry fee Ψ increases the total costs required to initiate the trade and results in a higher effective reference point. However, under a prospect theory framework, the agent's risk attitude is tied to the level of the reference point. When the purchase price is kept the same, a large Ψ will put the agent deeper in the loss territory over which he becomes highly risk-seeking. Thus he will give a higher valuation to the opportunity to enter the speculative trade and hence is more prone to immediate trade participation.

Of course, increasing Ψ will also decrease the potential profit of the trade, which is economically unfavourable. As the fixed cost further increases, say from $\Psi = 4$ to $\Psi = 8$, the price will eventually enter the no-trade region again. Hence the precise effect of Ψ on the trading decision is ambiguous and governed by the two opposing forces of increasing the agent's risk appetite versus decreasing profitability. When the economy is consisting of multiple agents with heterogeneous preferences, it is unclear whether increasing the fixed transaction costs can uniformly discourage trading participation for all agents.

The non-monotonicity of p_1^* with respect to λ, the proportional transaction cost on purchases, also implies that an increase in λ can potentially bring certain prices from the no-trade region to the purchase region. The rationale is the same as above that λ partly determines the cost of purchase and in turn the reference point. Hence increasing λ might actually make an agent find a speculative opportunity attractive.

6 Extensions

In this section, we briefly discuss several variations of our baseline model.

6.1 Risky asset with negative drift and voluntary stop-loss

Among all the non-trivial strategies derived in our baseline setup, the agent will never voluntarily realise a loss. This is not consistent with real world trading behaviour given the prevalent usage among market participants of stop-loss orders. One way to enable the model to generate stop-loss behaviour is to allow the excess return of the asset to be negative as inspired by Henderson [9]. A negative excess return is equivalent to $\beta := 1 - \frac{2\mu}{\sigma^2} > 1$ in our setup.

Lemma 6.1 *Suppose the model parameters are such that $\beta > 1$. For the exit problem* (3.2), *the agent will sell the asset when its price level first exits the interval* $\left(\frac{c_1 H}{\gamma}, \frac{c_2 H}{\gamma}\right)$, *where $0 < c_1 < 1 < c_2$ are the solutions to the system of equations*

$$\frac{k\alpha}{\beta}c_1^{1-\beta}(1-c_1)^{1-\alpha} = \frac{\alpha}{\beta}c_2^{1-\beta}(c_2-1)^{\alpha-1} = \frac{(c_2-1)^\alpha + k(1-c_1)^\alpha}{c_2^\beta - c_1^\beta}.$$

The value function is

$$V_1(p; H) = \begin{cases} -k(H - \gamma p)^\alpha & \text{for } p < \frac{c_1 H}{\gamma}, \\ -kH^\alpha(1-c_1)^\alpha \\ \quad + \frac{H^{\alpha-\beta}((c_2-1)^\alpha + k(1-c_1)^\alpha)((\gamma p)^\beta - c_1^\beta H^\beta)}{c_2^\beta - c_1^\beta} & \text{for } \frac{c_1 H}{\gamma} \leq p \leq \frac{c_2 H}{\gamma}, \\ (\gamma p - H)^\alpha & \text{for } p > \frac{c_2 H}{\gamma}. \end{cases} \quad (6.1)$$

Proof This follows from a slight extension of [9, Proposition 3]. □

Given that a purchase of the asset has occurred at some time τ which determines the reference level for the exit problem via $H = \lambda P_\tau + \Psi + R$, the agent is willing to impose a stop-loss level at $\frac{c_1 H}{\gamma}$ if the asset has a negative drift. Of course, it is not clear upfront whether the agent is willing to purchase an asset with negative drift in the first place. To understand the purchase behaviour, one needs to solve the entry problem

$$V_2(p) := \sup_{\tau \in \mathcal{T}} \mathbb{E}[\max\{V_1(P_\tau; \lambda P_\tau + \Psi + R), U(-R)\}|P_0 = p],$$
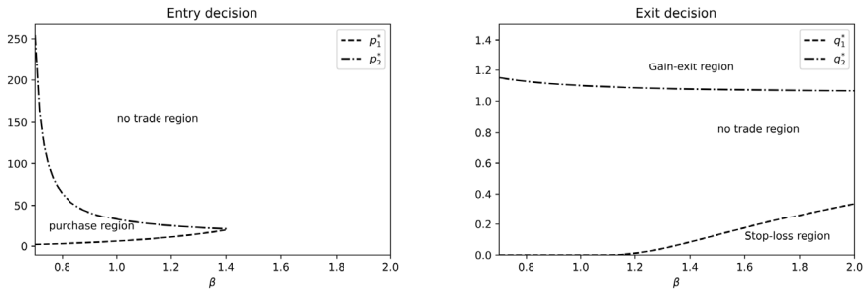
(a) Optimal purchase (entry) boundaries as a function of β.

(b) Optimal sale (exit) boundaries as a function of β.

Fig. 4 Optimal purchase and sale decisions as β varies; $\beta < 1$ (resp. $\beta > 1$) indicates that the asset has a positive (resp. negative) drift. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $R = 1$, $\lambda = 1.05$, $\gamma = 0.95$, $\Psi = 1$

where V_1 is now given by (6.1). Although the principle of the martingale method still applies, we have so far not managed to analyse the problem thoroughly to explicitly characterise different possible shapes of the scaled payoff function under different combinations of model parameters. We hence opt to obtain numerical solutions by solving the underlying variational inequality

$$\min\left\{-\frac{\sigma^2 p^2}{2}V_2''(p) - \mu p V_2'(p),\, V_2(p) - G_2(p)\right\} = 0$$

by the standard projected successive over-relaxation (PSOR) method and infer the optimal purchase (stopping) and no-trade (continuation) regions by numerically identifying the sets $\{p \geq 0 : V_2(p) = G_2(p)\}$ and $\{p \geq 0 : V_2(p) > G_2(p)\}$, respectively.

Figure 4(a) shows the optimal entry decisions as β varies. When $\beta \leq 1$, the optimal purchase boundaries p_1^* and p_2^* are obtained semi-analytically from Propositions 4.7 and 4.8, while the values for $\beta > 1$ are obtained numerically by the PSOR method. The agent is willing to purchase the asset if and only if the current price level is between p_1^* and p_2^*. As the drift of the asset becomes more and more negative (i.e., β increases), the purchase region $[p_1^*, p_2^*]$ shrinks and eventually vanishes when β is around 1.4. Beyond this critical level, the agent will not purchase the asset at any price level because of its poor quality.

In parallel, Fig. 4(b) plots the optimal sale boundaries in the form $q_i^* := \frac{c_i}{\gamma}$ so that if the asset has been purchased at level p, it will be sold when its price level leaves the interval (Hq_1^*, Hq_2^*), where $H = \lambda p + \Psi + R$. If $\beta \leq 1$, it is never optimal to voluntarily realise a loss, which is equivalent to $q_1^* = 0$. But for $\beta > 1$, q_1^* becomes strictly positive which represents a stop-loss level. Figure 4 demonstrates that there exist some combinations of model parameters such that the agent is willing to purchase the asset at some price level and subsequently liquidate it at a loss. This happens when β is between 1 and 1.4 in our numerical example. Explicitly characterising conditions on the model parameters where this phenomenon occurs is an interesting research question.
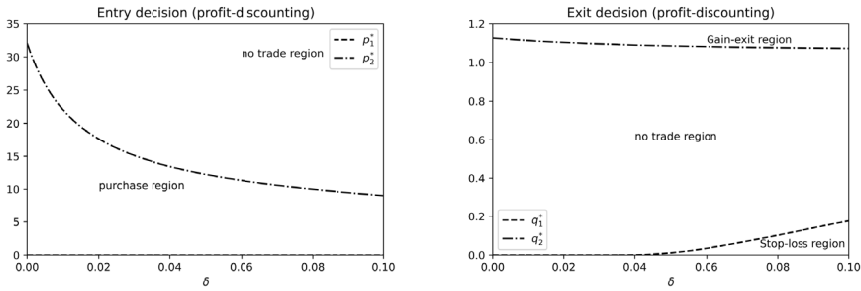
(a) Optimal purchase (entry) boundaries as a function of δ.

(b) Optimal sale (exit) boundaries as a function of δ.

Fig. 5 Optimal purchase and sale decisions as δ varies under the profit-discounting criterion. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $R = 1$, $\lambda = 1.05$, $\gamma = 0.95$, $\Psi = 0$, $\mu = 0.025$, $\sigma = 0.5$

6.2 Subjective discounting

To investigate the effect of impatience on the optimal strategy, it is constructive to consider a version of the problem with discounting. However, to the best of our knowledge, there is no consensus in the literature how discounting should be incorporated within a prospect theory framework with intertemporal cashflows. We briefly present two possible modelling choices.

6.2.1 Profit-discounting

The first idea is that the S-shaped utility function is applied to the net present value of the trading proceed, which we term as "*profit-discounting*". We consider the problem

$$\sup_{\tau, \nu \in \mathcal{T}: \tau \leq \nu} \mathbb{E}\big[U\big(\gamma e^{-\delta \nu} P_\nu \mathbb{1}_{\{\tau < \infty, \nu < \infty\}} - e^{-\delta \tau}(\lambda P_\tau + \Psi)\mathbb{1}_{\{\tau < \infty\}} - R\big)\big| P_0 = p\big],$$

where $\delta > 0$ is the subjective discount factor. If we further assume $\Psi = 0$ (which is perhaps relevant in the context of retail trading where the fixed transaction cost is insignificant), then this problem is the same as the undiscounted one except that the drift of the asset is now replaced by $\mu - \delta$ under the geometric Brownian motion assumption for the asset price. For small δ such that $\delta \leq \mu$, our baseline results under the standing assumption $\beta \leq 1$ apply. Otherwise, when $\delta > \mu$, the analysis becomes similar to the one in Sect. 6.1. Increasing δ has the same effect as increasing β, where an endowed asset tends to be liquidated sooner (lower gain-exit level and possibly higher stop-loss level) while the purchase region shrinks; see Fig. 5. This result is quite interesting because impatience affects sale and purchase decisions somewhat differently. Increasing impatience will cause an agent who already owns the asset to sell sooner, which is in line with common intuition; but surprisingly, a higher δ will also delay the purchase decision. The economic rationale is that an increasing discount rate makes the opportunity to sell the asset less valuable, and hence the agent is more reluctant to enter the trade in the first place.

Remark 6.2 If we insist on $\Psi > 0$, then the objective function has an explicit dependence on τ which will make the entry problem time-inhomogeneous. Such a problem is more difficult to be analysed analytically or numerically.

6.2.2 Utility-discounting

The second possibility to incorporate discounting is to assume the utility of the round-trip proceed is discounted by a single discount factor evaluated at the liquidation date. We call this approach "*utility-discounting*". The problem is formulated as

$$\sup_{\tau,\nu \in \mathcal{T}:\tau \leq \nu} \mathbb{E}\big[e^{-\delta\nu}U\big(P_\nu \mathbb{1}_{\{\tau<\infty,\nu<\infty\}} - (\lambda P_\tau + \Psi)\mathbb{1}_{\{\tau<\infty\}} - R\big)\big|P_0 = p\big]. \quad (6.2)$$

The downside of this approach is that it does not properly take into the account the timing of the cash outflow (incurred at τ) and inflow (incurred at ν), but an advantage is that this formulation resembles a standard discounted optimal stopping problem.

It turns out that introducing discounting in this fashion will drastically change the agent's entry behaviour, as summarised by the result below whose proof is given in Appendix A.2.

Proposition 6.3 *Suppose $\delta > 0$ and let $\omega_1 < 0 < \omega_2$ be the two distinct real roots of the quadratic equation $\frac{\sigma^2}{2}x^2 + (\mu - \frac{\sigma^2}{2})x - \delta = 0$. Under the assumption $\alpha < \beta$, the pair of stopping times*

$$\tau^* = 0, \qquad \nu^* = \inf\left\{t \geq \tau^* : P_t \geq \frac{\omega_2(\lambda P_{\tau^*} + \Psi + R)}{\gamma(\omega_2 - \alpha)}\right\}$$

is optimal for (6.2). *In other words, the agent always enters the trade immediately at any price level and subsequently adopts a gain-exit strategy.*

Proposition 6.3 applies to the case $\beta > 1$ as well. Unlike profit-discounting, the agent will never stop-loss under utility-discounting even when the asset drift is negative. More remarkably, the entry strategy now becomes trivial, provided that the standing assumption $\alpha < \beta$ holds. It is also interesting to point out that the optimal strategy is not continuous at $\delta = 0$, in the sense that letting $\delta \downarrow 0$ in Proposition 6.3 does not recover the no-discounting baseline results in Propositions 4.7 or 4.8. See Fig. 6 for an illustration where we show the optimal sale boundary as a function of δ under utility-discounting. Once discounting is applied to the utility term (no matter how small δ is), the impact of a poor trading performance in the form of negative utility can be mitigated by indefinitely deferring the realisation of a loss. The agent is then effectively protected from negative outcomes, and there is no downside to taking risk regardless of the asset quality or level of the transaction costs. The agent is impatient so that it is optimal to enter the trade as early as possible, and there is no reason to realise a loss thereafter because it can be discounted away.

In view of the above results, profit-discounting seems to yield more reasonable and flexible predictions of the agent's optimal trading behaviour. Nonetheless, a proper understanding of the implications behind the two discounting approaches is of interest to the field of behavioural economics.
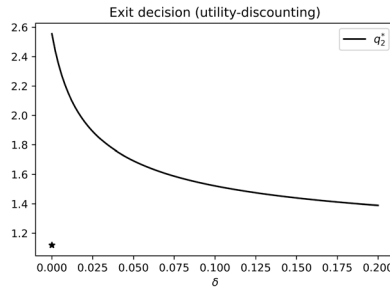
Fig. 6 Optimal sale decisions as δ varies under the utility-discounting criterion, where the agent will sell the endowed asset whenever the price level is at or above $q_2^* H$. There is a discontinuity at $\delta = 0$, where the star $*$ marks the optimal threshold for the undiscounted problem. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $\beta = 0.85$, $\lambda = 1.05$, $\gamma = 0.95$, $\Psi = 1$

6.3 Multiple round-trip trades

We have exclusively focused on the case that the agent's utility is derived from a single round-trip trade. But what if the agent can repeatedly purchase and sell the asset? Similarly to Sect. 6.2, we discuss two possible modelling choices how utility over multiple trades can be computed.

6.3.1 Optimisation of utility over total trading profit

The first possibility is to consider utility derived from the total net present value of the trading proceeds, which is similar to the profit-discounting idea in Sect. 6.2. Suppose the agent can perform N round-trip trades; then the objective function can be written as

$$V(p) := \sup_{\tau_i, v_i \in \mathcal{T}: i=1,\ldots,N} J\big(p; (\tau_i)_{i=1,\ldots,N}, (v_i)_{i=1,\ldots,N}\big) \tag{6.3}$$

with

$$J\big(p; (\tau_i)_{i=1,\ldots,N}, (v_i)_{i=1,\ldots,N}\big)$$
$$:= \mathbb{E}\left[U\left(\sum_{i=1}^{N}\big(\gamma P_{v_i}\mathbb{1}_{\{\tau_i<\infty, v_i<\infty\}} - (\lambda P_{\tau_i} + \Psi)\mathbb{1}_{\{\tau_i<\infty\}}\big) - R\right)\bigg|P_0 = p\right].$$

Here $(\tau_i)_{i=1,\ldots,N}$ (resp. $(v_i)_{i=1,\ldots,N}$) is an increasing sequence of stopping times representing the entry (resp. exit) time of the ith trade with

$$0 \leq \tau_1 \leq v_1 \leq \tau_2 \leq v_2 \leq \tau_3 \leq \cdots \leq \tau_N \leq v_N \leq +\infty.$$

In particular, the agent's utility is now derived from the total trading profits from the N available round-trip trading opportunities.

The problem can still be approached by a similar backward induction principle which we outline below based on heuristics. Define

$$
V_b^{(n)}(p; Q) := \sup_{(\tau_i, v_i)_{i=1,\ldots,n}} \mathbb{E}\left[U\left(\sum_{i=1}^{n} \left(\gamma P_{v_i} \mathbb{1}_{\{\tau_i < \infty, v_i < \infty\}} \right. \right. \right.
$$

$$
\left. \left. \left. - (\lambda P_{\tau_i} + \Psi) \mathbb{1}_{\{\tau_i < \infty\}} \right) - Q \right) \middle| P_0 = p \right]
$$

and

$$
V_s^{(n)}(p; H) := \sup_{v_1, (\tau_i, v_i)_{i=2,\ldots,n}} \mathbb{E}\left[U\left(\sum_{i=2}^{n} \left(\gamma P_{v_i} \mathbb{1}_{\{\tau_i < \infty, v_i < \infty\}} - (\lambda P_{\tau_i} + \Psi) \mathbb{1}_{\{\tau_i < \infty\}} \right) \right. \right.
$$

$$
\left. \left. + \gamma P_{v_1} \mathbb{1}_{\{v_1 < \infty\}} - H \right) \middle| P_0 = p \right].
$$

Here $V_b^{(n)}(p; Q)$ represents the value function when there are n purchase and sale opportunities available under some reference point Q, and $V_s^{(n)}(p; H)$ represents the value function when the agent already owns the asset and there are $n - 1$ purchase and n sale opportunities available under some reference point H. Then heuristically, we expect from the dynamic programming principle that

$$
V_b^{(n)}(p; Q) = \sup_{\tau_1} \mathbb{E}\left[\mathbb{1}_{\{\tau_1 = \infty\}} U(-Q) + \mathbb{1}_{\{\tau_1 < \infty\}} \sup_{v_1, (\tau_i, v_i)_{i=2,\ldots,n}} \Upsilon \middle| P_0 = p \right],
$$

where

$$
\Upsilon := \mathbb{E}\left[U\left(\sum_{i=2}^{n} \left(\gamma P_{v_i} \mathbb{1}_{\{\tau_i < \infty, v_i < \infty\}} - (\lambda P_{\tau_i} + \Psi) \mathbb{1}_{\{\tau_i < \infty\}} \right) \right. \right.
$$

$$
\left. \left. + \gamma P_{v_1} \mathbb{1}_{\{v_1 < \infty\}} - (\lambda P_{\tau_1} + \Psi) - Q \right) \middle| \mathcal{F}_{\tau_1} \right].
$$

Hence we expect that

$$
V_b^{(n)}(p; Q)
$$

$$
= \sup_{\tau_1} \mathbb{E}\left[\mathbb{1}_{\{\tau_1 = \infty\}} U(-Q) + \mathbb{1}_{\{\tau_1 < \infty\}} V_s^{(n)}(P_{\tau_1}; \lambda P_{\tau_1} + \Psi + Q) \middle| P_0 = p \right] \quad (6.4)
$$

$$
= \sup_{\tau_1} \mathbb{E}\left[\max\{V_s^{(n)}(P_{\tau_1}; \lambda P_{\tau_1} + \Psi + Q), U(-Q)\} \middle| P_0 = p \right].
$$

The first equality is due to the definition of $V_s^{(n)}$, and the second is expected to hold because for the optimal stopping problem in (6.4), it is clearly suboptimal to stop the process when $V_s^{(n)}(P_t; \lambda P_t + \Psi + Q) < U(-Q)$ and hence the payoff function can be replaced by $\max\{V_s^{(n)}(P_{\tau_1}; \lambda P_{\tau_1} + \Psi + Q), U(-Q)\}$.

Based on a similar reasoning, we expect for $n > 1$ that

$$V_s^{(n)}(p; H) = \sup_{\nu_1} \mathbb{E}\left[\mathbb{1}_{\{\nu_1=\infty\}} U(-H) + \mathbb{1}_{\{\nu_1<\infty\}} \sup_{(\tau_i,\nu_i)_{i=2,\dots,n}} \Theta \middle| P_0 = p\right],$$

where

$$\Theta := \mathbb{E}\left[U\left(\sum_{i=2}^{n}\left(\gamma P_{\nu_i}\mathbb{1}_{\{\tau_i<\infty,\nu_i<\infty\}} - (\lambda P_{\tau_i} + \Psi)\mathbb{1}_{\{\tau_i<\infty\}}\right) + \gamma P_{\nu_1} - H\right)\middle|\mathcal{F}_{\nu_1}\right].$$

Then we expect that

$$V_s^{(n)}(p; H)$$

$$= \sup_{\nu_1} \mathbb{E}\left[\mathbb{1}_{\{\nu_1=\infty\}} U(-H) + \mathbb{1}_{\{\nu_1<\infty\}} V_b^{(n-1)}(P_{\nu_1}; -\gamma P_{\nu_1} + H)\middle| P_0 = p\right]$$

$$= \sup_{\nu_1} \mathbb{E}\left[V_b^{(n-1)}(P_{\nu_1}; -\gamma P_{\nu_1} + H)\middle| P_0 = p\right].$$

The last equality holds because $V_b^{(n)}(p; Q)$ is obviously increasing in p and decreasing in Q so that $V_b^{(n)}(p; -\gamma p + H) \geq V_b^{(n)}(0; H) = U(-H)$ for all p and n. Finally, we obviously have

$$V_s^{(1)}(p; H) = \sup_{\nu} \mathbb{E}[U(\gamma P_\nu \mathbb{1}_{\{\nu_1<\infty\}} - H)|P_0 = p]$$

$$= \sup_{\nu} \mathbb{E}[U(\gamma P_\nu - H)|P_0 = p].$$

In summary, $V_s^{(n)}$ and $V_b^{(n)}$ should satisfy the recursion

$$V_s^{(n)}(p; H) = \sup_{\nu} \mathbb{E}[U(\gamma P_\tau - H)|P_0 = p], \qquad\qquad n = 1,$$

$$V_b^{(n)}(p; Q) = \sup_{\tau} \mathbb{E}\left[\max\{V_s^{(n)}(P_\tau; \lambda P_\tau + \Psi + Q), U(-Q)\}\middle| P_0 = p\right], \quad n \geq 1,$$

$$V_s^{(n)}(p; H) = \sup_{\nu} \mathbb{E}\left[V_b^{(n-1)}(P_\nu; -\gamma P_\nu + H)\middle| P_0 = p\right], \qquad n \geq 2.$$

The required value function for (6.3) with N round-trip trading opportunities is given by $V_b^{(N)}(p; R)$. A formal verification of the above assertion as well as a thorough analysis of this recursive problem is beyond the scope of the current paper.

6.3.2 Optimisation of total utilities from individual trading episodes

One may also assume that a burst of utility is derived upon completion of each round-trip trade, i.e., the objective function with N round-trip trading opportunities is

$$J\left(p; (\tau_i)_{i=1,\dots,N}, (\nu_i)_{i=1,\dots,N}\right)$$

$$:= \mathbb{E}\left[\sum_{i=1}^{N} U\left(\gamma P_{\nu_i}\mathbb{1}_{\{\tau_i<\infty,\nu_i<\infty\}} - (\lambda P_{\tau_i} + \Psi)\mathbb{1}_{\{\tau_i<\infty\}} - R\right)\middle|P_0 = p\right].$$

Unlike the criterion in Sect. 6.3.1 where a single utility is derived from the total profit of all trades, utility is now realised upon completion of each round-trip trade and the agent's goal is to optimise the sum of those utilities. Define

$$V_b^{(n)}(p) :=$$

$$\sup_{(\tau_i, v_i)_{i=1,\ldots,n}} \mathbb{E}\left[\sum_{i=1}^n U\left(\gamma P_{v_i} \mathbb{1}_{\{\tau_i < \infty, v_i < \infty\}}\right) - (\lambda P_{\tau_i} + \Psi)\mathbb{1}_{\{\tau_i < \infty\}} - R\right) \Big| P_0 = p\right]$$

and

$$V_s^{(n)}(p; H) := \sup_{v_1, (\tau_i, v_i)_{i=2,\ldots,n}} \mathbb{E}\left[U(\gamma P_{v_1} \mathbb{1}_{\{v_1 < \infty\}} - H)\right.$$

$$+ \sum_{i=2}^n U\left(\gamma P_{v_i} \mathbb{1}_{\{\tau_i < \infty, v_i < \infty\}}\right.$$

$$\left. - (\lambda P_{\tau_i} + \Psi)\mathbb{1}_{\{\tau_i < \infty\}} - R\right) \Big| P_0 = p\right].$$

Here $V_b^{(n)}(p)$ is the value function with n purchase and sale opportunities remaining, and $V_s^{(n)}(p; H)$ represents the value function when agent has an endowed asset (with some given reference point H for the first trading episode) and there are $n - 1$ purchase and n sale opportunities remaining.

Based on the dynamic programming principle, one can heuristically write down the recursive system satisfied by the value functions as

$$V_s^{(1)}(p; H) := \sup_v \mathbb{E}[U(\gamma P_v - H)|P_0 = p], \tag{6.5}$$

$$V_b^{(n)}(p) := \sup_\tau \mathbb{E}\left[\max\{V_s^{(n)}(P_\tau; \lambda P_\tau + \Psi + R), nU(-R)\}\big| P_0 = p\right], \tag{6.6}$$

$$V_s^{(n)}(p; H) := \sup_v \mathbb{E}\left[U(\gamma P_v - H) + V_b^{(n-1)}(P_v)\big| P_0 = p\right], \tag{6.7}$$

where (6.6) holds for $n \geq 1$ and (6.7) holds for $n \geq 2$. For the "payoff function" in (6.7), the first term $U(\gamma P_v - H)$ represents the utility burst when the endowed asset is sold, and the second term $V_b^{(n-1)}(P_v)$ reflects the maximal sum of expected utilities from the remaining $n - 1$ round-trip trading opportunities. Otherwise, if the agent does not own the asset to begin with and decides to purchase at time τ, the maximal expected utility attainable is given by $V_s^{(n)}(P_\tau; \lambda P_\tau + \Psi + R)$, where a new trading episode is initiated with reference point set to $H = \lambda P_\tau + \Psi + R$ (the total cost of purchase at time τ plus the exogenous aspiration level). But the agent can also choose not to purchase at all and forgo all the n remaining trading opportunities. This will result in a payoff of $U(-R)$ for each trading opportunity given up. Hence in (6.6) the "payoff function" for the entry problem is given by $\max\{V_s^{(n)}(P_\tau; \lambda P_\tau + \Psi + R), nU(-R)\}$. When $n = 1$, (6.5) and (6.6) agree with the sequential optimal stopping problem deduced in Sect. 3.

This generalisation is conceptually close to the realisation utility model in the literature. The canonical formulation (in our notation) of such a model is

$$\sup_{(\tau_i, \nu_i)_{i=1,2,3,\ldots}} \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-\delta \nu_i} U(G_{\nu_i -}, Q_{\nu_i -})\right],$$

where $Q = (Q_t)_{t \geq 0}$ is the reference point process so that Q_t is the benchmark to be used for performance evaluation at time t, and $G = (G_t)_{t \geq 0}$ is the gain-and-loss process with $G_t := \gamma P_t - Q_t$ representing the size of realised gain-and-loss if the agent liquidates an owned asset at time t. Typically, the function $U(G, Q)$ is assumed to be homogeneous in Q so that $U(G, Q) = Q^\eta u(G/Q)$ for some $\eta \in (0, 1]$ and $u(\cdot)$ is S-shaped.

There are many choices for the reference point process Q. Ingersoll and Jin [14] consider $Q_t = P_{\tau_i}$ for $t \in [\tau_i, \tau_{i+1})$ (up to a constant multiplier), which is simply the most recent purchase price of the asset. Barberis and Xiong [2] and Dai et al. [3] take $Q_t = P_{\tau_i} e^{r(t - \tau_i)}$ for $t \in [\tau_i, \tau_{i+1})$, which is the most recent purchase price growing at the risk-free rate. He and Yang [8] incorporate an additional term which asymmetrically adapts to the paper gain-and-loss. Kong et al. [17] study a path-dependent reference point which is a weighted average of the asset prices throughout the current trading episode. A common feature of the papers cited above is that Q_t is proportional to P_{τ_i} over $t \in [\tau_i, \tau_{i+1})$, and hence a dimension reduction is possible via introducing the new state variable $X_t := P_t/Q_t$. This greatly simplifies the entry problem, but it also trivialises the optimal strategy—one either immediately enters the trade again after a sale or never enters the trade in the first place. This observation remains the same even if one introduces additional modelling elements such as a Poisson random termination time and an extra utility term over final wealth; see Proposition 3.4 of He and Yang [8].

Our formulation can be seen as a version of the realisation utility model with a finite number of trading opportunities, where the reference level process is (see Remark 6.4 below as well)

$$Q_t := \begin{cases} \lambda P_{\tau_i} + \Psi + R & \text{for } t \in [\tau_i, \nu_i), \\ R & \text{for } t \in [\nu_i, \tau_{i+1}), \end{cases} \qquad (6.8)$$

the gain-and-loss process is $G_t := \gamma P_t \mathbb{1}_{\{t < \infty\}} - Q_t$, the utility function is given by $U(G, Q) = Q^\alpha u(G/Q)$ with $u(x) := x^\alpha \mathbb{1}_{\{x \geq 0\}} - k|x|^\alpha \mathbb{1}_{\{x < 0\}}$, and the discount rate δ is set to zero. The most important distinction of our framework from the existing realisation utility models is that our reference point consists of a constant component $\Psi + R$ reflecting a fixed transaction cost and some baseline aspiration level. Without this component, the reference level over a particular trading episode is always proportional to the asset value at the beginning of the episode. Specifically, if $\Psi = R = 0$, then using (6.5)–(6.7), the scaling property of U and the geometric Brownian motion assumption of P, we can inductively deduce for all n that $V_s^{(n)}(p; H) = H^\alpha V(\frac{p}{H}; 1)$

and

$$V_b^{(n)}(p) = \sup_\tau \mathbb{E}\big[\max\{V_s^{(n)}(P_\tau; \lambda P_\tau), 0\}\big|P_0 = p\big]$$

$$= \sup_\tau \mathbb{E}\big[\max\{\lambda^\alpha P_\tau^\alpha V_s^{(n)}(1/\lambda; 1), 0\}\big|P_0 = p\big].$$

If $V_s^{(n)}(1/\lambda; 1) < 0$, then $\tau \equiv +\infty$ is optimal and the associated value function for the entry problem is $V_b^{(n)}(p) = 0$. If instead $V_s^{(n)}(1/\lambda; 1) > 0$, then we have $V_b^{(n)}(p) = \lambda^\alpha V_s^{(n)}(1/\lambda; 1) \sup_\tau \mathbb{E}[P_\tau^\alpha|P_0 = p] = \lambda^\alpha V_s^{(n)}(1/\lambda; 1)p^\alpha$ under the standing assumption $\alpha \leq \beta$. The corresponding optimal entry strategy is $\tau \equiv 0$.

The above observations in conjunction with our main theoretical results for $n = 1$ suggest that incorporating a constant component within the dynamic reference point (e.g. in the form of a fixed transaction cost or a default aspiration level) might enable a realisation utility model to generate more realistic and non-trivial purchase behaviours. For the sale decision, recall that in the baseline model with $n = 1$, the optimal sale strategy is a simple gain-exit rule (Lemma 4.3). We expect this to change when $n > 1$, since the effective payoff function of the exit problem (6.7) now contains an additional term $V_b^{(n-1)}(p)$ which will drastically change the convexity/concavity of the scaled exit payoff function. We leave the full analysis of this problem for future work.

Remark 6.4 At first sight, (6.8) looks more complicated than the models proposed in the existing literature because our value of Q_t depends on whether the agent is inside a trading episode holding the asset ($t \in [\tau_i, \nu_i)$ for some i) or outside a trading episode without any asset ($t \in [\nu_i, \tau_{i+1})$ for some i). From an economic point of view, the reference point should not depend on P_{τ_i} any more once the ith trading episode is complete, and hence should be reset to the baseline aspiration level until the start of the $(i + 1)$th trading episode. In the special case $\tau_{i+1} = \nu_i$ for all i so that there is no time delay between the exit of an existing trade and the entry of a new trade, (6.8) simplifies to $Q_t = \lambda P_{\tau_i} + \Psi + R$ for $t \in [\tau_i, \tau_{i+1})$, which resembles the usual definition in the literature. In the absence of a constant component within the reference point process, the entry decision is trivial as $\tau_{i+1} = \nu_i$ for all i is indeed optimal because of the scaling property discussed previously; so there is no need to "correctly" specify Q_t over $t \in [\nu_i, \tau_{i+1})$. But this result is no longer true when R or Ψ is non-zero, and hence it is necessary to model the reference point process more carefully.

6.4 Endogenous aspiration level

Among all the model parameters, the "aspiration level" R of the agent is the hardest one to be interpreted and estimated. It can be a purely psychological parameter representing the agent's subjective threshold which distinguishes gains and losses. In some applications such as delegated portfolio management, R can also be the performance target imposed on the agent by a manager. Instead of calibrating R where the exercise can be very context-specific, one may seek to endogenise this parameter by introducing a further optimality criterion.

Fig. 7 The principal's certainty equivalent $-\ln(-\tilde{V}(R))/\eta$ as a function of R when $\tilde{U}(x) = -e^{-\eta x}$. Base parameters used are $\alpha = 0.5$, $k = 2.25$, $\lambda = 1.05$, $\gamma = 0.95$, $\Psi = 1$, $\eta = 0.01$, $P_0 = 30$
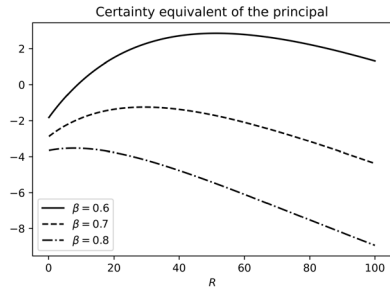
Consider a principal–agent setup as an example. The principal imposes an aspiration level on the agent in the form of a performance target. For a given R, the agent's optimal trading rule $(\tau^*(R), v^*(R))$ can be obtained by solving (2.3). If the principal has a utility function $\tilde{U}(\cdot)$ over the trading profit, then a particular choice of R will bring the principal an expected utility level of

$$\tilde{V}(R) := \mathbb{E}\big[\tilde{U}\big(\gamma P_{v^*(R)}\mathbb{1}_{\{\tau^*(R)<\infty, v^*(R)<\infty\}} - (\lambda P_{\tau^*(R)} + \Psi)\mathbb{1}_{\{\tau^*(R)<\infty\}}\big)\big|P_0 = p\big],$$

where we have suppressed the dependence of $\tilde{V}$ on p. The above can be maximised (numerically) with respect to R, where the solution in general depends on the initial asset price P_0. In Fig. 7, we consider a risk-averse principal with utility function $\tilde{U}(x) = -e^{-\eta x}$, where $\eta > 0$ is the constant absolute risk-aversion level and $\tilde{V}(R)$ can be maximised at some interior R in this particular example. The higher the Sharpe ratio of the asset (equivalent to a lower level of β), the higher the level of endogenous aspiration. In other words, a more aggressive performance goal is set in a bullish market. However, there are also examples where the principal's maximisation problem is degenerate (e.g. $\tilde{V}(R)$ being monotonically increasing or decreasing in R). We leave the complete analysis of such principal–agent problems for future research.

An alternative consideration to endogenise R is to modify the agent's utility function such that a round-trip profit of x leads to a utility value of $U(x - R, R)$, where $U(\cdot, \cdot)$ is increasing in the first and decreasing in the second argument. The second argument of U can reflect the agent's desire for "self-improvement" and "self-enhancement" which is achieved by choosing a high aspiration level R; see for example Falk and Knell [7]. The optimal R can then be determined alongside with the agent's optimal trading rule.

7 Concluding remarks

This paper considers a dynamic trading model under prospect theory preference with transaction costs. By solving a sequential optimal stopping problem, we find that the optimal trading strategy can have various forms depending on the model parameters and the price level of the asset. The impact of transaction costs is subtle. In contrast to conventional wisdom, increasing transaction costs does not necessarily deter economic agents from trading participation because the agents may face a higher

reference point and in turn be more risk-aggressive in an expensive trading environment. These results could potentially be useful to policy makers to better understand how undesirable speculative trading behaviour in certain markets can be curbed effectively.

Our key mathematical results are derived under a somewhat stylised modelling specification. In particular, asymmetry of the degree of risk-aversion/seeking over gains/losses, a fixed transaction cost on sales and a negative aspiration level are currently omitted from the baseline analysis. While these omissions allow us to derive a sharp characterisation and comparative statics of the optimal trading rules, it will nonetheless be constructive to extend the model to further examine the impact of other economic factors. Section 6 also highlights a number of possible extensions for further rigorous mathematical analysis in the presence of a negative drift, subjective discounting, repeated trading opportunities and an endogenous aspiration level. In particular, a natural extension is to further explore the implications of our results for the literature of realisation utility, where we believe that a different specification of the reference point process (e.g. incorporation of a constant component) can lead to more realistic predictions of purchase behaviour.

A more ambitious goal is to further incorporate probability weighting within our continuous-time optimal stopping model (as in Xu and Zhou [28] and Henderson et al. [10]) to fully reflect the features of the *cumulative prospect theory* framework of Tversky and Kahneman [27]. However, technical difficulties are likely to arise due to the time-inconsistency brought by probability weighting. A precise formulation of the problem as well as the development of appropriate mathematical techniques should be an another interesting proposal for future research.

Finally, the surprising comparative statics documented in this paper also reveal that the economic interaction between market frictions and behavioural preferences can be subtle or even counter-intuitive. The model considered in this paper is just one of the many possible motivations that these two topics should not be studied in isolation, and this could potentially open up a new strand of literature to advance our understanding towards trading behaviour in a more realistic setup.

Appendix

A.1 Proofs of Propositions 4.7, 4.8 and 5.1

We start with two useful lemmas.

Lemma A.1 *Write $\xi := \frac{\lambda}{\gamma}$. For the function f defined in (4.5), we have*

$$\lim_{x \to \infty} f(x) = \begin{cases} +\infty & for \ \xi < (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}, \\ 0 & for \ \xi = (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}, \\ -\infty & for \ \xi > (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}. \end{cases}$$

Moreover:

1) *Suppose $\alpha < \beta < 1$. Then:*

(a) *If $\xi \leq (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$, then f is an increasing concave function.*

(b) *If $\xi > (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$, then f is concave increasing on $[0, x_2^*]$, concave decreasing on $[x_2^*, \tilde{x}]$, and convex decreasing on $[\tilde{x}, \infty)$. Here x_2^* and $\tilde{x}$ are respectively the solutions to the equation*

$$c^{1-\beta}(c-1)^{\alpha-1}\left(x^{-\frac{1}{\beta}} + \frac{\xi\alpha}{\beta}\right) - k\xi(x^{-\frac{1}{\beta}} + \xi)^\beta = 0 \tag{A.1}$$

and

$$0 = c^{1-\beta}(c-1)^{\alpha-1}\left(-\frac{\xi\alpha}{\beta^2}(\beta-\alpha) + \frac{1}{\beta}\left(\frac{\alpha}{\beta} - \beta + \alpha - 1\right)x^{-\frac{1}{\beta}}\right)$$
$$- k\left(\xi + x^{-\frac{1}{\beta}}\right)^\beta\left(-\xi\left(1 - \frac{\alpha}{\beta}\right) + \left(\frac{1}{\beta} - 1\right)x^{-\frac{1}{\beta}}\right). \tag{A.2}$$

2) *Suppose $\alpha < \beta = 1$. Then:*

(a) *If $\xi \leq \frac{\alpha}{k}(c-1)^{\alpha-1}$, then f is an increasing concave function.*

(b) *If $\frac{\alpha}{k}(c-1)^{\alpha-1} < \xi \leq \frac{1}{k}(c-1)^{\alpha-1}$, then f is concave increasing on $[0, x_2^*]$, concave decreasing on $[x_2^*, \tilde{x}]$, and convex decreasing on $[\tilde{x}, \infty)$ with*

$$x_2^* := \frac{(c-1)^{\alpha-1} - k\xi}{\xi(k\xi - \alpha(c-1)^{\alpha-1})}, \qquad \tilde{x} := \frac{2(c-1)^{\alpha-1} - k\xi}{\xi(k\xi - \alpha(c-1)^{\alpha-1})}.$$

(c) *If $\xi > \frac{1}{k}(c-1)^{\alpha-1}$, then f is a decreasing function.*

Proof We can rewrite f as

$$f(x) = \frac{\frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1} - k(\xi + x^{-\frac{1}{\beta}})^\beta}{(\xi + x^{-\frac{1}{\beta}})^{\beta-\alpha}} x^{\frac{\alpha}{\beta}}$$

so that $\lim_{x\to\infty} f(x) = \pm\infty$ when $\xi \gtrless (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$. The corner case of $\xi = (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$ can be analysed by a simple application of L'Hôpital's rule.

We now derive the shapes of f by first focusing on the case $\beta \neq 1$. Direct differentiation gives

$$f'(x) = \frac{\alpha}{\beta} \frac{c^{1-\beta}(c-1)^{\alpha-1}(\frac{\xi\alpha}{\beta}x^{\frac{1}{\beta}} + 1) - k\xi x^{\frac{1}{\beta}-1}(\xi x^{\frac{1}{\beta}} + 1)^\beta}{(\xi x^{\frac{1}{\beta}} + 1)^{\beta-\alpha+1}}$$

$$= \frac{\alpha x^{\frac{1}{\beta}} h_1(x^{-\frac{1}{\beta}})}{\beta(\xi x^{\frac{1}{\beta}} + 1)^{\beta-\alpha+1}} \tag{A.3}$$

with

$$h_1(z) := c^{1-\beta}(c-1)^{\alpha-1}\left(z + \frac{\xi\alpha}{\beta}\right) - k\xi\,(z+\xi)^\beta,$$

and

$$f''(x) = \frac{\xi\alpha x^{\frac{1}{\beta}-2}}{\beta(\xi x^{\frac{1}{\beta}}+1)^{\beta-\alpha+2}}$$
$$\times \left(c^{1-\beta}(c-1)^{\alpha-1}x\left(-\frac{\xi\alpha}{\beta^2}(\beta-\alpha)x^{\frac{1}{\beta}} + \frac{1}{\beta}\left(\frac{\alpha}{\beta}-\beta+\alpha-1\right)\right)\right.$$
$$\left. - k\left(\xi x^{\frac{1}{\beta}}+1\right)^\beta\left(-\xi\left(1-\frac{\alpha}{\beta}\right)x^{\frac{1}{\beta}} + \frac{1}{\beta}-1\right)\right)$$
$$= \frac{\xi\alpha x^{\frac{2}{\beta}-1}h_2(x^{-\frac{1}{\beta}})}{\beta(\xi x^{\frac{1}{\beta}}+1)^{\beta-\alpha+2}},$$

where

$$h_2(z) := c^{1-\beta}(c-1)^{\alpha-1}\left(-\frac{\xi\alpha}{\beta^2}(\beta-\alpha) + \frac{1}{\beta}\left(\frac{\alpha}{\beta}-\beta+\alpha-1\right)z\right)$$
$$- k(\xi+z)^\beta\left(-\xi\left(1-\frac{\alpha}{\beta}\right) + \left(\frac{1}{\beta}-1\right)z\right).$$

We first investigate the convexity/concavity of f by studying the sign of $f''(x)$, which is determined by that of $h_2(x^{-\frac{1}{\beta}})$. Check that

$$h_2(0) = \xi\left(1-\frac{\alpha}{\beta}\right)\left(-\frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1} + k\xi^\beta\right),$$
$$h_2'(0) = c^{1-\beta}(c-1)^{\alpha-1}\frac{1}{\beta}\left(\frac{\alpha}{\beta}-\beta+\alpha-1\right) - k\xi^\beta\left(\frac{1}{\beta}-\beta+\alpha-1\right)$$

and

$$h_2''(z) = -k(\xi+z)^{\beta-2}\left((1-\beta)(1+\beta)z + \xi(1-\beta)(2+\beta-\alpha)\right) < 0$$

for all $z > 0$ since $\alpha \le \beta \le 1$, and thus h_2 is concave. Then there are two possibilities.

If $\xi \leq (\frac{\alpha}{\beta k} c^{1-\beta} (c-1)^{\alpha-1})^{\frac{1}{\beta}}$, then $h_2(0) \leq 0$ and

$$
\begin{aligned}
h_2'(0) &= c^{1-\beta}(c-1)^{\alpha-1} \frac{1}{\beta} \left(\frac{\alpha}{\beta} - \beta + \alpha - 1 \right) - k\xi^\beta \left(\frac{1}{\beta} - \beta + \alpha - 1 \right) \\
&< c^{1-\beta}(c-1)^{\alpha-1} \frac{1}{\beta} \left(\frac{\alpha}{\beta} - \beta + \alpha - 1 \right) - k\xi^\beta \left(\frac{\alpha}{\beta} - \beta + \alpha - 1 \right) \\
&= k \left(\frac{\alpha}{\beta} - \beta + \alpha - 1 \right) \left(\frac{c^{1-\beta}(c-1)^{\alpha-1}}{\beta k} - \xi^\beta \right) \\
&\leq k \left(\frac{\alpha}{\beta} - \beta + \alpha - 1 \right) \left(\frac{\alpha c^{1-\beta}(c-1)^{\alpha-1}}{\beta k} - \xi^\beta \right) \\
&\leq 0,
\end{aligned}
$$

where we have used the facts that $\alpha < 1$ and $\frac{\alpha}{\beta} - \beta + \alpha - 1 < \alpha - \beta \leq 0$. Since h_2 is concave, we must have $h_2(z) \leq 0$ for all $z > 0$. Hence $f_2''(x) \leq 0$ for all $x \geq 0$, i.e., f is a concave function.

If instead $\xi > (\frac{\alpha}{\beta k} c^{1-\beta} (c-1)^{\alpha-1})^{\frac{1}{\beta}}$, then $h_2(0) > 0$ and

$$
\lim_{z \to \infty} \frac{h_2(z)}{z^{\beta+1}} = -k \left(\frac{1}{\beta} - 1 \right) < 0
$$

so that $h_2(z) \to -\infty$ as $z \to \infty$. As h_2 is concave, we must have $z \mapsto h_2(z)$ down-crossing zero exactly once on $(0, \infty)$. Hence $f_2''(x) \propto h_2(x^{-\frac{1}{\beta}})$ has exactly one sign change from negative to positive, i.e., f is concave for small x and convex for large x with a unique inflection point $\tilde{x}$ which is given by the solution to $h_2(x^{-\frac{1}{\beta}}) = 0$. This corresponds to (A.2).

Now we look at the monotonicity of f via the sign of $f'(x)$ which in turn is determined by that of $h_1(x^{-\frac{1}{\beta}})$. Check that

$$
h_1(0) = \xi \left(\frac{\alpha}{\beta} c^{1-\beta} (c-1)^{\alpha-1} - k\xi^\beta \right),
$$

$$
h_1'(z) = c^{1-\beta}(c-1)^{\alpha-1} - \frac{k\xi\beta}{(z+\xi)^{1-\beta}}.
$$

Observe that h_1' is increasing and thus h_1 is convex. There are two cases.

If $\xi \leq (\frac{\alpha}{\beta k} c^{1-\beta} (c-1)^{\alpha-1})^{\frac{1}{\beta}}$, then $h_1(0) \geq 0$ and

$$
h_1'(0) = c^{1-\beta}(c-1)^{\alpha-1} - k\beta\xi^\beta \geq \alpha c^{1-\beta}(c-1)^{\alpha-1} - k\beta\xi^\beta \geq 0.
$$

As h_1 is convex, we must have $h_1(z) \geq 0$ for all $z > 0$. Hence $f'(x) \geq 0$ for all $x \geq 0$, i.e., f is an increasing function. Together with the consideration of f'' in this parameter regime, f is an increasing concave function.

If $\xi > (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$, then we have $h_1(0) < 0$ instead. We also have

$$\lim_{z \to \infty} \frac{h_1(z)}{z} = c^{1-\beta}(c-1)^{\alpha-1} > 0 \qquad (A.4)$$

and hence $h_1(z) \to \infty$ as $z \to \infty$. Since h_1 is convex, it must up-cross zero exactly once on $(0, \infty)$. Therefore $f'(x) \propto h_1(x^{-\frac{1}{\beta}})$ changes sign exactly once, from which we conclude that f is first increasing and then decreasing with a unique turning point x_2^*. Moreover, x_2^* is the solution to $h_1(x^{-\frac{1}{\beta}}) = 0$, which is equivalent to (A.1). Taking the behaviour of f'' into consideration, we conclude that f is increasing concave on $[0, x_2^*]$, decreasing concave on $[x_2^*, \tilde{x}]$, and decreasing convex on $[\tilde{x}, \infty)$.

The case $\beta = 1$ can be handled similarly. The key difference is that (A.4) no longer holds when $\beta = 1$, but we have instead

$$\lim_{z \to \infty} \frac{h_1(z)}{z} = (c-1)^{\alpha-1} - k\xi$$

which can be either positive or negative. If $\xi > \frac{(c-1)^{\alpha-1}}{k}$, we have $h_1(z) \to -\infty$ as $z \to \infty$. We can then deduce that $f'(x)$ is negative for all x and thus f is decreasing. $\qquad \square$

Lemma A.2 *If $\xi := \frac{\lambda}{\gamma} > (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$, then $f(x) < 0$ for all x, where f is defined in (4.5).*

Proof The result follows directly from the definition of f that

$$
\begin{aligned}
f(x) &= \left(\frac{\alpha}{\beta} c^{1-\beta}(c-1)^{\alpha-1} x \left(\frac{\lambda}{\gamma} x^{1/\beta} + 1 \right)^{-\beta} - k \right) \left(\frac{\lambda}{\gamma} x^{1/\beta} + 1 \right)^{\alpha} \\
&< \left(k \left(\frac{\lambda}{\gamma} \right)^{\beta} x \left(\frac{\lambda}{\gamma} x^{1/\beta} + 1 \right)^{-\beta} - k \right) \left(\frac{\lambda}{\gamma} x^{1/\beta} + 1 \right)^{\alpha} \\
&< \left(k \left(\frac{\lambda}{\gamma} \right)^{\beta} x \left(\frac{\lambda}{\gamma} x^{1/\beta} + 0 \right)^{-\beta} - k \right) \left(\frac{\lambda}{\gamma} x^{1/\beta} + 1 \right)^{\alpha} = 0. \qquad \square
\end{aligned}
$$

With the help of Lemmas A.1 and A.2, we now prove Propositions 4.7, 4.8 and 5.1.

Proof of Proposition 4.7 From the discussion in Sect. 3, we have to identify the smallest concave majorant $\bar{g}_2(\cdot)$ of the function

$$g_2(\theta) := \max \left\{ V_1 \left(s^{-1}(\theta); \lambda s^{-1}(\theta) + \Psi + R \right), U(-R) \right\}$$

$$= \max \{ V_1(\theta^{\frac{1}{\beta}}; \lambda \theta^{\frac{1}{\beta}} + \Psi + R), U(-R) \},$$

where V_1 is the value function of the exit problem given in Lemma 4.3. Since $c > 1$ and we assume that $R > 0$, $\Psi \geq 0$ and $\gamma \leq 1 \leq \lambda$, we have $c(\frac{\lambda \theta^{\frac{1}{\beta}} + \Psi + R}{\gamma}) \geq \theta^{\frac{1}{\beta}}$, and

hence the first regime in (4.3) will always apply when evaluating $V_1(\theta^{\frac{1}{\beta}}; \lambda\theta^{\frac{1}{\beta}} + \Psi + R)$, i.e.

$$
\begin{aligned}
v_1(\theta) &:= V_1\left(\theta^{\frac{1}{\beta}}; \lambda\theta^{\frac{1}{\beta}} + \Psi + R\right) \\
&= -k\left(\lambda\theta^{\frac{1}{\beta}} + \Psi + R\right)^\alpha + \frac{\alpha}{\beta}\left(\lambda\theta^{\frac{1}{\beta}} + \Psi + R\right)^{\alpha-\beta} c^{1-\beta}(c-1)^{\alpha-1}\gamma^\beta\theta \\
&= R^\alpha\left(1 + \frac{\Psi}{R}\right)^\alpha\left(\frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1}\left(\frac{\lambda}{\gamma}\frac{\gamma\theta^{\frac{1}{\beta}}}{R+\Psi} + 1\right)^{\alpha-\beta}\left(\frac{\gamma}{R+\Psi}\right)^\beta\theta \right. \\
&\qquad\qquad\qquad\qquad \left. - k\left(\frac{\lambda}{\gamma}\frac{\gamma\theta^{\frac{1}{\beta}}}{R+\Psi} + 1\right)^\alpha\right) \\
&= R^\alpha\left(1 + \frac{\Psi}{R}\right)^\alpha f\left(\left(\frac{\gamma}{R+\Psi}\right)^\beta\theta\right),
\end{aligned}
\tag{A.5}
$$

where f is defined in (4.5). The shape of f under different parameter combinations is given by Lemma A.1, and thus we have the following cases.

When $\xi := \frac{\lambda}{\gamma} \leq (\frac{\alpha}{\beta k}c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$, f is increasing concave and we have $\lim_{x\to\infty} f(x) = +\infty$. These properties are inherited by v_1. Furthermore,

$$
v_1(0) = R^\alpha(1 + \Psi/R)^\alpha f(0) = -kR^\alpha(1 + \Psi/R)^\alpha \leq -kR^\alpha
$$

and

$$
\lim_{\theta\to\infty} v_1(\theta) > 0 > -kR^\alpha.
$$

Thus g_2 is constructed by truncating an increasing concave function from below at $-kR^\alpha$. The smallest concave majorant of g_2 is formed by drawing a tangent line passing through $(0, -kR^\alpha)$ which touches v_1 at some θ_1^*; see Fig. 1(a) above. The exact form of the smallest concave majorant is

$$
\bar{g}_2(\theta) = \begin{cases} \frac{v_1(\theta_1^*)+kR^\alpha}{\theta_1^*}\theta - kR^\alpha & \text{for } \theta < \theta_1^*, \\ v_1(\theta) & \text{for } \theta \geq \theta_1^*, \end{cases}
$$

which is equivalent to (4.6) upon observing that $V_2(p) = \bar{g}_2(p^\beta)$.

The optimal strategy is to sell the asset when its transformed price Θ_t first reaches θ_1^* or above. The corresponding threshold in the original price scale is given by $p_1^* := s^{-1}(\theta_1^*) = (\theta_1^*)^{1/\beta}$. Since θ_1^* is the point of contact of the tangent line to v_1 which passes through $(0, -kR^\alpha)$, θ_1^* should solve

$$
v_1'(\theta) - \frac{v_1(\theta) + kR^\alpha}{\theta} = 0. \tag{A.6}
$$

Furthermore, we can deduce from a graphical inspection that the solution to (A.6) is a down-crossing. For the case of large proportional transaction costs where we have $\xi = \frac{\lambda}{\gamma} > (\frac{\alpha}{\beta k}c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$, the straight line passing $(0, -kR^\alpha)$ can touch v_1 at

two distinct locations. A simple geometric inspection tells us that the required root is the smaller one. Using the representation of $v_1(\theta)$ in (A.5), (A.6) can be rewritten as

$$0 = R^\alpha \left(1 + \frac{\Psi}{R}\right)^\alpha \left(\frac{\gamma}{R + \Psi}\right)^\beta f'\left(\left(\frac{\gamma}{R + \Psi}\right)^\beta \theta\right)$$
$$- \frac{R^\alpha \left(1 + \frac{\Psi}{R}\right)^\alpha f\left(\left(\frac{\gamma}{R + \Psi}\right)^\beta \theta\right) + k R^\alpha}{\theta}.$$

A further substitution of $x = \left(\frac{\gamma}{R+\Psi}\right)^\beta \theta$ leads to

$$\left(1 + \frac{\Psi}{R}\right)^\alpha \left(x f'(x) - f(x)\right) = k. \tag{A.7}$$

Then $p_1^* = (\theta_1^*)^{1/\beta} = \frac{R+\Psi}{\gamma}(x_1^*)^{1/\beta}$, where x_1^* is defined as the solution to (A.7) which is equivalent to (4.8).

When $\xi = \frac{\lambda}{\gamma} > (\frac{\alpha}{\beta k} c^{1-\beta}(c-1)^{\alpha-1})^{\frac{1}{\beta}}$, Lemma A.1 implies that v_1 is first concave increasing, reaching a global maximum at some θ_2^*, then concave decreasing and finally convex decreasing with $\lim_{\theta \to \infty} v_1(\theta) = -\infty$. There are two further possibilities.

If $v_1(\theta_2^*) > -kR^\alpha$, there must exist $0 \le \hat{\theta}_1 < \hat{\theta}_2$ such that $g_2(\theta) = -kR^\alpha$ on $[0, \hat{\theta}_1] \cup [\hat{\theta}_2, \infty)$ and $g_2(\theta) = v_1(\theta)$ on $[\hat{\theta}_1, \hat{\theta}_2]$. The smallest concave majorant of $g_2(\theta)$ is formed by a chord passing through $(0, -kR^\alpha)$ which touches v_1 at some $\theta_1^* < \theta_2^*$ on $\{\theta < \theta_1^*\}$, a horizontal line at level $g(\theta_2^*)$ on $\{\theta > \theta_2^*\}$, and the function g_2 itself on $\{\theta_1^* \le \theta \le \theta_2^*\}$; see Fig. 1(b) above. The smallest concave majorant is

$$\bar{g}_2(\theta) = \begin{cases} \frac{v_1(\theta_1^*)+kR^\alpha}{\theta_1^*}\theta - kR^\alpha & \text{for } \theta < \theta_1^*, \\ v_1(\theta) & \text{for } \theta_1^* \le \theta \le \theta_2^*, \\ v_1(\theta_2^*) & \text{for } \theta > \theta_2^*. \end{cases}$$

This gives the form of the value function in (4.7).

The optimal strategy is to purchase the asset when its transformed price Θ_t first enters the interval $[\theta_1^*, \theta_2^*]$. The boundary of the purchase regions in the original scale can be recovered via $p_i^* = (\theta_i^*)^{1/\beta}$ for $i = 1, 2$. Given that θ_2^* is the maximiser of $v_2(\theta)$, using the representation of (A.5), θ_2^* should then solve $f'((\frac{\gamma}{R+\Psi})^\beta \theta) = 0$. Using (A.3), $x_2^* := (\frac{\gamma}{R+\Psi})^\beta \theta_2^*$ is a solution to

$$h_1(x^{-\frac{1}{\beta}}) = c^{1-\beta}(c-1)^{\alpha-1}\left(x^{-\frac{1}{\beta}} + \frac{\xi\alpha}{\beta}\right) - k\xi\left(x^{-\frac{1}{\beta}} + \xi\right)^\beta = 0.$$

Then $p_2^* = (\theta_2^*)^{1/\beta} = \frac{R+\Psi}{\gamma}(x_2^*)^{1/\beta}$, where x_2^* is given by the solution to (4.9).

If $v_1(\theta_2^*) \le -kR^\alpha$ instead, then $v_1(\theta) \le -kR^\alpha$ for all θ. Thus $g_2(\theta) = -kR^\alpha$ which is a flat horizontal line, and it is also the smallest concave majorant of itself, i.e., $\bar{g}_2(\theta) = -kR^\alpha$. The optimal strategy is not to trade at all at any price level so that the utility received is always $U(-R) = -kR^\alpha$; see Fig. 1(c) above. The "never

purchase" case arises if and only if $v_1(\theta_2^*) \le -kR^\alpha$ or, equivalently,

$$R^\alpha \left(1 + \frac{\Psi}{R}\right)^\alpha f\left(\left(\frac{\gamma}{R+\Psi}\right)^\beta \theta_2^*\right) \le -kR^\alpha \iff \left(1 + \frac{\Psi}{R}\right)^\alpha f(x_2^*) \le -k,$$

where x_2^* is the maximiser of f introduced in Lemma A.1 and is independent of Ψ and R. Using the fact that $f(0) = -k$ and Lemma A.2, we deduce that $-k < f(x^*) < 0$ and hence there must exist $C := (-\frac{k}{f(x^*)})^{1/\alpha} - 1 > 0$ such that $(1 + \frac{\Psi}{R})^\alpha f(x^*) \le -k$ if and only if $\Psi/R \ge C$. $\qquad\square$

Proof of Proposition 4.8 Omitted since it is largely the same as the proof of Proposition 4.7. $\qquad\square$

Proof of Proposition 5.1 From the proof of Proposition 4.7, the required solution to (4.8) is a down-crossing. Then given that the left-hand side of (4.8) is increasing in Ψ (when evaluated at $x = x_1^*$), we can deduce that x_1^* and in turn p_1^* are both increasing in Ψ.

To show that p_1^* is decreasing in γ, consider the substitution $q = \frac{x^{1/\beta}}{\gamma}$. Then $p_1^* = (R + \Psi)q_1^*$, where q_1^* is the solution to

$$k = \left(1 + \frac{\Psi}{R}\right)^\alpha$$
$$\times \frac{k(\lambda q + 1)^\beta \left(\lambda\left(1 - \frac{\alpha}{\beta}\right)q + 1\right) - \frac{\alpha}{\beta}c^{1-\beta}(c-1)^{\alpha-1}\lambda\gamma^\beta\left(1 - \frac{\alpha}{\beta}\right)q^{\beta+1}}{(\lambda q + 1)^{\beta-\alpha+1}}, \quad \text{(A.8)}$$

where the right-hand side of (A.8) is decreasing in γ. Hence q_1^* and in turn p_1^* are both decreasing in γ.

The monotonicity of p_2^* with respect to Ψ is trivial because (4.9) which defines x_2^* does not depend on Ψ. To check the monotonicity with respect to γ, consider the substitution $q = \frac{x^{1/\beta}}{\gamma}$ again so that $p_2^* = (R + \Psi)q_2^*$, where q_2^* is defined as the solution to

$$c^{1-\beta}(c-1)^{\alpha-1}\left(\frac{1}{q} + \frac{\lambda\alpha}{\beta}\right) - \frac{k\lambda}{\gamma^\beta}\left(\frac{1}{q} + \lambda\right)^\beta = 0. \quad \text{(A.9)}$$

From the proof of Lemma A.1, the solution to $h_1(x^{-\frac{1}{\beta}}) = 0$ is a down-crossing. This property is inherited by (A.9). Moreover, the left-hand side of (A.9) is increasing in γ. Hence q_2^* and in turn p_2^* is increasing in γ.

Similarly, consider the substitution $y = \lambda^\beta x$. Then $p_2^* = \frac{R+\Psi}{\lambda\gamma}(y_2^*)^{1/\beta}$, where y_2^* is defined as the solution to

$$c^{1-\beta}(c-1)^{\alpha-1}\left(y^{-\frac{1}{\beta}} + \frac{\alpha}{\beta\gamma}\right) - \frac{k\lambda^\beta}{\gamma}\left(y^{-\frac{1}{\beta}} + \frac{1}{\gamma}\right)^\beta = 0. \quad \text{(A.10)}$$

The left-hand side of (A.10) is decreasing in λ, and hence y_2^* is decreasing in λ. Therefore p_2^* is decreasing in λ as well. $\qquad\square$

A.2 Proof of Proposition 6.3

We first show that the solution method in Sect. 3.1 can be extended to the problem with utility-discounting. A general exposition can be found in Dayanik and Karatzas [6], but we outline the key ideas under our specific model to introduce some notation to be used.

Let $\mathcal{A} := \frac{\sigma^2}{2}\frac{d^2}{dp^2} + \mu\frac{d}{dp}$ be the infinitesimal generator of P. The second order ordinary differential equation $\mathcal{A}u(p) = \delta u(p)$ with $\delta > 0$ admits $r_1(p) := p^{\omega_1}$ and $r_2(p) := p^{\omega_2}$ as two linearly independent solutions, where $\omega_1 < 0 < \omega_2$ are the distinct real roots to the quadratic equation

$$\frac{\sigma^2}{2}r^2 + \left(\mu - \frac{\sigma^2}{2}\right)r - \delta = 0. \tag{A.11}$$

Moreover, for $p \in [a, b] \subseteq \mathcal{J}$, the function $\varphi(p; a, b) := \mathbb{E}[e^{-\delta\tau_a}\mathbb{1}_{\{\tau_a < \tau_b\}}|P_0 = p]$ is the solution to $\mathcal{A}\varphi = \delta\varphi$ with boundary conditions $\varphi(a; a, b) = 1$ and $\varphi(b; a, b) = 0$, where $\tau_\ell := \inf\{t \geq 0 : P_t = \ell\}$ (likewise, $\vartheta(p; a, b) := \mathbb{E}[e^{-\delta\tau_b}\mathbb{1}_{\{\tau_b < \tau_a\}}|P_0 = p]$ has similar properties). From this, we obtain

$$\varphi(p; a, b) = \frac{r_2(b)r_1(p) - r_1(b)r_2(p)}{r_1(a)r_2(b) - r_2(a)r_1(b)}, \qquad \vartheta(p; a, b) = \frac{r_1(a)r_2(p) - r_2(a)r_1(p)}{r_1(a)r_2(b) - r_2(a)r_1(b)}.$$

Now consider a discounted optimal stopping problem of the form

$$V(p) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-\delta\tau}G(P_\tau)|P_0 = p].$$

As before, it is sufficient to look for an optimal stopping time which is of the form $\tau_{a,b} := \tau_a \wedge \tau_b$. Then

$$\begin{aligned}
J(p; \tau_{a,b}) &:= \mathbb{E}[e^{-\delta\tau_{a,b}}G(P_{\tau_{a,b}})|P_0 = p] \\
&= G(a)\mathbb{E}[e^{-\delta\tau_a}\mathbb{1}_{\{\tau_a < \tau_b\}}|P_0 = p] + G(b)\mathbb{E}[e^{-\delta\tau_b}\mathbb{1}_{\{\tau_b < \tau_a\}}|P_0 = p] \\
&= G(a)\varphi(p; a, b) + G(b)\vartheta(p; a, b) \\
&= G(a)\frac{r_2(b)r_1(p) - r_1(b)r_2(p)}{r_1(a)r_2(b) - r_2(a)r_1(b)} + G(b)\frac{r_1(a)r_2(p) - r_2(a)r_1(p)}{r_1(a)r_2(b) - r_2(a)r_1(b)} \\
&= r_1(p)\left(\frac{r_2(b)/r_1(b) - r_2(p)/r_1(p)}{r_2(b)/r_1(b) - r_2(a)/r_1(a)}\frac{G(a)}{r_1(a)}\right. \\
&\qquad\left. + \frac{r_2(p)/r_1(p) - r_2(a)/r_1(a)}{r_2(b)/r_1(b) - r_2(a)/r_1(a)}\frac{G(b)}{r_1(b)}\right) \\
&= r_1\left(s^{-1}(\theta)\right)\left(\frac{s(b) - s(p)}{s(b) - s(a)}\phi\left(s(a)\right) + \frac{s(p) - s(a)}{s(b) - s(a)}\phi\left(s(b)\right)\right),
\end{aligned}$$

where $s(x) := r_2(x)/r_1(x) = x^{\omega_2 - \omega_1}$, $\theta := s(p)$ and

$$\phi(x) := (G/r_1)\left(s^{-1}(x)\right) = x^{-\frac{\omega_1}{\omega_2 - \omega_1}}G\left(x^{1/(\omega_2 - \omega_1)}\right).$$

The optimal stopping rule can be deduced by maximising the above with respect to a and b. Upon replacing the dummy variables via $a' = s(a)$ and $b' = s(b)$, we have

$$V(p) = \sup_{a,b:a \leq p \leq b} J(p; \tau_{a,b})$$

$$= r_1\left(s^{-1}(\theta)\right) \sup_{a',b:a' \leq \theta \leq b'} \left(\frac{b' - \theta}{b' - a'}\phi(a') + \frac{\theta - a'}{b' - a'}\phi(b')\right)$$

$$=: v(\theta).$$

The supremum in the second to last term can be characterised by the smallest concave majorant $\bar{\phi}$ of the scaled payoff function $\phi = \frac{G}{r_1} \circ s^{-1}$, and the value function in the original coordinates is given by $V(p) = v(s(p)) = r_1(p)\bar{\phi}(s(p))$.

Proof of Proposition 6.3 Similarly to the baseline problem, (6.2) which features utility-discounting can be solved by decomposing the problem into the sub-problems of exit and entry. The discounted exit problem is

$$V_1(p; H) := \sup_{v \in \mathcal{T}} \mathbb{E}[e^{-\delta v} U(\gamma P_v - H)|P_0 = p],$$

where $H \geq 0$ is some given constant. The scaled payoff function is given by

$$g_1(\theta) := \theta^{-\frac{\omega_1}{\omega_2 - \omega_1}} U\left(\gamma \theta^{\frac{1}{\omega_2 - \omega_1}} - H\right)$$

$$= \begin{cases} -k\theta^{-\frac{\omega_1}{\omega_2 - \omega_1}}(H - \gamma \theta^{\frac{1}{\omega_2 - \omega_1}})^{\alpha} & \text{for } 0 \leq \theta < (\frac{H}{\gamma})^{\omega_2 - \omega_1}, \\ \theta^{-\frac{\omega_1}{\omega_2 - \omega_1}}(\gamma \theta^{\frac{1}{\omega_2 - \omega_1}} - H)^{\alpha} & \text{for } \theta \geq (\frac{H}{\gamma})^{\omega_2 - \omega_1}. \end{cases}$$

Note that $g_2(0) = g_2((\frac{H}{\gamma})^{\omega_2 - \omega_1}) = 0$ and $g_2(\theta) < 0$ on $\{0 < \theta < (\frac{H}{\gamma})^{\omega_2 - \omega_1}\}$. Furthermore, on $\{\theta > (\frac{H}{\gamma})^{\omega_2 - \omega_1}\}$, we have

$$g_1'(\theta) = \theta^{-\frac{\omega_2}{\omega_2 - \omega_1}}\left(\gamma \theta^{\frac{1}{\omega_2 - \omega_1}} - H\right)^{\alpha - 1}\left(-\frac{\omega_1(\gamma \theta^{\frac{1}{\omega_2 - \omega_1}} - H)}{\omega_2 - \omega_1} + \frac{\alpha \gamma \theta^{\frac{1}{\omega_2 - \omega_1}}}{\omega_2 - \omega_1}\right) > 0$$

and

$$g_1''(\theta) = \frac{\theta^{-\frac{\omega_1}{\omega_2 - \omega_1} - 2}(\gamma \theta^{\frac{1}{\omega_2 - \omega_1}})^2(\gamma \theta^{\frac{1}{\omega_2 - \omega_1}} - H)^{\alpha - 2}}{(\omega_2 - \omega_1)^2} h\left(1 - H\gamma^{-1}\theta^{-\frac{1}{\omega_2 - \omega_1}}\right),$$

where

$$h(z) := \omega_1\omega_2 z^2 + \alpha(1 - \omega_1 - \omega_2)z - \alpha(1 - \alpha).$$

We now show that on $[0, 1]$, the quadratic function $z \mapsto h(z)$ is strictly negative, which in turn will imply that g_1 is a strictly increasing and concave function for

$\theta \geq (\frac{H}{\gamma})^{\omega_2 - \omega_1}$. Note that $\omega_1 + \omega_2 = 1 - \frac{2\mu}{\sigma^2} = \beta$ by considering the sum of the roots of the quadratic equation (A.11). Then we have $h(0) = -\alpha(1 - \alpha) < 0$ and

$$h(1) = \omega_1 \omega_2 + \alpha(1 - \omega_1 - \omega_2) - \alpha(1 - \alpha) = (\alpha - \omega_1)(\alpha - \omega_2) < 0,$$

since $\omega_1 < 0$ and $0 < \alpha < \beta = \omega_1 + \omega_2 < \omega_2$. If $\beta \geq 1$, then

$$0 \geq \alpha(1 - \beta) = \alpha(1 - \omega_1 - \omega_2) = h'(0) > h'(1)$$

and h must be decreasing and in turn negative on $[0, 1]$. If $\beta < 1$ and $\alpha \geq -\frac{2\omega_1 \omega_2}{1-\beta}$, then

$$h'(0) > h'(1) = 2\omega_1 \omega_2 + \alpha(1 - \omega_1 - \omega_2) = 2\omega_1 \omega_2 + \alpha(1 - \beta) \geq 0$$

and h must be increasing and thus negative on $[0, 1]$. Finally, if $\beta < 1$ and $\alpha < -\frac{2\omega_1 \omega_2}{1-\beta}$, we can compute the discriminant of the quadratic function h as

$$\begin{aligned}
\triangle &:= \alpha^2 (1 - \omega_1 - \omega_2)^2 + 4\omega_1 \omega_2 \alpha(1 - \alpha) \\
&= \alpha^2 (1 - \beta)^2 + 4\omega_1 \omega_2 \alpha(1 - \alpha) \\
&< \alpha\big(-2\omega_1 \omega_2(1 - \beta) + 4\omega_1 \omega_2(1 - \alpha)\big) \\
&= -2\omega_1 \omega_2 \alpha\big((\alpha - 1) - (\alpha - \beta)\big) < 0,
\end{aligned}$$

and hence $h(z) < 0$ for all z.

Now, the smallest concave majorant of g_1 can be formed by drawing a straight line from $(0, 0)$ which touches g_1 at some $\theta^* > (\frac{H}{\gamma})^{\omega_2 - \omega_1}$. The point of contact is given by the unique θ^* satisfying $\frac{g_1(\theta^*)}{\theta^*} = g_1'(\theta^*)$. The required θ^* is thus given by the solution to the equation

$$\frac{\theta^{-\frac{\omega_1}{\omega_2 - \omega_1}}(\gamma \theta^{\frac{1}{\omega_2 - \omega_1}} - H)^\alpha}{\theta} = (\gamma \theta^{\frac{1}{\theta_2 - \theta_1}} - H)^{\alpha - 1} \theta^{-\frac{\omega_2}{\omega_2 - \omega_1}}$$

$$\times \left(\frac{\alpha \gamma \theta^{\frac{1}{\omega_2 - \omega_1}}}{\omega_2 - \omega_1} - \frac{\omega_1}{\omega_2 - \omega_1}(\gamma \theta^{\frac{1}{\omega_2 - \omega_1}} - H)\right)$$

which admits the explicit solution

$$\theta^* = \left(\frac{\omega_2 H}{\gamma(\omega_2 - \alpha)}\right)^{\omega_2 - \omega_1}.$$

The smallest concave majorant of g_1 is then

$$\begin{aligned}
\bar{g}_1(\theta) &= \bar{g}_1(\theta; H) \\
&= \begin{cases}
(\frac{\omega_2}{\gamma(\omega_2 - \alpha)})^{-\omega_2}(\frac{\alpha}{\omega_2 - \alpha})^\alpha H^{\alpha - \omega_2} \theta & \text{for } \theta < (\frac{\omega_2 H}{\gamma(\omega_2 - \alpha)})^{\omega_2 - \omega_1}, \\
\theta^{-\frac{\omega_1}{\omega_2 - \omega_1}}(\gamma \theta^{\frac{1}{\omega_2 - \omega_1}} - H)^\alpha & \text{for } \theta \geq (\frac{\omega_2 H}{\gamma(\omega_2 - \alpha)})^{\omega_2 - \omega_1}.
\end{cases}
\end{aligned}$$

The value function of the exit problem is thus

$$V_1(p; H) = r_1(p)\bar{g}_1(s(p)) = p^{\omega_1}\bar{g}_1(p^{\omega_2 - \omega_1}; H).$$

From the form of the value function, the optimal exit strategy is to sell the asset whenever its price level is at or above $p^* = s^{-1}(\theta^*) = \frac{\omega_2 H}{\gamma(\omega_2 - \alpha)}$.

Now we look at the entry problem. Unlike the problem without discounting whose objective function is (2.2), the strategy $\tau \equiv \infty$ now yields a value of zero rather than $U(-R)$ under utility-discounting. Hence the payoff function for the entry problem is $\max\{V_1(P_\tau; \lambda P_\tau + \Psi + R), 0\}$ rather than $\max\{V_1(P_\tau; \lambda P_\tau + \Psi + R), U(-R)\}$. The entry problem is thus

$$V_2(p) := \sup_\tau \mathbb{E}[e^{-\delta\tau}\max\{V_1(P_\tau; \lambda P_\tau + \Psi + R), 0\}|P_0 = p].$$

It is also clear that V_1 is nonnegative. Then the scaled payoff function of the entry problem is

$$g_2(\theta) := \theta^{-\frac{\omega_1}{\omega_2 - \omega_1}} V_1\left(\theta^{\frac{1}{\omega_2 - \omega_1}}; \lambda\theta^{\frac{1}{\omega_2 - \omega_1}} + \Psi + R\right)$$

$$= \bar{g}_1\left(\theta; \lambda\theta^{\frac{1}{\omega_2 - \omega_1}} + \Psi + R\right)$$

$$= \left(\frac{\omega_2}{\gamma(\omega_2 - \alpha)}\right)^{-\omega_2}\left(\frac{\alpha}{\omega_2 - \alpha}\right)^\alpha \left(\lambda\theta^{\frac{1}{\omega_2 - \omega_1}} + \Psi + R\right)^{\alpha - \omega_2}\theta.$$

The last equality holds because

$$\left(\frac{\omega_2}{\gamma(\omega_2 - \alpha)}\right)^{\omega_2 - \omega_1}\left(\lambda\theta^{\frac{1}{\omega_2 - \omega_1}} + \Psi + R\right)^{\omega_2 - \omega_1} \geq \left(\frac{\omega_2}{\gamma(\omega_2 - \alpha)}\right)^{\omega_2 - \omega_1}\lambda^{\omega_2 - \omega_1}\theta$$

$$= \left(\frac{\omega_2}{\omega_2 - \alpha}\right)^{\omega_2 - \omega_1}\left(\frac{\lambda}{\gamma}\right)^{\omega_2 - \omega_1}\theta$$

$$> \theta$$

due to the facts that $R > 0$, $\Psi \geq 0$, $\omega_1 < \omega_2$ and $\gamma \leq 1 \leq \lambda$, and thus the linear regime of $\bar{g}_1$ always applies when evaluating $\bar{g}_1(\theta; \lambda\theta^{\frac{1}{\omega_2 - \omega_1}} + \Psi + R)$. It remains to identify the smallest concave majorant $\bar{g}_2$ of g_2. But by similar (and indeed less tedious) calculations as for the exit problem, one can verify that $\theta \mapsto (\lambda\theta^{\frac{1}{\omega_2 - \omega_1}} + \Psi + R)^{\alpha - \omega_2}\theta$ is an increasing concave function for all $\theta \geq 0$ under the condition $\alpha < \beta$. We hence must have $g_2 = \bar{g}_2$ and the strategy $\tau \equiv 0$ is optimal. $\qquad \square$

Declarations

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