Imperial College London

Robust Model Predictive Control for Linear Systems Subject to Norm-Bounded Model Uncertainties and Disturbances: An Implementation to Industrial Directional Drilling System

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Statement of originality

I hereby declare that this thesis is the product of my own endeavour, and that any ideas or quotations from the work of other people, published or otherwise, are appropriately referenced.

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Abstract

Model Predictive Control (MPC) refers to a class of receding horizon algorithms in which the current control action is computed by solving online, at each sampling instant, a constrained optimization problem. MPC has been widely implemented within the industry, due to its ability to deal with multivariable processes and to explicitly consider any physical constraints within the optimal control problem in a straightforward manner. However, the presence of uncertainty, whether in the form of additive disturbances, state estimation error or plant-model mismatch, and the robust constraints satisfaction and stability, remain an active area of research. The family of predictive control algorithms, which explicitly take account of process uncertainties/disturbances whilst guaranteeing robust constraint satisfaction and performance is referred to as Robust MPC (RMPC) schemes.

In this thesis, RMPC algorithms based on Linear Matrix Inequality (LMI) optimization are investigated, with the overall aim of improving robustness and control performance, while maintaining conservativeness and computation burden at low levels. Typically, the constrained RMPC problem with state-feedback parameterizations is nonlinear (and nonconvex) with a prohibitively high computational burden for online implementation. To remedy this issue, a novel approach is proposed to linearize the state-feedback RMPC problem, with minimal conservatism, through the use of semidefinite relaxation techniques and the Elimination Lemma. The proposed algorithm computes the state-feedback gain and perturbation online by solving an LMI optimization that, in comparison to other schemes in the literature is shown to have a substantially reduced computational burden without adversely affecting the tracking performance of the controller.

In the case that only (noisy) output measurements are available, an output-feedback RMPC algorithm is also derived for norm-bounded uncertain systems. The novelty lies in the fact that, instead of using an offline estimation scheme or a fixed linear observer, the past input/output data is used within a Robust Moving Horizon Estimation (RMHE) scheme to compute (tight) bounds on the current state. These current state bounds are then used within the RMPC control algorithm. To reduce conservatism, the output-feedback control gain and control perturbation are both explicitly considered as decision variables in the online LMI optimization.

Finally, the aforementioned robust control strategies are applied in an industrial directional drilling configuration and their performance is illustrated by simulations. A rotary steerable system (RSS) is a drilling technology that has been extensively studied over the last 20 years in hydrocarbon ex-

ploration and is used to drill complex curved borehole trajectories. RSSs are commonly treated as dynamic robotic actuator systems, driven by a reference signal and typically controlled by using a feedback loop control law. However, due to spatial delays, parametric uncertainties, and the presence of disturbances in such an unpredictable working environment, designing such control laws is not a straightforward process. Furthermore, due to their inherent delayed feedback, described by delay differential equations (DDE), directional drilling systems have the potential to become unstable given the requisite conditions. To address this problem, a simplified model described by ordinary differential equations (ODE) is first proposed, and then taking into account disturbances and system uncertainties that arise from design approximations, the proposed RMPC algorithm is used to automate the directional drilling system.

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CHAPTER 1

Introduction

Over the past few decades MPC has been exploited for many researches and its application in the industrial processes has found great success [1]. MPC is inarguably the most widely accepted modern optimal control strategy [2]. This is mainly due to the fact that, compared to many traditional control algorithms (for example PID controller), it explicitly considers process constraints within its formulation. This allows the MPC algorithm to operate a plant closer to their constraint boundaries (without any violation) which enables optimal performance. Other advantages of MPC include its ability to handle multivariable, non-minimal phase and unstable processes, as well as a comparably straightforward way of tuning the controller. Numerous aspects of this class of advanced control algorithms have been the subject of extensive research over the last two decades for both linear and nonlinear systems [2–5]. For instance, stability conditions of the MPC schemes have been investigated through the use of terminal cost or by extending the prediction horizon of the optimal control problem [3]. A number of predictive control schemes within the literature have been proposed for deterministic systems (see for example [3, 6–8] and the references therein).

Although the MPC scheme is less conservative compared to other schemes, it is computationally com-

plex and requires more online time to find a solution which makes it inappropriate for fast dynamic systems. Another disadvantage of this model-based optimal control scheme is that the effectiveness of the controller relies to a large extent on the accuracy of the dynamic model utilised to characterise the plant. However, uncertainty, in the form of additive disturbances, state estimation error, a plant-model mismatch is generally present in most industrial implementations. Therefore, robust constraint satisfaction and stability remain active areas of research [2, 9–13]. In this respect, the field of Robust MPC (RMPC) is wide open for research to come up with novel ideas that will close the gap with respect to optimal performance, robustness, and computation time and its application to industrial processes.

1.1 Background and Motivation

In this section, a brief overview on state and output feedback robust MPC schemes, estimation algorithms and the industrial application of directional drilling problem are presented.

1.1.1 State-feedback Robust Model Predictive Control

The family of predictive control algorithms, which explicitly take account of process uncertainties/disturbances whilst guaranteeing robust constraint satisfaction and performance is referred to as robust MPC schemes [2]. An obvious approach to extend the MPC problem for uncertain systems is to solve an open-loop optimal control problem as is done with nominal MPC. Whilst this is attractive from a computational complexity perspective, it often leads to infeasibility and suboptimality [3]. A more effective method is to consider a state-feedback control law, as shown in [12], where state-feedback parameterization has been used in RMPC for systems subject to additive disturbances. By considering future inputs as linear/nonlinear functions of current and future predicted states, feedback RMPC schemes mitigate the effect of uncertainties whilst potentially avoiding the infeasibility issues. Nonlinear feedback schemes enjoy reduced conservatism, however, their main drawback is the excessive online computational burden due to the combinatorial nature of the optimization. Therefore, this

work focuses on linear state-feedback RMPC schemes. The three main types of RMPC schemes proposed in the literature include feedback min-max MPC, tube-MPC and LMI based RMPC schemes. The min-max MPC method computes the optimal control sequence that satisfies the constraints and steers the uncertain system to a robust positively invariant set whilst guarding against the worst-case uncertainty [9, 10]. The second approach, which has received significant attention in the recent years, is the tube-MPC (TMPC) class of algorithms [11]. These algorithms, instead of considering the worst-case uncertainty, decomposes RMPC into an offline robust controller design (calculation of "tubes" based on invariant sets) and online open-loop MPC problem based on a nominal system trajectory (without uncertainties). Then, this approach guarantees that all possible closed-loop state trajectories of the uncertain system lie inside a “tube” around the future prediction nominal trajectory, where the tube is computed offline by using the uncertainty bounds [6, 11]. Although min-max MPC and TMPC methods have a reduced online computational burden, they both rely on offline calculations that can lead to conservatism within the overall robust control scheme. An alternative approach to RMPC is to use semidefinite programming to compute, online, an optimal control sequence by solving an LMI optimization problem [14–16]. The two main advantages of the LMI based RMPC method are the explicit incorporation of uncertainty description within the optimization and the polynomial time that the optimization problem requires for its solution, which, although still high compared with min-max MPC and TMPC methods, allows online implementation [17] for certain problems. Further details around the implementation issues and trade-offs between min-max RMPC, TMPC and MPC via LMI techniques have been reviewed and quantified in [18]. Ideally, to reduce conservatism due to offline calculations, the desirable approach in linear state-feedback RMPC is to directly consider the control feedback gains as decision variables in the online optimization. However, as noted in [12], formulating such an RMPC problem in the standard way leads to sequences of predicted states and inputs that are nonlinear functions of the state-feedback gains, which renders the problem nonlinear and nonconvex. A solution to this problem has been proposed in [19, 20] where the state-feedback gains are computed through sequential online optimization based, in part, on the principles of dynamic programming. In most of the work described above, the focus has been on systems that involve only disturbances/noise or simple scalar uncertainties. A generalization of RMPC to systems subject to structured uncertainties and disturbances was proposed in tube-MPC format in [21–23] and in ongo-

ing research on System Level Synthesis (SLS) [24,25]. An LMI based RMPC approach was proposed in [14] and used in an industrial directional drilling application in [26]. In this LMI based scheme, the state-feedback gain and control perturbation are computed online whilst avoiding the nonconvexity issues. Although this approach shows significant performance improvement, it introduces a large online computational burden, which makes it unsuitable for fast dynamical systems. Therefore, computationally efficient RMPC formulation without sacrificing the robust control performance remain an active area of research by the control community.

1.1.2 Robust Estimation

In most industrial applications the states that characterise the dynamics of a system are not physically measurable and only noisy output measurements through sensors are available. Thus, state estimation plays an important role in different engineering areas such as feedback control, fault detection, system monitoring, as well as system optimization. One of the most popular approaches for state estimation, in a general context, is Kalman filtering, which is based on the minimization of the variance of the estimation error [27]. However, the main assumptions in the standard Kalman filter approach are that the state-space model of the linear system does not include any uncertainty and thus it accurately represents the real system, and also there are no constraints on the states. As these premises are not satisfied in many industrial applications, the standard Kalman filter may not have robust properties against an uncertain model with disturbances [28].

Recent studies in the literature, which investigate output-feedback robust control schemes, mostly employ a fixed stable linear observer, such as a Luenberger observer, to compute an estimate of the linear system-state, which is subsequently used within the control scheme (see for example [29–32]). The main assumption in [29] and [31] is that the observer has to run for a sufficiently long time before implementing the control scheme, in order to allow the estimation error to enter an invariant set. It is clear that the choice of observer gain has an impact on the estimation error bounds and, therefore, on the overall control algorithm. However, in most of the aforementioned schemes the observer is designed offline (to ensure stability). Consequently, all of the aforementioned offline calculations can potentially add to the conservatism of the corresponding control algorithm.

A very promising online approach to the estimation problem is the so called Moving Horizon Estimation (MHE). Originally proposed by [33] in the early 90s, the estimation scheme suggests estimating the state of a dynamic system by using only the input/output information of the system over the most recent time interval. MHE is a filtering scheme that can be solved online and it can successfully overcome the previously mentioned problems introduced by offline calculations. In the last decade, MHE has become a very popular topic of investigation and its application to linear and non-linear systems has achieved significant success [34–39].

Despite the plethora of MHE algorithms proposed in the literature, the contributions when the system is uncertain are negligible. In [40] the minimization of an upper bound on a worst-case quadratic cost defined over a moving horizon window allows one to construct a filter for uncertain linear systems. This design method is based on the solution of min-max regularized least-squares problems [41]. However, robust least-squares problems are known to have computational difficulties reaching a solution, since they are in general NP-hard [42].

1.1.3 Output-feedback Robust Model Predictive Control

As mentioned above, in most practical systems of interest, only (noisy) measurements of output are available. Predictive control algorithms for such systems are known as Output-Feedback MPC (OF-MPC) schemes and we discuss these next. Most of the OF-MPC studies proposed in the literature use the system output signal to estimate the state which is subsequently used within a state-feedback RMPC scheme, due to its ability to tolerate state estimation error (see e.g. [29–31, 43, 44]). Since the exact value of the estimation error is unknown, an offline estimation policy is used to replace state estimation error by its outer bound. State estimation error is generally assumed to be bounded by an invariant set and is considered as a source of disturbance within the system. One of the major advantages of schemes such as [29] and [31] is that their online computational complexity is similar to that of (full-state) nominal MPC schemes. An OF-MPC approach based on LMI/BMI optimization for systems subject to norm-bounded parametric uncertainty and disturbances is developed in [45], where the estimation bound is pre-specified (offline) as a constraint of the optimization problem. Løvaas et. al. proposed an OF-MPC approach for system subject to unstructured model uncertainty,

where the feedback control gain is pre-specified offline and is used as a known parameter for the online optimization problem [46]. In the dynamic output feedback RMPC approach proposed in [47], the state estimation gain and control gain are considered as a decision variables and the ellipsoid estimation bound is refreshed at each sample time. Although the closed-loop system in this method is proven to be quadratic-bounded, the resulting optimization problem is computationally demanding since the optimization problem needs to be solved by an iterative cone complementary method. An extension to the above work is presented in [48], where the estimation state matrix instead of estimation gain is considered as a decision variable and the optimization problem is expressed in an LMI form. Finally, Vilaivannaporn el. at. have recently proposed a new robust output feedback predictive controller for systems subject to disturbances and measurement noise, where adaptive invariant tubes for both estimation and control error, as well as, observer gains are updated at each sampling time [49].

It is clear that the choice of observer gain has an impact on the estimation error bounds and, therefore, on the overall control algorithm. However, in most of the aforementioned schemes, the observer is simply designed offline (to ensure stability). Moreover, in some cases the control feedback gain K is also assumed to be known and fixed. Both of these factors can potentially add to the conservatism of the corresponding robust control algorithm. Furthermore, it is common in the output-feedback RMPC algorithm to use state estimates to calculate the control signal instead of directly using the output measurement. As a result, the controller's performance is constrained by the estimation accuracy.

1.1.4 Directional Drilling Problem

For more than a century now, oil and gas industry has constantly searched for more economic and more efficient technologies to exploit hydrocarbon energy resources [50, 51]. The process to obtain and extract hydrocarbon energy resources such as oil and gas which remain the major fuels for powering today's society, experience two major difficulties. Firstly, access to energy resources sometimes requires boreholes with complex curves, which is not a simple task with conventional drilling systems. Secondly, deep-seated and offshore hydrocarbon explorations are commonly under an unpredictable environment and extreme working conditions, while targeting resource locations in the crust of earth

[21]. The solution of these difficulties could be tackled by drilling systems called Rotary Steerable Systems (RSS) [52]. In the beginning, when RSS system was introduced, actuation commands for steering the drilling mechanism to a pre-defined path were determined by experienced professionals using the tool past performance data and available real-time measurements. While it matured, to avoid humans error, RSS technology started to be treated like robotic actuator systems, and the trajectory tracking problem today is solved by control automation, typically controlled by a control unit using feedback loop control law [53, 54]. Unfortunately, designing such a control law that tracks a pre-defined reference signal is not a straightforward process. The main difficulty of developing a control algorithm for the RSS system is the lack of knowledge about the dynamic system which characterises the behaviour of the system. Previous research studies considered empirical or numerical models with conventional controllers such as PID [55], however, these models could not fully reflect the dynamic behaviour and variations of the system. As a result, the control law is very difficult to be applied in real-time in directional drilling applications. Another important issue for the design of a controller of the RSS systems is that conventional control laws are not able to handle model uncertainties and disturbances, which are caused by design approximations on the system's model and unpredictable working environment, respectively. Recently, researchers investigating the behaviour of RSS systems in directional drilling applications have proposed a three-dimensional analytical model using non-linear delay differential equation (DDE) [56]. The analytical model of RSS has been very promising since it can characterise the behaviour of the system with minimum error. Using the framework of the RSS analytical model, the aim of this study is to develop an appropriate control law that can guarantee robustness and stability in the presence of the aforementioned uncertainties and disturbances, while physical and safety constraints are preserved.

1.2 Challenges and Contribution

1.2.1 Research Challenges

The most important challenges addressed in this thesis can be classified into two categories as listed bellow:

1. Control and Estimation designs for Uncertain System

   - *Robust performance under structured uncertainties and disturbances.* Most of the algorithms presented in the literature consider either external disturbance signals or structured uncertainties within the formulation of RMPC scheme, but not both at the same time, due to the complex nature of the optimization problem that arises.
   - *Computation time.* Traditional online RMPC algorithms have a heavy computational burden. Therefore, the existing RMPC algorithms are barely used in fast dynamic systems. Finding feasible solutions to the RMPC problem, while minimising the computation effort, is significantly important.
   - *Optimised estimation error.* Output feedback RMPC algorithms are normally associated with offline observer schemes. Offline calculations lead to a larger state estimation error which can impact on the overall control performance. Estimating the states of the system by solving an optimization problem online can provide better control accuracy and less conservativeness.
   - *Problem feasibility subject to constraints.* Computing an initial feasible solution and ensuring the recursive feasibility of a constrained optimization problem is a very important condition in control synthesis.

2. Directional drilling automation

   - *Dynamic model for directional drilling system.* Conventional Directional drilling system representations based on literature utilised either less accurate kinematic system models, or comprehensive dynamic models that are far more complex and are presented in terms of delay differential equation (DDE).
   - *Implementation.* Factors such as the location of the controller (down-hole or at the surface), location of sensors and available data, time delays, computational time, and safety constraints are some of the key features which have to be considered in the control design and implementation. Addressing those factors on the control formulation that is currently located on the surface or embedding a control unit down-hole, are extremely challenging topics.

1.2.2 Contribution and Thesis Configuration

The main objective of this thesis is to develop Robust MPC strategies based on LMI optimization to automated complex industrial systems, based on approximated models subject to uncertainties and additive disturbances. A brief description and the contribution of each of the upcoming chapters are provided below.

Chapter 2 presents some fundamental concepts from optimization theory such as convex optimization, quadratic and semidefinite programming, linear matrix inequalities, and Schur complement. We also discuss the S-procedure which is an effective technique to re-formulate non-convex optimizations into (convex) LMI problems and is of key importance to the developments of the following chapters. Lastly, a summary of the two main identification procedures used to model an uncertain system is presented.

In Chapter 3 the problem of RMPC of linear-time-invariant discrete-time systems subject to structured uncertainty and bounded disturbances is investigated. Typically, the constrained RMPC problem with state-feedback parameterizations is nonlinear (and nonconvex) with a prohibitively high computational burden for online implementation. To tackle this issues, a novel linearization procedure is proposed for the state-feedback RMPC problem, with minimal conservatism, through the use of Elimination Lemma and semidefinite relaxation techniques. The proposed algorithm computes the state-feedback gain and perturbation online by solving an LMI optimization, where is shown to have a substantially reduced computational burden without adversely affecting the tracking performance of the controller. To improve the scalability of the control algorithm for systems with faster dynamics, a single LMI sufficient condition for all the constraints is provided, to further reduce the online computation time of the control algorithm. Additionally, an offline strategy that guarantees feasibility on the RMPC problem is presented. The effectiveness of the proposed scheme is demonstrated through numerical examples from the literature.

Chapter 4 investigates the problem of state estimation for LTI discrete-time systems subject to structured feedback uncertainty and bounded disturbances. The proposed robust moving horizon estimation scheme computes at each sample time tight bounds on the uncertain states by solving an LMI

optimization problem based on the available noisy input and output data. In comparison with conventional approaches that use offline calculation for the estimation, the suggested scheme achieves an acceptable level of performance with reduced conservativeness, while the online computational time is maintained relatively low. The effectiveness of the proposed estimation method is assessed via a numerical example.

Using the results from Chapters 3 and 4, an output-feedback RMPC scheme for norm-bounded uncertain systems is proposed in Chapter 5, considering that only the noisy output measurements are available. The novelty lies in the fact that, instead of using an offline state estimation scheme or a fixed linear observer, the past input/output data is used within a RMHE scheme to compute tight bounds on the current state. These current state bounds are then used within the output-feedback RMPC control algorithm. To reduce conservatism, the output-feedback control gain and control perturbation are both explicitly considered as decision variables in the LMI optimization. Additionally, an offline strategy that guarantees feasibility on the RMPC problem is presented. Numerical examples from the literature are used to demonstrate the advantages of the proposed scheme.

In Chapter 6, the proposed robust control and estimation schemes presented in this thesis, are utilized to automate an industrial directional drilling system. The complex bottom hole assembly rotary steerable system is first approximated by a simplified model described by ordinary differential equations in a state-space closed-form representation. Then disturbances and system uncertainties that arise from design approximations are considered within the formulation of RMPC in order to avoid constraints violation imposed by the problem specifications. The stability and computational efficiency of the scheme are improved by a state feedback strategy computed offline using Robust Positive Invariant (RPI) sets control approach and model reduction techniques. A crucial advantage of the proposed control scheme is that it computes an optimal control input considering physical and designer constraints. The control strategy is applied in an industrial directional drilling configuration represented by a DDE model and its performance is illustrated by simulations.

Finally, in Chapter 7 an overall summary of the main contributions of the thesis is provided and potential future research directions are suggested.

1.2.3 Publications

The research presented in this thesis is subject to the following publication:

- A. Georgiou, S. A. Evangelou, I. M. Jaimoukha, and G. Downton, “Tracking control for directional drilling systems using robust feedback model predictive control,” *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 11 974–11 981, 2020, 21st IFAC World Congress.
- A. Georgiou, F. Tahir, S. A. Evangelou, I. M. Jaimoukha,"Robust Moving Horizon State Estimation for Uncertain Linear Systems using Linear Matrix Inequalities", *59th IEEE Conference on Decision and Control (CDC)*, Jeju-Korea, 2020
- A. Georgiou, F. Tahir, S. A. Evangelou, I. M. Jaimoukha, “Computationally Efficient Robust Model Predictive Control for Uncertain System using Causal State-Feedback Parameterization.” *IEEE Transaction in Automatic Control. Under review*
- A. Georgiou, F. Tahir, S. A. Evangelou, I. M. Jaimoukha, “Robust Output-feedback Model Predictive Control using Input/output Data” *IEEE Transaction in Automatic Control. In Preparation.*

Contribution to Robust control in other publications not included in this thesis:

- S. Yu, X. Pan, A. Georgiou, B. Chen, I.M. Jaimoukha and S. A. Evangelou, “Robust Model Predictive Control Framework for Energy-Optimal Adaptive Cruise Control of Battery Electric Vehicles”, *Accepted in European Control Conference, London, 2021*
- Z. Feng, A. Georgiou, M. Yu, S. A. Evangelou, I. M. Jaimoukha and D. Dini, “LMI-based Robust Model Predictive Control for a quarter car model of series active variable geometry suspension subject to model uncertainties and safety constraints.” *IEEE Transaction on Control System Technology. In Preparation.*

1.3 Notation

The notation used is fairly standard. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the space of n-dimensional real (column) vectors, $\mathbb{R}^{n\times m}$ denotes the space of $n\times m$ real matrices, and $\mathbb{D}^n$ denotes the space of diagonal matrices in $\mathbb{R}^{n\times n}$. For $A\in\mathbb{R}^{n\times m}$, A^T denotes the transpose of A and for $A\in\mathbb{R}^{n\times n}$, $\mathscr{H}(A):=A+A^T$. If $A\in\mathbb{R}^{n\times n}$ is symmetric, $\underline{\sigma}(A)$ denotes the smallest eigenvalue of A. We call $A=A^T$ positive semidefinite and we write $A\succeq 0$ if $\underline{\sigma}(A)\geq 0$ and we call A positive definite and write $A\succ 0$ if $\underline{\sigma}(A)>0$. Analogous definitions apply to the largest eigenvalue $\overline{\sigma}(A)$, with respect to $A\preceq 0$ (negative semidefinite) and $A\prec 0$ (negative definite). For $x,y\in\mathbb{R}^n$, the inequality $x<y$ (and similarly $\leq$, $>$ and $\geq$) is interpreted element-wise. The notation I_q denotes the $q\times q$ identity matrix with the subscript omitted when it can be inferred from the context. For matrices $A_1,\ldots,A_m$, $\mathrm{diag}(A_1,\ldots,A_m)$ denotes a block diagonal matrix whose i-th diagonal block is A_i. The symbol e_i denotes the i-th column of the identity matrix of appropriate dimension. If $\boldsymbol{U}\subseteq\mathbb{R}^{p\times q}$ is a subspace, then $\mathscr{B}\boldsymbol{U}=\{U\in\boldsymbol{U}:UU^T\preceq I\}$ denotes the unit ball of $\boldsymbol{U}$. Finally, for matrices A and B, $A\otimes B$ denotes the Kronecker product.

CHAPTER 2

Background Theory

In this chapter, theoretical background material that is relevant to the context of MPC formulations is presented. In particular, the basic concept of convex optimization is briefly discussed in Section 2.1, followed by an introduction to quadratic programming, which is given in Section 2.2. The branch of convex optimization called Semi-definite Programming is presented in Section 2.3, along with some important techniques used throughout this project, such as linear matrix inequalities and Schur complement. In Section 2.4, a procedure to transform a nonconvex optimization problem to semi-definite programming is illustrated. Finally, in Section 2.5 the two main identification procedures used to model an uncertain system are presented.

2.1 Convex Optimization

As discussed in Chapter 1, MPC is an optimization-based control technique. In particular, an optimization problem is solved online, at each sampling instant, to compute the optimal control sequence.

Therefore, it is essential that the formulated optimization problem is such that it can be solved in an efficient manner - within the sampling interval. One such class of problems are the convex optimization problems [57].

Recall that convex optimization problems are of the general form:

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq d_i, \quad i = 1, \ldots, m \end{aligned} \tag{2.1}$$

where $g_i(x) \leq d_i$ represents convex constraints and $f(x)$ is the convex cost function to be minimized. These two components of optimization in (2.1) are quite significant and we briefly discuss each of them below.

Definition 2.1.1 *A set C is convex if, for any x_1, $x_2 \in C$, and a such that $0 \leq a \leq 1$, the following relation holds*

$$ax_1 + (1-a)x_2 \in C \tag{2.2}$$

Similarly, a convex function can be defined as follows.

Definition 2.1.2 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if its domain is a convex set and if for every pair of points x_1, x_2 in the domain of f, and α such that $0 \leq \alpha \leq 1$, the following inequality is satisfied:*

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \tag{2.3}$$

Recall that in the context of (2.1), the advantage of minimizing a convex function subject to convex constraints is that any local minimum of the problem is also a global minimum. Furthermore, for strictly convex functions (i.e. functions for which inequality (2.3) is strict), the minimum (if it exists) is unique. Algorithms, such as interior point methods [11], exploit these properties and are thus able to solve convex problems in an efficient, fast and reliable manner.

Convex optimization methods also play an important role in solving nonconvex problems. Algorithms for solving nonconvex and nonlinear optimization problems are generally inefficient. One approach

to solving such problems is to consider local optimization methods which yield a locally optimal solution. However, these methods require an initial solution of the decision variables as a starting point, which is a critical factor in the algorithm convergence. In such cases, an approximate convex formulation can be obtained for the original nonconvex problem (see Section 2.4). Then, the solution of the (approximate) convex problem, which is easily computed, can be used as the initial condition for the local optimization.

Convex optimization subsumes a large class of problems. For example, an important type of problems are the so-called Linear Programs (LP). These are of the form:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && \alpha_i^T x \leq b_i, \;\; i = 1, \ldots, m \end{aligned} \tag{2.4}$$

where the vectors c, $\alpha_i \in \mathbb{R}^n$ and scalars $b_i \in \mathbb{R}$. Note that the cost function and constraints in (2.4) are both linear and, therefore, convex. Another key class of optimization problems are the convex Quadratic Programs (QP), and is presented next.

2.2 Quadratic programming

Quadratic programming is an optimization program for minimising a quadratic cost function subject to linear equality and inequality constraints [58]. The quadratic programming problem has the following structure:

$$\begin{aligned} &\min_x \;\; \frac{1}{2} x^T H x + f^T x \\ &subject \;\; to \begin{cases} Ax \leq b \\ A_{eq} x = b_{eq} \\ lb \leq x \leq ub \end{cases} \end{aligned} \tag{2.5}$$

where $Ax \leq b$ and $A_{eq}x = b_{eq}$ denote the linear inequality and linear equality constraints on variable x, respectively; lb and ub represent the lower and upper bounds on x. In order to ensure that only one unique solution exists for this problem, the matrix $H = H^T \in \mathbb{R}^{n\times n}$ is required to be positive definite $(H \succ 0)$. Quadratic programming is widely used in the control and estimation fields, and is the most common method to solve effectively the nominal Model Predictive Control problem.

2.3 Semidefinite Programming

In the context of robust optimization and RMPC formulations, a particularly important class of convex optimization problems are the so-called semidefinite programs, which we discuss next.

Semidefinite programming has attracted substantial research interest over the past few decades [59]. This is because semidefinite programs (SDPs) have extensive application in system and control theory as well as other fields such as combinatorial and robust optimization. Also, importantly, there exist efficient algorithms to solve SDPs, for instance interior point methods [60]. SDPs are convex optimization problems which involve the minimization of a linear function subject to a constraint that requires a symmetric matrix - which is affine in the decision variables - to be positive semidefinite. In particular, an SDP can be written as:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && F(x) \succeq 0 \end{aligned} \tag{2.6}$$

with

$$F(x) := F_0 + \sum_{i=1}^{n} x_i F_i \tag{2.7}$$

where $x \in \mathbb{R}^n$ is the decision variable, with x_i denoting the i-th entry of x, and symmetric matrices $F_0, F_i \in \mathbb{R}^{m\times m}$, are given for all i. Note that for the case when all the matrices $F_0, \ldots, F_n$ are diagonal, the constraint in (2.6) becomes equivalent to m linear inequalities. Hence, in this case, the SDP problem simply reduces to a linear program of the form given in (2.4). The constraint in (2.6) is more

generally known as a Linear Matrix Inequality and we briefly discuss these next.

2.3.1 Linear Matrix Inequalities

Linear Matrix Inequality (LMI) techniques play an important role in the formulation of various problems within system and control theory [17]. For instance, one of the most widely used LMI conditions is the Lyapunov inequality for establishing stability [61, Section 2.5.2].

The robust predictive control algorithms proposed in this thesis are also mostly based on LMI constraints, which are formally defined as:

$$F(x) \succeq 0, \quad F(x) := F_0 + \sum_{i=1}^{n} x_i F_i. \tag{2.8}$$

Note that the symmetric matrix $F(x)$ is affine in variable $x \in \mathbb{R}^n$ and is required to be positive semidefinite, i.e. $y^T F(x) y \succeq 0, \forall y$. Furthermore, (2.8) represents a convex constraint on x. Strict inequalities (i.e. positive definite or negative definite) or negative semidefinite inequalities can also be defined analogously. In certain cases, optimization problems involve multiple LMI constraints, for instance:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && F^k(x) \succeq 0, \quad k = 1, \ldots, p \end{aligned} \tag{2.9}$$

with

$$F^k(x) := F_0^k + \sum_{i=1}^{n} x_i F_i^k, \quad i = 1, \ldots, n \tag{2.10}$$

Such problems can be readily transformed to an SDP of standard form (2.6), as follows:

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && \mathscr{L}(x) := \operatorname{diag}(F^1(x),\ F^2(x), \ldots,\ F^p(x)) \succeq 0 \end{aligned} \tag{2.11}$$

Finally, an important LMI result, which will be used extensively in the development throughout this thesis is known as the Schur complement [62]. This is a result to represent some convex nonlinear matrix inequalities in the form of LMIs without any conservatism, and is given by the following

lemma [62]:

Lemma 2.3.1 *Define matrices $A = A^T$, $C = C^T$ and B of appropriate dimensions and let*

$$\mathscr{L} := \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

Then, for $C \succ 0$, the matrix $\mathscr{L} \succeq 0$ if and only if $A - BC^{-1}B^T \succeq 0$. Similarly, for $A \succ 0$, the matrix $\mathscr{L} \succeq 0$ if and only if $C - B^T A^{-1} B \succeq 0$. Furthermore, the following three statements are also equivalent

(i) $\mathscr{L} \succ 0$

(ii) $C \succ 0$ *and* $A - BC^{-1}B^T \succ 0$

(iii) $A \succ 0$ *and* $C - B^T A^{-1} B \succ 0$

2.4 Semidefinite relaxation

In various fields of engineering, such as robust control design, communications and signal processing, one often encounters many important optimization problems that are computationally intractable (for example nonlinear nonconvex problems). For such optimizations, it is generally very difficult to compute the (global) solution, that is if one even exists [63]. In these cases, semidefinite relaxation provides a useful technique to obtain an (approximate) convex formulation for the original nonconvex optimization problem, in the form of an SDP (2.6), see e.g. [64,65]. The solution of the SDP generally serves as a good approximation to the actual optimal solution for the nonconvex problem. In fact, under certain conditions, semidefinite relaxation does not introduce any conservatism and hence, the SDP solution corresponds exactly to the optimal solution. As we will show in this thesis, feedback RMPC formulations for uncertain systems of the form (1.10) also result in optimization problems which are nonlinear and nonconvex in the decision variables (the control gain K). Therefore, we propose to obtain convexity through the application of semidefinite relaxation techniques to derive

RMPC algorithms based on SDP problems. Such an approach has the advantage that the resulting SDPs are solved very efficiently using interior point methods [17]. This, therefore means that the proposed RMPC control law can easily be computed online in polynomial time [15].

2.4.1 The S-procedure

The S-procedure is a technique that is used to relax some nonlinear, nonconvex optimizations and obtain their SDP approximations [61, 66]. It has found great application in many problem areas within control theory. The S-procedure can formally be defined as follows [61, Section 2.6.3].

Lemma 2.4.1 *Let $F_0, \ldots, F_p$ be quadratic functions of the variable $x \in \mathbb{R}^n$ such that:*

$$F_i := x^T T_i x + 2u_i^T x + \upsilon_i, \quad i = 0 \ldots, \ p \tag{2.12}$$

where $T_i = T_i^T$. Then, the following condition

$$F_0(x) \geq 0 \ \ \forall x, \text{ such that } F_i(x) \geq 0, \ i = 1, \ldots, p \tag{2.13}$$

holds if there exist $\tau_i \geq 0, \ldots, \ \tau_p \geq 0$ such that

$$\begin{bmatrix} T_0 & u_0 \\ u_0^T & \upsilon_0 \end{bmatrix} - \sum_{i=1}^{p} \tau_i \begin{bmatrix} T_i & u_i \\ u_i^T & \upsilon_i \end{bmatrix} \succeq 0 \tag{2.14}$$

Furthermore, when $p = 1$, the converse also holds provided that there exists an x_1 such that $F_1(x_1) > 0$.

Remark 1 *If the functions F_i, $i = 0, \ldots, p$, are convex in x, then* (2.13) *and* (2.14) *become equivalent. This is the so-called Farkas' Theorem [67]. Furthermore, if the function F_i are affine, then the equivalence of* (2.13) *and* (2.14) *is known as the Farkas' Lemma.*

2.4.2 S-procedure application example

In this section, let us consider an example problem so as to clarify the application of the S-procedure. Let

$$x^+ = [(A+B_uK)x+B_ww] \tag{2.15}$$

where $x \in \mathscr{X} := \{x \in \mathbb{R}^n : -d \leq x \leq d\}$, $w \in \mathscr{W} := \{w \in \mathbb{R}^{n_w} : -u \leq w \leq u\}$ are bounded signals, $A,\ B_u,\ B_w$ are given matrices and $d,\ u$ are known vectors. The main objective is to compute a matrix K, if it exists, such that

$$e_i^T[(A+B_uK)x+B_ww]-\gamma \leq 0, \quad \forall x \in \mathscr{X},\ \forall w \in \mathscr{W},\ i=1,\ldots,n. \tag{2.16}$$

The feasibility problem would require finding a K for a given γ, whereas the optimization problem consists of computing a K that minimizes γ. However, in both cases, it is clear that (2.16) requires nonlinear optimization techniques. To address this issue, we now use the S-procedure to obtain an equivalent SDP formulation for the above problem (see also Remark 2).

Theorem 2.4.2 *There exists K and γ such that* (2.16) *is satisfied if and only if there exist diagonal positive semidefinite matrices $D_x \in \mathbb{R}^{n\times n}$ and $D_w \in \mathbb{R}^{n_w\times n_w}$ such that the following LMI is satisfied:*

$$\mathscr{L}(\gamma,K,D_x,D_w) := \begin{bmatrix} D_w & 0 & \frac{1}{2}(A+B_uK)^Te_i \\ * & D_w & \frac{1}{2}B_w^Te_i \\ * & * & \gamma-d^TD_xd-u^TD_wu \end{bmatrix} \succeq 0. \tag{2.17}$$

Proof: For any $D_x \in \mathbb{R}^{n\times n}$ and $D_w \in \mathbb{R}^{n_w\times n_w}$, the left hand side of inequality in (2.16) can be written as

$$\begin{aligned} e_i^T[(A+B_uK)x+B_ww]-\gamma = &-(d-x)D_x(x+d)-(u-w)D_w(u+w) \\ &-[-(d-x)^TD_x(x+d)-(u-w)^TD_w(u+w)-e_i^T(A+B_uK)x-e_i^TB_w+\gamma] \end{aligned}$$

Using matrix manipulation for the terms inside the square-bracket in the above equation we can

rewrite it in a matrix forms as:

$$e_i^T[(A+B_uK)x+B_ww]-\gamma=\underbrace{-(d-x)D_x(x+d)}_{J_x}\underbrace{-(u-w)D_w(u+w)}_{J_w}$$
$$-\begin{bmatrix} x^T & w^T & 1 \end{bmatrix}\mathscr{L}(\gamma,K,D_x,D_w)\begin{bmatrix} x \\ w \\ 1 \end{bmatrix} \tag{2.18}$$

where $\mathscr{L}(\gamma,K,D_x,D_w)$ is the matrix defined in (2.17).

Notice that $J_x \leq 0$ and $J_w \leq 0$ for all $x \in \mathscr{X}$ and for all $w \in \mathscr{W}$ for any diagonal, positive semidefinite matrices D_x and D_w. Then, using the S-procedure (Farkas' Theorem) [81], it follows that the existence of such D_x and D_w such that $\mathscr{L}(\gamma,K,D_x,D_w) \succeq 0$, is a necessary and sufficient condition for (2.16). Therefore, the result follows. ■

Remark 2 *To clarify the above findings, note the following:*

- *In order to simplify the presentation, a step has been skipped in Theorem 2.4.2, where defining the functions F_0 and F_i to represent (2.16) in the form (2.13), which allows the use of Lemma 2.4.1, along with Remark 1, to arrive at (2.16) that corresponds to (2.14).*
- *The diagonal entries of D_x and D_w simply correspond to the τ_i in (2.14).*
- *Sufficiency of* (2.17) *for* (2.16) *follows from* (2.18) *without the need to reference the S-Procedure. Necessity follows from the S-Procedure since F_0 and F_i are affine functions of the variables.*
- *Even though the theorem dealt with the feasibility problem of* (2.16)*, the optimization problem follows since the cost function is linear in the variable γ.*

2.5 Uncertain System Models

As discussed above, real world processes may have a very complex plant description, which cannot be captured by the designed system model. To avoid using very complex models for control application,

uncertain systems are utilised to capture any plant-model miss-match. In this section the two main uncertain model representations, which arise from two different modeling and identification procedures commonly used in robust control are presented. The first model under consideration is known as 'Polytopic' or 'multi-model', and the second which is more frequently used is called 'structured feedback uncertainty' robust control model. In general linear uncertain systems are given as follows:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= Cx(k) \\ [A(k) \;\; B(k)] &\in \Omega \end{aligned} \tag{2.19}$$

where $u(k) \in \mathbb{R}^{n_u}$ is the control input, $x(k) \in \mathbb{R}^{n_x}$ is the state of the plant and $y(k) \in \mathbb{R}^{n_y}$ is the plant output, and Ω is some pre-specified set.

2.5.1 Polytopic uncertainty

In the polytopic system representation of uncertain system, the set Ω is a polytopic set:

$$\Omega = Co\{[A_1,\ B_1], [A_2,\ B_2], \ldots, [A_L,\ B_L]\}, \tag{2.20}$$

where Co denotes the convex hull. Therefore, $[A,\ B] \in \Omega$ if and only if there exist some nonnegative $\lambda_1, \lambda_2, \ldots, \lambda_L$, which are summing to one, such that

$$[A,\ B] = \sum_{i=1}^{L} \lambda_i [A_i,\ B_i].$$

Note that when $L = 1$ the polytopic uncertain system corresponds to the nominal LTI system. Methodologies on how a polytopic system models can be developed are presented on the next paragraph. Suppose that for the (possibly nonlinear) system under consideration, we have input/output data sets at different operating points, or at different times. From each data set, we develop a number of linear models (for simplicity, we assume that the various linear models involve the same state vector). Then it is reasonable to assume that any analysis and design methods for the polytopic system (2.19), (2.20) with vertices given by the linear models will apply to the real system. Alternatively, suppose that the

equilibrium point for the nonlinear system is equal to $x = 0$, $u = 0$ and the Jacobian $[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial u}]$ of a nonlinear discrete time varying system $x(k+1) = f(x(k), u(k), k)$ lies inside the polytope Ω. Then it can be shown that every trajectory (x, u) of the original nonlinear system is also a trajectory of (2.19) for some LTV system in Ω [68]. Thus the original nonlinear system can be approximated (possibly conservatively) by a polytopic uncertain LTV system. Equivalently, it can be shown that bounds on impulse response coefficients of single input/single output FIR plants can be transformed to a polytopic uncertainty description on the state-space matrices. Thus, this polytopic uncertainty description is convenient for many problems of engineering significance.

2.5.2 Norm-bounded structured feedback uncertainty

A second, more popular uncertain system model representation for robust control consists of an LTI system with uncertainties or perturbations appearing in the feedback loop:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_p p(k) \\ y(k) &= Cx(k) \\ q(k) &= C_q x(k) + D_{qu} u(k) \\ p(k) &= \Delta q(k) \end{aligned} \tag{2.21}$$

The matrix Δ typically has a block-diagonal structure:

$$\Delta = \begin{bmatrix} \Delta_1 & 0 & \cdots & 0 \\ 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_r \end{bmatrix} \tag{2.22}$$

where $\Delta_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$. The matrix Δ can represent either a memoryless time-varying matrix or a convolution operator, with the operator norm induced by l_2-norm less than 1 (for e.g., a stable LTI dynamic system).

Each Δ_i is assumed to be either a *repeated scalar* block or a *full* block, where it can model uncertain-

ties for number of factors, such as nonlinearities, dynamics or parameters, that are unknown, unmodeled or neglected. A number of control systems with uncertainties can be recast in this framework (see for example [14,15,26,69]. For ease of reference, we shall refer to such systems as systems with structured uncertainty. Note that in this case, the uncertainty set Ω is defined by (2.21) and (2.22). For the LTV case, it is easy to show through routine algebraic manipulations that the system (2.21) corresponds to the system (2.19) with

$$\begin{aligned}\Omega = \{[A + B_p \Delta C_q \quad B + B_p \Delta D_{qu}] :& \\ \Delta \text{ satisfies (2.22) with } \bar{\sigma}(\Delta_i) \leq 1\}.&\end{aligned} \tag{2.23}$$

When $\Delta = 0$ and $p(k) = 0 \ \ \forall\, k \geq 0$, the system corresponds to the nominal LTI system.

The issue of whether to model a system as a polytopic system or a system with structured uncertainty depends on a number of factors, such as the underlying physical model of the system, available model identification and validation techniques. For example, nonlinear systems can be modeled either as polytopic systems or as systems with structured perturbations. We shall not concern ourselves with such issues here: instead we shall assume that the structured feedback uncertainty model is available.

CHAPTER 3

Computationally Efficient State-Feedback Robust Model Predictive Control for Uncertain System

3.1 Introduction

In this chapter the problem of RMPC of linear-time-invariant (LTI) discrete-time systems subject to norm-bounded structured uncertainty and bounded disturbances is considered. Typically, the constrained RMPC problem with state-feedback parameterizations is nonlinear (and nonconvex) with a prohibitively high computational burden for online implementation. To remedy this issues, a novel approach is proposed to linearize the state-feedback RMPC problem, with minimal conservatism, through the use of semidefinite relaxation techniques. The proposed algorithm computes the state-feedback gain and perturbation online by solving an LMI optimization that, in comparison to other schemes in the literature is shown to have a substantially reduced computational burden without adversely affecting the tracking performance of the controller. Additionally, an offline strategy that guarantees feasibility on the RMPC problem is presented. The effectiveness of the proposed RMPC

algorithm compared to other schemes in the literature is demonstrated through numerical examples. The contributions of this chapter are summarized as follows. Firstly, a new LMI-based RMPC scheme is proposed in Section 3.2 for systems subject to structured uncertainty and disturbances. The feedback gain and control perturbation are considered as decision variables whilst nonlinearities are circumvented using a novel linearization procedure (see Section 3.3). This substantially reduces the online computations, while it improves performance due to its less restrictive nature (see Remark 23) without reducing the feasibility region. Secondly, to reduce the online computation time further, an extension is proposed in Section 3.4 which derives a single LMI sufficient condition for all the constraints. This improves the scalability of the algorithm. Finally, Section 3.5 proposes an offline initialization strategy to guarantee recursive feasibility for the problem. The formulation and results presented in this chapter are mainly based on the result presented in [70].

The following lemma has been used throughout this work to deal with norm-bounded feedback uncertainty structure.

Lemma 3.1.1 *Let $H_{11}=H_{11}^T, H_{12}, H_{21}$, and H_{22} be real matrices. Let $\widehat{\mathbf{\Delta}}$ be a linear subspace and define the linear subspace:*

$$\widehat{\Psi}=\{(S,R,G): S=S^T\succ 0, R=R^T\succ 0,\, S\Delta=\Delta R,\, \mathscr{H}(\Delta G)=0\,\forall\Delta\in\widehat{\mathbf{\Delta}}\}. \tag{3.1}$$

Then $det(I-H_{22}\Delta)\neq 0$ and $H_{11}+\mathscr{H}\left(H_{12}\Delta(I-H_{22}\Delta)^{-1}H_{21}\right)\succ 0$ for every $\Delta\in\mathscr{B}\widehat{\mathbf{\Delta}}$ if there exists $(S,R,G)\in\widehat{\Psi}$ such that:

$$\begin{bmatrix} H_{11} & H_{21}^T+H_{12}G^T & H_{12}S \\ * & R+\mathscr{H}\left(H_{22}G^T\right) & H_{22}S \\ * & * & S \end{bmatrix}\succ 0. \tag{3.2}$$

Note that the above Lemma is based on the results presented in [71, Lemma 3.2] followed by some rearrangements using Schur complement argument.

3.2 Problem Statement

In this section, we are first presenting the system description including control dynamics, constraints and cost signal. Then, utilizing a causal state feedback control law and recasting disturbances as structure bounded uncertainties similar to [14], the RMPC control problem is presented. Lastly, the difficulties to solve this optimization problem using linear optimization solvers are highlighted.

3.2.1 System Description

The following linear discrete-time system, subject to bounded disturbances and norm-bounded structured uncertainty, is considered (see e.g. [15]):

$$\begin{aligned}
\begin{bmatrix} x_{k+1} \\ q_k \\ f_k \\ z_k \end{bmatrix} &= \begin{array}{c} \\ n \\ n_q \\ n_f \\ n_z \end{array} \begin{array}{c} \begin{array}{cccc} n & n_u & n_p & n_w \end{array} \\ \begin{bmatrix} A & B_u & B_p & B_w \\ C_q & D_{qu} & 0 & 0 \\ C_f & D_{fu} & D_{fp} & D_{fw} \\ C_z & D_{zu} & D_{zp} & D_{zw} \end{bmatrix} \end{array} \begin{bmatrix} x_k \\ u_k \\ p_k \\ w_k \end{bmatrix}, \quad p_k = \Delta_k q_k, \\
\begin{bmatrix} q_N \\ f_N \\ z_N \end{bmatrix} &= \begin{bmatrix} \hat{C}_q & 0 \\ \hat{C}_f & \hat{D}_{fp} \\ \hat{C}_z & \hat{D}_{zp} \end{bmatrix} \begin{bmatrix} x_N \\ p_N \end{bmatrix}, \qquad p_N = \Delta_N q_N,
\end{aligned} \tag{3.3}$$

where $x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^{n_u}, w_k \in \mathbb{R}^{n_w}, f_k \in \mathbb{R}^{n_f}, z_k \in \mathbb{R}^{n_z}, p_k \in \mathbb{R}^{n_p}$ and $q_k \in \mathbb{R}^{n_q}$ are the state, input, disturbance, constraint, cost, and input and output uncertainty vectors, respectively, with $k \in \mathscr{N} := \{0, 1, \ldots, N-1\}$, where N is the horizon length. It is assumed that the state x_k is measurable. Note that the description includes terminal cost and state constraints to ensure closed-loop stability [4]. The symbols in capital letters denote coefficient matrices with the dimensions indicated for ease of reference.

Furthermore, $\Delta_k \in \mathscr{B}\mathbf{\Delta}$ where $\mathbf{\Delta} \subseteq \mathbb{R}^{n_p \times n_q}$ is a subspace that captures the uncertainty structure. Finally, the disturbance w_k is assumed to belong to the set $\mathscr{W}_k = \{w_k \in \mathbb{R}^{n_w} \colon -\bar{d}_k \leq w_k \leq \bar{d}_k\}$, where the disturbance's bound $\bar{d}_k > 0$ is given.

Remark 3 *Note that we allow uncertainty in all the problem data including the constraints and the cost signal. It is easy to verify that the dynamics in (3.3) can be rewritten in the form:*

$$\begin{bmatrix} x_{k+1} \\ f_k \\ z_k \end{bmatrix} = \begin{bmatrix} A+B_p\Delta_k C_q & B_u+B_p\Delta_k D_{qu} & B_w \\ C_f+D_{fp}\Delta_k C_q & D_{fu}+D_{fp}\Delta_k D_{qu} & D_{fw} \\ C_z+D_{zp}\Delta_k C_q & D_{zu}+D_{zp}\Delta_k D_{qu} & D_{zw} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ w_k \end{bmatrix}$$

$$\begin{bmatrix} f_N \\ z_N \end{bmatrix} = \begin{bmatrix} \hat{C}_f+\hat{D}_{fp}\Delta_N\hat{C}_q \\ \hat{C}_z+\hat{D}_{zp}\Delta_N\hat{C}_q \end{bmatrix} x_N.$$

3.2.2 Algebraic formulation

To simplify the presentation, with a slight abuse of notation we re-parameterize the disturbance as uncertainty by redefining $\mathscr{W}_k := \{\Delta_k^w \bar{d}_k \colon \Delta_k^w \in \mathscr{B}\mathbf{\Delta}^w\}$, where $\mathbf{\Delta}^w = \mathbb{D}^{n_w}$,

$$B_p := \begin{bmatrix} B_p & B_w \end{bmatrix}, C_q := \begin{bmatrix} C_q \\ 0 \end{bmatrix}, D_{qu} := \begin{bmatrix} D_{qu} \\ 0 \end{bmatrix}, \bar{d}_k := \begin{bmatrix} 0 \\ \bar{d}_k \end{bmatrix}, p_k := \begin{bmatrix} p_k \\ w_k \end{bmatrix},$$

$$q_k := C_q x_k + D_{qu} u_k + \bar{d}_k,$$

where new uncertainty dimensions are $n_p := n_p + n_w$ and $n_q := n_q + n_w$. The vector $\bar{z}_k$ is assumed to be given and defines the reference trajectory. The constraint and terminal constraint signals are defined by $\bar{f}_k$ and $\bar{f}_N$, respectively, and are assumed to be known. They are chosen to satisfy polytopic constraints on the input and state signals, and terminal state signals, respectively. The only assumption that is imposed here is that the terminal constraints presented here by $\bar{f}_N$ are define a Robust Control Invariant set (RCI) [72]. This is used to derive conditions for recursive feasibility on the proposed control scheme (see Remark 8).

By defining the stacked vectors,

$$\mathbf{u} = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{N_u}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{N_n}, \ \boldsymbol{\zeta} = \begin{bmatrix} \zeta_0 \\ \vdots \\ \zeta_N \end{bmatrix} \in \mathbb{R}^{N\zeta},$$

where $\boldsymbol{\zeta}$ stands for $\mathbf{f},\bar{\mathbf{f}},\mathbf{p},\mathbf{q},\mathbf{z},\bar{\mathbf{z}}$ or $\bar{\mathbf{d}}$ and $N_n = Nn$, $N_u = Nn_u$ and $N_\zeta = (N+1)n_\zeta$, we get

$$\begin{bmatrix}\mathbf{x}\\ \mathbf{q}\\ \mathbf{f}\\ \mathbf{z}\end{bmatrix} = \begin{array}{c} \\ N_n \\ N_q \\ N_f \\ N_z \end{array}\!\!\begin{array}{c} \begin{array}{cccc} n & N_u & N_p & 1 \end{array} \\ \begin{bmatrix}\mathbf{A} & \mathbf{B}_u & \mathbf{B}_p & 0\\ \mathbf{C}_q & \mathbf{D}_{qu} & \mathbf{D}_{qp} & \bar{\mathbf{d}}\\ \mathbf{C}_f & \mathbf{D}_{fu} & \mathbf{D}_{fp} & 0\\ \mathbf{C}_z & \mathbf{D}_{zu} & \mathbf{D}_{zp} & 0\end{bmatrix}\end{array}\begin{bmatrix}x_0\\ \mathbf{u}\\ \mathbf{p}\\ 1\end{bmatrix}, \quad \mathbf{p} = \hat{\Delta}\mathbf{q}, \tag{3.4}$$

with $\hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}} \subset \mathbb{R}^{N_p \times N_q}$ where,

$$\hat{\mathbf{\Delta}} = \{\mathrm{diag}(\Delta_0, \Delta_0^w, \ldots, \Delta_{N-1}, \Delta_{N-1}^w, \Delta_N) : \Delta_k \in \mathbf{\Delta}, \Delta_k^w \in \mathbf{\Delta}^w\},$$

and where the stacked matrices in (3.4) (shown in bold) have the indicated dimensions and are readily obtained from iterating the dynamics in (3.3) and the re-definitions in this section. The input signal u_i is considered as a causal state feedback that depends only on states $x_0, \ldots, x_i$ (see e.g. [73]). Thus

$$\mathbf{u} = K_0 x_0 + K\mathbf{x} + \boldsymbol{\upsilon}, \tag{3.5}$$

where $\boldsymbol{\upsilon} \in \mathbb{R}^{N_u}$ is the (stacked) control perturbation vector and K_0, K are the current and predicted future state feedback gains. Causality is preserved by restricting $[K_0 \;\; K] \in \mathscr{K} \subset \mathbb{R}^{N_u \times N_n}$, where $\mathscr{K}$ is the set of $N_u \times N_n$ lower block triangular matrices with $n_u \times n$ blocks. K_0, K and $\boldsymbol{\upsilon}$ are considered as decision variables. Note that, while K_0 is redundant for a given x_0 as it can be absorbed in $\boldsymbol{\upsilon}$, we keep it for when we tackle the case of variable x_0 in Section 3.5. Substituting the expression of $\mathbf{x}$ in (3.4) into (3.5) gives,

$$\mathbf{u} = \hat{K}_0 x_0 + \hat{K}\mathbf{B}_p\mathbf{p} + \hat{\upsilon}, \tag{3.6}$$

where $\begin{bmatrix}\hat{K}_0 & \hat{K} & \hat{\upsilon}\end{bmatrix} = (I - K\mathbf{B}_u)^{-1}\begin{bmatrix}K_0 + K\mathbf{A} & K & \boldsymbol{\upsilon}\end{bmatrix}$.

Note that u is affine in $\hat{K}_0, \hat{K}$ and $\hat{\upsilon}$ which have the same structure and dimensions as K_0, K and $\boldsymbol{\upsilon}$, respectively. Note also that

$$\begin{bmatrix}K_0 & K & \boldsymbol{\upsilon}\end{bmatrix} = (I + \hat{K}\mathbf{B}_u)^{-1}\begin{bmatrix}\hat{K}_0 - \hat{K}\mathbf{A} & \hat{K} & \hat{\upsilon}\end{bmatrix}, \tag{3.7}$$

and so $\begin{bmatrix}\hat{K}_0 & \hat{K} & \hat{\upsilon}\end{bmatrix}$ will be used as the decision variables instead. Using (3.6) to eliminate $\mathbf{u}$ from (3.4) and re-arranging x_0 gives

$$\begin{aligned}\begin{bmatrix}\mathbf{q}\\ \mathbf{f}\\ \mathbf{z}-\bar{\mathbf{z}}\end{bmatrix} &= \left[\begin{array}{c|c}\mathbf{D}_{qp}^{\hat{K}} & \mathbf{D}_{q}^{\hat{K}_0,\hat{\upsilon}}\\ \hline \mathbf{D}_{fp}^{\hat{K}} & \mathbf{D}_{f}^{\hat{K}_0,\hat{\upsilon}}\\ \hline \mathbf{D}_{zp}^{\hat{K}} & \mathbf{D}_{z}^{\hat{K}_0,\hat{\upsilon}}\end{array}\right]\begin{bmatrix}\mathbf{p}\\ 1\end{bmatrix},\\ &:= \left[\begin{array}{c|c}\mathbf{D}_{qp}+\mathbf{D}_{qu}\hat{K}\mathbf{B}_p & \mathbf{D}_{qu}\hat{\upsilon}+(\mathbf{C}_q+\mathbf{D}_{qu}\hat{K}_0)x_0+\bar{d}\\ \hline \mathbf{D}_{fp}+\mathbf{D}_{fu}\hat{K}\mathbf{B}_p & \mathbf{D}_{fu}\hat{\upsilon}+(\mathbf{C}_f+\mathbf{D}_{fu}\hat{K}_0)x_0\\ \hline \mathbf{D}_{zp}+\mathbf{D}_{zu}\hat{K}\mathbf{B}_p & \mathbf{D}_{zu}\hat{\upsilon}+(\mathbf{C}_z+\mathbf{D}_{zu}\hat{K}_0)x_0-\bar{z}\end{array}\right]\begin{bmatrix}\mathbf{p}\\ 1\end{bmatrix}.\end{aligned} \tag{3.8}$$

Note that all the coefficient matrices in (3.8) are affine in $\hat{K}_0$, $\hat{K}$ and $\hat{\upsilon}$. Finally, eliminating p using $p = \hat{\Delta}q$ we get

$$\begin{bmatrix}\mathbf{f}\\ \mathbf{z}-\bar{\mathbf{z}}\end{bmatrix} = \begin{bmatrix}\mathbf{D}_{f}^{\hat{K}_0,\hat{\upsilon}} + \mathbf{D}_{fp}^{\hat{K}}\hat{\Delta}(I-\mathbf{D}_{qp}^{\hat{K}}\hat{\Delta})^{-1}\mathbf{D}_{q}^{\hat{K}_0,\hat{\upsilon}}\\ \mathbf{D}_{z}^{\hat{K}_0,\hat{\upsilon}} + \mathbf{D}_{zp}^{\hat{K}}\hat{\Delta}(I-\mathbf{D}_{qp}^{\hat{K}}\hat{\Delta})^{-1}\mathbf{D}_{q}^{\hat{K}_0,\hat{\upsilon}}\end{bmatrix}. \tag{3.9}$$

For convenience, we write $\mathbf{f} = \mathscr{F}(\hat{K}_0,\hat{K},\hat{\upsilon},\hat{\Delta})$ and $(\mathbf{z}-\bar{\mathbf{z}})^T(\mathbf{z}-\bar{\mathbf{z}}) = \mathscr{Z}(\hat{K}_0,\hat{K},\hat{\upsilon},\hat{\Delta})$ to emphasize dependence on the variables.

3.2.3 RMPC problem

Given the initial state x_0, the RMPC problem is then to find a feedback law u_k for all $k \in \mathscr{N}$ such that the cost function, is minimized, while the constraint signals satisfy $f_k \leq \bar{f}_k$ and $f_N \leq \bar{f}_N$ for all $w_k \in \mathscr{W}_k$ and all $\Delta_k \in \mathscr{B}\mathbf{\Delta}$ and for all $k \in \mathscr{N}$. The RMPC problem can be posed as a min-max problem [9], where the objective is to find a feasible $(\hat{K}_0,\hat{K},\hat{\upsilon})$ that solves

$$\mathbf{J} = \min_{(\hat{K}_0,\hat{K},\hat{\upsilon})\in\mathscr{U}} \max_{\hat{\Delta}\in\mathscr{B}\mathbf{\Delta}} \mathscr{Z}(\hat{K}_0,\hat{K},\hat{\upsilon},\hat{\Delta}), \tag{3.10}$$

where $\mathscr{U}$ is defined to be the set of all feasible control variables $(\hat{K}_0, \hat{K}, \hat{\upsilon})$ such that all the problem constraints are satisfied:

$$\mathscr{U} := \{([\hat{K}_0\ \hat{K}], \hat{\upsilon}) \in \mathscr{K} \times \mathbb{R}^{N_u} : \mathscr{F}(\hat{K}_0, \hat{K}, \hat{\upsilon}, \hat{\Delta}) \leq \bar{\mathbf{f}}, \forall \hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}}\}.$$

K_0, K and υ can be computed online and applied in the usual receding horizon MPC manner, where the first input of the control sequence u is applied to the plant, the time window is shifted by 1, the current state is read and the process is repeated. Since the optimization in (3.10) is nonconvex, a semidefinite relaxation is used by introducing an upper bound γ^2 on the cost function. Using Lemma 3.1.1 and a Schur complement argument, the next result derives nonlinear conditions for solving (3.10).

Theorem 3.2.1 *Let all the variables be defined as above. Then $\mathscr{L}(\hat{K}_0, \hat{K}, \hat{\upsilon}, \hat{\Delta}) \leq \gamma^2$ and $\mathscr{F}(\hat{K}_0, \hat{K}, \hat{\upsilon}, \hat{\Delta}) \leq \bar{\mathbf{f}}$ are satisfied for all $\hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}}$ if there exists a solution to the nonlinear matrix inequalities*

$$T_1 + \mathscr{H}(T_2 \hat{K} \mathbf{B}_p T_3) \succ 0, \tag{3.11}$$

$$T_1^i + \mathscr{H}(T_2^i \hat{K} \mathbf{B}_p T_3^i) \succ 0,\ i = 1, \ldots, N_f, \tag{3.12}$$

where

$$\begin{bmatrix} T_1^i & T_2^i \\ T_3^i & 0 \end{bmatrix} = \begin{array}{c} \\ 1 \\ N_q \\ N_p \\ N_p \end{array} \begin{array}{c} \begin{array}{cccc} 1 & N_q & N_p & N_u \end{array} \\ \left[\begin{array}{cccc|c} e_i^T(\bar{\mathbf{f}} - \mathbf{D}_f^{\hat{K}_0,\hat{\upsilon}}) & (\mathbf{D}_q^{\hat{K}_0,\hat{\upsilon}})^T - \frac{e_i^T}{2}\mathbf{D}_{fp}G_i^T & -\frac{e_i^T}{2}\mathbf{D}_{fp}S_i & -\frac{e_i^T}{2}\mathbf{D}_{fu} \\ * & R_i + \mathscr{H}\left(\mathbf{D}_{qp}G_i^T\right) & \mathbf{D}_{qp}S_i & \mathbf{D}_{qu} \\ * & * & S_i & 0 \\ \hline 0 & G_i^T & S_i & 0 \end{array}\right] \end{array},$$

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & 0 \end{bmatrix} = \begin{array}{c} \\ N_z \\ 1 \\ N_q \\ N_p \\ N_p \end{array} \begin{array}{c} \begin{array}{ccccc} N_z & 1 & N_q & N_p & N_u \end{array} \\ \left[\begin{array}{cccc|c} I & \mathbf{D}_z^{\hat{K}_0,\hat{\upsilon}} & \mathbf{D}_{zp}G^T & \mathbf{D}_{zp}S & \mathbf{D}_{zu} \\ * & \gamma^2 & (\mathbf{D}_q^{\hat{K}_0,\hat{\upsilon}})^T & 0 & 0 \\ * & * & R + \mathscr{H}\left(\mathbf{D}_{qp}G^T\right) & \mathbf{D}_{qp}S & \mathbf{D}_{qu} \\ * & * & * & S & 0 \\ \hline 0 & 0 & G^T & S & 0 \end{array}\right] \end{array},$$

where $([\hat{K}_0\ \hat{K}], \hat{\upsilon}) \in \mathscr{K} \times \mathbb{R}^{N_u}$ and (S,R,G), $(S_i,R_i,G_i) \in \hat{\Psi}$, $i \in \mathscr{N}_f := \{1,\ldots,N_f\}$ are slack variables with $\hat{\Psi}$ defined in (3.1).

In the sequel, we will occasionally write $T_1(\gamma^2, \hat{K}_0, \hat{\upsilon}, S, R, G)$ etc. to emphasise dependence on the variables. It follows that the relaxed RMPC problem can be summarized as:

$$\begin{aligned} \min\{\gamma^2 : &([\hat{K}_0\ \ \hat{K}], \hat{\upsilon}) \in \mathscr{K} \times \mathbb{R}^{N_u}, (3.11), (3.12) \text{ are satisfied,} \\ &(S,R,G), (S_i,R_i,G_i) \in \widehat{\Psi},\ i \in \mathscr{N}_f\}. \end{aligned} \tag{3.13}$$

Definitions (3.8)-(3.9) verify that (3.13) is nonlinear due to terms of the form $\hat{K}\mathbf{B}_p Z^T$ where Z stands for S, S_i, G and G_i. Note that (3.13) is linear for fixed K and RMPC schemes with fixed K have been proposed [30]. However, this introduces conservatism depending on the choice of K. A linearization scheme is proposed in [14], which uses an S-procedure to separate $\hat{K}$. However, this scheme has a high computational burden. Furthermore, some of the introduced linearization variables are restricted to a specific form. To overcome these two limitations, a new linearization procedure for (3.13) is proposed, which substantially reduces the computational complexity at the expense of only minor conservatism in the formulation.

3.3 Linearization scheme for the relaxed RMPC problem

As mentioned in Section 1.1, although min-max MPC and TMPC are more suitable for fast dynamic system due to their lower online computational cost, both schemes rely heavily on offline calculation to achieve robustness which can lead to performance conservatism within the overall robust control scheme. Moreover, both methods are more complex to implement due firstly to the complexity of calculating robust invariant sets (necessary for both methods) and secondly to the not straightforward process of tuning all the parameters of the MPC [18]. On the other hand, LMI-based methods reduce conservatism by explicitly incorporating uncertainty within the online optimization problem. However, this method suffers from its heavy computational cost, which makes it impractical for systems with a high number of states or fast dynamics. Considering the control gains and perturbation as

decision variables in the optimization problem in the case of systems subject to both norm bounded model uncertainties and additive disturbances (shown in Section 3.2), the problem presented in (3.10) is nonlinear and nonconvex. Subsequently, linearization methods using extended S-procedure proposed lead to conservatism and a significant increase in the computational cost. In this section, utilizing the Elimination lemma (see Lemma 3.3.1), a novel linearization procedure is proposed to overcome the nonlinearity and nonconvexity for the LMI based RMPC problem presented in (3.13), while conservativeness and computation burden are maintained at low levels. The following form of the Elimination Lemma will be used in this section.

Lemma 3.3.1 *(Elimination Lemma) Let $Q = Q^T \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times p}$ be given matrices and let $B^\perp$ and $C^\perp$ denote orthogonal complements of B and C, respectively. Then the following two statements are equivalent:*

(i) $(B^\perp)^T Q (B^\perp) \succ 0$ & $(C^\perp)^T Q (C^\perp) \succ 0$

(ii) $\exists\, Z \in \mathbb{R}^{p\times m}: \;\; Q + \mathscr{H}(CZB^T) \succ 0,$

where $B^\perp$ and $C^\perp$ denote orthogonal complements of B and C, respectively. The proof and some applications of the Elimination Lemma can be found in [61, 74, 75]. The next result uses the Elimination Lemma to derive LMI sufficient conditions for the nonlinear matrix inequality conditions of Theorem 3.2.1.

Theorem 3.3.2 *Let all variables be as defined Section 3.2. Then, $\mathscr{Z}(\hat{K}_0, \hat{K}, \hat{\upsilon}, \hat{\Delta}) \leq \gamma^2$ and $\mathscr{F}(\hat{K}_0, \hat{K}, \hat{\upsilon}, \hat{\Delta}) \leq \bar{\mathbf{f}}$ for all $\hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}}$ if there exist solutions $([\hat{K}_0 \;\; \hat{K}], \hat{\upsilon}) \in \mathscr{K} \times \mathbb{R}^{N_u}$, $X \in \mathbb{R}^{N_n \times N_n}$, with X lower block-diagonal with $n \times n$ blocks, (S, R, G), $(S_i, R_i, G_i) \in \widehat{\Psi}$, $\forall i \in \mathscr{N}_f$ to the following LMIs:*

$$\begin{bmatrix} T_1 + \mathscr{H}\left(T_2 \bar{K} Y^*\right) & * \\ \left(\mathbf{B}_p T_3 - \bar{K}^T T_2^T\right) - XY^* & X + X^T \end{bmatrix} \succ 0 \tag{3.14}$$

$$\begin{bmatrix} T_1^i + \mathscr{H}\left(T_2^i \bar{K} Y_i^*\right) & * \\ \left(\mathbf{B}_p T_3^i - \bar{K}^T (T_2^i)^T\right) - XY_i^* & X + X^T \end{bmatrix} \succ 0, \tag{3.15}$$

for any $Y^* \in \mathbb{R}^{N_n \times (N_z + 1 + N_q + N_p)}, Y_i^* \in \mathbb{R}^{N_n \times (1 + N_q + N_p)}$ and where $\bar{K} := \hat{K}X \in \mathscr{K}$.

Furthermore, suppose that (3.11) *and* (3.12) *have feasible solutions for* $(\gamma^2, \hat{K}_0, \hat{K}, \hat{\upsilon}, R, S, G, R_i, S_i, G_i) =$

$(\gamma^{2*}, \hat{K}_0^*, \hat{K}^*, \hat{\upsilon}^*, R^*, S^*, G^*, R_i^*, S_i^*, G_i^*)$ *so that*

$$T_1(\gamma^{2*}, \hat{K}_0^*, \hat{\upsilon}^*, S^*, R^*, G^*) + \mathscr{H}\left(T_2\hat{K}^*\mathbf{B}_p T_3(S^*, G^*)\right) \succ 0, \tag{3.16}$$
$$T_1^i(\hat{K}_0^*, \hat{\upsilon}^*, R_i^*, S_i^*, G_i^*) + \mathscr{H}\left(T_2^i\hat{K}^*\mathbf{B}_p T_3^i(S_i^*, G_i^*)\right) \succ 0,$$

and let $Y^* = \mathbf{B}_p T_3(S^*, G^*) + (T_2\hat{K}^*)^T$ *and* $Y_i^* = \mathbf{B}_p T_3^i(S_i^*, G_i^*) + (T_2^i\hat{K}^*)^T$. *Then* (3.14) *and* (3.15) *are feasible.*

Proof: We prove the first part by proving that the LMIs in (3.14) and (3.15) are sufficient for the nonlinear matrix inequalities in (3.11) and (3.12), respectively. We first use the Elimination Lemma to give an equivalent form to (3.11). In order to separate $\hat{K}$ from T_3, the inequality in (3.11) can be rearranged as:

$$\begin{bmatrix} I & T_2\hat{K} \end{bmatrix} \overbrace{\begin{bmatrix} T_1 & T_3^T\mathbf{B}_p^T \\ \mathbf{B}_p T_3 & 0 \end{bmatrix}}^{Q} \overbrace{\begin{bmatrix} I \\ \hat{K}^T T_2^T \end{bmatrix}}^{C^\perp} \succ 0. \tag{3.17}$$

Then, applying the Elimination Lemma 3.3.1 on (5.36) (with $B = I$) shows that (5.36), hence (3.11) is equivalent to

$$\begin{bmatrix} T_1 & T_3^T\mathbf{B}_p^T \\ \mathbf{B}_p T_3 & 0 \end{bmatrix} + \overbrace{\begin{bmatrix} -T_2\hat{K} \\ I \end{bmatrix}}^{C} \overbrace{\begin{bmatrix} Y & X \end{bmatrix}}^{Z} + \begin{bmatrix} Y^T \\ X^T \end{bmatrix} \begin{bmatrix} -\hat{K}^T T_2^T & I \end{bmatrix} \succ 0, \tag{3.18}$$

where Y and X are free slack variables. Since $\mathscr{H}(X) \succ 0$, X is nonsingular and we can define $\bar{K} := \hat{K}X$ as a new variable. To preserve the structure of $\hat{K}$ which ensures causality, we restrict X to be block lower triangular (with $n \times n$ blocks). To preserve linearity, we restrict Y to have the form $Y = -XY^*$ with Y^* free (but not a variable). Substituting $Y = -XY^*$ into (5.37) proves that (3.14) is sufficient for (3.11) (but not necessary due to the restrictions on Y and X). A similar procedure proves that (3.15) are sufficient for (3.12).

Next, we prove feasibility of (3.14) and (3.15). To show that (3.14) has a feasible solution, set $(\gamma^2, \hat{K}_0, \hat{K}, \hat{\upsilon}, R, S, G, X) = (\gamma^{2*}, \hat{K}_0^*, \hat{K}^*, \hat{\upsilon}^*, R^*, S^*, G^*, I)$. Then the LHS of (3.14) becomes

$$T^* := \begin{bmatrix} T_1^* + \mathscr{H}\left(T_2\hat{K}^*\left(\mathbf{B}_p T_3^* + (\hat{K}^*)^T T_2^T\right)\right) & * \\ -2(\hat{K}^*)^T T_2^T & 2I \end{bmatrix}$$

where $T_1^* := T_1(\gamma^{2*}, \hat{K}_0^*, \hat{\upsilon}^*, S^*, R^*, G^*)$ and $T_3^* := T_3(S^*, G^*)$. Applying a Schur complement on T^* shows that $T^* \succ 0$ if and only if (3.16) is satisfied. It follows that (3.14) is feasible if (3.16) is. A similar procedure proves the feasibility of (3.15). ■

Remark 4 *Theorem 3.3.2 provides sufficient LMI conditions for the initial nonconvex RMPC problem. Therefore, K_0, K and υ can be computed online and applied in the usual MPC manner, where the first input of the control sequence u is applied to the plant, the time window is shifted by 1, the current state is read and the process is repeated.*

Remark 5 *In comparison to [14], the novelty of the proposed linearization procedure is that it does not restrict the structure of the slack variables (R, S, G) and (R_i, S_i, G_i) beyond the requirements of $\widehat{\Psi}$, and therefore it is less conservative.*

3.4 Single LMI approach for handling constraints signal for RMPC problem

Instead of solving multiple matrix inequalities for the constraints (one for each of the N_f constraints (3.12) or (3.15)), we propose a strategy to combine all within a single inequality. This results in reduced computational complexity and improved algorithm scalability. Our algorithm is based on the following result which uses an S-procedure to derive one LMI condition that is sufficient for a set of elementwise inequalities.

Theorem 3.4.1 *Let $\tilde{f} \in \mathbb{R}^{N_f}$ and let $e \in \mathbb{R}^{N_f}$ be the vector of ones. Then $\tilde{f} \geq 0$ if there exist $\mu \in \mathbb{R}$ and $M \in \mathbb{D}^{N_f}$ such that,*

$$\mathscr{L} := \begin{bmatrix} 2\mu & \left(\tilde{f} - Me - e\mu\right)^T \\ * & M + M^T \end{bmatrix} \succeq 0. \tag{3.19}$$

Proof: Let $\mathbf{\Omega} := \{\mathrm{diag}(\delta_1, \ldots, \delta_{N_f}) : \delta_i \in \{0,1\}, \sum_{i=1}^{N_f} \delta_i = 1\}$. Then,

$$\tilde{f} \geq 0 \Leftrightarrow e^T \Delta \tilde{f} + \tilde{f}^T \Delta^T e \geq 0 \; \forall \Delta \in \mathbf{\Omega}. \tag{3.20}$$

Let $\Delta \in \mathbf{\Omega}$. Since $\delta_i \in \{0,1\}$ and $\sum_{i=1}^{N_f} \delta_i = 1$, then

$$\begin{aligned} M_\Delta &:= \Delta M + M^T \Delta^T - \Delta(M+M^T)\Delta^T = 0 \; \forall M \in \mathbb{D}^{N_f}, \\ \mu_\Delta &:= e^T \Delta e \mu + \mu e^T \Delta^T e - 2\mu = 0 \; \forall \mu \in \mathbb{R}, \end{aligned} \tag{3.21}$$

respectively. It is straightforward to verify the identity,

$$e^T \Delta \tilde{f} + \tilde{f}^T \Delta^T e = e^T M_\Delta e + \mu_\Delta + \begin{bmatrix} 1 & e^T \Delta \end{bmatrix} \mathscr{L} \begin{bmatrix} 1 \\ \Delta^T e \end{bmatrix}.$$

The proof now follows from (3.20) and (3.21). ■

Theorem 3.4.1 enables us to give sufficient conditions for the constraints in (3.12) in the form of a single matrix inequality.

Theorem 3.4.2 *Let all variables be as defined Section 3.2. Then,* $\mathscr{Z}(\hat{K}_0, \hat{K}, \hat{\upsilon}, \hat{\Delta}) \leq \gamma^2$ *and* $\mathscr{F}(\hat{K}_0, \hat{K}, \hat{\upsilon}, \hat{\Delta}) \leq \bar{\mathbf{f}}$ *for all* $\hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}}$ *if there exist solutions* $([\hat{K}_0 \;\; \hat{K}], \hat{\upsilon}) \in \mathscr{K} \times \mathbb{R}^{N_u}$, $(S,R,G), (\tilde{S},\tilde{R},\tilde{G}) \in \widehat{\Psi}$, $\mu \in \mathbb{R}$ *and* $M \in \mathbb{D}^{N_f}$ *to* (3.11) *and,*

$$\tilde{T}_1 + \mathscr{H}(\tilde{T}_2 \hat{K} \mathbf{B}_p \tilde{T}_3) \succ 0, \tag{3.22}$$

where $\begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \\ \tilde{T}_3 & 0 \end{bmatrix} =$

$$\begin{array}{c} \\ 1 \\ N_f \\ N_q \\ N_p \\ N_p \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & N_f & N_q & N_p & N_u \end{array} \\ \left[\begin{array}{cccc|c} 2\mu(\bar{\mathbf{f}} - \mathbf{D}_f^{\hat{K}_0,\hat{\upsilon}} - Me - e\mu)^T & (\mathbf{D}_q^{\hat{K}_0,\hat{\upsilon}})^T & 0 & & 0 \\ * & M+M^T & -\mathbf{D}_{fp}\tilde{G}^T & -\mathbf{D}_{fp}\tilde{S} & -\mathbf{D}_{fu} \\ * & * & \tilde{R} + \mathscr{H}(\mathbf{D}_{qp}\tilde{G}^T) & \mathbf{D}_{qp}\tilde{S} & \mathbf{D}_{qu} \\ * & * & * & \tilde{S} & 0 \\ \hline 0 & 0 & \tilde{G}^T & \tilde{S} & 0 \end{array}\right] \end{array}.$$

Proof: We only need to prove that (3.22) is sufficient for $\tilde{f} := \bar{\mathbf{f}} - \mathbf{f} \geq 0$ for all $\hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}}$, where $\mathbf{f}$ is defined in (3.9). Using Theorem 3.4.1 and rearranging (3.19) verifies that a sufficient condition for

the constraints is

$$H_{11}+\mathscr{H}(H_{12}\hat{\Delta}(I-H_{22}\hat{\Delta})^{-1}H_{21})\succ 0,\ \forall\hat{\Delta}\in\mathscr{B}\hat{\mathbf{\Delta}}, \tag{3.23}$$

where we have used a strict inequality to avoid issues related to optimality and conditioning and where

$$\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] := \left[\begin{array}{cc|c} 2\mu & \left(\bar{\mathbf{f}}-\mathbf{D}_f^{\hat{K}_0,\hat{\upsilon}}-Me-\mu e\right)^T & 0 \\ * & M+M^T & -\mathbf{D}_{fp}^{\hat{K}} \\ \hline \mathbf{D}_{\hat{q}}^{\hat{K}_0,\hat{\upsilon}} & 0 & \mathbf{D}_{qp}^{\hat{K}} \end{array}\right].$$

Using Lemma 3.1.1 on (5.39) and the definition of (3.8) yields the matrix inequality (3.22) as a sufficient condition. ■

Using the linearization procedure in Section 3.3, we next derive sufficient LMI conditions for the problem stated in (3.13).

Theorem 3.4.3 *Let all variables be as defined Theorem 3.4.2. Then,* $\mathscr{L}(\hat{K}_0,\hat{K},\hat{\upsilon},\hat{\Delta})\leq\gamma^2$ *and* $\mathscr{F}(\hat{K}_0,\hat{K},\hat{\upsilon},\hat{\Delta})\leq\bar{\mathbf{f}}$ *for all* $\hat{\Delta}\in\mathscr{B}\hat{\mathbf{\Delta}}$ *if there exist* $([\hat{K}_0\ \hat{K}],\hat{\upsilon})\in\mathscr{K}\times\mathbb{R}^{N_u}$, $(S,R,G),(\tilde{S},\tilde{R},\tilde{G})\in\widehat{\Psi}$, $\mu\in\mathbb{R}$, $M\in\mathbb{D}^{N_f}$ *and* $X\in\mathbb{R}^{N_n\times N_n}$, *with* X *lower block-triangular with* $n\times n$ *blocks, to (3.14) and the following LMI:*

$$\begin{bmatrix} \tilde{T}_1+\mathscr{H}\left(\tilde{T}_2\bar{K}\tilde{Y}^*\right) & * \\ \left(\mathbf{B}_p\tilde{T}_3-\bar{K}^T\tilde{T}_2^T\right)-X\tilde{Y}^* & X+X^T \end{bmatrix}\succ 0, \tag{3.24}$$

for any $Y^*\in\mathbb{R}^{N_n\times(N_z+1+N_q+N_p)}$, $\tilde{Y}^*\in\mathbb{R}^{N_n\times(1+N_f+N_q+N_p)}$ *and where* $\bar{K}:=\hat{K}X\in\mathscr{K}$.

Furthermore, suppose that (3.11) *and* (3.22) *have feasible solutions for* $(\gamma^2,\hat{K}_0,\hat{K},\hat{\upsilon},R,S,G,\tilde{R},\tilde{S},\tilde{G})=(\gamma^{2*},\hat{K}_0^*,\hat{K}^*,\hat{\upsilon}^*,R^*,S^*,G^*,\tilde{R}^*,\tilde{S}^*,\tilde{G}^*)$ *so that* (3.16) *and*

$$\tilde{T}_1(\hat{K}_0^*,\hat{K}^*,\hat{\upsilon}^*,\tilde{R}^*,\tilde{S}^*,\tilde{G}^*)+\mathscr{H}\left(\tilde{T}_2\hat{K}^*\mathbf{B}_p\tilde{T}_3(\tilde{S}^*,\tilde{G}^*)\right)\succ 0. \tag{3.25}$$

are satisfied and let $Y^*=\mathbf{B}_pT_3(S^*,G^*)+(T_2\hat{K}^*)^T$ *and* $\tilde{Y}^*=\mathbf{B}_p\tilde{T}_3(\tilde{S}^*,\tilde{G}^*)+(\tilde{T}_2\hat{K}^*)^T$. *Then* (3.14) *and* (3.24) *are feasible.*

Proof: The result can be proved by applying the Elimination Lemma 3.3.1 on (3.22) in a similar procedure to that used in the proof of Theorem 3.3.2 and is omitted. ■

Remark 6 *Note that the LMI presented in* (3.19) *is based on semidefinite relaxation procedure and provides only sufficient conditions for* $\tilde{f} \geq 0$ *and can therefore be conservative. To reduce the conservativeness, we can add more redundant constraints in Theorem 3.4.1 For example, it can be shown that the redundant constraint* $\sum_{i=1}^{N_f} \delta_i^2 = 1$ *can be used to replace the LMI in* (3.19) *by the less conservative LMI*

$$\begin{bmatrix} 2\mu + \nu & \left(\tilde{f} - Me - e\mu\right)^T \\ * & M + M^T - \nu I \end{bmatrix} \succeq 0.$$

with $\nu \in \mathbb{R}$. *However, this is not pursued further in this work. Our numerical experimentation, including the examples presented in Section 3.6, indicates that in practice, the single LMI sufficient condition for the constraints provided by Theorem 3.4.3 (which is based on Theorem 3.4.1), performs as well as the multiple LMI sufficient conditions for the constraints provided by Theorem 3.3.2.*

3.5 Feasibility analysis

A major problem in MPC is to ensure that the constraints are feasible. Infeasibility may arise if the constraints are too tight or it may be due to the approximations used to obtain a practical solution. In the context of this work, to guarantee feasibility, Theorems 3.3.2 and 3.4.3 require initial feasible solutions to (3.11), and (3.12) (to compute Y^* and Y_i^*) or (3.11) and (3.22) (to compute Y^* and $\tilde{Y}^*$). On the other hand, (3.11) and (3.12) are nonlinear and difficult to solve and these computations need to be carried out online. In this section we develop algorithms that address these issues that involve extensive computations, which, however, are convex and can be carried out offline. We will concentrate on Theorem 3.4.3 since the procedure for Theorem 3.3.2 is similar. One approach is to use the solutions in step k as the initial solutions in step $k+1$. There is no guarantee that these solutions are feasible in step $k+1$ since x_k will be different from x_{k+1}.

Our approach is to find solutions to (3.11) and (3.22) offline that are feasible for every x_0 in a constrained set. Note that both (3.11) and (3.22) can be written as $M(x_0) := M_1 + \mathscr{H}(M_2 x_0 M_3) \succ 0$ in which M_3 is constant and M_1 and M_2 are independent of x_0, see (3.8). The next result uses an S-Procedure to derive sufficient conditions for $M(x_0) \succ 0$ for all x_0 in a polytopic set and forms the basis

for our Algorithm 1.

Theorem 3.5.1 *Let $M_1 = M_1^T \in \mathbb{R}^{m\times m}$, $M_2 \in \mathbb{R}^{m\times n}$, $M_3 \in \mathbb{R}^{1\times m}$, $C_0 \in \mathbb{R}^{p\times n}$, $\underline{c}_0 \le \bar{c}_0 \in \mathbb{R}^p$ be given. Then $M_1 + \mathscr{H}(M_2 x_0 M_3) \succ 0$ for all $x_0 \in \mathscr{X}_0 := \{x_0 \in \mathbb{R}^n : \underline{c}_0 \le C_0 x_0 \le \bar{c}_0\}$ if there exists $0 \preceq D_0 \in \mathbb{D}^p$ such that*

$$L := \begin{bmatrix} M_1 + \frac{1}{2}\mathscr{H}(M_3^T(\underline{c}_0^T D_0 \bar{c}_0)M_3) & * \\ M_2^T - \frac{1}{2}C_0^T D_0(\underline{c}_0 + \bar{c}_0)M_3 & C_0^T D_0 C_0 \end{bmatrix} \succ 0$$

Proof: A manipulation verifies the following identity

$$M_1 + \mathscr{H}(M_2 x_0 M_3) = M_0 + \begin{bmatrix} I_m & M_3^T x_0^T \end{bmatrix} L \begin{bmatrix} I_m \\ x_0 M_3 \end{bmatrix},$$

where $M_0 := M_3^T(C_0 x_0 - \underline{c}_0)^T D_0(\bar{c}_0 - C_0 x_0)M_3$. The result then follows from the constraints on x_0 and the structure and sign-definiteness of D_0 (which ensure that $M_0 \succ 0$ for all $x_0 \in \mathscr{X}_0$) and since $L \succ 0$ (which ensures that the second term on the RHS of the identity is positive definite for all $x_0 \in \mathbb{R}^n$). ■

Theorem 3.5.1 gives an LMI procedure for solving (3.11) and (3.22) for all $x_0 \in \mathscr{X}_0$ when $\hat{K}$ is given and for solving (3.14) and (3.24) for all $x_0 \in \mathscr{X}_0$ when an initial feasible solution for (3.11) and (3.22) is given. Algorithm 1 outlines the suggested offline policy for computing initial feasible solutions for Theorem 3.4.3.

Remark 7 *If $\beta = 1$ at the end of Step 2, we have feasible solutions to* (3.11) *and* (3.22) *for all $x_0 \in \mathscr{X}_0$ and we can use Theorem 3.4.3 online. If fewer online computations are needed, Theorem 3.4.2 can be used online with $\hat{K} = \hat{K}^*$ and $\hat{K}_0, \hat{\upsilon}$ can be used to minimize γ. If $\beta > 1$ at the end of Step 2, then, with minor modifications to Algorithm 1, Theorem 3.4.3 can still be used, although without a guaranteed feasible solution, but possibly a good initial solution if $\beta \sim 1$ since the conditions are not necessary. Alternatively, we may sub-divide $\mathscr{X}_0$ into subsets, find a feasible solution for each and use a look-up table to choose the initial solution given x_0 in the online implementation. See Example 1 for more details.*

Algorithm 1: Initial feasible solutions for Theorem 3.4.3

Result: Lookup Table contain $Y^*(S^*, G^*, \hat{K}^*)$ and $\tilde{Y}^*(\tilde{S}^*, \tilde{G}^*, \hat{K}^*)$, $\forall x \in \mathscr{X}_0$

Step 1:
In the (3.22) LMI fix $\hat{K}$ (e.g. $\hat{K}=0$) and replace $\bar{f}$ by $\beta\bar{f}$.
Minimize β such that (3.22) is satisfied for all $x_0 \in \mathscr{X}_0$ (using the finding of Theorem 3.5.1).
Record variables β, $\tilde{S}$ and $\tilde{G}$ and let $\hat{K}^*=0, \tilde{S}^*=\tilde{S}, \tilde{G}^*=\tilde{G}, i=1, \beta_i=\beta$.
Select a maximum number of iterations $i_{\max}$ and tolerance $tol_\beta < 1$;
Step 2:
while *($\beta > 1$) & ($i < i_{\max}$)* **do**
  In the (3.24) LMI, replace $\bar{f}$ by $\beta\bar{f}$ and to find the smallest $\beta \geq 1$ such that (3.24) is satisfied for all $x_0 \in \mathscr{X}_0$ (using the finding of Theorem 3.4.3 and 3.5.1).
  Set $\beta_{i+1} = \beta$ and update $\hat{K}^* := \hat{K}$, $\tilde{S}^* := \tilde{S}$ and $\tilde{G}^* := \tilde{G}$;
  if *($\frac{|\beta_{i+1}-\beta_i|}{\beta_{i+1}} < tol_\beta$)* **then**
    break; (convergence to a $\beta > 1$)
  end
  Set $i := i+1$.
end
Step 3:
if $\beta > 1$ **then**
  Sub-divide $\mathscr{X}_0$ into smaller sets;
  Go back to Step 2;
else
  In (3.11) fix $\hat{K}=\hat{K}^*$ and minimize γ^2 such that (3.11) is satisfied for all $x_0 \in \mathscr{X}_0$ (using the finding of Theorem 3.5.1).
  Record γ^2 and let $S^* = S$ and $G^* = G$;
end
Step 4:
Set $j = 1$, $\gamma_j^2 = \gamma^2$ and select $j_{\max}$ to be the maximum number of iterations and $tol_\gamma < 1$ to be a tolerance.
while *(($j < j_{\max}$)* **do**
  Minimize γ^2 such that (3.14) and (3.24) are satisfied for all $x_0 \in \mathscr{X}_0$ (using the finding of Theorems 3.4.3 and 3.5.1).
  Set $\gamma_{j+1}^2 = \gamma^2$ and update $\hat{K}^* := \hat{K}$, $S^* = S$, $G^* = G$, $\tilde{S}^* := \tilde{S}$ and $\tilde{G}^* := \tilde{G}$;
  if *($\frac{|\gamma_{j+1}^2-\gamma_j^2|}{\gamma_{j+1}^2} < tol_\gamma$)* **then**
    break;
  end
  Set $j := j+1$.
end

Remark 8 *Note here that recursive feasibility of the proposed schemes could be ensured using the standard shifting arguments and the assumption that the invariant terminal set defined by $\bar{f}_N$. In particular, under the conditions given in [76], the optimal control sequence computed at time k can be shifted and appended with the terminal control law u_{f_N} to yield : $\{u(k+1 \mid k), \cdots, u(k+N \mid k), u_{f_N}\}$ which remains feasible at next time step $k+1$ (see [3, 14] for further details). However, since our proposed linearization procedure derives only sufficient (and not necessary and sufficient) conditions further investigation in the topic needed.*

3.6 Numerical Examples and Simulations

In this section the effectiveness of the proposed algorithms is illustrated by two benchmark examples taken from the literature. The simulations in both examples are performed using MOSEK LMI/SDP solver within the CVX package [77], in MATLAB R2019b on a computer with 2.40 GHz Intel Xeon(R) CPU and 64.0 GB memory.

3.6.1 Example 1

The first system is a variation on a system proposed in [14, 19, 20]. It is a second order unstable process subject to time-invariant uncertainty as well as external bounded disturbances. The system's dynamics are represented in the form of (3.3) with the following distribution matrices:

$$A = \begin{bmatrix} 1 & 0.8 \\ 0.5 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$D_{qu} = B_u, \qquad C_q = A, \quad \hat{C}_q = 0.$$

The uncertainty is considered to be time-invariant of the form $\mathbf{\Delta} := \{\delta I_2 : \delta \in \mathbb{R}\}$ and the disturbances set is taken to be $\mathscr{W} := \{w \in \mathbb{R}^{n_w} : -1 \leq w \leq 1\}$. For illustration, we give the structure of the sets $\hat{\mathbf{\Delta}}$

and $\widehat{\Psi}$ for a prediction horizon $N = 2$, even though we will be using $N = 5$ in our simulations:

$$\hat{\mathbf{\Delta}} = \{\mathrm{diag}\,(\delta I_2, \delta_0^w, \delta I_2, \delta_1^w) : \delta, \delta_0^w, \delta_1^w \in \mathbb{R}\}$$

$$\widehat{\Psi} = \{(S,R,G) \colon R = S = S^T = \begin{bmatrix} S_{11} & 0 & S_{13} & 0 \\ 0 & s_{22} & 0 & 0 \\ S_{13}^T & 0 & S_{33} & 0 \\ 0 & 0 & 0 & s_{44} \end{bmatrix} \succ 0,$$

$$G = -G^T = \begin{bmatrix} G_{11} & 0 & G_{13} & 0 \\ 0 & 0 & 0 & 0 \\ -G_{13}^T & 0 & G_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\},$$

where $S_{11} = S_{11}^T,\ S_{33} = S_{33}^T,\ G_{11} = -G_{11}^T,\ G_{33} = G_{33}^T,\ S_{13},\ G_{13} \in \mathbb{R}^{2\times 2}$ and $s_{22},\ s_{44} \in \mathbb{R}$. Moreover, the input and state constraints are given by $-\bar{u} \le u_k \le \bar{u} = 8, k = 0, \ldots, N-1$ and $-\bar{x} \le x_k \le \bar{x} = [7\ 7]^T, k = 0, \ldots, N$, respectively ($C_f = [I\ 0\ \ -I\ 0]^T, D_{fu} = [0\ 1\ 0\ \ -1]^T, \bar{f}_k = [\bar{x}^T\ \bar{u}\ \bar{x}^T\ \bar{u}]^T, \hat{C}_f = [I\ -I]^T, \bar{f}_N = [\bar{x}^T\ \bar{x}^T]^T, D_{fp} = \hat{D}_{fp} = 0, D_{fw} = 0$). Finally, the initial state is set to be at the boundaries of the state constraints $x_0 = \bar{x}$. Given the above process description, the control objective is to regulate the unstable system subject to uncertainties and disturbances into the origin whilst satisfying the input and state constraints. The states and inputs are equally weighted ($C_z = [I\ 0]^T, \hat{C}_z = I, D_{zu} = [0\ I]^T$). To accomplish the control objective, two robust algorithms presented in this paper are applied to the system. The first computationally efficient RMPC algorithm, CE_RMPC*#1* $(\hat{K}_0, \hat{K}, \hat{\upsilon})$, is described by Theorem 3.4.3, where the decision variable are $(\hat{K}_0, \hat{K}, \hat{\upsilon})$ and the initial feasible solutions $(Y^*, \tilde{Y}^*)$ are computed offline by Algorithm 1. The second robust algorithm, CE_RMPC*#2* $(\hat{K}_0, \hat{\upsilon})$, is described by Theorem 3.4.2, where $\hat{K}$ is fixed to $\hat{K}^*$ (computed offline by Algorithm 1) and the decision variables are $\hat{K}_0$ and $\hat{\upsilon}$. Note that for Example 1, the variable β in Algorithm 1 is greater than 1 if we take $\mathscr{X}_0$ to be the entire constrained state-space ($\mathscr{X}_0 := \{x_0 : -\bar{x} \le x_0 \le \bar{x}\}$). Thus $\mathscr{X}_0$ is divided into 25 smaller sets $\mathscr{X}_0^{i,j}$, with $\beta = 1$ for each of these sets, and a look-up table has been used to store $(Y^*, \tilde{Y}^*)$ for

each subset. The subset we used are

$$\mathscr{X}_0^{i,j} = \{ \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} : \underline{x}_{01}^i \leq x_{01} \leq \bar{x}_{01}^i,\ \underline{x}_{02}^j \leq x_{02} \leq \bar{x}_{02}^j \}, i, j = 1, \ldots, 5,$$

where for $k = 1, 2$,

$$[\underline{x}_{0k}^1, \bar{x}_{0k}^1] = 7[.75, 1],\ [\underline{x}_{0k}^2, \bar{x}_{0k}^2] = 7[.4, .75],\ [\underline{x}_{0k}^3, \bar{x}_{0k}^3] = 7[-.4, .4],$$
$$[\underline{x}_{0k}^4, \bar{x}_{0k}^4] = -7[.75, .4],\ [\underline{x}_{0k}^5, \bar{x}_{0k}^5] = -7[1, .75].$$

The time to compute the initial solutions $(Y^*, \tilde{Y}^*)$ for all subsets is 94.5 seconds. Note that offline computation time depends on the number of subsets and terminal iteration values i_{max} and j_{max}. The controller from [14] is also presented for comparison. All algorithms are simulated with prediction horizon $N = 5$.

As shown in Fig. 3.1, using the proposed computationally efficient robust algorithms from Theorem 3.4.3 or Theorem 3.4.2 with fix $\hat{K}$, robust control performance has been achieved, while both states (x_1, x_2) of the unstable system are converging faster to the origin compare to the results from [14]. Fig. 3.2 illustrates the control input computed by the robust control algorithms considered in this example, where it can be seen that the input computed by the proposing algorithms hit the constraint boundaries, in comparison with the input computed by the algorithm suggested in [14]. Therefore, it can be stated that the robust algorithms presented in this paper are less conservative, even in the case of a fix $\hat{K}$, due to the novel linearization procedure (see Remark 23). The most notable outcome using the proposed robust algorithms is that they required significantly low computation burden compare to the robust method from the literature. In particular, as shown in Table 3.1, using the CE_RMPC*#1* $(\hat{K}_0, \hat{K}, \hat{\upsilon})$ procedure results in the average and maximum computation cost per iteration being reduced by 94% and 92%, respectively, as compared to the algorithm proposed in [14]. Implementing CE_RMPC*#2* $(\hat{K}_0, \hat{\upsilon})$ results in the average and maximum computation cost per iteration being reduced by 96% and 94%, respectively, in comparison to the time required by the algorithm in [14]. The numerical values in Table 3.1 were realized using the same computer to solve the above regulation problem and repeated 10 times for each method.

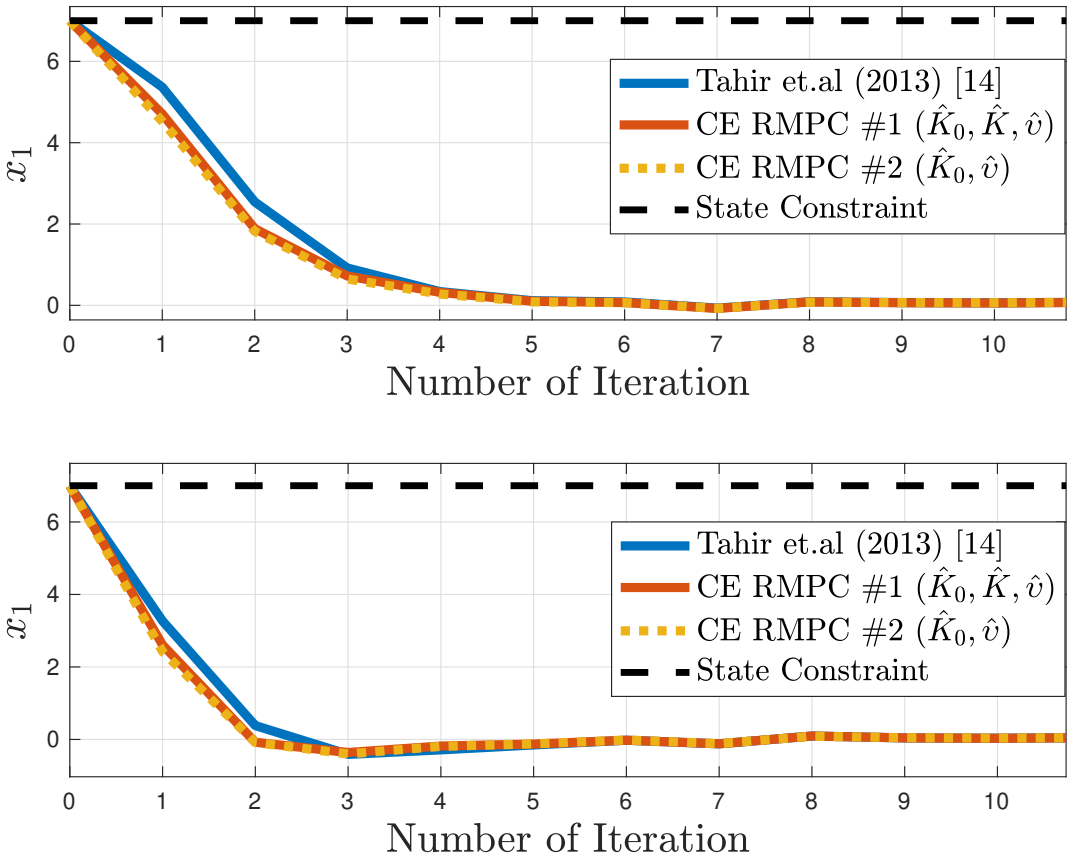

Figure 3.1: States trajectory simulation results observed by the proposed RMPC algorithms and the algorithm from the literature to the second order unstable system.

Table 3.1: Computation time per iteration

Horizon Length $N = 5$

Method	Mean $\pm$ Std Deviation	Maximum time
Tahir et.al (2013) [14]	23.4573 $\pm$ 2.2287 s	29.3438 s
CE RMPC *#1* $(\hat{K}_0, \hat{K}, \hat{\upsilon})$	1.4187 $\pm$ 0.4482 s	2.5313 s
CE RMPC *#2* $(\hat{K}_0, \hat{\upsilon})$	0.9187 $\pm$ 0.2686 s	1.7656 s

3.6.2 Example 2

In this section, the benchmark problem of control tracking of a coupled spring-mass system (see for example [14, 15]) is considered. In particular, the mechanical system consists of a two-mass-spring system as shown in Fig. 3.3. By discretizing the continuous-time equations of the system using Euler's first order approximation for the derivative and with sampling time of $T_s = 0.1s$, the following

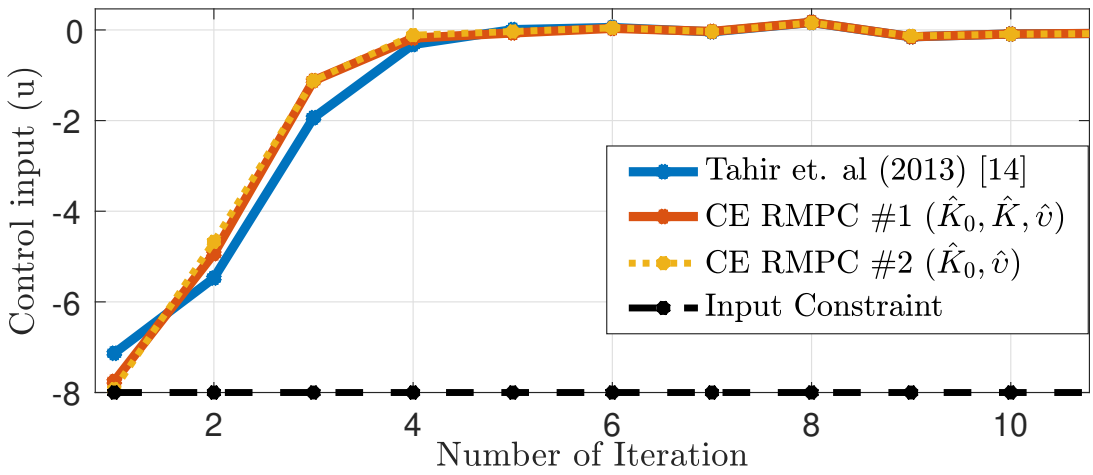

Figure 3.2: Control input sequence computed by the proposed RMPC algorithms and the algorithm from the literature considering the the model of the second order unstable system.

discrete-time state space equations are obtained [78]:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1\frac{K}{m_1} & 0.1\frac{K}{m_1} & 1 & 0 \\ 0.1\frac{K}{m_2} & -0.1\frac{K}{m_2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{0.1}{m_1} \\ 0 \end{bmatrix} u(k), \qquad y(k) = x_2(k),$$

where m_1 and m_2 are the two masses and K is the spring constant. The state variables x_1 and x_2 are the position (displacement) of mass 1 and 2 respectively, whereas x_3 and x_4 represent their respective velocities. For the nominal system, $m_1 = m_2 = K = 1$ with the appropriate units and control force u acting on m_1.

The objective for this problem is to compute a control law u (force that would be apply to the first mass m_1), such that the output (state x_2) to track a unit step whilst providing robustness against persistent variations in the spring constant K, as well as satisfying the states and input constraints:

$$-1 \leq u(k) \leq 1, \qquad -\bar{x} \leq x(k) \leq \bar{x}, \qquad \bar{x} = [1.5\ 1.5\ 1\ 1]^T,$$

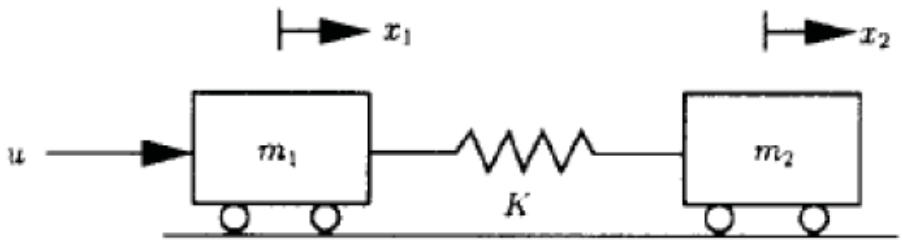

Figure 3.3: Configuration of coupled spring-mass system [14].

In this setup, the exact measurements of the state of the system are assumed to be available, while it is assumed that the spring constant K is uncertain within the range $K_{min} \leq K \leq K_{max}$, where $K_{min} = 0.5$ and $K_{max} = 10$, in appropriate units. Therefore, the uncertainty in K can be modeled as structured feedback uncertainty as presented in (3.3), by defining,

$$A = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1K_{nom} & 0.1K_{nom} & 1 & 0 \\ 0.1K_{nom} & -0.1K_{nom} & 0 & 1 \end{bmatrix}, \; B_u = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix},$$

$$B_p = \begin{bmatrix} 0 \\ 0 \\ -0.1 \\ 0.1 \end{bmatrix}, \; C_q = \begin{bmatrix} K_{dev} & -K_{dev} & 0 & 0 \end{bmatrix}, \; D_{qu} = 0,$$

where $\delta = \frac{K - K_{nom}}{K_{dev}}$, $K_{nom} = \frac{1}{2}(K_{max} + K_{min})$, and $K_{dev} = \frac{1}{2}(K_{max} - K_{min})$. Due to the time invariant structure of the uncertainty matrix $\Delta = \delta I$, the slack variables $S = R, \tilde{S} = \tilde{R}$ are full symmetric positive definite while G and $\tilde{G}$ are full skew symmetric. The weighted matrices in the cost function are set as $C_z = [5I \; 0]^T$, $D_{zu} = [0 \; I]^T$ and the prediction horizon is set as $N = 6$.

The output response in Fig. 3.5 shows that both proposed algorithms robustly steer the system to the reference signal. Fig. 3.5 also shows the responses of the infinite horizon methods in [15, 16] and the finite horizon method in [14], as well as NL-RMPC$(\hat{K}_0, \hat{K}, \hat{\upsilon})$ which is described by Theorem 3.3.2 (New Linearization RMPC with multiple LMIs) . It can be seen that our approaches converge much

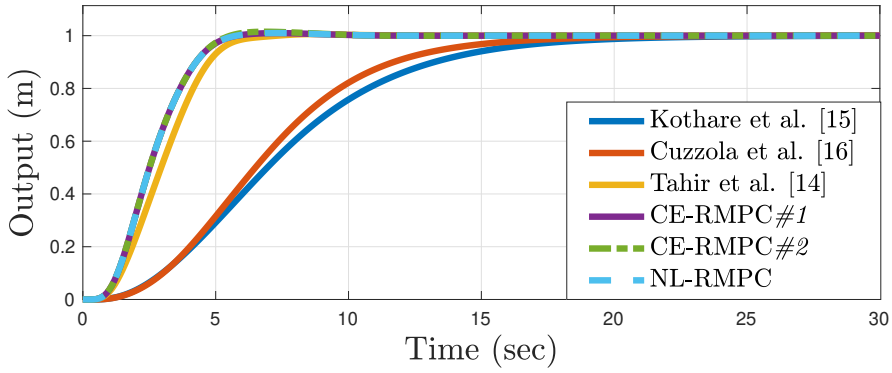

Figure 3.4: Time history of the output variable (x_2) using the proposed RMPC controllers that utilized new linearization with a single LMI to achieve step tracking. Algorithms from the literature based on infinite and finite horizon schemes are displayed for comparison.

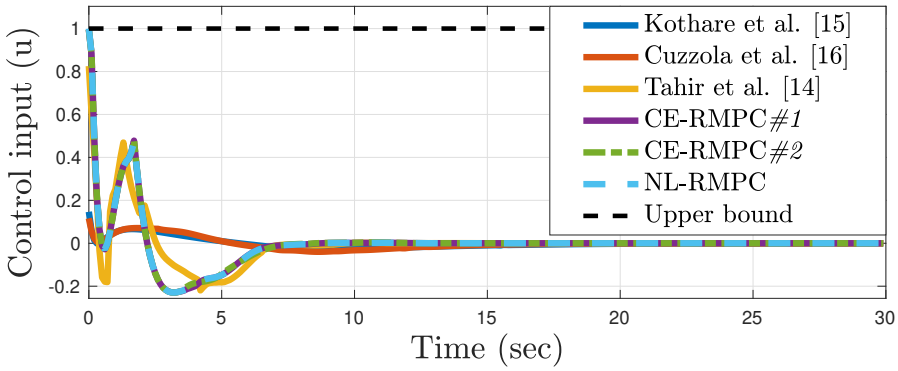

Figure 3.5: Time history of the input signal (u) computed by the proposed RMPC controllers that utilized new linearization with a single LMI to achieve step tracking. The control signal calculated by algorithms from the literature based on infinite and finite horizon schemes is displayed for comparison.

faster than the two infinite horizon methods. Comparing with [14], it can be seen that even though all algorithms have excellent tracking properties, the proposed controllers have slightly faster responses due to the less restrictive nature in the formulation (see Remark 23). Fig. 3.5 also shows that the control input calculated by the proposed algorithms is much faster and is closer to the upper bound. According to the finding in figure 3.6, our method also gives a much smaller cost function compared with the infinite horizon methods and similar cost compared with the finite horizon method in [14].

To quantify the effect of Theorem 3.4.3 compared with Theorem 3.3.2 with respect to feasibility domain, K_{max} was increased until infeasibility is observed. CE_RMPC#*1* and CE_RMPC#*2* can reach a solution for values up to $K_{max} = 20.5$, while NL-RMPC for values up to $K_{max} = 21$. Therefore we can conclude that the large computation time reduction using the suggested algorithms (CE_RMPC#*1* and CE_RMPC#*2*) comes with only a small reduction in the feasibility domain. Comparing the

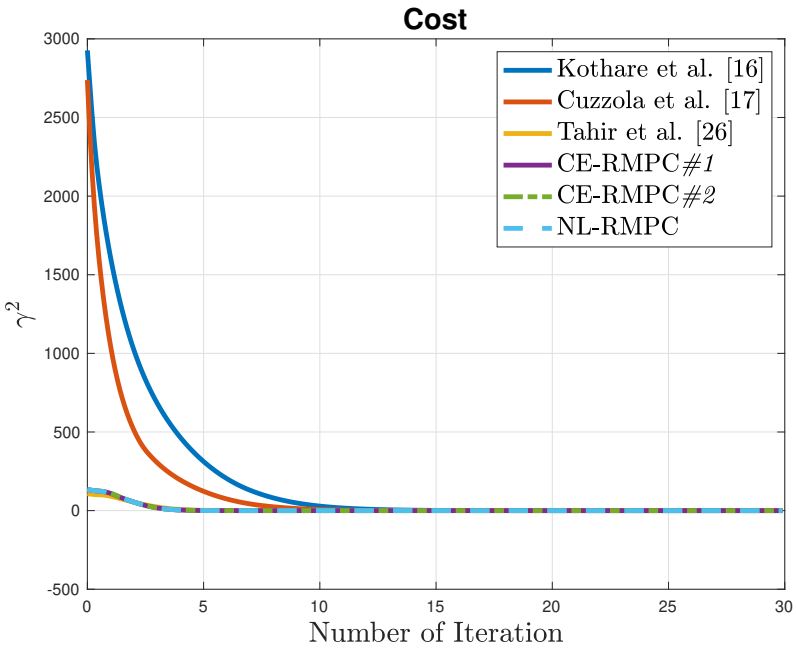

Figure 3.6: Cost signal history using the proposed RMPC controllers. Algorithms from the literature based on infinite and finite horizon schemes are displayed for comparison.

Table 3.2: Computation time per iteration for the two-mass-spring system subject to uncertainties.

Method	Mean ± Std Deviation	Max. time
Inf. horizon RMPC from [15]	1.0788 ± 0.3321 s	2.2813 s
Inf. horizon RMPC from [16]	1.0679 ± 0.3002 s	2.5625 s
RMPC from [14]	3.0672 ± 0.5137 s	5.3750 s
CE RMPC *#1*	1.0734 ± 0.2695 s	2.0156 s
CE RMPC *#2*	0.3502 ± 0.1044 s	0.7117 s
NL-RMPC	2.3547 ± 0.9762 s	3.7656 s

computational times in Table 3.2, it can be seen that CE_RMPC*#1* has a similar computational burden as [15, 16], and is much faster than [14] and the algorithm NL-RMPC. A significant computation time reduction can also be observed for CE_RMPC*#2*. Therefore our approach combines the fast online computational performance of the infinite horizon methods and the good performance of the finite horizon approaches. Note that in general, a larger prediction horizon increases the computation time, while stability is improved. Restricting the computational time to be similar to infinite horizon methods from the literature (on average 1 sec), the prediction horizon was set to $N = 6$ to allow a fair comparison with respect to performance. Horizon length $N = 7$ gives an average computational time $t = 2.1845$ sec, however, the control performance was not noticeably improved.

3.7 Synopsis

In this chapter, two algorithms are proposed to reduce the computational complexity of state-feedback RMPC for linear-time-invariant discrete-time systems, subject to structured uncertainty and bounded disturbances. In particular, a new linearization approach, based on the Elimination Lemma and S-Procedures, is developed to address the nonlinearity and nonconvexity associated with state-feedback RMPC, with minimal conservatism whilst resulting in a substantially lower computational burden as compared to similar methods in the literature. The approach requires initial feasible solutions to the nonlinear matrix inequalities, which however can be obtained offline. Further reduction in the computational complexity is achieved by a proposed algorithm that solves a single LMI for handling all the constraints in the RMPC problem.

The effectiveness of the proposed techniques is demonstrated through numerical examples taken from the literature. In particular, it has been shown that the proposed RMPC scheme can successfully calculate an optimal control signal up to 96% faster than other finite horizon RMPC, while being able to steer the system quicker to a predefined reference with minimum conservativeness compared with other RMPC approaches.

CHAPTER 4

Robust Moving Horizon State Estimation for Uncertain Linear Systems using Linear Matrix Inequalities

4.1 Introduction

As already mentioned in Section 1.1.2, moving horizon estimation algorithms can be solved online and they can successfully overcome the previously mentioned conservativeness problems introduced by offline calculations. Despite the plethora of MHE algorithms proposed in the literature, the contributions when the system is uncertain are scarce. One such contribution is [40], in which the minimization of an upper bound on a worst-case quadratic cost defined over a moving horizon window allows one to construct a filter for uncertain linear systems. This design method is based on the solution of min-max regularized least-squares problems [41]. However, robust least-squares problems are known to have computational difficulties reaching a solution, since they are in general NP-hard [42]. Reduction of the excessive online computational burden can be achieved by reformulating the optimization problem as an equivalent SDP problem using LMIs. SDP is concerned with optimization problems

that have solutions over the cone of all positive semidefinite matrices. SDP is a well-established methodology that allows the solution of a class of problems within a given accuracy in polynomial time using interior-point methods [79].

In the present work, instead of employing an offline linear observer, the past input/output data window is used, in a manner similar to Receding Horizon Estimation (RHE) described in [33], to compute online (tight) bounds on the current state. The main contribution of this chapter is the generalization of MHE from systems subject to disturbances only (investigated in [34]), to systems subject to structured feedback uncertainty, as well as external disturbances, which is more realistic for applications. In addition, the proposed estimation method reduces conservativeness as compared to observer based methods by solving online an optimization problem through LMIs, while keeping the computational burden low. Finally, in the proposed method, at every sample time hard bounds on the estimated state are given rather than only the estimated state values, which most of the estimation schemes in the literature compute. Very importantly, hard bounds on the estimated states can potentially be used in a control scheme and improve significantly the robust properties of the controller.

The remainder of this chapter is organized as follows. In Section 4.2 the estimation problem description is presented. The proposed state estimation is explained in Section 4.3. In Section 4.4 and 4.5 the overall proposed algorithm and simulation results for an exemplary case study from the literature involving a paper-making process are presented, respectively. Finally, conclusions are drawn in Section 4.6. The formulation and results presented in this chapter are mainly based on the result presented in [80].

4.2 Problem Statement

In this section, the system description including system dynamics, initial condition, disturbances and uncertain signals, is first provided. Then the problem of moving horizon estimation for discrete-time systems subject to bounded disturbances and structured uncertainties is presented as an optimization problem.

4.2.1 System Description

The following linear discrete-time system, subject to norm-bounded structured uncertainty and external disturbances, is considered (see for example [15]):

$$\begin{bmatrix} x_{k+1} \\ q_k \\ y_k \end{bmatrix} = \begin{bmatrix} A & B_u & B_w & B_p \\ C_q & D_{qu} & 0 & 0 \\ C_y & D_{yu} & D_{yw} & D_{yp} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ w_k \\ p_k \end{bmatrix}, \quad p_k = \Delta_k q_k, \tag{4.1}$$

where $k = 0,1,2,\ldots$ is the time instant, $\Delta_k \in \mathscr{B}\mathbf{\Delta}$, where $\mathbf{\Delta} \subseteq \mathbb{R}^{n_p \times n_q}$ is a subspace that captures the uncertainty structure. Furthermore, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, $w_k \in \mathbb{R}^{n_w}$ are the state, input, output and disturbance signal, respectively, at time instant k. Here $p_k \in \mathbb{R}^{n_p}$ and $q_k \in \mathbb{R}^{n_q}$ represent the uncertainty vectors and all other symbols in capital letters denote the appropriate distribution matrices. Only the input u_k and the noisy output y_k are measured and it is assumed that (A, C_y) is detectable and (A, B_u) is stabilizable.

Furthermore, lower and upper bounds $\underline{x}_0$ and $\overline{x}_0$ on the initial state are given a priori such that (see also Section 4.3):

$$x_0 \in \mathscr{X}_0 := \left\{ x \in \mathbb{R}^n : \underline{x}_0 \leq x \leq \overline{x}_0 \right\}. \tag{4.2}$$

Finally, the unmeasured additive disturbances w_k are bounded by a given nonnegative vector r so that

$$w_k \in \mathscr{W} := \left\{ w \in \mathbb{R}^{n_w} : -r \leq w \leq r \right\}. \tag{4.3}$$

Remark 9 *Note that uncertainty is allowed in all the problem data including the state and the output signal. It is easy to verify that the state dynamics in (4.1) can be re-written in the form:*

$$x_{k+1} = (A + B_p \Delta_k C_q) x_k + (B_u + B_p \Delta_k D_{qu}) u_k + B_w w_k, \quad \Delta_k \in \mathscr{B}\mathbf{\Delta}.$$

Remark 10 *For the sake of clarity of exposition, both the state-disturbance (η_k) and output-disturbance (ν_k) are combined into a single vector in (4.1), namely $w_k := [\eta_k^T \;\; \nu_k^T]^T$.*

4.2.2 Estimation Problem

The objective of the proposed RMHE algorithm is to compute tight upper/lower bounds on the states using a moving and fixed-size window of past input and output data. The information vectors for the inputs and output are defined as follows:

$$\begin{aligned} \mathbf{u} &= [u_{k-N_e}^T, \cdots, u_{k-1}^T]^T, \\ \mathbf{y} &= [y_{k-N_e}^T, \cdots, y_k^T]^T, \end{aligned} \tag{4.4}$$

where $N_e > 0$ denotes a given estimation horizon. The information vectors are updated every sample time by removing the oldest input/output data while the new output measurement and the latest control input are added. Then the estimation problem can be transform into an optimization problem as follows:

Problem 4.2.1 *At the time instant k, for given information vectors* $(\mathbf{u}, \mathbf{y})$ *and pre-computed state bounds values* $(\underline{x}_{k-N_e}, \bar{x}_{k-N_e})$*, it is required to find lower and upper bound* $(\underline{x}_k,\ \bar{x}_k)$ *on the current state that solve the min/max and max/min problems*

$$\max_{\underline{x}_k} \quad \min_{w_k \in \mathscr{W}_k,\ \Delta_k \in \mathscr{B}\mathbf{\Delta}} \quad e_i^T \underline{x}_k, \tag{4.5}$$

$$\min_{\bar{x}_k} \quad \max_{w_k \in \mathscr{W}_k,\ \Delta_k \in \mathscr{B}\mathbf{\Delta}} \quad e_i^T \bar{x}_k. \tag{4.6}$$

such that the dynamics in (4.1) *are satisfied.*

Remark 11 *Note here that decision variables in the above problem are the lower and upper bound of the current state and information vectors* $(\mathbf{u}, \mathbf{y})$ *and initial state bounds* $(\underline{x}_{k-N_e}, \bar{x}_{k-N_e})$ *are known parameters. For the lower bounds calculation minimum disturbances and uncertainty are considered and for the upper bound calculations the maximum bounded disturbances and uncertainty are considered.*

Such a strategy that solves the above estimation problem through LMI optimization is developed in this chapter as described in more detail in Section 4.3.

4.3 Estimation Problem Formulation using LMIs

This section formulates an optimization problem which uses the past N_e inputs and outputs (as well as the current output y_k) to compute upper and lower state bounds ($\underline{x}_k$ and $\overline{x}_k$), as briefly presented in Section 4.2.2.

We start by iterating the process dynamics in (4.1) to obtain:

$$\begin{bmatrix} x_k \\ \mathbf{q} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_u & \mathbf{B}_w & \mathbf{B}_p \\ \mathbf{C}_q & \mathbf{D}_{qu} & \mathbf{D}_{qw} & \mathbf{D}_{qp} \\ \mathbf{C}_y & \mathbf{D}_{yu} & \mathbf{D}_{yw} & \mathbf{D}_{yp} \end{bmatrix} \begin{bmatrix} x_{k-N_e} \\ \mathbf{u} \\ \mathbf{w} \\ \mathbf{p} \end{bmatrix}, \quad \mathbf{p} = \mathbf{\Delta q}, \tag{4.7}$$

where the input/output data vectors $\mathbf{u}$ and $\mathbf{y}$ (defined in (4.4)) are known, and $\mathbf{w} = [w_{k-N_e}^T \cdots \; w_k^T]^T$, $\mathbf{q} = [q_{k-N_e}^T \cdots \; q_{k-1}^T]^T$, $\mathbf{p} = [p_{k-N_e}^T \cdots \; p_{k-1}^T]^T$ and $\mathbf{\Delta} = \mathrm{diag}(\Delta_{k-N_e}, \cdots, \Delta_{k-1})$. All the bold matrices in (4.7) are the stacked coefficient matrices, which can be computed through iteration over the estimation horizon N_e using (4.1).

By using the definition of $\mathbf{q}$ in (4.7), the vector $\mathbf{p}$ $(:= \mathbf{\Delta q})$ can be rearranged as:

$$\mathbf{p} = \mathbf{\Delta}(I - \mathbf{D}_{qp}\mathbf{\Delta})^{-1}(\mathbf{C}_q x_{k-N_e} + \mathbf{D}_{qu}\mathbf{u} + \mathbf{D}_{qw}\mathbf{w}). \tag{4.8}$$

Then, using (4.8) to eliminate $\mathbf{p}$ from (4.7) gives:

$$\begin{bmatrix} x_k \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} A_d + \mathbf{B}_p\overline{\Delta}\mathbf{C}_d & \mathbf{B}_u + \mathbf{B}_p\overline{\Delta}\mathbf{D}_{qu} \\ \mathbf{C}_{yd} + \mathbf{D}_{yp}\overline{\Delta}\mathbf{C}_d & \mathbf{D}_{yu} + \mathbf{D}_{yp}\overline{\Delta}\mathbf{D}_{qu} \end{bmatrix} \begin{bmatrix} d \\ \mathbf{u} \end{bmatrix}, \tag{4.9}$$

where $\overline{\Delta} := \mathbf{\Delta}(I - \mathbf{D}_{qp}\mathbf{\Delta})^{-1}$, $A_d := [\mathbf{A} \;\; \mathbf{B}_w]$, $\mathbf{C}_{yd} := [\mathbf{C}_y \;\; \mathbf{D}_{yw}]$, $\mathbf{C}_d := [\mathbf{C}_q \;\; \mathbf{D}_{qw}]$ and $d := [x_{k-N_e}^T \;\; \mathbf{w}^T]^T$ such that

$$\begin{bmatrix} \underline{x}_{k-N_e} \\ -\mathbf{r} \end{bmatrix} =: \underline{d} \leq d \leq \overline{d} := \begin{bmatrix} \overline{x}_{k-N_e} \\ \mathbf{r} \end{bmatrix}, \tag{4.10}$$

where $\mathbf{r} = \mathbf{1} \otimes r$ and where $\mathbf{1}$ represents the N_e–dimensional vector of ones.

By using (4.9) and (4.10), upper- and lower-bounds on x_k are derived in the following theorem.

Theorem 4.3.1 *Let all variables be as defined above. Then, an upper-bound on the i-th element of x_k, i.e. $e_i^T x_k$, can be computed by minimizing $e_i^T \overline{x}_k$ subject to the existence of $(\bar{S}_i, \bar{G}_i) \in \widehat{\Psi}$, $\mu_i \in \mathbb{R}^{N_e n_y}$, $0 \prec \overline{D}_x^i \in \mathbb{D}$, $\forall i \in \mathcal{N}_n := \{1, \cdots, n\}$ and the LMI*

$$\begin{bmatrix} \overline{D}_x^i & \Pi_{12} & \mathbf{C}_d^T \overline{G}_i^T & \mathbf{C}_d^T \overline{S}_i \\ \star & \Pi_{22} & \mu_i^T \mathbf{D}_{yp} - \frac{1}{2} e_i^T \mathbf{B}_p + (\mathbf{D}_{qu}\mathbf{u})^T \overline{G}_i & (\mathbf{D}_{qu}\mathbf{u})^T \overline{S}_i \\ \star & \star & \overline{S}_i + \mathbf{D}_{qp}^T \overline{G}_i^T + \overline{G}_i \mathbf{D}_{qp} & \mathbf{D}_{qp}^T \overline{S}_i \\ \star & \star & \star & \overline{S}_i \end{bmatrix} \succeq 0, \tag{4.11}$$

where $\Pi_{12} = -\frac{1}{2}\overline{D}_x^i(\overline{d} + \underline{d}) - \frac{1}{2}A_d^T e_i + \mathbf{C}_{yd}^T \mu_i$ and $\Pi_{22} = e_i^T \overline{x}_k + \overline{d}^T \overline{D}_x^i \underline{d} - e_i^T \mathbf{B}_u \mathbf{u} - 2\mu_i^T y^{\mathbf{u}}$.

Similarly, a lower-bound on $e_i^T x_k$ can be computed by maximizing $e_i^T \underline{x}_k$ subject to the existence of $(\underline{S}_i, \underline{G}_i) \in \widehat{\Psi}$, $\mu_i \in \mathbb{R}^{N_e n_y}$, $0 \succ \underline{D}_x^i \in \mathbb{D}$, $\forall i \in \mathcal{N}_n := \{1, \cdots, n\}$ and the LMI

$$\begin{bmatrix} \underline{D}_x^i & \Lambda_{12} & \mathbf{C}_d^T \underline{G}_i^T & \mathbf{C}_d^T \underline{S}_i \\ \star & \Lambda_{22} & \mu_i^T \mathbf{D}_{yp} - \frac{1}{2} e_i^T \mathbf{B}_p + (\mathbf{D}_{qu}\mathbf{u})^T \underline{G}_i & (\mathbf{D}_{qu}\mathbf{u})^T \underline{S}_i \\ \star & \star & \underline{S}_i + \mathbf{D}_{qp}^T \underline{G}_i^T + \underline{G}_i \mathbf{D}_{qp} & \mathbf{D}_{qp}^T \underline{S}_i \\ \star & \star & \star & \underline{S}_i \end{bmatrix} \succeq 0, \tag{4.12}$$

where $\Lambda_{12} = -\frac{1}{2}\underline{D}_x^i(\overline{d} + \underline{d}) + \frac{1}{2}A_d^T e_i + \mathbf{C}_{yd}^T \mu_i$ and $\Lambda_{22} = -e_i^T \underline{x}_k + \overline{d}^T \underline{D}_x^i \underline{d} + e_i^T \mathbf{B}_u \mathbf{u} - 2\mu_i^T y^{\mathbf{u}}$, and where $y^{\mathbf{u}} = \mathbf{y} - \mathbf{D}_{yu}\mathbf{u}$.

Proof: In order to take account of the available past input/output data $(\mathbf{u}, \mathbf{y})$ in the proposed formulation, the following equality constraint is considered, based on the expression for $\mathbf{y}$ in (4.9):

$$y^{\mathbf{\Delta}} - C_d^{\mathbf{\Delta}} d = 0, \tag{4.13}$$

where $y^{\mathbf{\Delta}} := \mathbf{y} - (\mathbf{D}_{yu} + \mathbf{D}_{yp}\overline{\Delta}\mathbf{D}_{qu})\mathbf{u}$ and $C_d^{\mathbf{\Delta}} := (\mathbf{C}_{yd} + \mathbf{D}_{yp}\overline{\Delta}\mathbf{C}_d)$. Now by considering $\overline{x}_k$ as an upper-

bound on x_k in (4.9), it is required for all $i \in \mathscr{N}_n$:

$$e_i^T x_k - e_i^T \bar{x}_k = e_i^T (A_d^{\mathbf{\Delta}} d + B_{\mathbf{u}}^{\mathbf{\Delta}} \mathbf{u}) - e_i^T \bar{x}_k \leq 0, \tag{4.14}$$

where $A_d^{\mathbf{\Delta}} := A_d + \mathbf{B}_p \overline{\Delta} \mathbf{C}_d$ and $B_{\mathbf{u}}^{\mathbf{\Delta}} = \mathbf{B}_u + \mathbf{B}_p \overline{\Delta} \mathbf{D}_{qu}$.

By incorporating (4.13), it can then be verified that for any diagonal $\overline{D}_x^i \succ 0$ and $\mu_i \in \mathbb{R}^{N_e n_y}$

$$\begin{aligned} e_i^T x_k - e_i^T \bar{x}_k = & -(\overline{d} - d)^T \overline{D}_x^i (d - \underline{d}) \\ & - \left(\mu_i^T (y^{\mathbf{\Delta}} - C_d^{\mathbf{\Delta}} d) + (y^{\mathbf{\Delta}} - C_d^{\mathbf{\Delta}} d)^T \mu_i \right) \\ & - \hat{d}^T \overline{\mathscr{L}}_i (\overline{D}_x^i, \mathbf{\Delta}, \mu_i) \hat{d}, \ \forall i \in \mathscr{N}_n, \end{aligned} \tag{4.15}$$

where $\hat{d} := [d^T \ 1]^T$ and $\overline{\mathscr{L}}_i(\overline{D}_x^i, \mathbf{\Delta}, \mu_i)$ is defined as:

$$\begin{bmatrix} \overline{D}_x^i & -\frac{1}{2}\overline{D}_x^i(\overline{d} + \underline{d}) - \frac{1}{2}(A_d^{\mathbf{\Delta}})^T e_i + (C_d^{\mathbf{\Delta}})^T \mu_i \\ \star & e_i^T \bar{x}_k - e_i^T B_{\mathbf{u}}^{\mathbf{\Delta}} \mathbf{u} + \overline{d}^T \overline{D}_x^i \underline{d} - 2\mu_i^T y^{\mathbf{\Delta}} \end{bmatrix} \tag{4.16}$$

By using the constraints (4.10) and (4.13) in (4.15), together with the S-procedure (Farkas' Theorem) [67], it follows that $\overline{\mathscr{L}}_i(\overline{D}_x^i, \mathbf{\Delta}, \mu_i) \succ 0$, $\forall i \in \mathscr{N}_n$, is a sufficient condition for (4.14). Applying a Schur complement argument followed by a re-arrangement, shows that, for all $i \in \mathscr{N}_n$, this sufficient condition can be written as:

$$R_i + F_i \mathbf{\Delta} (I - H\mathbf{\Delta})^{-1} E + E^T (I - \mathbf{\Delta}^T H^T)^{-1} \mathbf{\Delta}^T F_i^T \succ 0, \tag{4.17}$$

where

$$\left[\begin{array}{c|c} R_i & F_i \\ \hline E & H \end{array}\right] := \left[\begin{array}{cc|c} \overline{D}_x^i & -\frac{1}{2}\overline{D}_x^i(\overline{d} + \underline{d}) - \frac{1}{2}A_d^T e_i + \mathbf{C}_{yd}^T \mu_i & 0 \\ \star & e_i^T \bar{x}_k + \overline{d}^T \overline{D}_x^i \underline{d} - e_i^T \mathbf{B}_u \mathbf{u} - 2\mu_i^T y^{\mathbf{u}} & \mu_i \mathbf{D}_{yp}^T - \frac{1}{2} e_i^T \mathbf{B}_p \\ \hline \mathbf{C}_d & \mathbf{D}_{qu}\mathbf{u} & \mathbf{D}_{qp}^T \end{array}\right].$$

Using Lemma 3.1.1 yields the LMI (4.11) as a sufficient condition for (4.17) for all $\mathbf{\Delta}$. A similar procedure can be used to derive LMI (4.12) for the lower-bound i.e. $-e_i^T x_k \leq -e_i^T \underline{x}_k$, $\forall i \in \mathscr{N}_n$. ■

Remark 12 *The estimated value of the state $\hat{x}_k$ is selected to be the mid-point of the upper and lower bounds of the state computed by the LMIs* (4.11) *and* (4.12)*, i.e $\hat{x}_k = \frac{1}{2}(\bar{x}_k + \underline{x}_k)$. Note that, at the time $k = 0$ the initial estimated value $\hat{x}_0$ is arbitrarily selected to be the mid-point of the known a priori initial bounds ($\bar{x}_0, \underline{x}_0$).*

Remark 13 *Note that the LMIs* (4.11) *and* (4.12) *always have feasible solutions since they are used to evaluate upper bounds on $e_i^T \underline{x}_k$ and lower bounds on $e_i^T \bar{x}_k$. The main issue is the tightness of these bounds. The quality of the bounds is illustrated in the example below.*

Remark 14 *For systems with only disturbances (i.e. no uncertainty), tight lower/upper bounds on x_k can easily be computed through a simple Linear Program (LP) given by minimizing/maximizing $e_i^T(A_d d + \tilde{B}_u \tilde{u})$ subject to the constraints $\underline{d} \leq d \leq \overline{d}$ and $\tilde{C}_{yd} d = \tilde{y} - \tilde{D}_{yu}\tilde{u}$ (see also Section 5.4.2).*

Remark 15 *Note that whilst a large value of N_e means a more accurate computation of the state-bounds (due to greater amount of data being considered in the moving window), it can be computationally expensive (particularly in the presence of model-uncertainty which lead to an LMI problem instead of an LP) since the estimation problem is solved online at every time step. Hence, the choice of N_e is problem-dependent and should be made in a way so as to find a balance between the conflicting requirements of computational complexity versus state-bound accuracy.*

4.4 Overall RMHE Algorithm

4.4.1 Implementation Strategy

The proposed estimation scheme computes online hard upper and lower bounds on the state x_k based on past input/output data. However, at sample time $k=0$ there is no past data to compute the state bounds and the state estimation value. Thus, at the time point $k=0$ the a priori bounds on x_0 are used and $\hat{x}_0$ is computed (see Remark 12). Subsequently, while more data is collected from the input/output at each iteration, the estimation horizon $\tilde{N}_e$ is incremented until it reaches the pre-specified estimation horizon N_e. During this period the current state bounds $\underline{x}_k$, $\bar{x}_k$ and the estimated state $\hat{x}_k$ are computed by considering all available past data. By the time that $\tilde{N}_e$ is equal to N_e the bounds and the estimated state are calculated by the moving horizon framework presented in Section 4.3. The overall approach can therefore be outlined as follows.

Algorithm 2: Robust Moving Horizon Estimation scheme

Result: $\underline{x}_{k-N_e}$, $\bar{x}_{k-N_e}$
Step 1:
Initially at $k=0$, given a priori bounds on x_0 compute the estimated state $\hat{x}_0$. Then apply the first control action u_0 onto the system.
Step 2:
Update the vectors **u**, **y** with the newly available input/output data from the current and previous step (u_{k-1}, y_k).
Step 3:
if $\tilde{N}_e < N_e$ **then**
  $\tilde{N}_e$=$\tilde{N}_e$+1
else
  $\tilde{N}_e = N_e$
end
Step 4:
Using vectors **u**, **y** and state bounds $\underline{x}_{k-N_e}$, $\bar{x}_{k-N_e}$ solve the LMI problem in Theorem 4.3.1 multiple times to compute each element of state bounds ($2n$ times) and estimated state $\hat{x}_k$ of the current state x_k.
Step 5:
Return to step (2).

Remark 16 *Note here that if the initial states bounds $\underline{x}_{k-N_e}$, $\bar{x}_{k-N_e}$ satisfy the inequality $\underline{x}_{k-N_e} \leq x_{k-N_e}$, $x_{k-N_e} \leq \bar{x}_{k-N_e}$, then using the Algorithm 2 the actual state value is always between the estimated lower/upper bounds $\underline{x}_k \leq x_k \leq \bar{x}_k$.*

4.5 Numerical Example

The benchmark problem of the control of a paper-making process (see for example [20, 81, 82]) is considered in this subsection to investigate the performance of the proposed estimation scheme. The system, shown in Fig. 4.1, consists of process states $x = [H_1\ H_2\ N_1\ N_2]^T$, where H_1 and N_1 denote liquid level and composition of the feed tank, respectively, and H_2 and N_2 denote liquid level and composition of the headbox, respectively. The control input vector is given by $u = [G_p\ G_w]^T$, where G_p is the flow rate of stock entering the feed tank and G_w is the recycled white water flow rate. All variables are normalized (i.e. they are zero at steady state) and only noisy measurements of H_2 and N_2 are available. The consistency and composition of white water is a source of uncertainty within the dynamics, particularly in the state N_1 and input G_w. Moreover, disturbance ζ_k affects all four states and v_k denotes the output measurement noise (see Remark 10 to describe the system as shown in (4.1)).

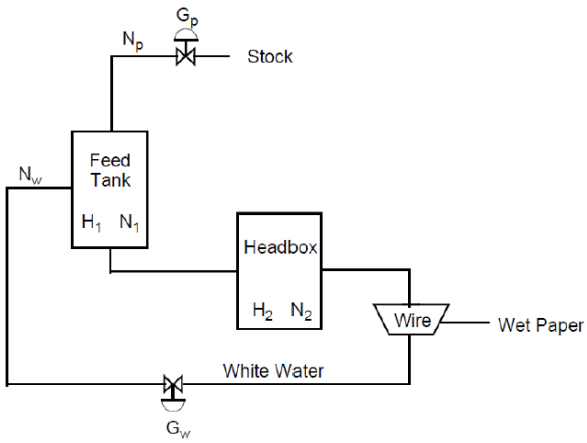

Figure 4.1: Schematic of Paper Machine Headbox [82].

The discrete-time dynamics (including uncertainty description), sampled at 2 minutes (see [81]), are given by (4.1) with:

$$A = \begin{bmatrix} 0.0211 & 0 & 0 & 0 \\ 0.1062 & 0.4266 & 0 & 0 \\ 0 & 0 & 0.2837 & 0 \\ 0.1012 & -0.6688 & 0.2893 & 0.4266 \end{bmatrix}$$

$$B_u = \begin{bmatrix} 0.6462 & 0.6462 \\ 0.2800 & 0.2800 \\ 1.5237 & -0.7391 \\ 0.9929 & 0.1507 \end{bmatrix}, \; B_w = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \; B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 0 & 0 & 0.2 & 0 \end{bmatrix}, \; D_{qu} = \begin{bmatrix} 0 & 0.2 \end{bmatrix}$$

$$C_y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; D_{yw} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

The process disturbance and output measurement noise are respectively characterized by the sets:

$$\zeta_k \in Z := \left\{ \zeta \in \mathbb{R} : -0.1 \leq \zeta \leq 0.1 \right\}, \quad v_k \in V := \left\{ v \in \mathbb{R} : -0.05 \leq v \leq 0.05 \right\}$$

Finally, the estimation horizon for the above set-up is set at $N_e = 15$ and the initial state bounds are $\underline{x}_0 = [-0.1, -1.5, 0, -1.5]^T$ and $\bar{x}_0 = [0.1, 1.5, 1, 1.5]^T$. For an arbitrary control input sequence u (see for example 4.2), the objective of the estimation scheme is to compute tight bounds on the states of the system, while the estimation error $(error_i = x_i - \hat{x}_i)$ between the actual states and the estimated states is computed.

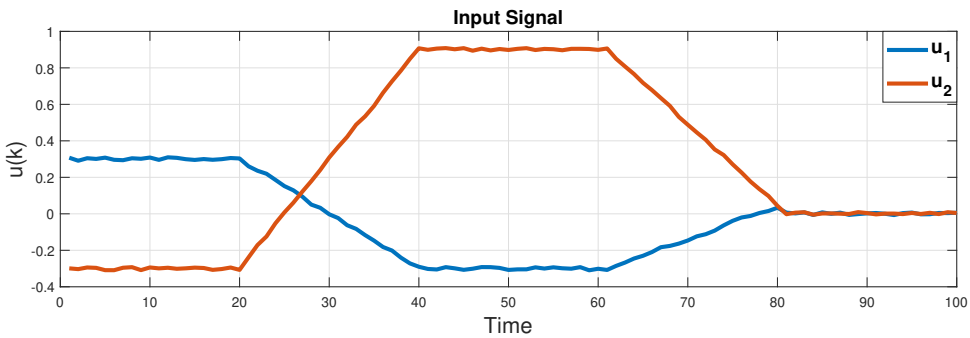

Figure 4.2: Pre-specified control input applied in the paper making process to access the proposed robust estimation algorithm.

Figures 4.3 to 4.5 show the simulation results. For the sake of comparison with previous works, the classic Luenberger observer and Receding Horizon Estimation (RHE) method proposed by Alessandri in [34], are considered. The control input signal applied to the paper making machine for all estimation algorithms under consideration in this case study is presented in Fig. 4.2. The value of Δ_k

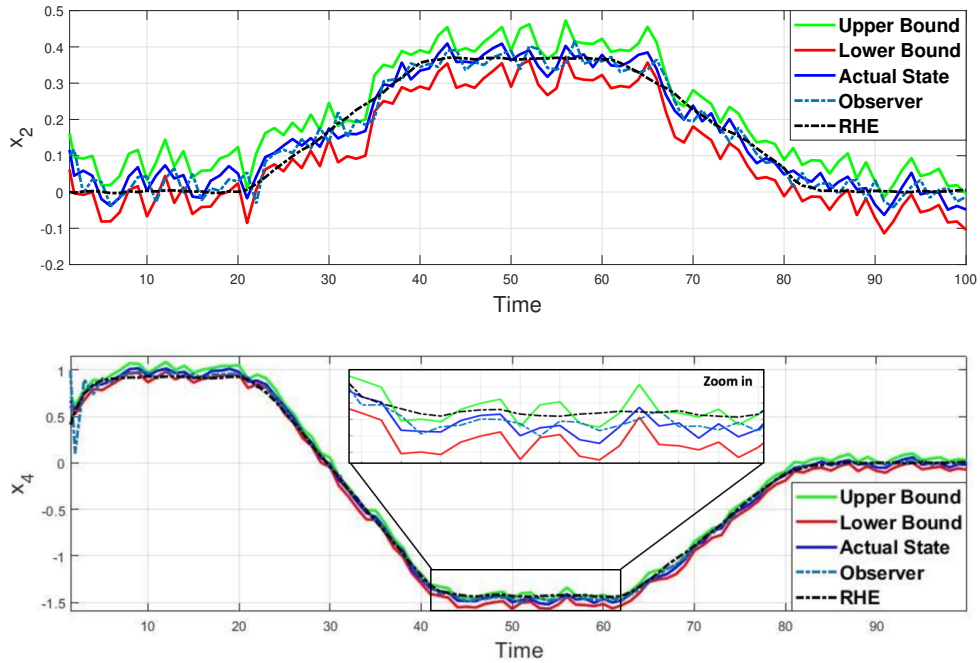

Figure 4.3: The observed states x_2 and x_4 for two different estimation schemes (Luenberger observer, RHE), as well as the actual states evolution with their respective computed upper and lower bounds by the proposed RMHE.

is set equal to 0.5 for all k. Figure 4.3 shows the state bounds for the measured states (x_2 and x_4), while Fig. 4.4 illustrates the state bounds for the unmeasured states (x_1 and x_3). For comparison purposes, in these figures the estimated state by utilizing Luenberger observer (dashed light blue) and RHE (dashed black), as well as the actual states (solid blue lines) of the process (not measurable in real time) are also included in these plots. It is noted that the computed bounds almost touch the actual states at some points, which demonstrates their tightness and the effectiveness of the new estimation scheme. It is also important to observe that for both algorithms considered from the literature, sometimes the estimated states are outside the hard bounds provided by the proposed MHE algorithm, which again demonstrates the superiority of the proposed scheme. Figures 4.5 and 4.6 show the state estimation error for the measured and unmeasured states, respectively, where it can be seen that the estimation error using the proposed RMHE converges faster and into a smaller set around zero as compared to the other methods under consideration from the literature. All the simulations are performed using MOSEK LMI/SDP solver within CVX pakage in MATLAB R2017b on a computer with 2.40GHz Intel Xeon(R) CPU and 64.0GB memory. The average online computation time at each sampling time (2 minutes) for the MHE estimation problem at the presented example is 0.8 seconds. Note that the

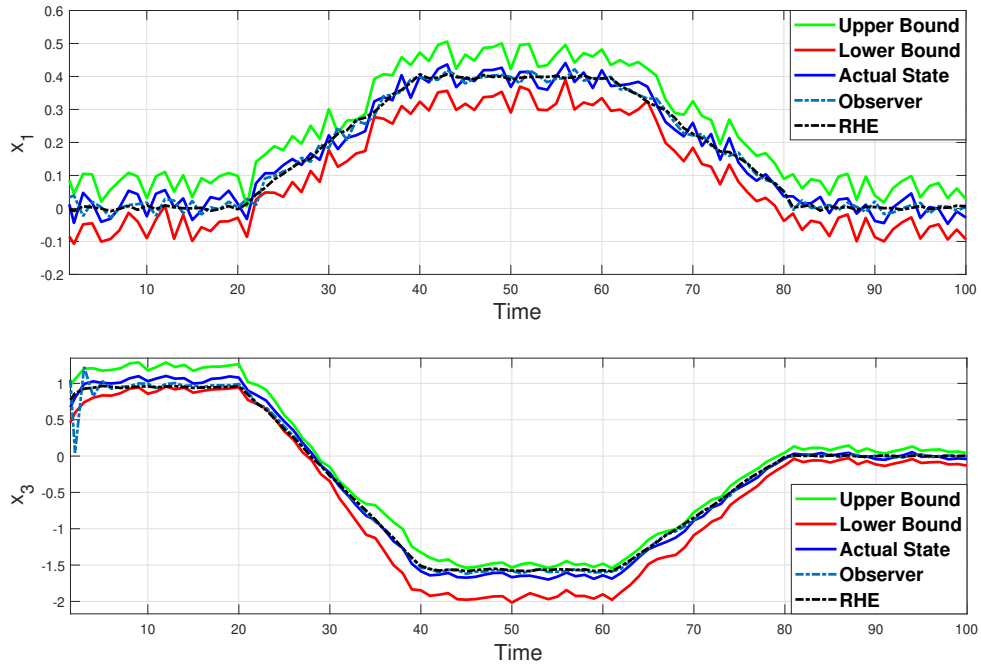

Figure 4.4: The unobserved states x_1 and x_3 for two different estimation schemes (Luenberger observer, RHE), as well as the actual states evolution with their respective computed upper and lower bounds by the proposed RMHE.

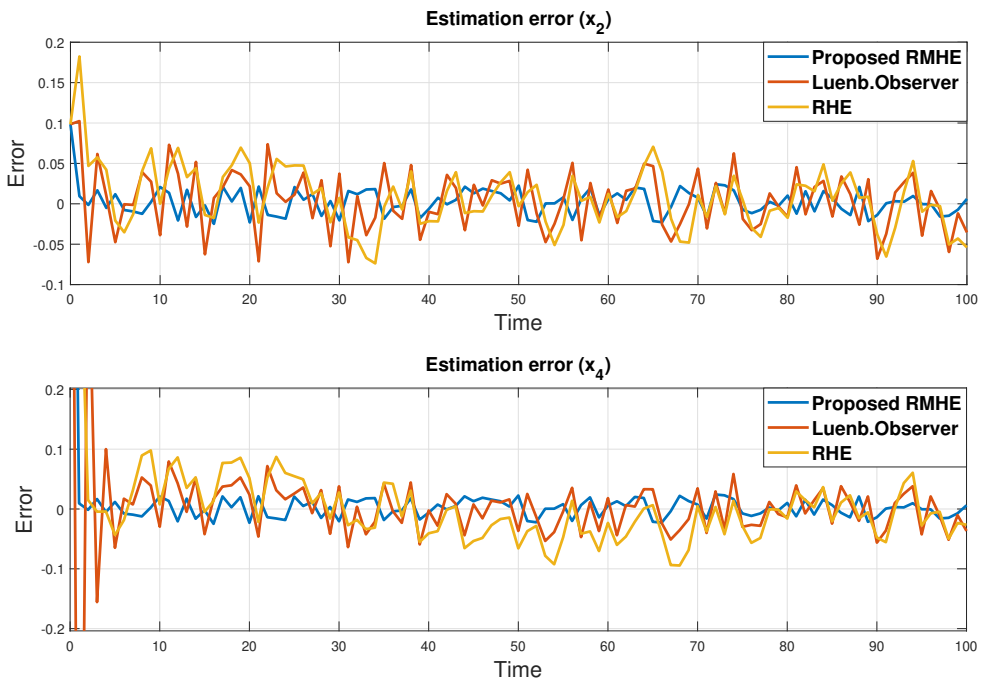

Figure 4.5: Estimation error for the observed states x_2 and x_4 using the proposed RMHE method, Luenberger observer and Receding Horizon Estimation from the literature.

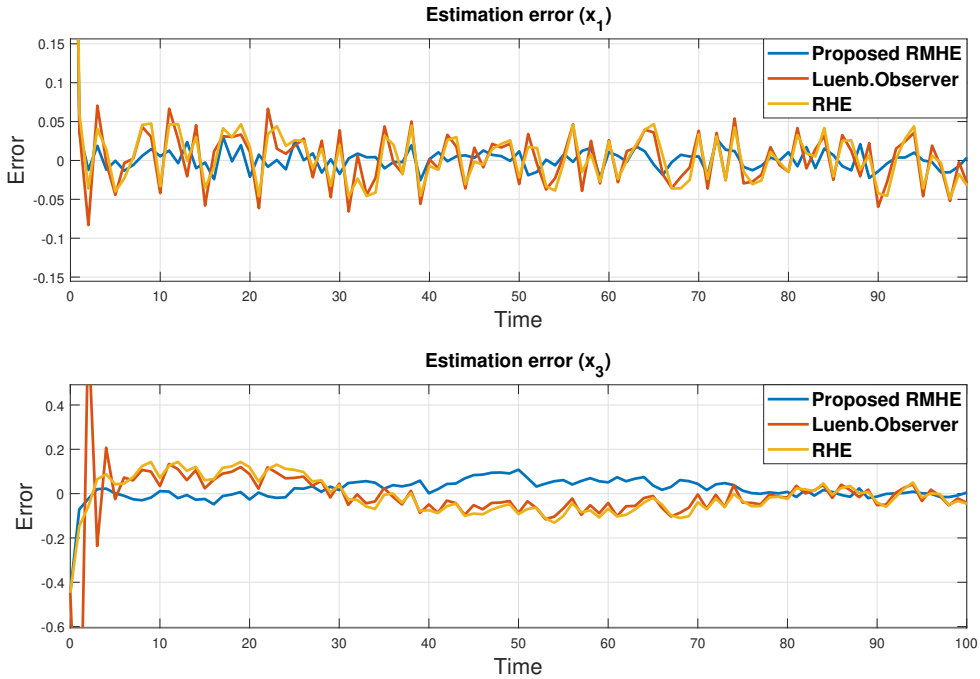

Figure 4.6: Estimation error for the unobserved states x_1 and x_3 using the proposed RMHE method, Luenberger observer and Receding Horizon Estimation.

estimation horizon is directly related with the estimation error and computational burden. Although selecting a short estimation horizon results in less online computation time, the estimation error is larger due to the lack of information considered to the estimation problem. On the other hand, continuing to increase the estimation horizon does not improve further the estimation error due to data overfeeding. Therefore, choosing a suitable value of estimation horizon depends on the sampling time of the system (maximum available computation time) and the estimation error improvement that you get by increasing the estimation horizon. In the presented example the maximum estimation horizon is $N_e = 45$, however it is chosen to be $N_e = 15$ since there is no improvement in the estimation error above this value.

4.6 Synopsis

In this chapter an investigation of the estimation problem based on past input/output data of linear discrete-time systems subject to model-uncertainties and bounded disturbances is presented. An on-line algorithm that computes estimates of the state alongside with tight bounds is suggested, while conservativeness is reduced and computation complexity is maintained low. Importantly, the proposed robust moving horizon estimation algorithm is formulated in a convex form and optimality is guaranteed at every sample time by solving an LMIs optimization problem. Finally, the effectiveness and superior performance of the proposed MHE algorithm as compared to state-of-the-art algorithms in the literature is demonstrated by an industrial process example.

CHAPTER 5

Output-feedback Robust MPC using Input/Output Data

5.1 Introduction

In this chapter, the output-feedback RMPC constrained control problem for linear discrete-time systems subject to norm-bounded model uncertainties, additive disturbances and noisy measurement is considered. The developments on this chapter is an extension to the results of a state-feedback RMPC for uncertain systems presented in Chapter 3, where the states of the system were assumed to be measurable and available. Using the findings of RMHE presented in Chapter 4, instead of employing a commonly used offline observer, we use the past input/output data window, to compute (tight) bounds on the current state which are then used within the output feedback control algorithm, rather than using estimation error bounds. Furthermore, to reduce conservatism, the feedback gain (K) and control perturbation (v) are both explicitly considered as decision variables in the online optimization. Similar to the state feedback RMPC approach demonstrated in Chapter 3, the nonlinearity associated with such a formulation is resolved by using the Elimination Lemma and the S-procedure to develop an algorithm based on LMI optimizations.

The contributions of this chapter can be summarized as follows. A new OF-RMPC scheme, based on convex LMI optimization is proposed to generalize the OF-RMPC problem to systems subject to structured feedback uncertainty, as well as external disturbances (see Section 5.2). In particular, to reduce conservatism, the feedback gain (K) and control perturbation (υ) are explicitly considered as decision variables within the OF-RMPC optimization problem, whilst nonlinearities are circumvented by using the Elimination Lemma and the S-procedure. Importantly, the proposed linearization method substantially improves performance due to its less restrictive nature without reducing the feasibility region of the problem (see Remark 23). To reduce the online computation time of the controller, analogous to the single LMI approach presented in Section 3.4, an extension is proposed to derive a single LMI-based sufficient condition for all the problem constraints (instead of solving multiple LMIs for each constraint within the problem). This in turn helps to improve the scalability of the proposed algorithm. Moreover, instead of employing an offline observer to directly estimate the states of the system, past input/output data window is considered to compute (tight) bounds on the current state which are then used within the output feedback control algorithm. Finally, an initialization strategy, computed offline, is proposed to guarantee feasibility for the online OF-RMPC problem, see Section 5.3.

This chapter is organized as follows. Section 5.2 provides a description of the system and formulates the output-feedback RMPC problem subject to uncertainties/disturbances. A feasibility analysis is discussed in Section 5.3 where an offline policy to guarantee recursive feasibility is provided. In Section 5.4, a summary of the overall algorithm is proposed and numerical examples to highlight the effectiveness of the proposed OF-RMPC scheme are presented. Finally, in Section 5.5, some concluding thoughts, as well as potential future work directions, are presented.

5.2 Output-feedback RMPC Problem

In this section, we first provide a description of the system including control dynamics, constraints and cost function. Then, we formulate the output-feedback RMPC problem. Note here that the formulation procedure followed for the output-feedback case mirrors the state-feedback case in Chapter 3.

5.2.1 System Description

As before, we consider the following linear discrete-time system subject to norm-bounded uncertainty and additive disturbances. The main difference from the system considered in 3.3 is that output measurements affected by sensor noise are now available.

$$\begin{bmatrix} x_{k+1} \\ q_k \\ y_k \\ f_k \\ z_k \end{bmatrix} = \begin{array}{c} \\ n_q \\ n_y \\ n_f \\ n_z \end{array} \begin{array}{c} \begin{array}{cccc} n & n_u & n_p & n_w \end{array} \\ \begin{bmatrix} A & B_u & B_p & B_w \\ C_q & D_{qu} & 0 & 0 \\ C_y & D_{yu} & D_{yp} & D_{yw} \\ C_f & D_{fu} & D_{fp} & D_{fw} \\ C_z & D_{zu} & D_{zp} & D_{zw} \end{bmatrix} \end{array} \begin{bmatrix} x_k \\ u_k \\ p_k \\ w_k \end{bmatrix}, \; p_k = \Delta_k q_k, \tag{5.1}$$

$$\begin{bmatrix} q_N \\ f_N \\ z_N \end{bmatrix} = \begin{bmatrix} \hat{C}_q & 0 \\ \hat{C}_f & \hat{D}_{fp} \\ \hat{C}_z & \hat{D}_{zp} \end{bmatrix} \begin{bmatrix} x_N \\ p_N \end{bmatrix}, \qquad p_N = \Delta_N q_N,$$

with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, $w_k \in \mathbb{R}^{n_w}$, $f_k \in \mathbb{R}^{n_f}$, $z_k \in \mathbb{R}^{n_z}$ are the state, input, output, disturbance, constrained signal, and cost signal, respectively, at prediction step k, where $k \in \mathscr{N} := \{0, 1, \cdots, N-1\}$ and N denotes the control horizon. Note that the description includes terminal cost and state constraints to ensure closed-loop stability [4]. The symbols in capital letters denote coefficient matrices of the system, with the dimensions indicated for ease of reference. Furthermore, $\Delta_k \in \mathscr{B}\mathbf{\Delta}$ where $\mathbf{\Delta} \subseteq \mathbb{R}^{n_p \times n_q}$ is a subspace that captures the uncertainty structure. Here $p_k \in \mathbb{R}^{n_p}$ and $q_k \in \mathbb{R}^{n_q}$ represent the uncertainty vectors and all other symbols denote the appropriate distribution matrices. Only the noisy output y_k is measured and we assume that the pair (A, C_y) is detectable and (A, B_u) stabiliz-

able. Moreover, it is assumed that bounds on the initial state are given a priori such that:

$$x_0 \in \mathscr{X}_0 := \left\{ x \in \mathbb{R}^n : \underline{x}_0 \leq x \leq \overline{x}_0 \right\}. \tag{5.2}$$

Remark 17 *Alternatively, initial state bounds ($\underline{x}_0, \overline{x}_0$), can be computed using inital sensor measurement y_0 through equation 5.1, considering the maximum output-disturbance/feedback uncertainty.*

Finally, the (unmeasured) additive disturbances belong to the bounded set:

$$w_k \in \mathscr{W} := \left\{ w \in \mathbb{R}^{n_w} : -r \leq w \leq r \right\}, \tag{5.3}$$

where $0 < r \in \mathbb{R}^{n_w}$ is a predefined vector that captures the maximum (state/output) disturbance values. The objective of the online optimization problem is to find a feedback law u_k, for all $k \in \mathscr{N}$, such that the future constrained outputs satisfy $f_k \leq \bar{f}_k$, $f_N \leq \bar{f}_N$ for all $w_k \in \mathscr{W}$ and $\Delta \in \mathscr{B}\mathbf{\Delta}$, and the cost function

$$J = \max_{w_k \in \mathscr{W}_k,\ \Delta \in \mathscr{B}\mathbf{\Delta}} \sum_{k=0}^{N} (z_k - \bar{z}_k)^T (z_k - \bar{z}_k) \tag{5.4}$$

is minimized.

Similar to Chapter 3 the vector $\bar{z}_k \in \mathbb{R}^{n_z}$ is assumed to be known and defines the reference trajectory. The constraint and terminal constraint signals are defined by $\bar{f}_k$ and $\bar{f}_N$, respectively, and are assumed to be known. They are chosen to satisfy polytopic constraints on the input and state signals, and terminal state signals, respectively. The only assumption that is imposed here is that the terminal constraints defined by $\bar{f}_N$ are within a polytopic invariant set [72]. This is used to derive conditions for recursive feasibility on the proposed control scheme (see Remark 27).

Remark 18 *Note that we allow uncertainty in all the problem data including the constraints and the cost signal. It is easy to verify that the dynamics in (5.1) can be rewritten in the form:*

$$\begin{bmatrix} x_{k+1} \\ y_k \\ f_k \\ z_k \end{bmatrix} = \begin{bmatrix} A+B_p\Delta_k C_q & B_u+B_p\Delta_k D_{qu} & B_w \\ C_y+D_{yp}\Delta_k C_q & D_{yu}+D_{yp}\Delta_k D_{qu} & D_{yw} \\ C_f+D_{fp}\Delta_k C_q & D_{fu}+D_{fp}\Delta_k D_{qu} & D_{fw} \\ C_z+D_{zp}\Delta_k C_q & D_{zu}+D_{zp}\Delta_k D_{qu} & D_{zw} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ w_k \end{bmatrix}$$

$$\begin{bmatrix} f_N \\ z_N \end{bmatrix} = \begin{bmatrix} \hat{C}_f+\hat{D}_{fp}\Delta_N \hat{C}_q \\ \hat{C}_z+\hat{D}_{zp}\Delta_N \hat{C}_q \end{bmatrix} x_N.$$

Remark 19 *For clarity of exposition, we have combined both the state-disturbance (η_k) and output-disturbance (ν_k) into a single vector in (5.1), namely $w_k := [\eta_k^T \;\; \nu_k^T]^T$.*

5.2.2 Algebraic formulation

Following the standard predictive control formulation the stacked vectors are defined as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{N_n}, \; \mathbf{u} = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \in \mathbb{R}^{N_u},$$

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix} \in \mathbb{R}^{N_{n_\xi}}, \; \boldsymbol{\zeta} = \begin{bmatrix} \zeta_0 \\ \vdots \\ \zeta_N \end{bmatrix} \in \mathbb{R}^{N_\zeta},$$

where $\boldsymbol{\xi}$ stands for $\mathbf{y}$, $\mathbf{w}$, $\mathbf{p}$, $\mathbf{q}$ and $\boldsymbol{\zeta}$ stands for $\mathbf{f}$, $\bar{\mathbf{f}}$, $\mathbf{z}$, $\bar{\mathbf{z}}$, and where the dimensions of the stack matrices are $N_n = Nn$, $N_u = Nn_u$, $N_\xi = Nn_\xi$ and $N_\zeta = (N+1)n_\zeta$. Then the dynamic system over the

horizon N can be written as:

$$\begin{bmatrix}\mathbf{x}\\ \mathbf{q}\\ \mathbf{y}\\ \mathbf{f}\\ \mathbf{z}\end{bmatrix} = \begin{array}{c} \\ N_n \\ N_q \\ N_y \\ N_f \\ N_z \end{array}\!\!\begin{array}{c} \begin{array}{cccc} n & N_u & N_p & N_w \end{array} \\ \begin{bmatrix}\mathbf{A} & \mathbf{B}_u & \mathbf{B}_p & \mathbf{B}_w\\ \mathbf{C}_q & \mathbf{D}_{qu} & \mathbf{D}_{qp} & \mathbf{D}_{qw}\\ \mathbf{C}_y & \mathbf{D}_{yu} & \mathbf{D}_{yp} & \mathbf{D}_{yw}\\ \mathbf{C}_f & \mathbf{D}_{fu} & \mathbf{D}_{fp} & \mathbf{D}_{fw}\\ \mathbf{C}_z & \mathbf{D}_{zu} & \mathbf{D}_{zp} & \mathbf{D}_{zw}\end{bmatrix}\end{array}\begin{bmatrix}x_0\\ \mathbf{u}\\ \mathbf{p}\\ \mathbf{w}\end{bmatrix}, \quad \mathbf{p} = \hat{\Delta}\mathbf{q}, \tag{5.5}$$

with $\hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}} \subset \mathbb{R}^{N_p \times N_q}$ where,

$$\hat{\mathbf{\Delta}} = \{\mathrm{diag}(\Delta_0, \Delta_1, \ldots, \Delta_N) : \Delta_k \in \mathbf{\Delta}\},$$

and where the stacked matrices in (5.5) (shown in bold) have the indicated dimensions and are readily obtained from iterating the dynamics in (5.1).

By defining a vector $[x_0^T \;\; \mathbf{w}^T]^T = d \in \mathbb{R}^{N_d}$ such that

$$\begin{bmatrix}\underline{x}_0\\ -\mathbf{1}\otimes r\end{bmatrix} =: \underline{d} \leq d \leq \overline{d} := \begin{bmatrix}\overline{x}_0\\ \mathbf{1}\otimes r\end{bmatrix},$$

where $\mathbf{1}$ represents a vector of ones and $\otimes$ is the Kronecker Tensor Product, equation (5.5) can be written as:

$$\begin{bmatrix}\mathbf{q}\\ \mathbf{y}\\ \mathbf{f}\\ \mathbf{z}\end{bmatrix} = \begin{array}{c} \\ N_q \\ N_y \\ N_f \\ N_z \end{array}\!\!\begin{array}{c} \begin{array}{ccc} N_d & N_u & N_p \end{array} \\ \begin{bmatrix}\mathbf{D}_{qd} & \mathbf{D}_{qu} & \mathbf{D}_{qp}\\ \mathbf{D}_{yd} & \mathbf{D}_{yu} & \mathbf{D}_{yp}\\ \mathbf{D}_{fd} & \mathbf{D}_{fu} & \mathbf{D}_{fp}\\ \mathbf{D}_{zd} & \mathbf{D}_{zu} & \mathbf{D}_{zp}\end{bmatrix}\end{array}\begin{bmatrix}d\\ \mathbf{u}\\ \mathbf{p}\end{bmatrix}, \quad p = \hat{\Delta}q, \tag{5.6}$$

with $\mathbf{D}_{gd} := [\mathbf{C}_g \;\; \mathbf{D}_{gw}]$, where g stands for q, y, f and z above. Using the information of the upper and lower bounds of the initial states $(\underline{x}_0, \overline{x}_0)$ and the bounds of disturbances, following simple vector

manipulations, we can express the vector d as a norm-bounded uncertain signal:

$$d = \Delta^d \hat{d} + d_o \tag{5.7}$$

where $\Delta^d := \mathrm{diag}(\Delta^x, \Delta^w)$ with $||\Delta^d|| \leq 1$, $\hat{d} := [(\frac{1}{2}(\bar{x}_0 - \underline{x}_0)^T,\ \mathbf{1}^T \otimes r^T]^T$, and $d_o := [\frac{1}{2}(\bar{x}_0 + \underline{x}_0)^T,\ 0]^T$. Δ^x represents the initial uncertainty of the system due to model mismatch and Δ^w is the uncertainty introduced by reparameterizing the disturbances into uncertainties. Note that by definition $||\Delta^x|| \leq 1$ and $||\Delta^w|| \leq 1$, therefore it is verified that $||\Delta^d|| \leq 1$.

5.2.3 Output-feedback RMPC

For the proposed output-feedback RMPC scheme the following form of a control law is considered:

$$\mathbf{u} = K\mathbf{y} + \mathbf{v} \tag{5.8}$$

where K is the output-feedback control gain and $\mathbf{v} \in \mathscr{V} \subset \mathbb{R}^{N_u}$ is the control perturbation. To ensure causality (i.e. u_i depends only on y_j, $j = 0, \ldots, i$), we impose that $K \in \mathscr{K} \subset \mathbb{R}^{N_u \times N_n}$, where $\mathscr{K}$ is the set of $N_u \times N_{n_y}$ lower block triangular matrices with $n_u \times n_y$ blocks. Substituting the equation for $\mathbf{y}$ in (5.6) into (5.8) yields the following expression for $\mathbf{u}$:

$$\mathbf{u} = \hat{K}\mathbf{D}_{yd} d + \hat{K}\mathbf{D}_{yp}\mathbf{p} + \hat{v} \tag{5.9}$$

where

$$[\hat{K} \quad \hat{v}\,] := (I - K\mathbf{D}_{yu})^{-1}[K \quad \mathbf{v}].$$

Note that $\mathbf{u}$ is affine in the new variables $(\hat{K}, \hat{v})$ and that the original control variables $(K, \mathbf{v})$ can easily be recovered from $(\hat{K}, \hat{v})$ as follows:

$$[K \quad \mathbf{v}\,] := (I + \hat{K}\mathbf{D}_{yu})^{-1}[\hat{K} \quad \hat{v}\,]. \tag{5.10}$$

Remark 20 *Note that an alternative way to formulate the RMPC problem is to consider state-feedback control law of the form of $u = K_0\hat{x}_0 + Kx + v$ (see Chapter 3), where $\hat{x}_0$ is the state estimated value computed using the state bounds $\hat{x}_0 = \frac{1}{2}(\underline{x}_0 + \overline{x}_0)$. However, by utilizing the same estimation scheme the two control options (state-feedback, output-feedback) have similar performance in the overall control scheme. The state-feedback option could potentially add conservativeness to the problem since it simply considers the state as a midpoint of the bounds without using any information from the output measurements.*

The aim of the rest of this section is to obtain a representation of vectors $\mathbf{y}$, $\mathbf{f}$ and $\mathbf{z}$ in terms of the (new) decision variables $\hat{K}$ and $\hat{v}$. To this end, by using the control structure in (5.9), we can eliminate $\mathbf{u}$ from (5.6) to yield

$$\begin{aligned}
\begin{bmatrix} \mathbf{y} \\ \mathbf{q} \\ \mathbf{f} \\ \mathbf{z}-\bar{\mathbf{z}} \end{bmatrix} &= \left[\begin{array}{cc|c} (I+\mathbf{D}_{yu}\hat{K})\mathbf{D}_{yp} & (I+\mathbf{D}_{yu}\hat{K})D_{yd} & \mathbf{D}_{yu}\hat{v} \\ \mathbf{D}_{qp}+\mathbf{D}_{qu}\hat{K}\mathbf{D}_{yp} & \mathbf{D}_{qd}+\mathbf{D}_{qu}\hat{K}D_{yd} & \mathbf{D}_{qu}\hat{v} \\ \mathbf{D}_{fp}+\mathbf{D}_{fu}\hat{K}\mathbf{D}_{yp} & \mathbf{D}_{fd}+\mathbf{D}_{fu}\hat{K}D_{yd} & \mathbf{D}_{fu}\hat{v} \\ \mathbf{D}_{zp}+\mathbf{D}_{zu}\hat{K}\mathbf{D}_{yp} & \mathbf{D}_{zd}+\mathbf{D}_{zu}\hat{K}D_{yd} & \mathbf{D}_{zu}\hat{v}-\bar{z} \end{array}\right] \left[\begin{array}{c} \mathbf{p} \\ \mathbf{d} \\ \hline 1 \end{array}\right] \\
&:= \left[\begin{array}{cc|c} \mathbf{D}_{yp}^{\hat{K}} & D_{yd}^{\hat{K}} & \mathbf{D}_{y}^{\hat{v}} \\ \mathbf{D}_{qp}^{\hat{K}} & D_{qd}^{\hat{K}} & \mathbf{D}_{q}^{\hat{v}} \\ \mathbf{D}_{fp}^{\hat{K}} & D_{fd}^{\hat{K}} & \mathbf{D}_{f}^{\hat{v}} \\ \mathbf{D}_{zp}^{\hat{K}} & D_{zd}^{\hat{K}} & \mathbf{D}_{z}^{\hat{v}} \end{array}\right] \left[\begin{array}{c} \mathbf{p} \\ \mathbf{d} \\ \hline 1 \end{array}\right] \\
&:= \begin{bmatrix} \mathbf{D}_{y\hat{p}}^{\hat{K}} & \mathbf{D}_{y}^{\hat{v}} \\ \mathbf{D}_{q\hat{p}}^{\hat{K}} & \mathbf{D}_{q}^{\hat{v}} \\ \mathbf{D}_{f\hat{p}}^{\hat{K}} & \mathbf{D}_{f}^{\hat{v}} \\ \mathbf{D}_{z\hat{p}}^{\hat{K}} & \mathbf{D}_{z}^{\hat{v}} \end{bmatrix} \begin{bmatrix} \hat{p} \\ 1 \end{bmatrix}
\end{aligned} \tag{5.11}$$

with $\hat{p} := [\mathbf{p}^T,\ \mathbf{d}^T]^T$ such that

$$\hat{p} = \hat{\Delta}\hat{q} + q_o, \tag{5.12}$$

were $q_o := [0,\ d_o^T]^T$ and

$$\hat{q} := \begin{bmatrix} \mathbf{q} \\ \hat{d} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{D}_{qp}^{\hat{K}} & \mathbf{D}_{qd}^{\hat{K}} \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}} \hat{p} + \underbrace{\begin{bmatrix} \mathbf{D}_{q}^{\hat{v}} \\ \hat{d} \end{bmatrix}}_{\mathbf{D}_{\hat{q}}^{\hat{v}}} \tag{5.13}$$

Using (5.7), the new structured subspace $\hat{\Delta}$ is defined as:

$$\hat{\Delta} = \{\mathrm{diag}(\Delta_0, \Delta_0^d, \ldots, \Delta_{N-1}, \Delta_{N-1}^d, \Delta_N) : \Delta_k \in \mathbf{\Delta}, \Delta_k^d \in \mathbf{\Delta}^d\}.$$

For convenience, we also define

$$\mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}} = \hat{\mathbf{D}}_q + \mathbf{D}_{\hat{q}u}\hat{K}\hat{\mathbf{C}}_y, \tag{5.14}$$

$$\mathbf{D}_{f\hat{p}}^{\hat{K}} = \hat{\mathbf{D}}_f + \mathbf{D}_{fu}\hat{K}\hat{\mathbf{C}}_y, \tag{5.15}$$

$$\mathbf{D}_{z\hat{p}}^{\hat{K}} = \hat{\mathbf{D}}_z + \mathbf{D}_{zu}\hat{K}\hat{\mathbf{C}}_y, \tag{5.16}$$

where $\hat{\mathbf{D}}_f := [\mathbf{D}_{fp} \;\; \mathbf{D}_{fd}]$, $\hat{\mathbf{C}}_y := [\mathbf{D}_{yp} \;\; \mathbf{D}_{yd}]$, $\hat{\mathbf{D}}_z := [\mathbf{D}_{zp} \;\; \mathbf{D}_{zd}]$ and

$$\hat{\mathbf{D}}_q := \begin{bmatrix} \mathbf{D}_{qp} & \mathbf{D}_{qd} \\ 0 & 0 \end{bmatrix}, \;\; \mathbf{D}_{\hat{q}u} := \begin{bmatrix} \mathbf{D}_{qu} \\ 0 \end{bmatrix}.$$

Inserting $\hat{q}$ from (5.13) into (5.12) and simplifying yields:

$$\hat{p} = (I - \hat{\Delta}\mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}})^{-1}\hat{\Delta}(\mathbf{D}_{\hat{q}}^{\hat{v}} + \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}q_o) + q_o. \tag{5.17}$$

Then, using (5.17) to eliminate $\hat{p}$ from (5.11) gives

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{f} \\ \mathbf{z}-\bar{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{y\hat{p}}^{\hat{K}}\hat{\Delta}(I - \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}(\mathbf{D}_{\hat{q}}^{\hat{v}} + \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}q_o) + \mathbf{D}_{y\hat{p}}^{\hat{K}}q_o + \mathbf{D}_y^{\hat{v}} \\ \mathbf{D}_{f\hat{p}}^{\hat{K}}\hat{\Delta}(I - \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}(\mathbf{D}_{\hat{q}}^{\hat{v}} + \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}q_o) + \mathbf{D}_{f\hat{p}}^{\hat{K}}q_o + \mathbf{D}_f^{\hat{v}} \\ \mathbf{D}_{z\hat{p}}^{\hat{K}}\hat{\Delta}(I - \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}(\mathbf{D}_{\hat{q}}^{\hat{v}} + \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}q_o) + \mathbf{D}_{z\hat{p}}^{\hat{K}}q_o + \mathbf{D}_z^{\hat{v}} \end{bmatrix} \tag{5.18}$$

Now define

$$\bar{v} := \hat{v} + \hat{K}\mathbf{D}_{yd}d_o \tag{5.19}$$

and let α denote y, f, z. Then, it can be verified that

$$\mathbf{D}_{\alpha\hat{p}}^{\hat{K}}q_o + \mathbf{D}_{\alpha}^{\hat{v}} = \mathbf{D}_{\alpha d}d_o + \mathbf{D}_{\alpha u}\bar{v} - \bar{\alpha} := \mathbf{D}_{\alpha}^{\bar{v}} \tag{5.20}$$

where the $\bar{\alpha}$ term in (5.20) is only included in the definition for $\alpha = \mathbf{z}$, and denotes the reference trajectory $\bar{\mathbf{z}}$ which is given. Furthermore, we define

$$\mathbf{D}_{\hat{q}}^{\hat{v}} + \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}} q_o = \begin{bmatrix} \mathbf{D}_{qd} d_o + \mathbf{D}_{qu}\bar{v} \\ \hat{d} \end{bmatrix} := \mathbf{D}_{\hat{q}}^{\bar{v}} \tag{5.21}$$

Finally, using the redefinitions in (5.20) and (5.21), we can re-write (5.18) as

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{f} \\ \mathbf{z} - \bar{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{y\hat{p}}^{\hat{K}} \hat{\Delta}(I - \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}} \hat{\Delta})^{-1} \mathbf{D}_{\hat{q}}^{\bar{v}} + \mathbf{D}_{y}^{\bar{v}} \\ \mathbf{D}_{f\hat{p}}^{\hat{K}} \hat{\Delta}(I - \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}} \hat{\Delta})^{-1} \mathbf{D}_{\hat{q}}^{\bar{v}} + \mathbf{D}_{f}^{\bar{v}} \\ \mathbf{D}_{z\hat{p}}^{\hat{K}} \hat{\Delta}(I - \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}} \hat{\Delta})^{-1} \mathbf{D}_{\hat{q}}^{\bar{v}} + \mathbf{D}_{z}^{\bar{v}} \end{bmatrix} := \begin{bmatrix} \mathbf{D}_{y}^{\hat{K},\bar{v},\hat{\Delta}} \\ \mathbf{D}_{f}^{\hat{K},\bar{v},\hat{\Delta}} \\ \mathbf{D}_{z}^{\hat{K},\bar{v},\hat{\Delta}} \end{bmatrix} \tag{5.22}$$

5.2.4 Sufficient conditions for the constraints and cost

In this section, we derive sufficient conditions for the satisfaction of the constraints as well as an upper bound on the cost function. For convenience, we write $\mathbf{f} = \mathscr{F}(\hat{K}, \bar{v}, \hat{\Delta})$ and $(\mathbf{z} - \bar{\mathbf{z}})^T(\mathbf{z} - \bar{\mathbf{z}}) = \mathscr{Z}(\hat{K}, \bar{v}, \hat{\Delta})$ to emphasize the dependence of the constraints and the cost function on the variables. Using the notions that are presented above, the OF-RMPC problem can be posed as a min-max problem [9], where the objective is to find a feasible couple $(\hat{K}, \bar{v})$ that achieve the minimum,

$$\mathbf{J} = \min_{(\hat{K},\bar{v}) \in \mathscr{U}} \max_{\hat{\Delta} \in \mathscr{B}\mathbf{\Delta}} \mathscr{Z}(\hat{K}, \bar{v}, \hat{\Delta}), \tag{5.23}$$

where the set $\mathscr{U}$ is defined to be the set of all feasible control variables $(\hat{K}, \bar{v})$ such that all the problem constraints are satisfied:

$$\mathscr{U} = \{(\hat{K}, \bar{v}) : e_i^T \mathscr{F}(\hat{K}, \bar{v}, \hat{\Delta}) \leq e_i^T \bar{\mathbf{f}}, \forall i \in \mathscr{N}_f, \forall \hat{\Delta}\}. \tag{5.24}$$

The following theorem uses Lemma 3.1.1 to derive sufficient conditions for $(\hat{K}, \bar{v}) \in \mathscr{U}$ (necessary and sufficient in the case of unstructured uncertainties, see Remark 21) and an upper bound, call it γ^2, on the cost function in (5.23).

Theorem 5.2.1 *Let all variables be as defined above. Then, $J(\hat{K},\bar{v},\hat{\Delta}) \leq \gamma^2$ and $(\hat{K},\bar{v}) \in \mathscr{U}$ for all $\hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}}$, if there exist solutions $(S,R,G),(S_i,R_i,G_i) \in \widehat{\Psi}$, $\forall i \in \mathscr{N}_f := \{1,\cdots,(N+1)n_f\}$, to the following matrix inequalities*

$$T_1 + \mathscr{H}(T_2\hat{K}\hat{\mathbf{C}}_y T_3) \succ 0, \tag{5.25}$$

and

$$T_1^i + \mathscr{H}(T_2^i\hat{K}\hat{\mathbf{C}}_y T_3^i) \succ 0,\ i = 1,\ldots,N_f \tag{5.26}$$

where,

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & 0 \end{bmatrix} = \begin{array}{c} \\ 1 \\ N_z \\ N_q \\ N_p \\ N_p \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & N_z & N_q & N_p & N_u \end{array} \\ \left[\begin{array}{cccc|c} \gamma^2 & (\hat{\mathbf{D}}_z^{\bar{v}})^T & (\mathbf{D}_{\hat{q}}^{\bar{v}})^T & 0 & 0 \\ * & I & \hat{\mathbf{D}}_z G^T & \hat{\mathbf{D}}_z S & \mathbf{D}_{zu} \\ * & * & R + \mathscr{H}\left(\hat{\mathbf{D}}_q G^T\right) & \hat{\mathbf{D}}_q S & \mathbf{D}_{\hat{q}u} \\ * & * & * & S & 0 \\ \hline 0 & 0 & G^T & S & 0 \end{array}\right] \end{array},$$

$$\begin{bmatrix} T_1^i & T_2^i \\ T_3^i & 0 \end{bmatrix} =$$

$$\begin{array}{c} \\ 1 \\ N_q \\ N_p \\ N_p \end{array} \begin{array}{c} \begin{array}{cccc} 1 & N_q & N_p & N_u \end{array} \\ \left[\begin{array}{ccc|c} e_i^T(\overline{f} - \mathbf{D}_f^{\bar{v}}) & (\mathbf{D}_{\hat{q}}^{\bar{v}})^T - \frac{1}{2}e_i^T\hat{\mathbf{D}}_f G_i^T & -\frac{1}{2}e_i^T\hat{\mathbf{D}}_f S_i & -\frac{e_i^T}{2}\mathbf{D}_{fu} \\ \star & R_i + \mathscr{H}\left(\hat{\mathbf{D}}_q G_i^T\right) & \hat{\mathbf{D}}_q S_i & \mathbf{D}_{\hat{q}u} \\ * & * & S_i & 0 \\ \hline 0 & G_i^T & S_i & 0 \end{array}\right] \end{array}$$

where $(\hat{K},\bar{v}) \in (\mathscr{K},\upsilon)$ and (S,R,G), $(S_i,R_i,G_i) \in \hat{\Psi}, i \in \mathscr{N}_f := \{1,\ldots,N_f\}$, are slack variables with $\hat{\Psi}$ defined in (3.1).

Proof: The constraints in (5.24) can be written as

$$\begin{aligned} e_i^T\bar{\mathbf{f}} - e_i^T\mathbf{f} =& e_i^T\bar{\mathbf{f}} - e_i^T\mathbf{D}_{f\hat{p}}^{\hat{K}}\hat{\Delta}(I-\mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}\hat{\Delta})^{-1}\mathbf{D}_{\hat{q}}^{\bar{v}} - e_i^T\mathbf{D}_f^{\bar{v}} \geq 0 \\ & \forall i \in \mathscr{N}_f,\ \forall\hat{\Delta}\in\mathscr{B}\hat{\mathbf{\Delta}}. \end{aligned} \tag{5.27}$$

A re-arrangement verifies that (5.27) can be written in the form:

$$H_{i_{11}} + \mathscr{H}\left(H_{i_{12}}\Delta(I-H_{22}\hat{\Delta})^{-1}H_{21}\right) \succ 0,\ \ \forall i \in \mathscr{N}_f,\ \forall\hat{\Delta}\in\mathscr{B}\hat{\mathbf{\Delta}} \tag{5.28}$$

where

$$\left[\begin{array}{c|c} H_{i_{11}} & H_{i_{12}} \\ \hline H_{21} & H_{22} \end{array}\right] := \left[\begin{array}{c|c} e_i^T(\bar{\mathbf{f}}-\mathbf{D}_f^{\hat{v}}) & -\frac{1}{2}e_i^T\mathbf{D}_{f\hat{p}}^{\hat{K}} \\ \hline \mathbf{D}_{\hat{q}}^{\bar{v}} & \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}} \end{array}\right].$$

Using Lemma 3.1.1 on (5.28) yields the matrix inequality (5.26).

Next, we consider the cost function. Let γ^2 be an upper bound on the cost such that

$$\mathscr{Z}(\hat{K},\bar{v},\hat{\Delta}) := (\mathbf{z}-\bar{\mathbf{z}})^T(\mathbf{z}-\bar{\mathbf{z}}) \leq \gamma^2. \tag{5.29}$$

By taking the Schur complement argument, the inequality (5.29) can be written as

$$\begin{bmatrix} \gamma^2 & (\mathbf{z}-\bar{\mathbf{z}})^T \\ \star & I \end{bmatrix} \succ 0 \tag{5.30}$$

Using the definitions in (5.22), it is easy to verify that (5.30) can be re-arranged into the form

$$H_{11} + \mathscr{H}\left(H_{21}\Delta(I-H_{22}\hat{\Delta})^{-1}H_{21}\right), \tag{5.31}$$

$$\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] := \left[\begin{array}{cc|c} \gamma^2 & (\mathbf{D}_z^{\bar{v}})^T & 0 \\ \mathbf{D}_z^{\bar{v}} & I & \mathbf{D}_{z\hat{p}}^{\hat{K}} \\ \hline \mathbf{D}_{\hat{q}}^{\bar{v}} & 0 & \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}} \end{array}\right].$$

Finally, an application of Lemma 3.1.1 on (5.31) yields (5.25). ■

It follows that the output-feedback RMPC problem can be summarized as:

$$\begin{aligned}\min\{\gamma^2 :&(\hat{K},\bar{v})\in(\mathscr{K},\upsilon),\ ((5.25),(5.26) \text{ are satisfied}\\ &(S,R,G),(S_i,R_i,G_i)\in\widehat{\Psi},\ i\in\mathscr{N}_f\}\end{aligned} \tag{5.32}$$

By considering the definitions (5.14)-(5.16), it can be verified that problem (5.32) is highly nonlinear and non-convex in $\hat{K}$ due to terms of the form $\mathbf{D}_{\phi u}\hat{K}\hat{\mathbf{C}}_y\Phi$, where low case ϕ stands for $\hat{q}$, f and z and capital Φ stands for S, S_i, G, G_i, $i\in\mathscr{N}_f$. Here, note that optimization problem stated in (5.32) becomes convex for a fixed K. Output-feedback RMPC schemes with a fixed K have been proposed in the literature, see e.g. [30], [45] and the references therein. However, this introduces a degree of conservatism depending on the offline choice of K. To remedy this, we now use Lemma 3.3.1 to convexify problem (5.32) at the expense of only minor conservatism within the formulation.

Remark 21 *Note that there is no gap in Lemma 3.1.1 in the case of an unstructured Δ (see e.g. [71]). Therefore, conditions in (5.25), (5.26) become both necessary and sufficient for systems subject to unstructured uncertainties and/or additive disturbances.*

Theorem 5.2.2 *Let all variables be as defined Section 5.2. Then, $\mathscr{L}(\hat{K},\bar{v},\hat{\Delta})\leq\gamma^2$ and $\mathscr{F}(\hat{K},\bar{v},\hat{\Delta})\leq\bar{\mathbf{f}}$ for all $\hat{\Delta}\in\mathscr{B}\hat{\mathbf{\Delta}}$ if there exist solutions $(\hat{K},\bar{v})\in(\mathscr{K},\upsilon)$, $X\in\mathbb{R}^{N_n\times N_n}$, with X lower block-diagonal with $n\times n$ blocks, (S,R,G), $(S_i,R_i,G_i)\in\widehat{\Psi}$, $\forall i\in\mathscr{N}_f$ to the following LMIs:*

$$\begin{bmatrix} T_1+\mathscr{H}\left(T_2\bar{K}Y^*\right) & * \\ \left(\hat{\mathbf{C}}_yT_3-\bar{K}^TT_2^T\right)-XY^* & X+X^T\end{bmatrix}\succ 0, \tag{5.33}$$

$$\begin{bmatrix} T_1^i+\mathscr{H}\left(T_2^i\bar{K}Y_i^*\right) & * \\ \left(\hat{\mathbf{C}}_yT_3^i-\bar{K}^T(T_2^i)^T\right)-XY_i^* & X+X^T\end{bmatrix}\succ 0, \tag{5.34}$$

for some $Y^*\in\mathbb{R}^{N_n\times(N_z+1+N_q+N_p)}, Y_i^*\in\mathbb{R}^{N_n\times(1+N_q+N_p)}$ and where $\bar{K}:=\hat{K}X\in\mathscr{K}$.

Furthermore, suppose that (5.25) *and* (5.26) *have feasible solutions for $(\gamma^2,\hat{K},\bar{v},R,S,G,R_i,S_i,G_i)=$*

$(\gamma^{2*}, \hat{K}^*, \bar{v}^*, R^*, S^*, G^*, R_i^*, S_i^*, G_i^*)$ *so that*

$$T_1(\gamma^{2*}, \bar{v}^*, S^*, R^*, G^*) + \mathscr{H}\left(T_2\hat{K}^*\hat{\mathbf{C}}_y T_3(S^*, G^*)\right) \succ 0, \tag{5.35}$$
$$T_1^i(\bar{v}^*, R_i^*, S_i^*, G_i^*) + \mathscr{H}\left(T_2^i\hat{K}^*\hat{\mathbf{C}}_y T_3^i(S_i^*, G_i^*)\right) \succ 0,$$

and let $Y^* = \hat{\mathbf{C}}_y T_3(S^*, G^*) + (T_2\hat{K}^*)^T$ *and* $Y_i^* = \hat{\mathbf{C}}_y T_3^i(S_i^*, G_i^*) + (T_2^i\hat{K}^*)^T$. *Then* (5.33) *and* (5.34) *are feasible.*

Proof: We prove the first part by proving that the LMIs in (5.33) and (5.34) are sufficient for the nonlinear matrix inequalities in (5.25) and (5.26), respectively. In order to separate $\hat{K}$ from T_3, the inequality in (5.25) can be rearranged as:

$$\begin{bmatrix} I & T_2\hat{K} \end{bmatrix} \overbrace{\begin{bmatrix} T_1 & T_3^T\hat{\mathbf{C}}_y^T \\ \hat{\mathbf{C}}_y T_3 & 0 \end{bmatrix}}^{Q} \overbrace{\begin{bmatrix} I \\ \hat{K}^T T_2^T \end{bmatrix}}^{C^\perp} \succ 0. \tag{5.36}$$

Then, applying the Elimination Lemma 3.3.1 on (5.36) (with $B = I$) shows that (5.36) is equivalent to

$$\begin{bmatrix} T_1 & T_3^T\hat{\mathbf{C}}_y^T \\ \hat{\mathbf{C}}_y T_3 & 0 \end{bmatrix} + \overbrace{\begin{bmatrix} -T_2\hat{K} \\ I \end{bmatrix}}^{C} \overbrace{\begin{bmatrix} Y & X \end{bmatrix}}^{Z} + \begin{bmatrix} Y^T \\ X^T \end{bmatrix} \begin{bmatrix} -\hat{K}^T T_2^T & I \end{bmatrix} \succ 0, \tag{5.37}$$

where Y and X are free slack variables. Since $X + X^T \succ 0$, X is nonsingular and we can define $\bar{K} := \hat{K}X$ as a new variable. Then in order to preserve the structure of $\hat{K}$, we restrict X to be block lower triangular (with $n \times n$ blocks). In order to preserve linearity, we restrict Y to have the structure $Y = -XY^*$ with Y^* free (but not a variable) so that $\hat{K}Y = -\hat{K}XY^* = -\bar{K}Y^*$. Substituting $Y = -XY^*$ into (5.37) proves that (5.33) is sufficient for (5.25) (but not necessary due to the above restrictions on the slack variables Y and X). A similar procedure proves that (5.34) are sufficient for (5.26).

Next, we prove feasibility of (5.33) and (5.34) for the the given Y^* and Y_i^*. To show that (5.33) has a feasible solution, set $(\gamma^2, \hat{K}, \bar{v}, R, S, G) = (\gamma^{2*}, \hat{K}^*, \bar{v}^*, R^*, S^*, G^*)$ and let $X = I$. Then the LHS of

(5.33) becomes

$$T^* := \begin{bmatrix} T_1^* + \mathscr{H}\left(T_2\hat{K}^*\left(\hat{\mathbf{C}}_y T_3^* + (\hat{K}^*)^T T_2^T\right)\right) & * \\ -2(\hat{K}^*)^T T_2^T & 2I \end{bmatrix}$$

where we have defined $T_1^* := T_1(\gamma^{2*}, \bar{v}^*, S^*, R^*, G^*)$ and $T_3^* := T_3(S^*, G^*)$. Then applying a Schur complement on T^* shows that $T^* \succ 0$ if and only if (5.35) is satisfied. It follows that (5.33) is feasible if (5.35) is. The feasibility of (5.34) can be shown using a similar procedure. ■

Remark 22 *Theorem 5.2.2 provides sufficient LMI conditions for the initial nonconvex and nonlinear relaxed RMPC problem presented in (5.32). Therefore, the control gains K and control perturbation* **v** *can be computed online and applied in the usual receding horizon MPC manner, where the first input of the control sequence* **u** *is applied to the plant, the time window is shifted by 1, the current output measurement is read and the process is repeated. Note that $\hat{v}$ can be recovered from $\bar{v}$ and the expression in* (5.19)*, where K and* **v** *can be recovered from the variables $\hat{K}$ and $\hat{v}$ by the expression in (5.9) and (5.19).*

Remark 23 *The novelty of the proposed linearization procedure is that it does not restrict the structure of the slack variables (R, S, G) and (R_i, S_i, G_i) beyond the requirements of $\widehat{\Psi}$, and is therefore less conservative compared to other LMI-based Robust MPC approaches suggested in the literature [14, 75].*

Remark 24 *When the system is subject only to additive disturbance (and no model-uncertainty), the matrix inequalities (5.25), (5.26) become linear. To see this, note that in such a case, C_q, D_{qu} become zero and therefore, $\mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}}$ and $\mathbf{D}_{\hat{q}}^{\bar{v}}$ are no longer functions of variables $(\hat{K}, \hat{v})$. In addition, the variables G, G_i become zero since Δ is now purely diagonal. Then, effecting the congruence transformation* $\mathrm{diag}(I, I, S^{-1}, S^{-1})$ *on (5.25), and considering S^{-1} as a variable, renders (5.25) linear in $(\hat{K}, \hat{v})$. A similar procedure can be adopted to linearize (5.26). Hence, these LMIs become necessary and sufficient conditions for the cost and constraints. Therefore, the output-feedback RMPC problem for systems with additive disturbances becomes convex with no additional conservatism.*

Remark 25 *It is worth mentioning here that a simple procedure for linearizing the inequalities (5.25), (5.26) is to set $S = S_i = \lambda I_{N_p}$, and $G = G_i = 0$, $\forall i$, for a variable $\lambda \in \mathbb{R}$, and subsequently take $\lambda \hat{K}$ as the variable. Though this may seem attractive from a computational point of view, the problem is the excessive conservatism potentially associated with such a restriction which, in turn, is likely to render the problem infeasible (examples for such scenario can be found in [83]).*

In general industrial control applications involve a large number of constraints to preserve operational safety. Similar to the observation made in Section 3.4, the LMI optimization problem presented in Theorem 5.2.2 will require a high computational burden, due to the large number of constraints associated with the MPC formulation. Instead of solving multiple matrix inequalities for the constraints (one for each of the N_f constraints (5.26) or (5.34)), using Theorem 3.4.1 we propose a strategy to combine all constrains LMIs within a single inequality.

Theorem 5.2.3 *Let all variables be as defined above. Then, $\mathscr{Z}(\hat{K},\hat{v},\hat{\Delta}) \leq \gamma^2$ and $\mathscr{F}(\hat{K},\hat{v},\hat{\Delta}) \leq \bar{\mathbf{f}}$ for all $\hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}}$ if there exist solutions $(\hat{K},\hat{v}) \in (\mathscr{K},\upsilon)$, $(S,R,G),(\tilde{S},\tilde{R},\tilde{G}) \in \widehat{\Psi}$, $\mu \in \mathbb{R}$ and $M \in \mathbb{D}^{N_f}$ to* (5.25) *and,*

$$\tilde{T}_1 + \mathscr{H}(\tilde{T}_2 \hat{K} \hat{\mathbf{C}}_y \tilde{T}_3) \succ 0, \tag{5.38}$$

where

$$\begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \\ \tilde{T}_3 & 0 \end{bmatrix} =$$

$$\begin{array}{c} \\ 1 \\ N_f \\ N_q \\ N_p \\ \hline N_p \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & N_f & N_q & N_p & N_u \end{array} \\ \left[\begin{array}{cccc|c} 2\mu(\bar{\mathbf{f}} - \mathbf{D}_f^{\hat{v}} - Me - e\mu)^T & (\mathbf{D}_q^{\hat{v}})^T & 0 & & 0 \\ * & M+M^T & -\hat{\mathbf{D}}_f \tilde{G}^T & -\hat{\mathbf{D}}_f \tilde{S} & -\mathbf{D}_{fu} \\ * & * & \tilde{R} + \mathscr{H}(\hat{\mathbf{D}}_q \tilde{G}^T) & \hat{\mathbf{D}}_q \tilde{S} & \mathbf{D}_{\hat{q}u} \\ * & * & * & \tilde{S} & 0 \\ \hline 0 & 0 & \tilde{G}^T & \tilde{S} & 0 \end{array}\right] \end{array}.$$

Proof: We only need to prove that (5.38) is sufficient for the constraints

$$\tilde{f} := \bar{\mathbf{f}} - \mathbf{f} = \bar{\mathbf{f}} - \mathbf{D}_{f\hat{p}}^{\hat{K}} \hat{\Delta}(I - \mathbf{D}_{\hat{q}\hat{p}}^{\hat{K}} \hat{\Delta})^{-1} \mathbf{D}_{\hat{q}}^{\bar{v}} - \mathbf{D}_f^{\bar{v}} \geq 0 \; \forall \hat{\Delta} \in \mathscr{B}\hat{\mathbf{\Delta}}.$$

Using Theorem 3.4.1 and a rearrangement of (3.19) verifies that a sufficient condition for these constraints is:

$$H_{11}+\mathscr{H}(H_{12}\hat{\Delta}(I-H_{22}\hat{\Delta})^{-1}H_{21})\succ 0,\ \forall\hat{\Delta}\in\mathscr{B}\hat{\mathbf{\Delta}}, \tag{5.39}$$

where,

$$\left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] := \left[\begin{array}{cc|c} 2\mu & \left(\bar{\mathbf{f}}-\mathbf{D}_f^{\hat{v}}-M\mathbf{1}-\mathbf{1}\mu\right)^T & 0 \\ * & M+M^T & -\hat{\mathbf{D}}_f \\ \hline \mathbf{D}_q^{\hat{v}} & 0 & \hat{\mathbf{D}}_q \end{array}\right].$$

Using Lemma 3.1.1 on (5.39) and some rearrangement yields the matrix inequality (5.38) as a sufficient condition. ■

Using the linearization procedure presented in Theoren 5.2.2, we next derive sufficient LMI conditions (signle LMI for constrained signal) for the problem stated in (5.32).

Theorem 5.2.4 *Let all variables be as defined as above. Then, $\mathscr{Z}(\hat{K},\hat{v},\hat{\Delta})\leq\gamma^2$ and $\mathscr{F}(\hat{K},\hat{v},\hat{\Delta})\leq\bar{\mathbf{f}}$ for all $\hat{\Delta}\in\mathscr{B}\hat{\mathbf{\Delta}}$ if there exist solutions $(\hat{K},\hat{v})\in(\mathscr{K},\upsilon)$, $(S,R,G),(\tilde{S},\tilde{R},\tilde{G})\in\widehat{\Psi}$, $\mu\in\mathbb{R}$ and $M\in\mathbb{D}^{N_f}$ and $X\in\mathbb{R}^{N_n\times N_n}$, with X lower block-diagonal with $n\times n$ blocks, to (5.33) and the following LMI:*

$$\begin{bmatrix} \tilde{T}_1+\mathscr{H}\left(\tilde{T}_2\bar{K}\tilde{Y}^*\right) & * \\ \left(\hat{\mathbf{C}}_y\tilde{T}_3-\bar{K}^T\tilde{T}_2^T\right)-X\tilde{Y}^* & X+X^T \end{bmatrix}\succ 0, \tag{5.40}$$

for some $Y^*\in\mathbb{R}^{N_n\times(N_z+1+N_q+N_p)}$, $\tilde{Y}^*\in\mathbb{R}^{N_n\times(1+N_f+N_q+N_p)}$ and where $\bar{K}:=\hat{K}X\in\mathscr{K}$.
Furthermore, suppose that (5.25) *and* (5.38) *have feasible solutions for $(\gamma^2,\hat{K},\hat{v},R,S,G,\tilde{R},\tilde{S},\tilde{G})=(\gamma^{2*},\hat{K}^*,\hat{v}^*,R^*,S^*,G^*,\tilde{R}^*,\tilde{S}^*,\tilde{G}^*)$ so that* (5.35) *and*

$$\tilde{T}_1(\hat{K}^*,\hat{v}^*,\tilde{R}^*,\tilde{S}^*,\tilde{G}^*)+\mathscr{H}\left(\tilde{T}_2\hat{K}^*\hat{\mathbf{C}}_y\tilde{T}_3(\tilde{S}^*,\tilde{G}^*)\right)\succ 0. \tag{5.41}$$

are satisfied and let $Y^*=\hat{\mathbf{C}}_yT_3(S^*,G^*)+(T_2\hat{K}^*)^T$ and $\tilde{Y}^*=\hat{\mathbf{C}}_y\tilde{T}_3(\tilde{S}^*,\tilde{G}^*)+(\tilde{T}_2\hat{K}^*)^T$. Then (5.33) *and* (5.40) *are feasible.*

Proof: The result can be proved by applying the Elimination Lemma 3.3.1 on (5.38) in a similar manner to the proof of Theorem 5.2.2 and is therefore omitted. ■

It follows that the output-feedback RMPC problem can now be given by the following LMI optimization:

$$\begin{aligned}\overline{\phi} = \min\{\gamma^2 : (5.33) \text{ and } (5.40) \text{ are satisfied for } (\hat{K}, \hat{v}) \in (\mathscr{K}, \upsilon),\\ (S, R, G), (\tilde{S}, \tilde{R}, \tilde{G}) \in \widehat{\Psi}. \end{aligned} \tag{5.42}$$

5.3 Feasibility analysis

A very critical point in RMPC schemes is to ensure feasibility of the optimization problem subject to constraints, first at the initial step ($k = 0$) and second recursive feasibility of the LMI optimization problem. Infeasibility may arise if the constraints are too tight or it may be due to the approximations used to obtain a practical solution, especially with RMPC.

In the context of this work, to guarantee feasibility, the solutions we provide in Theorems 5.2.2 and 5.2.4 require initial feasible solutions to the matrix inequalities in (5.25), and (5.26) (to compute Y^* and Y_i^*) or (5.25) and (5.38) (to compute Y^* and $\tilde{Y}^*$). On the other hand, the matrix inequalities in (5.25), (5.26) and (5.38) are nonlinear and difficult to solve. Furthermore, all these computations need to be carried out online. In this section we develop algorithms that address these issues that involve carrying out extensive computations, which, however, are convex and can be carried out offline. We will concentrate on Theorem 5.2.4 (Single LMI for constrained signal) since the procedure for Theorem 5.2.2 is similar.

One approach is to solve offline the LMIs in (5.25), and (5.26), with fix $\hat{K} = 0$ (feasible but not optimal solution), for the entire constrained state-space as initial state bounds and use the results $(S, G, \tilde{S}, \tilde{G})$ as initial guess in variables (Y^* and Y_i^*) to solve the problem (5.42) at time $k = 0$. Then use the solutions computed in time step k as the initial solutions in time step $k+1$. There is no guarantee that these solutions are feasible in step $k+1$ since x_k and the associated stated bounds ($\underline{x}_k$ and $\overline{x}_k$) will be different from x_{k+1} and next sample time state bounds ($\underline{x}_{k+1}$, $\overline{x}_{k+1}$), respectively.

Another approach, which we demonstrate in detail here, is to find solutions to (5.25) and (5.38) offline

that are feasible for every x_0 in the state space constrained set ($\mathscr{F}_x$):

$$x_0 \in \mathscr{F}_x := \left\{ x \in \mathbb{R}^n : x_l \leq x \leq x_h \right\}, \tag{5.43}$$

where x_l and x_h represent the minimum and the maximum value that belongs inside the state constrained set.

Algorithm 3 outlines the suggested offline policy that creates offline a lookup table that contains initial feasible solutions for Theorem 5.2.4 for any $x_0 \in \mathscr{F}_x$.

Algorithm 3: Offline computation of Initial feasible solutions for Theorem 5.2.4

Result: $Y^*(S^*, G^*, \hat{K}^*)$ and $\tilde{Y}^*(\tilde{S}^*, \tilde{G}^*, \hat{K}^*)$

Step 1:
In Theorem 5.2.3, fix $\hat{K}$ (e.g. $\hat{K} = 0$), replace $\bar{f}$ by $\beta\bar{f}$ and minimize β such that (5.38) is satisfied for all $x_0 \in \mathscr{F}_x$. Record β, $\tilde{S}$ and $\tilde{G}$ and let $\hat{K}^* = 0$, $\tilde{S}^* = \tilde{S}$ and $\tilde{G}^* = \tilde{G}$. Set $i = 1$, $\beta_i = \beta$ and select $i_{\max}$ to be the maximum number of iterations and $tol_\beta < 1$ to be a tolerance;
Step 2:
while *($\beta > 1$) & ($i < i_{\max}$)* **do**
  In Theorem 5.2.4, replace $\bar{f}$ by $\beta\bar{f}$ and find the smallest $\beta \geq 1$ such that (5.40) is satisfied for all $x_0 \in \mathscr{F}_x$. Set $\beta_{i+1} = \beta$ and update $\hat{K}^* := \hat{K}$, $\tilde{S}^* := \tilde{S}$ and $\tilde{G}^* := \tilde{G}$;
  if *($\frac{|\beta_{i+1} - \beta_i|}{\beta_{i+1}} < tol_\beta$)* **then**
    break; (convergence to a $\beta > 1$)
  Set $i := i + 1$.
Step 3:
if $\beta > 1$ **then**
  Sub-divide $\mathscr{F}_x$ into smaller sets;
  Go back to Step 2;
else
  In Theorem 5.2.3 fix $\hat{K} = \hat{K}^*$ and minimize γ^2 such that (5.25) is satisfied for all $x_0 \in \mathscr{F}_x$. Record γ^2 and let $S^* = S$ and $G^* = G$;
Step 4:
Set $j = 1$, $\gamma_j^2 = \gamma^2$ and select $j_{\max}$ to be the maximum number of iterations and $tol_\gamma < 1$ to be a tolerance.
while *(($j < j_{\max}$)* **do**
  In Theorem 5.2.4, minimize γ^2 such that (5.33) and (5.40) are satisfied for all $x_0 \in \mathscr{X}_0$. Set $\gamma_{j+1}^2 = \gamma^2$ and update $\hat{K}^* := \hat{K}$, $S^* = S$, $G^* = G$, $\tilde{S}^* := \tilde{S}$ and $\tilde{G}^* := \tilde{G}$;
  if *($\frac{|\gamma_{j+1}^2 - \gamma_j^2|}{\gamma_{j+1}^2} < tol_\gamma$)* **then**
    break;
  Set $j := j + 1$.
return $(\hat{K}^*, S^*, G^*, \tilde{S}^*, \tilde{G}^*)$

Remark 26 *If $\beta = 1$ at the end of Step 2 of Algorithm 3, then we have feasible solutions $\hat{K}^*$, S^*, G^*, $\tilde{S}^*$ and $\tilde{G}^*$ to* (5.25) *and* (5.38) *for all $x_0 \in \mathscr{F}_x$ and so we can use Theorem 5.2.4 online since it is guaranteed to have a feasible solution. If fewer online computations are required, then Theorem 5.2.3 can be used online with $\hat{K}$ fixed at $\hat{K}^*$ and the degrees of freedom in $\hat{\upsilon}$ can be used to minimize γ for the given $x_0 \in \mathscr{X}_0$. In the case that $\beta > 1$ at the end of Step 2 of Algorithm 3, then, with suitable modifications to the rest of Algorithm 3, Theorem 5.2.4 can still be used as above, although without a guaranteed feasible solution, but possibly a good initial solution if β is close to 1. Alternatively, we may sub-divide $\mathscr{F}_x$ into smaller sets, find a feasible solution for each of these subsets and use a look-up table to choose the initial solution depending on x_0 in the online implementation.*

Remark 27 *Recursive feasibility of the proposed schemes can be ensured due to incorporating the invariant terminal set defined by $\bar{f}_N$. In particular, under the conditions given in [76], using their notation, the control sequence computed at time k can be shifted and appended with the terminal control law u_{f_N} to yield $\{u(k+1 \mid k), \cdots, u(k+N-1 \mid k), u_{f_N}\}$ which remains feasible at next time step $k+1$. See [3, 14] for further details.*

5.4 Overall Output-feedback RMPC Algorithm Outline

In this section the overall proposed strategy, which combines the control and RMHE schemes discussed in Section 5.2 and Section 4.3 respectively, is presented and its effectiveness is demonstrated by a benchmark example.

5.4.1 Implementation Strategy

The proposed output-feedback RMPC scheme relies on the state estimation upper/lower bounds $(\bar{x}_k, \underline{x}_k)$ based on past input/output data. However, at sample time $k = 0$ there are no past data to compute the state bounds and the state estimation value. Thus, at the time point $k = 0$ the a priori bounds on x_0 $(\bar{x}_0, \underline{x}_0)$ are assumed to be known and are used to compute the first control sequence u, where only the first control value u_0 is applied to the system. Subsequently, while more data is

collected from the input/output at each iteration, the estimation horizon $\tilde{N}e$ is incremented until it reaches the pre-specified estimation horizon N_e. During this period the current state bounds $\underline{x}_k, \bar{x}_k$ and the estimated state $\hat{x}_k$ are computed by considering all available past data. By the time that $\tilde{N}_e$ is equal to N_e the state bounds are calculated by the moving horizon framework presented in Section 4.3. The overall approach can therefore be outlined as follows.

Algorithm 4: Output-feedback RMPC scheme

Offline calculation:
Create a look-up table for all $x_0 \in \mathscr{F}_x$ using Alg.3, where initial feasible solution $Y^*(S^*, G^*, \hat{K}^*)$ and $\tilde{Y}^*(\tilde{S}^*, \tilde{G}^*, \hat{K}^*)$ are stored.

Online calculation:

(1) *Initialization:* At sample time $k = 0$, given y_0 calculate the bounds on x_0 based on equation (3), considering the maximum output-disturbance. Then, solve (5.42) and set as control action u_k the first value of the control sequence $\mathbf{u}$.

(2) *Data collection:* Update the vectors $\tilde{u}$, $\tilde{y}$ with the newly available input/output data from the previous step.

(3) *Estimation scheme:* If $\tilde{N}_e < N_e$, increment $\tilde{N}_e$, else fix $\tilde{N}_e = N_e$. Then, using vectors $\tilde{u}$ and $\tilde{y}$ solve the LMI optimization problem stated in Theorem 4.3.1 to compute bounds of the current state x_k.

(4) *Control scheme:* Set the control action u_k to be the first value of the control sequence $\mathbf{u}$ by solving the optimization problem stated in (5.42) and loop back to step (2).

5.4.2 Numerical Example

In this subsection the effectiveness of the proposed algorithms is illustrated by two benchmark examples taken from the literature. The simulations in both examples are performed using MOSEK LMI/SDP solver within the CVX package [77], in MATLAB R2019b on a computer with 2.40 GHz Intel Xeon(R) CPU and 64.0 GB memory.

Example 1

The first system under consideration is the double integrator example taken from [29] . The system is affected only by additive disturbances and measurement noise. In particular the discrete-time system can be described as follows:

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \eta_k, \quad y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + v_k$$

The disturbance and measurement noise respectively belong to the sets:

$$\eta_k \in Z := \left\{ \eta \in \mathbb{R}^2 : -0.1 \leq \eta \leq 0.1 \right\},$$
$$v_k \in V := \left\{ v \in \mathbb{R} : -0.05 \leq v \leq 0.05 \right\}.$$

The input constraints are given by: $-3 \leq u_k \leq 3$, and we consider (tightened) state constraints given by $[-12 \;\; -12]^T \leq x \leq [3 \;\; 3]^T$. The cost signal is selected as $z_k := [x_k \; u_k]^T$ and the states and inputs are equally weighted ($C_z = [I \; 0]^T, \hat{C}_z = I, D_{zu} = [0 \; I]^T$). The control and estimation horizons are selected as $N = 5$ and $N_e = 8$, respectively. Finally, to remain consistent with [29], the initial state bounds are set: $[-3.02; -8.02] \leq x_0 \leq [-2.98; -7.98]$.

Since the system is subject to disturbances only, the output-feedback RMPC problem presented in (5.32) becomes linear and it can express in LMI form as illustrated in Remark 24. In Fig. 5.1 the state evolution of the system using the proposed robust control approach is presented, where it can be seen that the system is regulated to the origin. The estimated state trajectory is shown in red dash line and the actual state trajectory is presented by the blue line. Note that the actual state values are not available while utilising the suggested algorithm and is presented here simply for reference purposes. Despite the action of persistent state disturbance and measurement noise, a robust control performance has been achieved by utilising the presented control algorithm without violating any state constraints (red area in Fig. 5.1). The Fig. 5.1 also shows the state estimation bounds (cyan rectangles), computed using the results of RMHE in Section 4.3 and in particular for the uncertainty-free system of the double integrator through the linear program (see Remarks 14). Comparing the

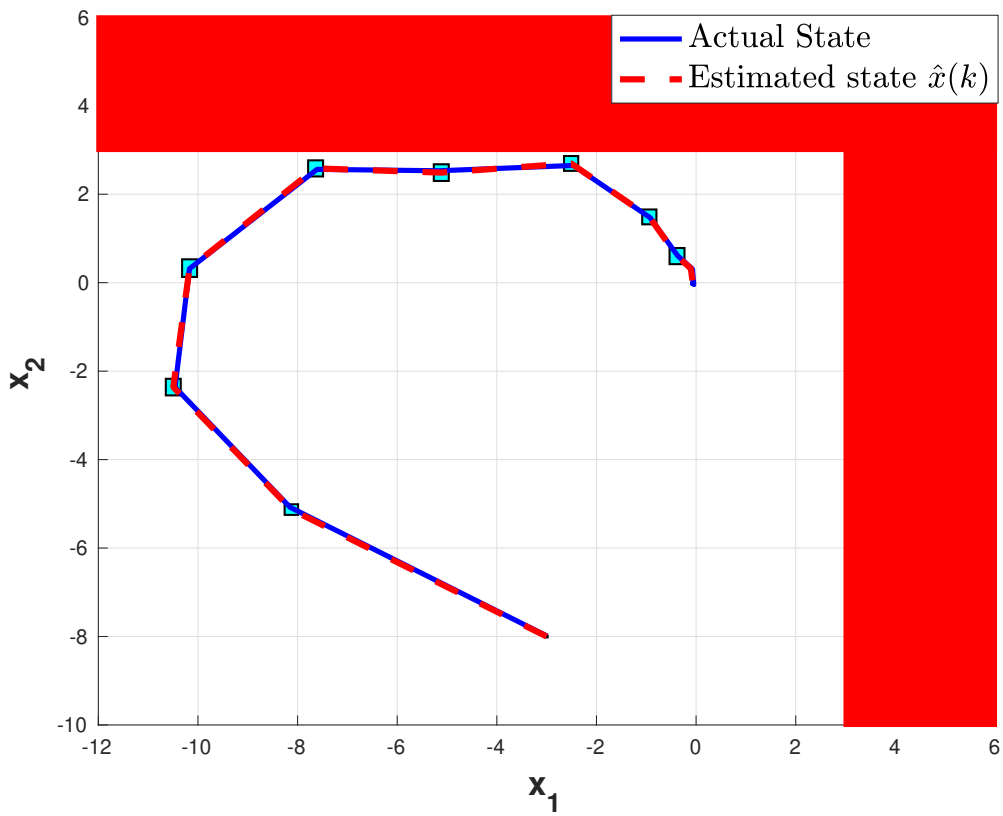

Figure 5.1: State evolution history for the double integrator example using the proposed OF-RMPC algorithm.

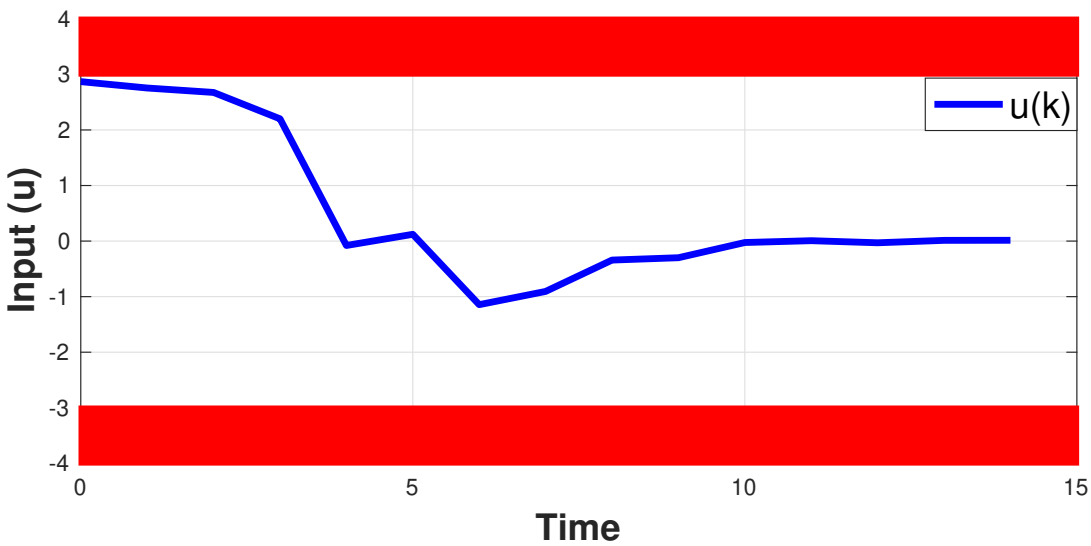

Figure 5.2: Control input history for the double integrator example using the proposed OF-RMPC algorithm.

actual state with the estimated states based on the bounds, it can be confirmed that the computed bounds are very accurate (they are in fact tight in this case) and they play an important role on the

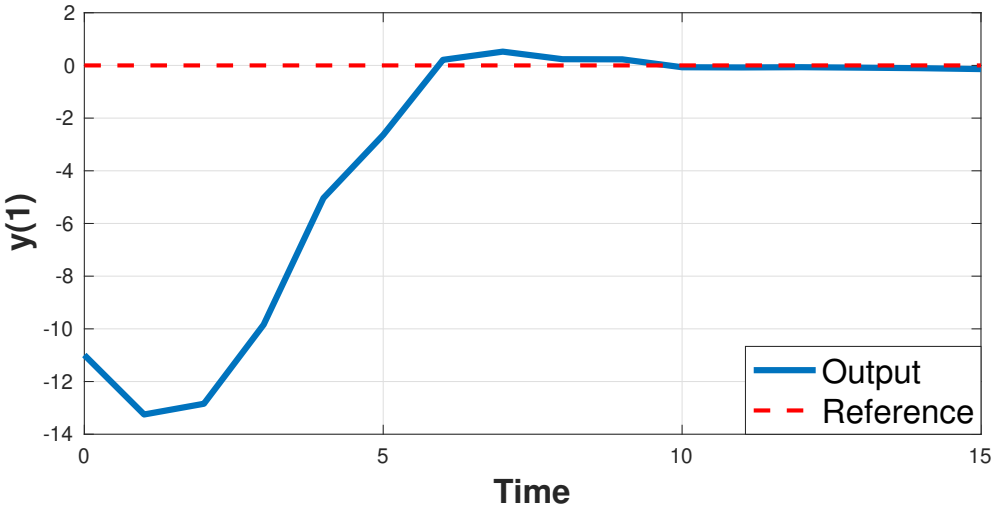

Figure 5.3: Output trajectory for the double integrator example using the proposed OF-RMPC algorithm.

fast regulation performance. Figure 5.2 illustrates the control input evolution, which is also within the predefined constraints. Note here that the control input at $t = 0$ is just below the constraint boundary, which verifies that constraints have indeed been incorporated in the formulation in a non-conservative manner. Similarly, fig. 5.3 illustrates the output signal evolution, where it tracks the reference signal and verifies the robust performance of the suggested control algorithm.

Example 2

To investigate the performance of the proposed OF-RMPC scheme, for this example the benchmark problem of the control of a paper-making process (see section 4.5) is again considered. To preserve continuity on this chapter the system description and state-space representation is once more summarised. The system, consists of process states $x = [H_1\ H_2\ N_1\ N_2]^T$, where H_1 and N_1 denote liquid level and composition of the feed tank, respectively, and H_2 and N_2 denote liquid level and composition of the headbox, respectively. The control input vector is given by $u = [G_p\ G_w]^T$, where G_p is the flow rate of stock entering the feed tank and G_w is the recycled white water flow rate. All variables are normalized (i.e. they are zero at steady state) and only noisy measurements of H_2 and N_2 are available. The consistency and composition of white water is a source of uncertainty within the dynamics, particularly in the state N_1 and input G_w.

The discrete-time dynamics (including uncertainty description), sampled at 2 minutes (see [81]), are given by (5.1) with:

$$A = \begin{bmatrix} 0.0211 & 0 & 0 & 0 \\ 0.1062 & 0.4266 & 0 & 0 \\ 0 & 0 & 0.2837 & 0 \\ 0.1012 & -0.6688 & 0.2893 & 0.4266 \end{bmatrix}$$

$$B_u = \begin{bmatrix} 0.6462 & 0.6462 \\ 0.2800 & 0.2800 \\ 1.5237 & -0.7391 \\ 0.9929 & 0.1507 \end{bmatrix},\ B_w = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 0 & 0 & 0.2 & 0 \end{bmatrix},\ D_{qu} = \begin{bmatrix} 0 & 0.2 \end{bmatrix}$$

$$C_y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ D_{yw} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Moreover, disturbance η_k affects all four states and ν_k denotes the output measurement noise (see Remark 19 to describe the system as shown in (5.1)). The process disturbance and output measurement

noise are respectively characterized by the sets:

$$\eta_k \in Z := \left\{ \eta \in \mathbb{R} : -0.1 \leq \eta \leq 0.1 \right\}$$

$$v_k \in V := \left\{ v \in \mathbb{R} : -0.05 \leq v \leq 0.05 \right\}$$

The prediction horizon and the estimation horizon for the above set-up are set $N = 3$ and $N_e = 5$, respectively. Similar to Example 5.4.2 the states and inputs are equally weighted ($C_z = [I\ 0]^T, \hat{C}_z = I, D_{zu} = [0\ I]^T$) and the cost signal is defined as $z_k := \left[y_k^T\ u_k^T\right]^T$.

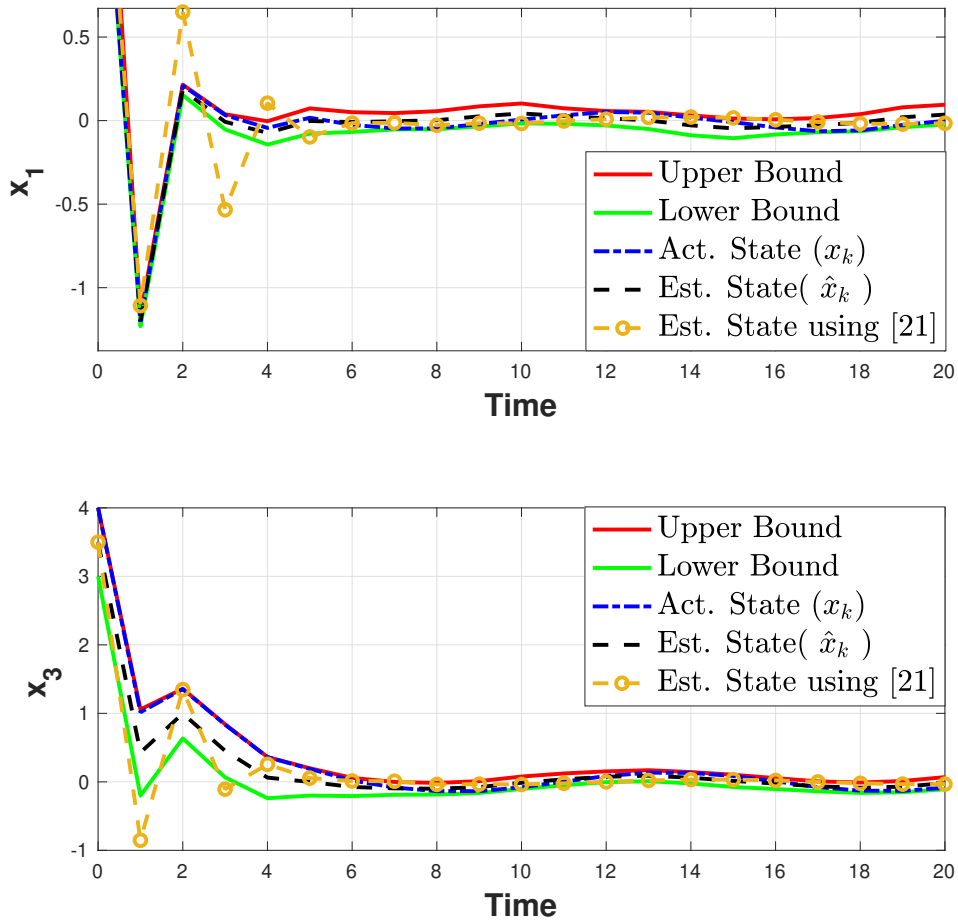

Figure 5.4: State evolution for Example 2.

Simulation results for the paper making process example under persist worst-case uncertainty ($\Delta = I$) and randomly distributed disturbances are shown in figures 5.4-5.7.

Figure 5.4 presents the state evolution over time, where at each sample time the upper and lower bound of the unmeasured state (x_1 and x_3) are computed through the presented estimation scheme. For comparison, estimation values of the unmeasured states using the algorithm presented in [48] shown in dash yellow line, where it can be clearly seen that estimation values at initial steps are way outside our given tight bounds. Although large estimation error, in general, can be lead to either constraint violation or infeasibility on the control face, in figure 5.3, it can be seen that the proposed algorithm and the considered algorithm from [48] have similar tracking performance.

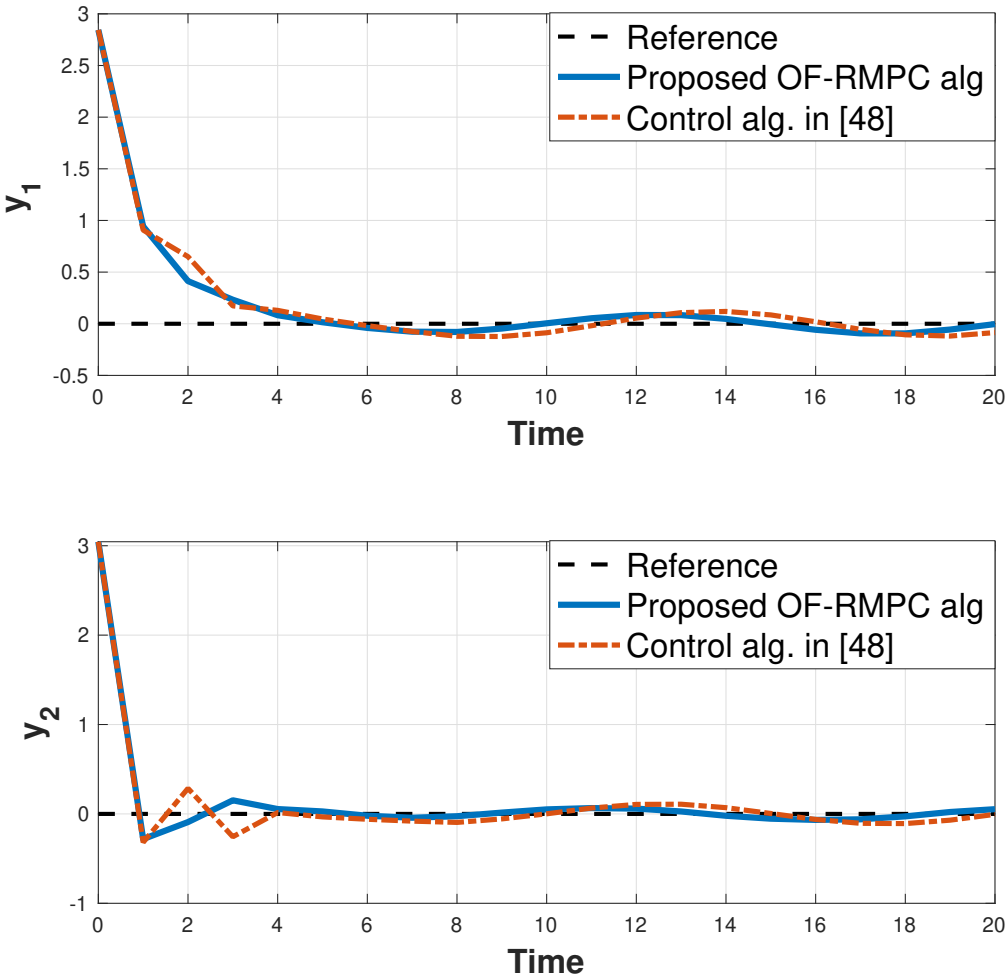

Figure 5.5: Output trajectory for Example 2.

Looking at Figures 5.6 and 5.7, where the control input signal and the cost signal are illustrated respectively, it can be concluded that imprecise estimation values can lead to conservative controllers. In more detail, utilizing the suggested OF-RMPC scheme the control actions computed by the con-

troller are more aggressive (at $k = 0$ touch the constraint) compare to the algorithm presented in [48]. Looking at the figure 5.7 it can be verified that the suggested OF-RMPC scheme minimises the cost function faster than the algorithm in [48].

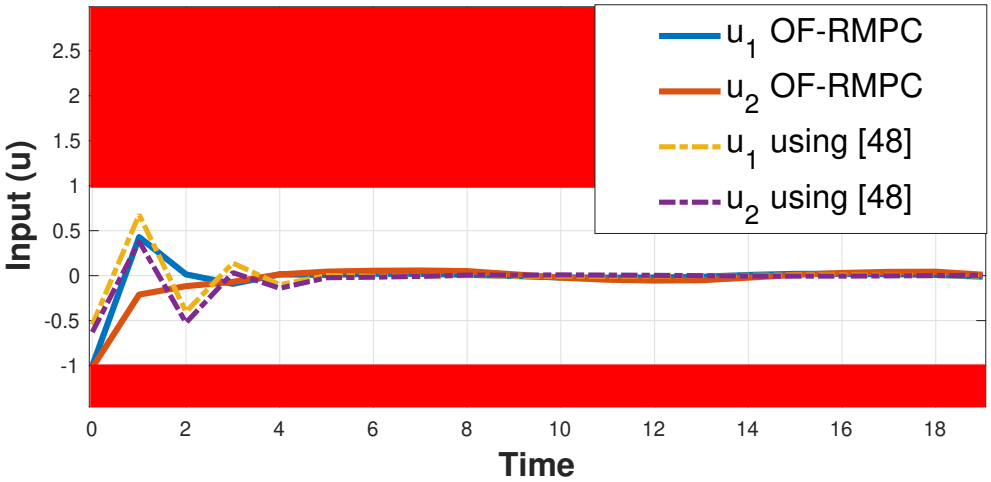

Figure 5.6: Computed Control input for Example 2 using the proposed OF-RMPC algorithm.

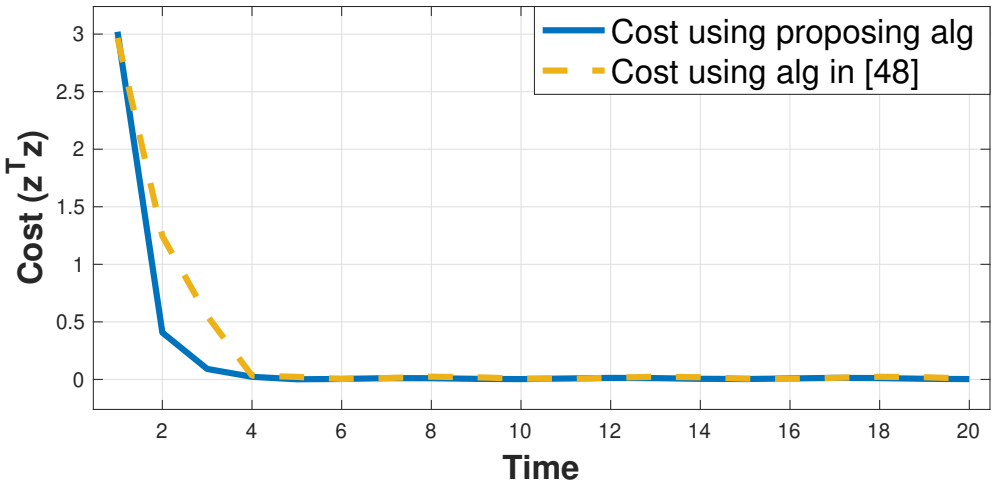

Figure 5.7: Cost signal for Example 2.

5.5 Synopsis

In this work, a new algorithm based on LMI's is proposed for the formulation of an OF-MPC of linear discrete-time systems subject to norm-bounded model-uncertainties, additive state disturbances and measurement noise.

The novelty lies in the fact that the algorithm computes, online, both the output-feedback gain and a control perturbation through an LMI optimization. A significant reduction in the computational complexity of the control problem is achieved by a proposed algorithm that solves a single LMI for handling the constraints in the RMPC problem. Moreover, unlike most output-feedback MPC schemes from the literature which use a fixed (linear) state observer, the presented algorithm, in order to reduce conservativeness, uses a past input/output data window - in a manner similar to Moving Horizon Estimation - to compute (tight) bounds on the current state which are then used within the control scheme. Note that, the suggested control approach requires initial feasible solutions to the nonlinear matrix inequalities, which however can be obtained offline without compromising the performance of the controller.

The effectiveness of the proposed techniques, in terms of robust control performance as well as estimation accuracy, is demonstrated through numerical examples taken from the literature. The natural follow-up for the proposed robust control scheme is to be implemented and tested in an industrial-scale application.

CHAPTER 6

Tracking Control for Directional Drilling Systems Using Robust Model Predictive Control

6.1 Introduction

The oil and gas industry has constantly searched for more economic and efficient technologies to exploit hydrocarbon energy resources. The process for obtaining and extraction of energy resources such as oil and gas, which remain the major fuels for powering today's society, has two major difficulties. Firstly, access to energy resources most of the times requires boreholes with complex curves, which is not a simple task to achieve. Secondly, deep-seated and offshore hydrocarbon explorations commonly take place under an unpredictable environment and extreme working conditions while targeting resource locations in the crust of the Earth [84]. In many cases these challenges are being addressed by the introduction of Rotary Steerable Systems [85]. This steering mechanism is a tool placed close to the drilling bit of a bottom hole assembly (BHA) as illustrated in Fig. 6.1. In this project we study a push-the-bit RSS that controls the direction of borehole propagation via force ac-

tuated pads mounted close to the bit. At the early stages, when the RSS tool was used, the control actuator commands are operated with major communication delays by professionals, where they are located at the surface close to the drilling rig, using complex data sets, such as location of the reservoir, rock layer geometry, mud phase telemetry and etc. Human errors and communication delays could be minimized by automating the steering commands by developing a closed-loop controller using real-time data from sensors located in the drill string.

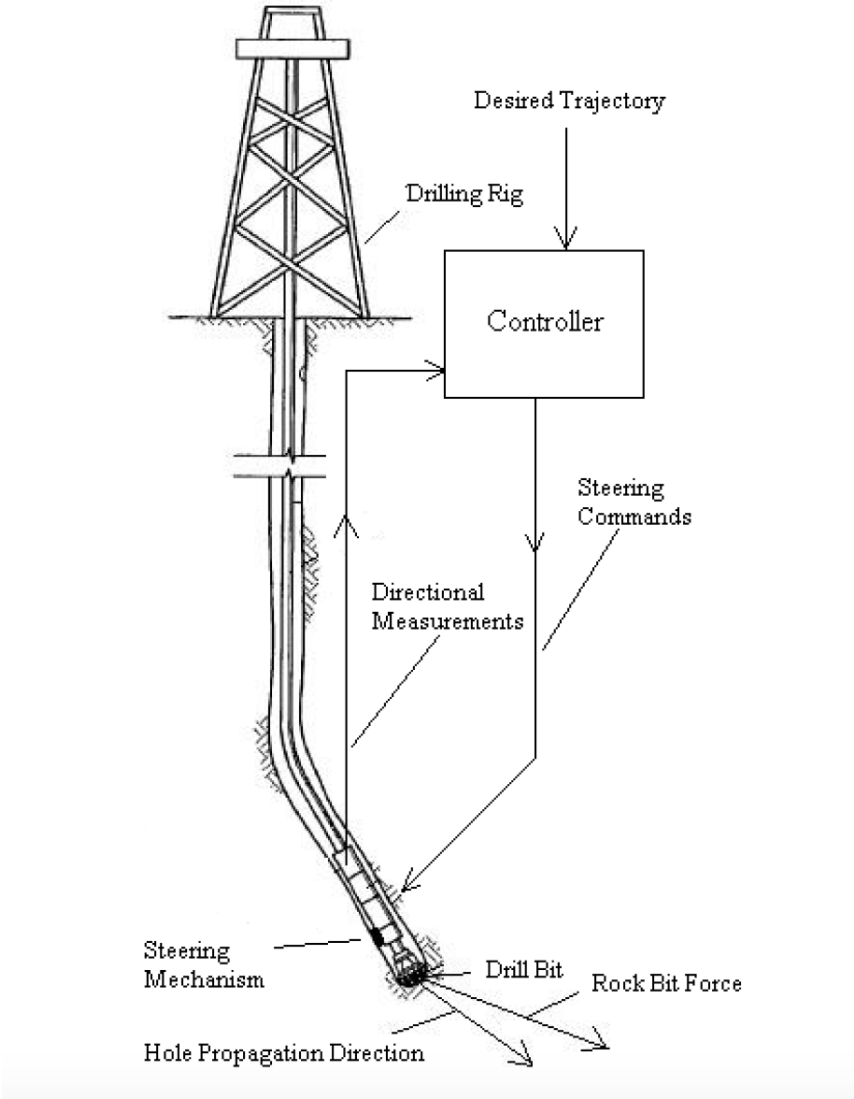

Figure 6.1: Directional drilling system [88].

The main difficulties of developing an automated RSS system are, firstly the unpredictable and harsh working environment, secondly, key parameters vary whilst drilling and lastly the poor communication between surface and downhole. Previous research studies considered empirical or numerical kinematics models using the assumption that the curvature of the BHA is directly linked to the force

applied by the RSS [86], however, these models could not fully reflect the dynamic behavior and variations of the system especially during transients. Downton *et. al.* suggested various novel RSS dynamic models described by linear spatial delay equations based on reasonable simplifications and assumptions [87, 88]. Based on the directional drilling model presented in [87], an $\mathscr{L}_1$ adaptive controller alongside state prediction is presented in [89]. Recently, Kremers *et. al.* investigated the behaviour of RSS system in directional drilling applications and have proposed a three-dimensional analytical model using non-linear delay differential equations [90]. However, in this approach it is assumed that all parameters remain constant while drilling, which is not generally a realistic assumption in drilling. Analytical models of RSS have been very promising since they can characterize the behavior of the system with minimum error.

By using the framework of RSS analytical modeling, the aim of this study is to develop an appropriate closed-loop feedback control law that can guarantee robustness and stability in the presence of the aforementioned uncertainties and disturbances. Since what is involved is a relatively slow dynamic system, and physical and design constraints which are very important for drilling operation safety, MPC type schemes are very suitable for designing controllers for this application [91, 92].

The contribution of this work can thus be divided as follows. Firstly, a dynamic model of the directional drilling system is proposed in terms of ordinary differential equations by a closed-form state-space representation, unlike conventional representations in the literature that utilize either less accurate kinematics system models [92], or with comprehensive dynamic models that are presented in terms of delay differential equations [88]. Very importantly, the present model is validated successfully against a high-fidelity industry grade finite element model developed by Schlumberger. Secondly, the present work advances the control solutions available in the literature and to industry for directional drilling automation by proposing a robust control strategy that can handle disturbances and uncertainties. The overall proposed control algorithm for this case study is synthesized based on the OF-RMPC control algorithm presented in Chapter 5, which makes it the first time that these strategies are applied successfully to a complex industrial level problem. The particular methodologies employed are OF-RMPC scheme, which is further combined with a Robust Positive Invariant (RPI) sets generated feedback control strategy [14, 20], to overcome the difficulties alluded to above regard-

ing automating RSS systems, while minimizing the trajectory tracking error during drilling. Although the proposed combined strategy requires high-performance computation, it provides an optimal solution at each sample time with limited conservatism in the formulation while safety constraints are preserved, unlike other works (see for example [91] which ignores disturbances).

The chapter is organized as follows. In Section 6.2, the analytical model of the directional drilling system is introduced and a simplified discrete-spatial uncertain system is suggested. In Section 6.3, in order to further reduce the computational burden of the controller while ensuring stability, an offline controller based on RPI set problem is presented. Following that, the overall control architecture that combines online OF-RMPC and the RPI-based offline controller applied to the directional drilling application is summarised. A case study in directional drilling using the proposed control approach is illustrated in Section 6.4, in which robustness and tracking performance of the controller is demonstrated by simulations. Finally, a summary of the findings is given in Section 6.5, along with potential future work. The formulation and results presented in this chapter are mainly based on the result presented in [26].

6.2 Directional Drilling System

The directional drilling system can be presented as a mechanical structure, where the centerline of the borehole can be expressed with respect to actuator stimuli by a quasi-polynomial transfer function. In this study, the complex push-the-bit RSS drilling model presented by [88] is used, where the average direction of drilling is normally assumed to be tangential to the m-*axis*, shown in Fig. 6.2. This assumption allow us to use a small angle approximation for displacements and angles in the system. The propagation of the BHA centerline can be computed by the lateral displacement rate ($\frac{dH(m)}{dm}$) with respect to distance drilled (m) as determined by:

$$\frac{dH(m)}{dm} = \tan(\alpha + \tan^{-1}(\frac{LWOR}{WOR}K_{anis})), \tag{6.1}$$

where α indicates the angle of the bit's rotation axis with respect to the m-*axis* (indicated by the slope of the blue line in Fig. 6.2), K_{anis} is the *anisotropy* of the bit which measures the rock removal capability ratio of the two axis (axial and lateral), and WOR and LWOR is the axial and lateral load on the rock, respectively. In this work the lateral displacement $H(m)$ is considered as the dependent variable and the distance drilled m as the independent variable.

By using the assumption that the deformation inside the borehole is small, the BHA can be statically treated as an Euler-Bernouli beam. Therefore, the general expression of a beam element under load for small angles is given by:

$$\frac{\partial^2}{\partial l^2}(EI(\frac{\partial y^2}{\partial l^2})) + \frac{\partial}{\partial l}(P(\frac{\partial y}{\partial l})) = w, \tag{6.2}$$

where y is the beam's lateral displacement, l is the length along the beam which is considered as the independent variable, EI is the bending stiffness, P is the axial-load along the beam and w is the beam's load per unit length.

The method of dividing the drilling string into smaller segments [93], according to the position of the stabilizers on the BHA, is followed. In this case, a BHA with four stabilizers is considered, with the parameters L_1, L_2, L_3 and L_4 defined as the distance of the first stabilizer from the bit and stabilizer i from stabilizer $i-1$, respectively (as shown in Fig. 6.2). As a result, four independent beam equations arise that are influenced by self-weights and applied load and moments for each element, with continuity constraints at the joints between any two beams.

After extensive algebra using these four equations, the parameter LWOR is expressed in terms of the forces and moments applied on the BHA. By substituting this in (6.1) and assuming constant terms for K_{anis}, end-moments (M_1 and M_2) and weight on the bit (WOB), as well as, that all stabilizers are located on the centerline of the borehole ($v_i = 0$ for $i = 2, \ldots, 5$), the final expression for lateral borehole propagation is given by the delay differential equation (DDE):

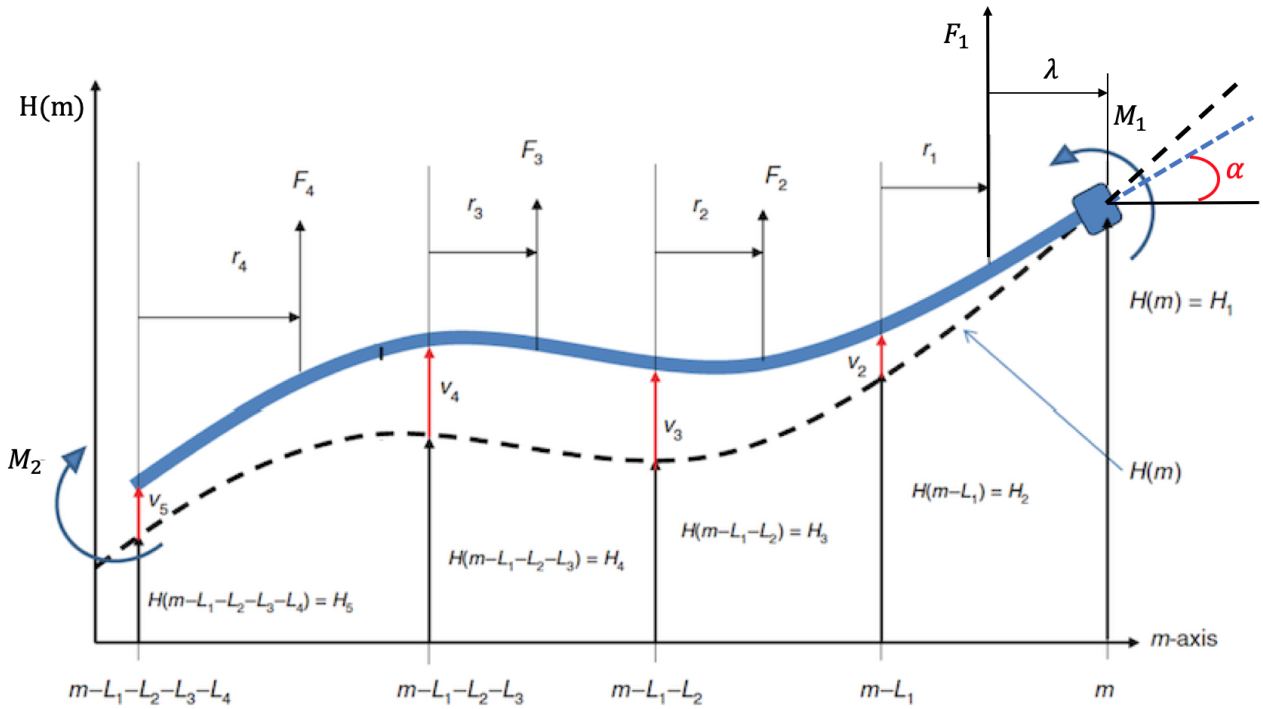

Figure 6.2: Generic BHA drilling system formulation based on lateral displacement H(m) with respect to the drilled distance m expressed in a locally tangent coordinate system [88]. The black dashed line represents the centerline of the borehole, while the blue line is the actual shaft shape and the dash blue line is the slope of the shaft. F_1 is the force applied by the steering mechanism (RSS) and is considered as the input of the system. F_2 up to F_4 and v_2 up to v_5 model the forces applied by the stabilizers to the sidewall of the borehole and the lateral displacement of each stabilizer with respect to the centerline, respectively. F_2 to F_4 and v_2 to v_5 are assumed zero for the present case study.

$$\frac{dH(m)}{dm} = -\left(\sum_{i=1}^{n_{stb}}(A_i \cdot H(m-\tau_i)) + \sum_{i=1}^{n_{beam}}(B_{w_i} \cdot w_i) + \sum_{i=1}^{2}(B_{M_i} \cdot M_i) + \sum_{i=1}^{n_{force}}(B_{F_i} \cdot F_{pad(m)_i})\right), \tag{6.3}$$

where A_i, B_{w_i}, B_{M_i} and B_{F_i} are the coefficient vectors computed by the BHA configuration, τ_i denotes the distance of stabilizer i with respect to the bit (e.g. $\tau_3 = L_1 + L_2 + L_3$), n_{stb} is the number of stabilizers under consideration, $n_{beam} = n_{stb} - 1$, and n_{force} is the number of (control and reaction) forces applied to the system. F_{pad} is the force applied to the sidewall of the borehole by the steering mechanism (shown as F_1 in Fig. 6.2). In this study we consider only one RSS system located at distance λ away from the bit and the effective stabilizers are the first four on the BHA (Fig. 6.2 shows up to the fourth stabilizer). Ideally, the supervisory trajectory control system should be embedded in the BHA to minimize communication delays. However, this method requires a high-performance processor

with insignificant size due to the limited space on the BHA, that can work efficiently at extreme environments with minimum power consumption. Also, it is important to note that the drilled distance measurement (m) is available only at the surface of drilling, which implies that the control unit at present must be considered to be located at the surface, since drilled distance (m) is the dependent variable in the control scheme.

For the purposes of this work, the sensors are assumed to be located at some distance from the bit, close to the rear stabilizers of the BHA. The sensors measure the tilt of the beam, which is related to the inclination, $dH(m)/dm$, instead of lateral displacement, $H(m)$, at the drill bit. Therefore, the borehole propagation DDE in (6.3) can be modified as follows:

$$\begin{aligned}\frac{dH(m)}{dm} = -\Bigg(&\sum_{i=1}^{n_{stb}}(A_i \cdot \frac{dH(m-\tau_i)}{dm} \cdot \tau_{n_{stb}-i}) + \sum_{i=1}^{n_{stb}-1}(B_{w_i} \cdot w_i) \\ &+ \sum_{i=1}^{2}(B_{M_i} \cdot M_i) + B_F \cdot F_{pad(m)}\Bigg),\end{aligned} \tag{6.4}$$

in which also the sum in the last term has been dropped, since in the present work only one steering mechanism (applying force F_{pad}) is considered.

6.2.1 Simplified Model

The general expression for lateral borehole propagation is transformed into an ODE and then reduced to a low order system in order to be computationally efficient for closed loop control formulation. The first step is to transform the DDE presented in (6.4) into state space form by considering the lateral displacement and inclination at the drill bit as the states of the system:

$$\mathbf{x}(m) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} H(m) \\ \frac{dH}{dm} \end{bmatrix}, \quad \dot{\mathbf{x}}(m) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{dH}{dm} \\ \frac{d^2H}{dm^2} \end{bmatrix},$$

$$\dot{\mathbf{x}}(m) = \begin{bmatrix} 0 & 1 \\ 0 & -A_1G_1 \end{bmatrix} \begin{bmatrix} x_1(m) \\ x_2(m) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -A_2G_2 \end{bmatrix} \begin{bmatrix} x_1(m-\tau_1) \\ x_2(m-\tau_1) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & -A_3G_3 \end{bmatrix} \begin{bmatrix} x_1(m-\tau_2) \\ x_2(m-\tau_2) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -A_4G_4 \end{bmatrix} \begin{bmatrix} x_1(m-\tau_3) \\ x_2(m-\tau_3) \end{bmatrix} \tag{6.5}$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} x_1(m-\tau_4) \\ x_2(m-\tau_4) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ BW & B_F \end{bmatrix} \begin{bmatrix} W \\ u_F \end{bmatrix},$$

$$\mathbf{y}(m) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(m) \\ x_2(m) \end{bmatrix}, \tag{6.6}$$

where $\dot{x}_2 = \frac{d^2H}{dm^2}$ represents the rate of inclination (curvature of the borehole trajectory calculated by differentiating (6.4)), G_1, G_2, G_3, and G_4 are constant coefficients depending on the structure of the BHA, and the parameter E is given by $E = A_1G_1 + A_2G_2 + A_3G_3 + A_4G_4$. BW and W represent the column vectors of B_{w_i} and of the spatial derivatives of w_i for $i = 1 \ldots n_{stb-1}$, respectively, and u_F is the spatial derivative of F_{pad}. M_1 and M_2 are assumed constant and therefore do not appear in (6.5).

In the second step, the state space form of the directional drilling system with delays (6.5) is transformed into an ODE by a rational approximation method. In this project the *Páde* approximation method is utilized [94]. The accuracy of this method can be improved by increasing the order of the approximation, however, doing so also increases the number of states of the ODE, which is not desirable in terms of computational efficiency of the RMPC method that will be employed. In order to minimize the approximation error while keeping the system's state number low, several values of the approximation order were evaluated. In the particular system under study a 9th order *Páde* approximation method is chosen, since it keeps the approximation error low while the number of states is not excessive (38 states). However, using a RMPC scheme with a system of this order, the online computational time is extremely high. Therefore, model reduction by balanced truncation is employed to reduce the number of states [95], from 38 states to an ODE with 3 states, for the specific BHA configuration studied in this work. The error of transforming the DDE system to a reduced order ODE is presented in Fig. 6.5 at Section 6.4. In order to compensate such approximations and unmodeled dynamics which may be left out either at the design process or after delays approximation

and model reduction, the directional drilling system is reformulated as a linear discrete-time system subject to feedback uncertainties and additive disturbances (see Section 2.5 and [15]), as shown in (6.7). It is assumed that the sample distance, λ, which is chosen to discretize the system, is equal to the distance between the bit and the RSS actuator, as shown in Fig. 6.2. An assumption of fully measurable states is not practically realistic since the states of the simplify model, after approximation and model reduction of the system, do not represent any physical quantities which can be measured. Using the available measurements (inclination angle ($dH(m)/dm$) and distance drilled (m)), the estimation strategy presented in Chapter 4 is employed followed by linear transformations to provide the state's value of the simplify model with a minimum error.

$$\begin{bmatrix} x_{k+1} \\ q_k \\ y_k \\ f_k \\ z_k \end{bmatrix} = \begin{bmatrix} A & B_u & B_w & B_p \\ C_q & D_{qu} & D_{qw} & 0 \\ C_y & D_{yu} & D_{yw} & D_{yp} \\ C_f & D_{fu} & D_{fw} & D_{fp} \\ C_z & D_{zu} & D_{zw} & D_{zp} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ w_k \\ p_k \end{bmatrix}, \quad p_k = \Delta q_k,$$
$$\begin{bmatrix} q_N \\ f_N \\ z_N \end{bmatrix} = \begin{bmatrix} \hat{C}_q & 0 \\ \hat{C}_f & \hat{D}_{fp} \\ \hat{C}_z & \hat{D}_{zp} \end{bmatrix} \begin{bmatrix} x_N \\ p_N \end{bmatrix}, \quad p_N = \Delta q_N, \tag{6.7}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, $w_k \in \mathbb{R}^{n_w}$, $f_k \in \mathbb{R}^{n_f}$, $z_k \in \mathbb{R}^{n_z}$, $p_k \in \mathbb{R}^{n_p}$ and $q_k \in \mathbb{R}^{n_q}$ are the state, input, output, disturbance, constraint, cost, and input and output uncertainty vectors, respectively, and where N is the prediction horizon. Similar to the previous chapters, $\Delta_k \in \mathscr{B}\mathbf{\Delta}$ where $\mathbf{\Delta} \subseteq \mathbb{R}^{n_p \times n_q}$ is a subspace that captures the uncertainty structure. All the coefficient matrices can be computed from the configuration of the BHA and its reduced order model approximation already explained. The constraints are imposed by physical factors, such as input actuator limits, or design preferences (see Section 6.4 for more details).

6.3 Tracking Control Approach

In this section, the overall RMPC methodology employed for the directional drilling tracking control problem is briefly summarized. To avoid repetition, the algebraic formulation of the estimation and control scheme presented in the previous chapters are not presented here. See Chapters 4 and 5 for full details on the robust moving horizon estimation and robust model predictive control schemes. On the other hand, the state feedback law based on an optimal RPI set problem [20], which is used to provide stability in the overall control algorithm, is briefly explained. The main advantages of using a combination of these controllers are insuring stability while robust properties are preserves with minimum of computation burden. At the end of this section an algorithm is presented to summarize the control strategy that is followed in the directional drilling application.

6.3.1 Offline controller using optimal RPI set

RPI sets found great success in robust analysis and synthesis of uncertain systems. In the case of RMPC, state feedback law based on RPI sets guarantees stability in uncertain systems and reduction of the computation time, since the feedback gain and the volume of the invariant set are computed off-line. A set is defined as an RPI set if the following statement is satisfied [96]:

Definition 6.3.1 *The set $Z \subset \mathbb{R}^n$ is a Robust Positively Invariant set of a system* (6.7) *if, by applying state-feedback control law $u = K_Z x$, then $A_\Delta Z \oplus B_\Delta K_Z Z \oplus B_w \mathscr{W} \subseteq Z$ is satisfied for all Δ, where $\oplus$ denotes the Minkowski sum. The definition $A_\Delta = A + B_p \Delta C_q$, $B_\Delta = B_u + B_p \Delta D_{qu}$ are used to simplify the notation and they represent the model feedback uncertainties of a system.*

Consequently, if the current state is inside the set Z, by applying the state feedback control law $u = K_Z x$ all the future states lie in the set Z in the presence of model uncertainties caracterised by A_Δ, B_Δ, and disturbances $w_k \in \mathscr{W}$. In this framework, a multi-objective problem is considered where the target is to maximize the volume of the polytopic invariant set Z of the form,

$$Z = \{x \in \mathbb{R}^n : -\mathbf{1} \leq Ex \leq \mathbf{1}\}, \tag{6.8}$$

where E is a matrix with appropriate dimension ($E \in \mathbb{R}^{n\times n}$) and $\mathbf{1}$ is column vectors with all entries one ($\mathbf{1} \in \mathbb{R}^n$). The problem can be expressed as an optimization problem as follows:

$$\begin{aligned} &\max_{K_Z, Z} \quad Volume(Z) \\ &subject\ to \begin{cases} Z \subseteq \mathscr{X}_I \\ K_Z Z \subseteq \mathscr{U}_I \\ A_\Delta Z \oplus B_\Delta K_Z Z \oplus B_w W \subseteq Z \end{cases} \end{aligned} \tag{6.9}$$

where $\mathscr{U}_I := \{u \in \mathbb{R}^{n_u} \ : \ \underline{u}_I \leq u \leq \overline{u}_I\}$ and $\mathscr{X}_I := \{x \in \mathbb{R}^{n_x} \ : \ \underline{x}_I \leq x \leq \overline{x}_I\}$ define the input and state constraints sets, respectively.

By considering the dynamic system (6.7), the invariant set constraint condition for the polytopic set (6.8) can be rewritten as:

$$-e_i^T \mathbf{1} \leq e_i^T E\left((A_\Delta + B_\Delta K_Z)x + B_w w\right) \leq e_i^T \mathbf{1}. \tag{6.10}$$

Since the polytopic invariant constraint is assumed to be symmetric, the relevant invariant conditions are computed using only the upper bound:

$$e_i^T E\left((A_\Delta + B_\Delta K_Z)x + B_w w\right) - e_i^T \mathbf{1} \leq 0.$$

Using the extended S-procedure (for further details refer to 2.4), the problem can be expressed as a convex LMI optimization problem, similar to the presentation in [20].

Following the description that is given for both offline and online controllers, the RMPC strategy that is proposed in this tracking control problem is summarized as follows:

Algorithm 5: RMPC controller strategy

Offline calculation:

(1) Compute the polytopic RPI set Z and the corresponding gain matrix K_Z, by solving the optimization problem described in Section 6.3.1.

(2) Create a look-up table for all $x_0 \in \mathscr{F}_x$ using Alg.3, where initial feasible solution $Y^*(S^*, G^*, \hat{K}^*)$ and $\tilde{Y}^*(\tilde{S}^*, \tilde{G}^*, \hat{K}^*)$ are stored.

Online calculation:

(1) *Initialization:*
At sample time $k = 0$, given y_0 calculate the initial bounds on x_0 as well as estimated state $\hat{x}_0$ based on equation 5.1, considering the maximum output-disturbance (see Remark 17).

(2) *Control scheme:*
If (the estimated state $\hat{x}_k$ lies inside the RPI set Z)
Switch to offline control and apply the state feedback controller $u = K_Z \hat{x}_k$.
Else
Solve LMI optimization problem for the output feedback RMPC express by equation (5.42) using upper and lower state bounds, and set as control action u_k the first value of the control sequence $\mathbf{u}(K, \mathbf{v})$(definition given at equation (5.8)).
end

(3) *Data collection:*
Update the vectors $\tilde{u}$, $\tilde{y}$ with the newly available input/output data from the previous step.

(4) *Estimation scheme:*
If $\tilde{N}_e < N_e$, increment $\tilde{N}_e$, else fix $\tilde{N}_e = N_e$. Then, using vectors $\tilde{u}$ and $\tilde{y}$ solve the LMI optimization problem stated in Theorem 4.3.1 to compute bounds of the current state x_k.

(5) *Termination:*
Using the state bounds from step (3), check if the state satisfy pre-specify terminal conditions. If so exit scheme, otherwise loop back to step (2).

6.4 Case Study

In this section a directional drilling application with an industrial BHA configuration is considered. Simulation results are presented to demonstrate the effectiveness of the proposed control strategy for directional drilling. At first we validate that the chosen BHA configuration is successfully described by the DDE model in (6.5). Validation is presented in Fig. 6.3 using curvature steady state values from a finite element industrial model provided by Schlumberger, which accurately describes the borehole's propagation with respect to distance drilled. In both models the same normalized input force (F_{pad}) is applied. As it can be seen, the steady-state curvature predicted by the DDE model converges to the curvature values calculated by the industrial model.

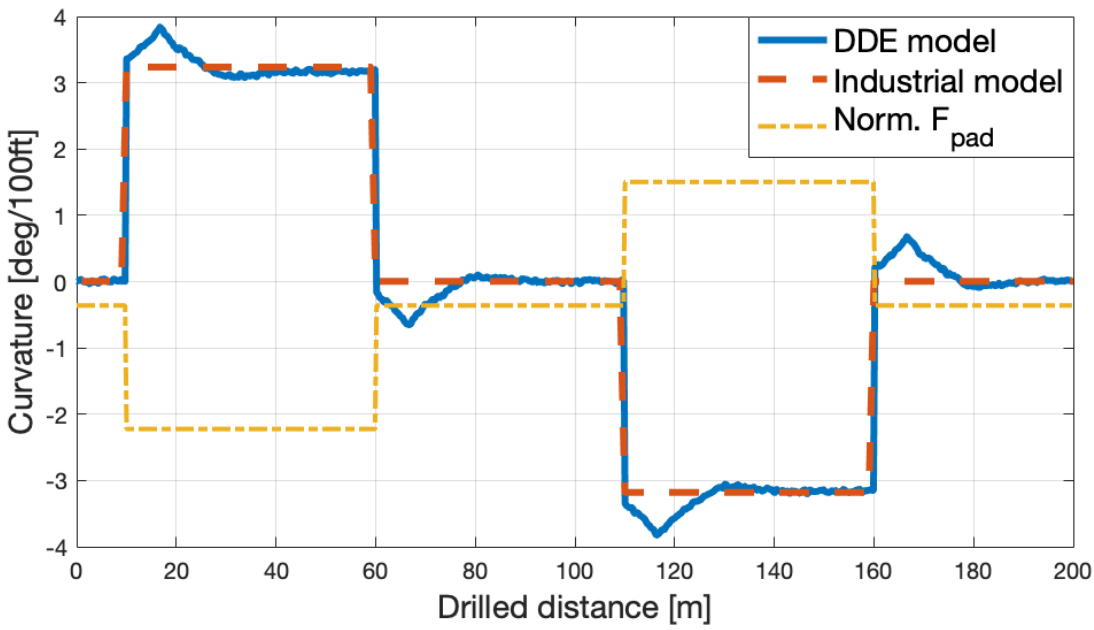

Figure 6.3: Open-loop response of curvature versus measured drilled distance predicted by the DDE model (6.5) and industrial model. The normalized F_{pad} input force applied to both models is also shown.

In order to test the model approximation strategy presented in Section 6.2.1, the same input signal (F_{pad} shown in Fig. 6.3) is also applied in an open-loop manner to the DDE and simplified ODE models. As shown in Figs. 6.4 and 6.5, the open-loop inclination responses for the two systems are very similar and the error between them remains below 0.3 degrees. Therefore, it is sufficient to steer the directional drilling system with minimum error, by developing closed-loop control using the simplified model and considering uncertainties on the model.

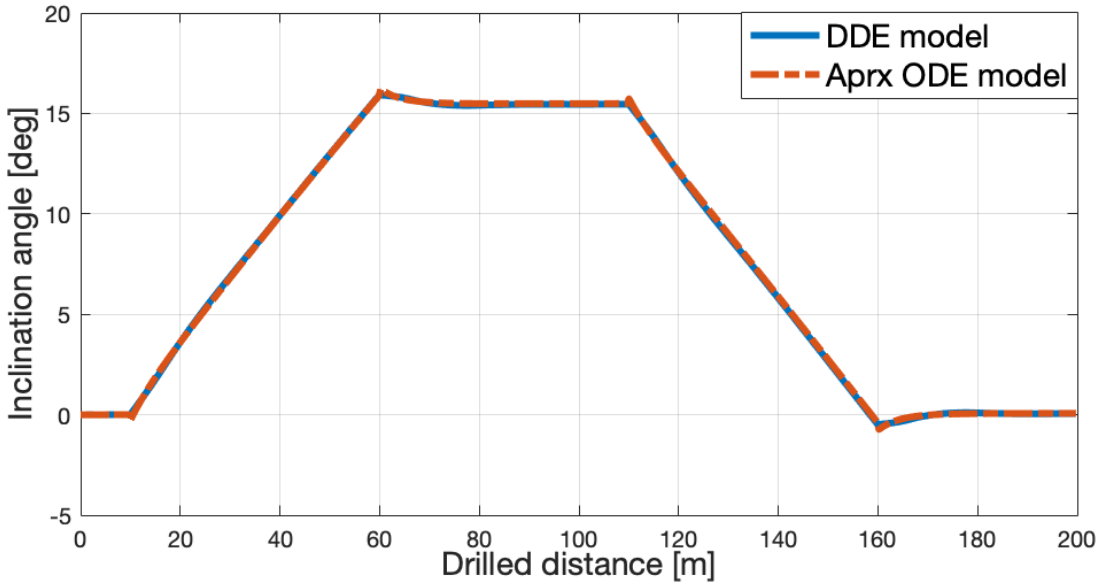

Figure 6.4: Open-loop inclination response versus measured drilled distance predicted by the DDE model (6.5) and the ODE simplified model. The normalized input force applied to both models is as shown in Fig. 6.3.

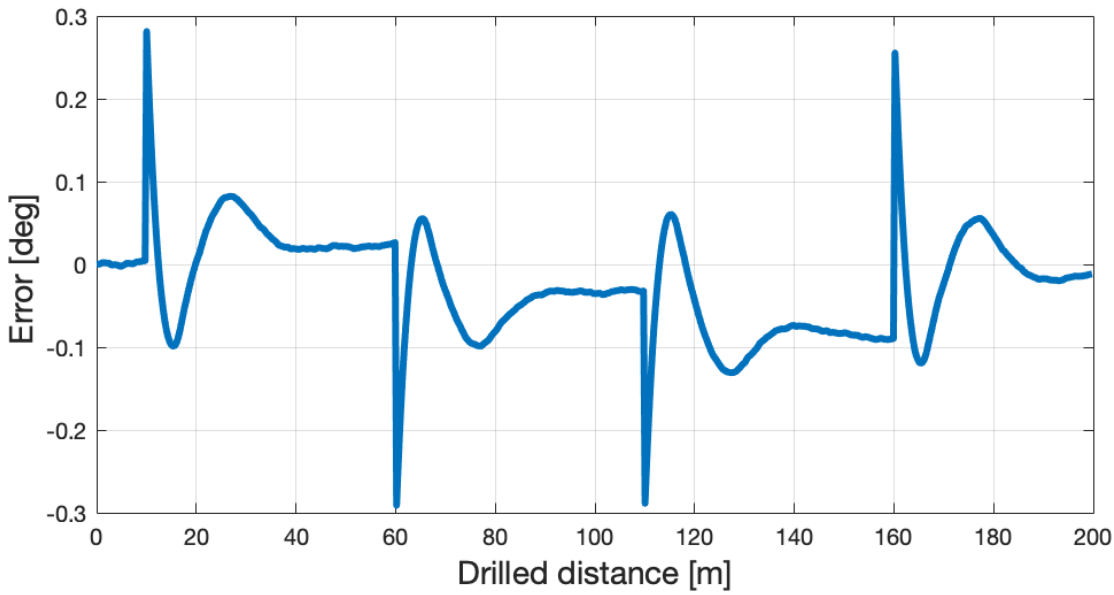

Figure 6.5: Inclination error between the responses of the DDE model (6.5) and the ODE simplified model, for the input force shown in Fig. 6.3.

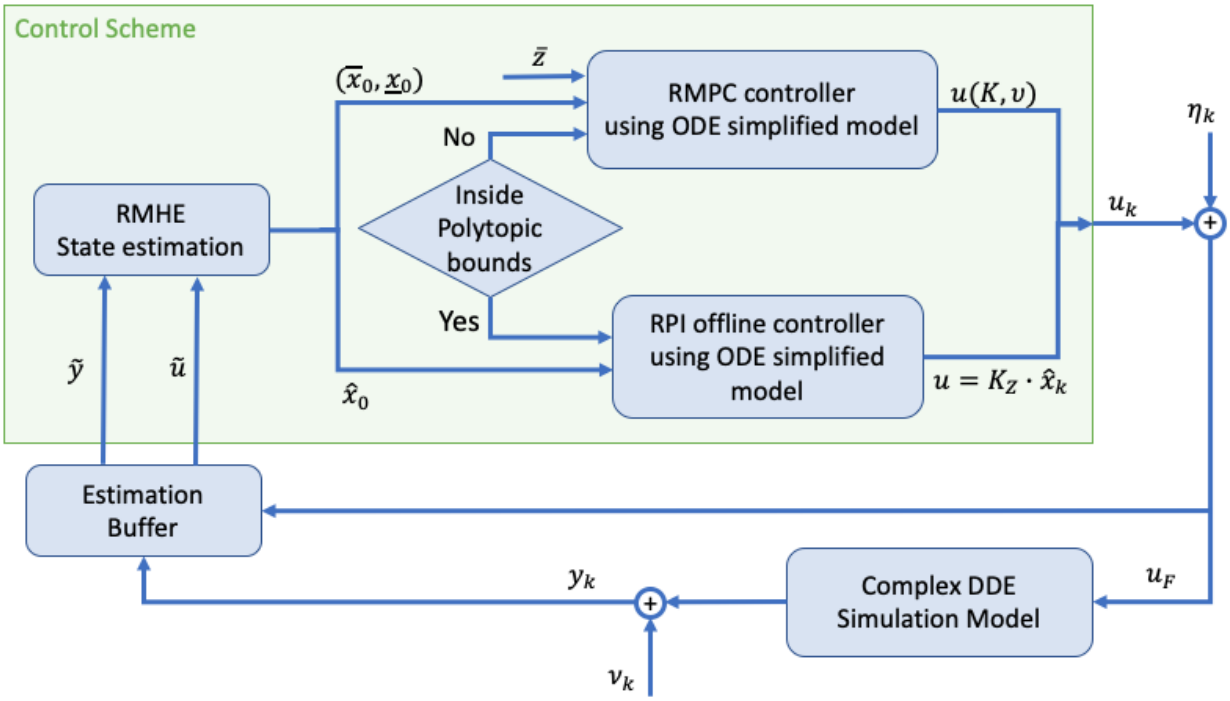

Figure 6.6: Block diagram of directional drilling closed-loop control and simulation scheme.

For the closed-loop control problem, the drilling system (DDE model in (6.5)) is required to track an inclination reference, while satisfying BHA bending limitations, which can be approximately translated to input constraints. Furthermore, it is assumed that the system is affected by disturbances at its input and output denoted by η_k and v_k, respectively. Input disturbance η_k aims to characterize the discrepancy between the desired and actual input value provided by the actuator due to physical losses and inability to measure the input directly by a sensor, and also to capture relevant signal noise. The disturbance signal η_k is assumed to be bounded by 10% of the maximum input constraint value (u_{max}) that is chosen by design at the RPMC formulation. Therefore, the distribution of η_k is assigned as white noise with zero mean and an appropriate standard deviation ($std = \frac{0.1 \cdot u_{max}}{3}$), such that 97% of the disturbance stays within the 10% of u_{max}. The output (inclination) disturbance v_k is due to inertial sensors accuracy and it is also white noise with zero mean and 0.33 standard deviation, such that 97% of the disturbance stays within 1 degree. The main design uncertainties arise from the knowledge that the WOB and K_{anis} values fluctuate during drilling. Therefore, to demonstrate that our proposed control scheme can successfully steer the system to the reference trajectory under the presence of uncertain variables that describe the system, we assign the parameters WOB and K_{anis} as uniformly varying through out the simulation while staying inside the following sets: $5000 \leq WOB \leq 15000$

(lbf) and $0.018 \leq K_{anis} \leq 0.043$. The bounded sets that describe the uncertainty of WOB and K_{anis} are selected based on past experimental data obtained by Schlumberger for a given rock formation set and known bit design [97]. The block diagram in Fig. 6.6 shows the closed-loop scheme of the controller (utilizing the simplified ODE model) and plant (complex DDE model in (6.5)), used for closed-loop simulations.

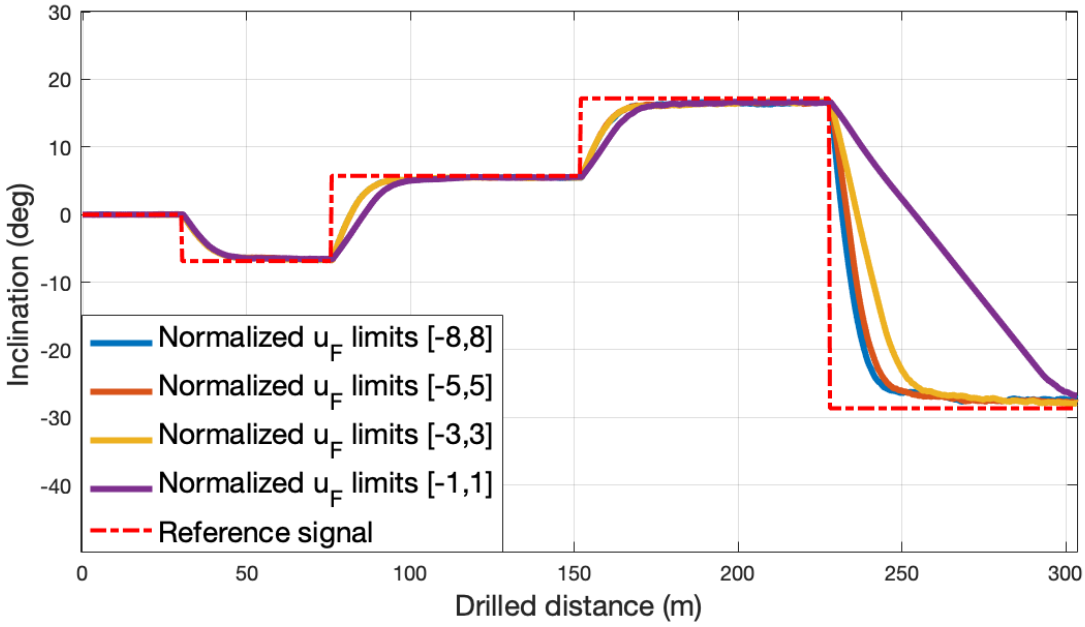

Figure 6.7: Closed-loop system inclination response versus drilled distance for a predefined inclination reference trajectory and various levels of normalized control input constraints, using the proposed closed-loop RMPC controller.

Figure 6.7 shows the inclination response of the closed-loop system, while tracking a given reference value, for various constraint levels of the control input, to assess the performance of the proposed control scheme. It can be seen that the inclination response is stable and the reference is tracked well despite the controller only uses a simplified model of the plant, and despite the presence of disturbances and constraints. It can also be seen that by tightening the input constraints, there is slower convergence to the steady-state value, as would be expected by the more limited flexibility of the BHA. Figure 6.8 shows the normalized control input for the case when its bounds are between [-3,3], demonstrating that constraints are satisfied.

Looking into details when the normalized input constraint limits are [-3,3], in figure 6.9 it can be seen that the tracking performance when the response is inside the RPI set Z (close to steady-state values)

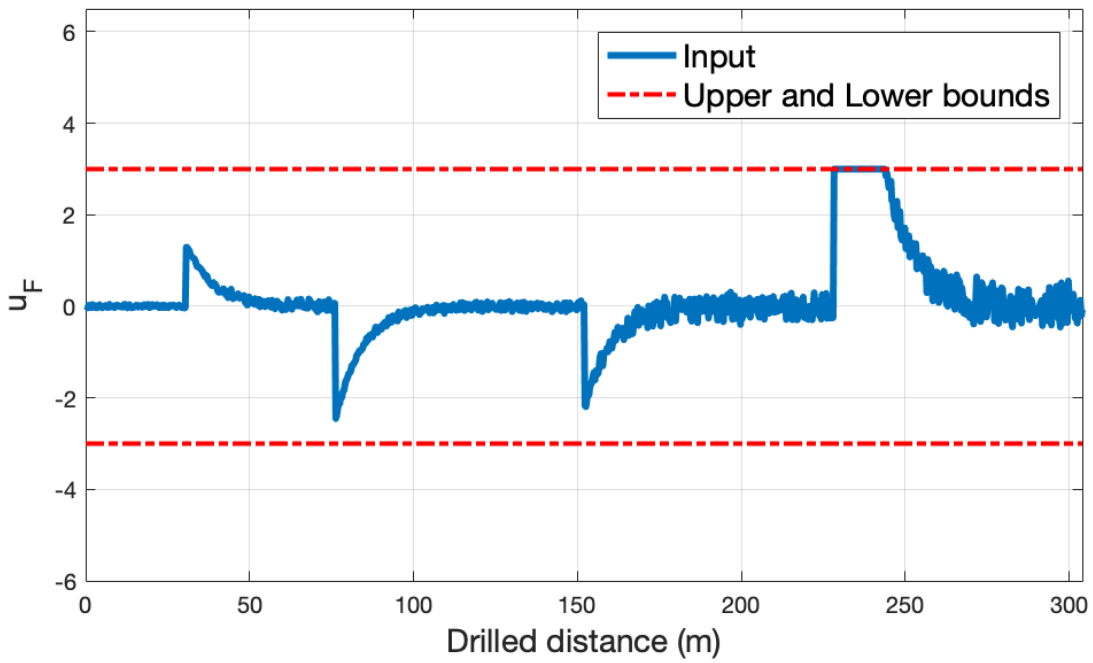

Figure 6.8: Normalized control input evolution versus drilled distance using the proposed closed-loop RMPC controller, when the normalized input constraint limits are [-3,3].

is decreased due to the offline computed controller, however robust performance is guaranteed due to the RPI set properties. The 2-D simulated drilling trajectory is also displayed in figure 6.10, where the blue line indicate the BHA trajectory and the two red lines define the constrains.

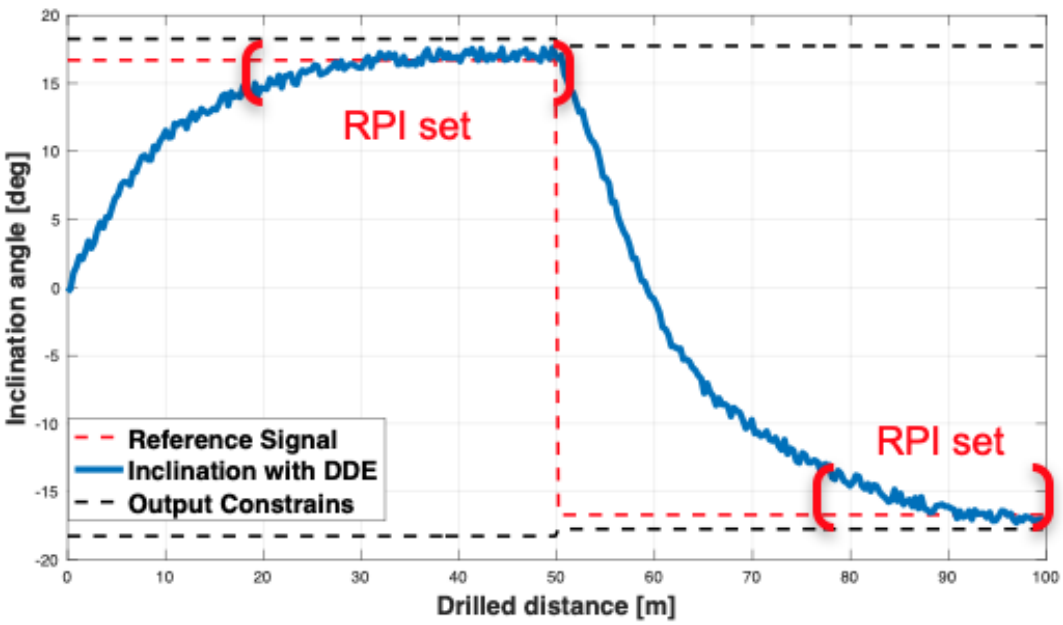

Figure 6.9: Closed-loop system inclination response versus drilled distance for a predefined inclination reference trajectory using the proposed closed-loop RMPC controller, when the input constraint limits are [-3,3].

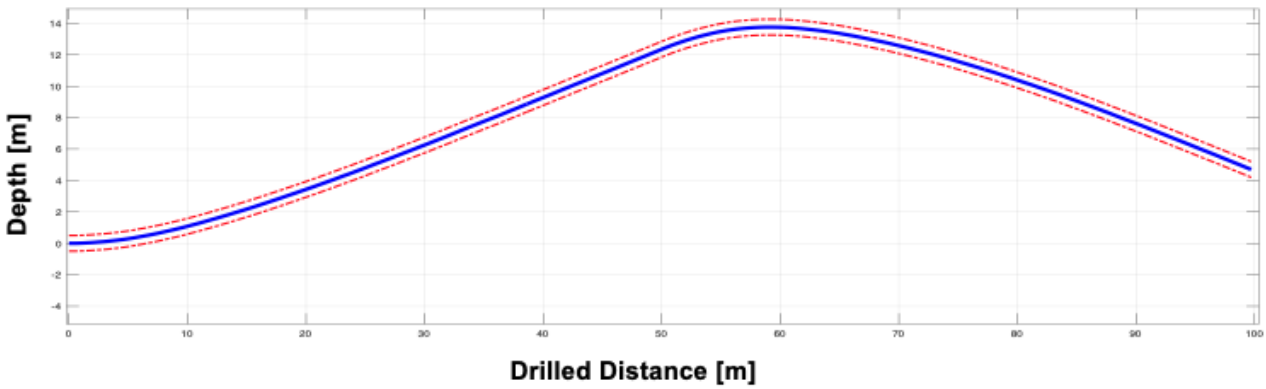

Figure 6.10: Simulated Drilling reference trajectory in 2D.

In terms of comparison of the proposed method with other MPC based control methods, a conventional MPC scheme has also been tested for the same closed-loop task shown in Fig. 6.6. However, the inclination response is found to diverge from the reference trajectory due to the systems mismatch (ODE and DDE) caused by the presence of disturbances and uncertainties. Consequently, the problem's constraints are violated and the solver is not able to provide a feasible solution to the problem. By comparing the online RMPC proposed in the present work with tube-based MPC described in [98], the tube-based MPC can effectively reduce the computing time of the optimization problem, however the main drawback of this approach is the additional conservatism on the optimization solution due to the state-observer estimation error that is calculated offline. Due to time limitations, comparison with tube-MPC is left for future work.

6.5 Synopsis

In this chapter, an effective way to simplify a directional drilling model which characterizes inclination and lateral displacement borehole assembly behavior is presented. On this basis, a robust model predictive control scheme is utilised that can effectively control the complex rotary steerable system using an uncertain system description, while system stability is preserved by the proposed robust positive invariant set. The work provides a promising method for effectively automating the inclination tracking control process in directional drilling applications, to replace the currently employed manual human-in-the-loop control processes.

Future work will focus on extending this work in 3-dimensional space by azimuth control, considering time delays on the steering input force, and spatial delay on the output signals at the formulation of the problem.

CHAPTER 7

Conclusions

In this chapter, we summarize the contributions of the thesis and also suggest some future research directions.

7.1 Summary of Thesis Achievements

The main objective of this research has been on the development of efficient algorithms - based on convex/LMI optimizations using Semidefinite Relaxation - for robust estimation and control subject to constraints, for norm-bounded structure feedback uncertain systems (presented in Subsection 2.5.2). Moreover, a particular interest of this thesis is the implementation of the robust algorithms developed in this research framework into the industrial application of the directional drilling tracking control problem. In this regard, the main contributions of the thesis are summarized below:

In Chapter 3, two strategies are proposed to reduce the computational complexity of state-feedback RMPC for linear-time-invariant discrete-time systems, subject to structured uncertainty and bounded

disturbances. In more details, a novel linearization procedure, based on the Elimination Lemma and S-Procedures, is developed to tackle the nonlinearity and nonconvexity associated with state-feedback RMPC, with minimal conservatism whilst resulting in a substantially lower computational burden as compared to similar methods in the literature. The approach requires initial feasible solutions to the nonlinear matrix inequalities, which however can be obtained offline. Further reduction in the computational complexity is achieved by the second developed algorithm that solves a single LMI for handling all the constraints in the RMPC problem. Through numerical examples, it has been demonstrated that the proposed algorithms improve the scalability of the control scheme to fast dynamic system due to its less demanding computational burden, without compromising its performance or robustness properties.

In Chapter 4, an investigation of the estimation problem based on past input/output data of linear discrete-time systems subject to model-uncertainties and bounded disturbances is presented. An online algorithm that computes estimates of the state along with tight bounds is suggested, while conservativeness is reduced and computation complexity is maintained low. Importantly, the proposed robust moving horizon estimation algorithm is formulated in a convex form and optimality is guaranteed at every sample time by solving an LMIs optimization problem. Finally, the effectiveness and superior performance of the proposed MHE algorithm as compared to state-of-the-art algorithms in the literature is demonstrated by an industrial process example. As shown in Chapter 5, the state bounds given by the Robust MHE scheme provide valuable information to develop an MPC based output-feedback control scheme with reduced conservativeness.

In Chapter 5 following similar steps as in Chapter 3, an OF-MPC algorithm based on LMIs is formulated for linear discrete-time systems subject to norm-bounded model-uncertainties, additive state disturbances and measurement noise. The novelty lies in the fact that the algorithm computes, online, both the output-feedback gain and a control perturbation through an LMI optimization. A significant reduction in the computational complexity of the control problem is achieved by a proposed algorithm that solves a single LMI for handling the constraints in the RMPC problem. Moreover, unlike most output-feedback MPC schemes from the literature which use a fixed (linear) state observer, the presented algorithm, in order to reduce conservativeness, uses a past input/output data window - in a

manner similar to Moving Horizon Estimation - to compute (tight) bounds on the current state which are then used within the control scheme. Note that the suggested control approach requires initial feasible solutions to the nonlinear matrix inequalities, which, however, can be obtained offline without compromising the performance of the controller. The effectiveness of the proposed techniques, in terms of robust control performance as well as estimation accuracy, is demonstrated through numerical examples taken from the literature.

Lastly in Chapter 6, an effective way to simplify a directional drilling model which characterizes inclination and lateral displacement borehole assembly behavior is presented. On this basis, a robust model predictive control scheme is utilised that can effectively control the complex rotary steerable system using an uncertain system description, while system stability is preserved by the proposed robust positive invariant set. The work provides a promising method for effectively automating the inclination tracking control process in directional drilling applications.

7.2 Future Research Directions

Based on the findings presented in this thesis, potential future contributions can be achieved by extending the theoretical framework involving uncertain systems or by its implementation to advance industrial applications. A list of potential research directions are outline here.

- **Theoretical extension**

    1. The focus of this research has been on developing robust algorithms using LMI optimization for LTI discrete-time systems subject to norm-bounded model uncertainties and additive input/output disturbances. However, it would be useful to extend these results for continues-time systems, systems subject to time/spatial delays (as shown in the directional drilling system in Chapter 6), and linear time-varying systems. In the case of an LTV system, the uncertainty matrix Δ becomes Δ_k and can be varied over time. However, the challenge is to convexify these problems in a minimally conservative manner.

2. Throughout this thesis, we have considered the uncertainty set Δ as a norm-bounded model-uncertainty. In theory, it should be possible to extend the results to formulate fault-tolerant RMPC schemes. In that case Δ can be taken to be a diagonal matrix where the diagonal elements can be considered as binary variables. So $\Delta_{ii} = 0$ could correspond to system faults such as an actuator failure or loss of signals. Research in this direction could yield some interesting results.

3. Computational complexity, as well as feasibility and stability of the overall control scheme, can be significantly improved by the use of RPI terminal sets, particularly in the context of RMPC. There exists a vast amount of literature for the computation of such sets in the case when all states are measured (see e.g. [99, 100] and the references therein). However, relatively few contributions have been made for the case when only noisy output measurements are available (see e.g. [101–103]). To the best of our knowledge, there are no algorithms in the literature for the computation of these so-called output-feedback RPI sets for systems subject to both norm-bounded uncertainty and disturbances. Therefore, a study on output-feedback RPI sets for uncertain systems could potentially lead to some interesting outcomes.

- **Application developments**

1. The case study presented in Chapter 6 mainly concentrates on the 2-dimensional (2-D) directional drilling problem, where the reference signal is defined by the inclination of the drilling bit. Future work will focus on extending this project to a more realistic 3-D space by the combination of inclination and azimuth control.

2. Following a very promising simulated result utilizing the developed robust control and estimation scheme in the directional drilling application, an ideal follow-up would be to embed these methods into a hardware component and test its performance on a full-scale prototype.

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