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The Dual Spherical Curves and Surfaces In Terms of Vectorial Moments

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Abstract

In the article, the parametric expressions of the dual ruled surfaces are expressed in terms of the vectorial moments of the Frenet vectors. The integral invariants of these surfaces are calculated. It is seen that the dual parts of these invariants can be stated by the real terms. Finally, we present examples of the ruled surfaces with bases such as helix and Viviani's curves.

Keywords: Closed ruled surface; Gauss curvature; Drall; Dual spherical curves; Dual angle of pitch; Vectorial moment

MSC 2010 No.: 53A04, 53A05

1. Introduction

A surface that includes at least one 1-parameter family of straight lines is the ruled surface in IR^3 (Do Carmo (1976)). By investigating one parameter spherical motion in dual indicatrice curves, Yaylı and Saraçoğlu (2011) obtain some special ruled surfaces. By considering Frenet and Blaschke frames, unit dual spherical curves are studied with ruled surfaces (Yaylı and Saraçoğlu

(2012)). Tunçer (2017) introduces the w-dual curves, constructed by the Frenet vectors of a regular curve, and gives some properties about these curves. Yanlin and Donghe (2016) define the notion of evolutes of dual spherical curves and find some results. In Oral and Kazaz (2017), the authors examine slant ruled surfaces as dual. Using the vectorial moment, the ruled surface drawn by the dual pole curve is examined in Şenyurt and Çalışkan (2020). İncesu (2021) finds interesting results for ruled surfaces corresponding to Bézier curves on the dual sphere and explains these results with examples. Aslan et al. (2021) examines the ruled surface with quaternions in both real and dual spaces, and the new expressions are obtained on the transforms of the geometric shapes. There exist studies by many authors on dual invariants of the ruled surface (Hacısalihoğlu (1972); Gürsoy (1990b); Güven et al. (2011); Rashad (2002); Rashad (2003)).

In this study, the ruled surface is examined in terms of dual spaces, and the dual parts of the corresponding ruled surface are expressed in real terms. The new results are found about the dual invariants of the ruled surface. In the examples, ruled surfaces drawn by special curves such as helix and Viviani's curves are given, and the invariants of these surfaces are examined.

2. Main Results

Let $f(\varpi)$, $g(\varpi)$ and $h(\varpi)$ be at least C^3- functions. $\alpha(\varpi)$ can be written in the form of

$$\alpha(\varpi) = f(\varpi)T(\varpi) + g(\varpi)N(\varpi) + h(\varpi)B(\varpi), \tag{1}$$

as a linear combination of the Frenet vectors. By differentiating both side of (1), we obtain

$$f'(\varpi) - g(\varpi)\kappa(\varpi) = 1,\ h'(\varpi) + g(\varpi)\tau(\varpi) = 0,\ g'(\varpi) + f(\varpi)\kappa(\varpi) - h(\varpi)\tau(\varpi) = 0. \tag{2}$$

The geometric location of $\widehat{T} = T + \varepsilon T^*$, $\widehat{N} = N + \varepsilon N^*$ and $\widehat{B} = B + \varepsilon B^*$ vectors draws closed curves on the dual sphere. These curves are called dual spherical curves. The dual expressions of the closed ruled surfaces are

$$\begin{aligned}
&\psi_{\widehat{T}}(\varpi, v) = \beta_T(\varpi) + vT(\varpi),\ \beta_T(\varpi) = T(\varpi) \wedge T^*(\varpi),\\
&\psi_{\widehat{N}}(\varpi, v) = \beta_N(\varpi) + vN(\varpi),\ \beta_N(\varpi) = N(\varpi) \wedge N^*(\varpi),\\
&\psi_{\widehat{B}}(\varpi, v) = \beta_B(\varpi) + vB(\varpi),\ \beta_B(\varpi) = B(\varpi) \wedge B^*(\varpi).
\end{aligned} \tag{3}$$

Vectorial moments of T, N, B are given, respectively, by

$$T^* = \alpha \wedge T = hN - gB,\quad N^* = \alpha \wedge N = -hT + fB,\quad B^* = \alpha \wedge B = gT - fN. \tag{4}$$

Using (4) in (3), we obtain ruled surfaces in terms of linear combination of Frenet vectors as follows:

$$\psi_{\widehat{T}}(\varpi, v) = T(\varpi) \wedge T^*(\varpi) + vT(\varpi) = vT(\varpi) + g(\varpi)N(\varpi) + h(\varpi)B(\varpi),$$

$$\psi_{\widehat{N}}(\varpi, v) = N(\varpi) \wedge N^*(\varpi) + vN(\varpi) = f(\varpi)T(\varpi) + vN(\varpi) + h(\varpi)B(\varpi),$$

$$\psi_{\widehat{B}}(\varpi, v) = B(\varpi) \wedge B^*(\varpi) + vB(\varpi) = f(\varpi)T(\varpi) + g(\varpi)N(\varpi) + vB(\varpi).$$

Theorem 2.1.

Let $\psi_{\widehat{T}}(\varpi, v), \psi_{\widehat{N}}(\varpi, v)$ and $\psi_{\widehat{T}}(\varpi, v)$ be the ruled surfaces corresponding to the $(\widehat{T})$, $(\widehat{N})$ and $(\widehat{B})$ curves. Dralls of the closed ruled surfaces are given by

$$P_{\widehat{T}} = 0,\ P_{\widehat{N}} = \frac{\tau}{\kappa^2 + \tau^2},\ P_{\widehat{B}} = \frac{1}{\tau}\cdot$$

Proof:

We know that the drall is calculated by

$$P_{\widehat{T}} = \frac{det((T \wedge T^*)', T, T')}{\|T'\|^2}\cdot \tag{5}$$

Using (2), it can be written that

$$(T \wedge T^*)' = (gN + hB)' = -g\kappa T - f\kappa N.$$

If above value is substituted into (5), we can write

$$\begin{aligned} P_{\widehat{T}} &= \frac{det((T \wedge T^*)', T, T')}{\|T'\|^2} \\ &= \frac{1}{\kappa^2} \begin{vmatrix} -g\kappa & -f\kappa & 0 \\ 1 & 0 & 0 \\ 0 & \kappa & 0 \end{vmatrix} \\ &= 0. \end{aligned}$$

Similarly, with the values of

$$(N \wedge N^*)' = (fT + hB)' = (1 + g\kappa)T - (-f\kappa + h\tau)N + (-g\tau)B,$$

$$(B \wedge B^*)' = (fT + gN)' = T + h\tau N + g\tau B,$$

dralls of the surfaces corresponding to the $(\widehat{N})$ curve and $(\widehat{B})$ are, respectively,

$$
\begin{aligned}
P_{\widehat{N}} &= \frac{det((N \wedge N^*)', N, N')}{\|N'\|^2} \\
&= \frac{1}{\kappa^2+\tau^2}\begin{vmatrix} 1+g\kappa & f\kappa - h\tau' & -g\tau \\ 0 & 1 & 0 \\ -\kappa & 0 & \tau \end{vmatrix} \\
&= \frac{\tau}{\kappa^2+\tau^2},
\end{aligned}
$$

$$
\begin{aligned}
P_{\widehat{B}} &= \frac{det((B \wedge B^*)', B, B')}{\|B'\|^2} \\
&= \frac{1}{\tau^2}\begin{vmatrix} 1 & h\tau & g\tau \\ 0 & 0 & 1 \\ 0 & -\tau & 0 \end{vmatrix} \\
&= \frac{1}{\tau}.
\end{aligned}
$$

■

Theorem 2.2.

Let $\psi_{\widehat{T}}(\varpi, v), \psi_{\widehat{N}}(\varpi, v)$ and $\psi_{\widehat{T}}(\varpi, v)$ be the ruled surfaces corresponding to the $(\widehat{T})$, $(\widehat{N})$ and $(\widehat{B})$ curves. The dual angles of pitch (d.a.p.) of closed ruled surfaces are

$$
\begin{aligned}
\Lambda_{\widehat{T}} &= \lambda_T - \varepsilon(g\lambda_B + \oint f'), \\
\Lambda_{\widehat{N}} &= -\varepsilon(h\lambda_T - f\lambda_B + \oint g'), \\
\Lambda_{\widehat{B}} &= \lambda_B - \varepsilon(-g\lambda_T + \oint h').
\end{aligned}
$$

Proof:

The d.a.p. of the surface $(\widehat{T})$ is

$$
\begin{aligned}
\Lambda_{\widehat{T}} &= -\langle D, \widehat{T}\rangle = -\langle d + \varepsilon d^*, T + \varepsilon(hN - gB)\rangle \\
&= -\oint \tau - \varepsilon(-g\oint \kappa + \oint f') \\
&= \lambda_T - \varepsilon(g\lambda_B + \oint f').
\end{aligned}
$$

Similarly, d.a.p. of the surfaces corresponding to the $(\widehat{N})$ and $(\widehat{B})$ are

$$\begin{aligned}\Lambda_{\widehat{N}} &= -\langle D, \widehat{N}\rangle = -\langle d+\varepsilon d^*, N+\varepsilon(-hT+fB)\rangle\\ &= -\varepsilon(h\lambda_T - f\lambda_B + \oint g'),\end{aligned}$$

$$\begin{aligned}\Lambda_{\widehat{B}} &= -\langle D, \widehat{B}\rangle = -\langle d+\varepsilon d^*, B+\varepsilon(gT-fN)\rangle\\ &= \lambda_B - \varepsilon(-g\lambda_T + \oint h').\end{aligned}$$

■

Example 2.1.

Let $\alpha(\varpi) = \frac{1}{\sqrt{2}}(-\cos\varpi, -\sin\varpi, \varpi)$ be a circular helix curve. Then, we obtain

$$\begin{aligned}T(\varpi) &= \frac{1}{\sqrt{2}}(\sin\varpi, -\cos\varpi, 1),\ \ N(\varpi) = (\cos\varpi, \sin\varpi, 0),\\ B(\varpi) &= \frac{1}{\sqrt{2}}(-\sin\varpi, \cos\varpi, 1),\\ \kappa(\varpi) &= \frac{1}{\sqrt{2}},\ \ \tau(\varpi) = \frac{1}{\sqrt{2}}.\end{aligned}$$

Considering the dual expression of a ruled surface, we get

$$\begin{aligned}\psi_{\widehat{T}}(\varpi, v) &= -\frac{\sqrt{2}}{4}\Big(\varpi\sin\varpi + 2\cos\varpi - 2v\sin\varpi, -\varpi\cos\varpi + 2\sin\varpi + 2v\cos\varpi, -\varpi - 2v\Big),\\ \psi_{\widehat{N}}(\varpi, v) &= (v\cos\varpi, v\sin\varpi, \frac{1}{\sqrt{2}}\varpi),\\ \psi_{\widehat{B}}(\varpi, v) &= \frac{\sqrt{2}}{4}\Big(w\sin\varpi - 2\cos\varpi - 2v\sin\varpi, -w\cos\varpi - 2\sin\varpi + 2v\cos\varpi, \varpi + 2v\Big).\end{aligned}$$

These ruled surfaces are shown in Figure 1. Let us find the functions $f(\varpi), g(\varpi), h(\varpi)$. From the equation (1), we can write

$$\begin{aligned}-\cos\varpi &= f(\varpi)\sin\varpi + g(\varpi)\sqrt{2}\cos\varpi - h(\varpi)\sin\varpi,\\ -\sin\varpi &= -f(\varpi)\cos\varpi + g(\varpi)\sqrt{2}\sin\varpi + h(\varpi)\cos\varpi,\\ \varpi &= f(\varpi) + g(\varpi).\end{aligned}$$

The solutions of $f(\varpi), g(\varpi), h(\varpi)$ are given by

$$f(\varpi) = \frac{\varpi}{2},\quad g(\varpi) = -\frac{1}{\sqrt{2}},\quad h(\varpi) = \frac{\varpi}{2}.$$

By taking into account consideration of the above equation, d.a.p. are given by

$$\begin{aligned}\Lambda_{\widehat{T}} &= \lambda_T + \varepsilon(\frac{1}{\sqrt{2}}\lambda_B - \frac{1}{2}\oint d\varpi)\\ &= \lambda_T + \varepsilon(\frac{1}{\sqrt{2}}\lambda_B - \frac{1}{2}L_T),\end{aligned}$$

$$\Lambda_{\widehat{N}} = \varepsilon\frac{\varpi}{2}(-\lambda_T + \lambda_B),$$

$$\begin{aligned}\Lambda_{\widehat{B}} &= \lambda_B - \varepsilon(\frac{1}{\sqrt{2}}\lambda_T + \frac{1}{2}\oint d\varpi)\\ &= \lambda_B - \varepsilon(\frac{1}{\sqrt{2}}\lambda_T + \frac{1}{2}L_T).\end{aligned}$$

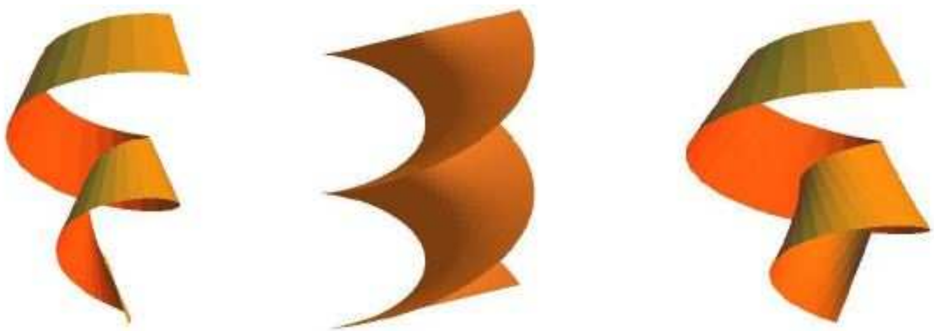

Figure 1. The figures (i), (ii) and (iii) show closed ruled surfaces corresponding to the $(\widehat{T})$, $(\widehat{N})$ and $(\widehat{B})$ dual spherical curves, respectively. Base curves of these surfaces are helix curve.

Example 2.2.

Let $\alpha(\varpi) = (\frac{4}{5}\cos\varpi, 1-\sin\varpi, -\frac{3}{5}\cos\varpi)$ be a curve. Then, it is easy to show that

$$T(\varpi) = \left(-\frac{4}{5}\sin\varpi, -\cos\varpi, \frac{3}{5}\sin\varpi\right),\ N(\varpi) = \left(-\frac{4}{5}\cos\varpi, \sin\varpi, \frac{3}{5}\cos\varpi\right),$$
$$B(\varpi) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right),\ \kappa(\varpi) = 1,\ \tau(\varpi) = 0.$$

Considering the dual expression of a ruled surface, we obtain closed ruled surfaces as

$$\begin{aligned}\psi_{\widehat{T}}(\varpi, v) = \Big(&-\frac{4}{5}\big((\sin\varpi - 1)\cos\varpi + v\sin\varpi\big), (\sin\varpi - 1)\sin\varpi - v\cos\varpi,\\ &\frac{3}{5}((\sin\varpi - 1)\cos\varpi + v\sin\varpi)\Big),\end{aligned}$$
$$\psi_{\widehat{N}}(\varpi, v) = \Big(\frac{4}{5}\cos\varpi\sin\varpi - v\frac{4}{5}\cos\varpi, \cos^2\varpi + v\sin\varpi, -\frac{3}{5}\cos\varpi\sin\varpi + v\frac{3}{5}\cos\varpi\Big),$$
$$\psi_{\widehat{B}}(\varpi, v) = \Big(\frac{4}{5}\cos^2\varpi - v\frac{3}{5}, 1 - \sin\varpi, -\frac{3}{5}\cos^2\varpi - v\frac{4}{5}\Big).$$

These ruled surfaces are shown in Figure 2. Let us find the functions $f(\varpi), g(\varpi), h(\varpi)$. From Equation (1), we can write

$$\begin{aligned} 4\cos\varpi &= -4f(\varpi)\sin\varpi - 4g(\varpi)\cos\varpi - 3h(\varpi), \\ 1-\sin\varpi &= -f(\varpi)\cos\varpi + g(\varpi)\sin\varpi, \\ -3\cos\varpi &= 3f(\varpi)\sin\varpi + 3g(\varpi)\cos\varpi - 4h(\varpi). \end{aligned}$$

The solutions of $f(\varpi), g(\varpi), h(\varpi)$ are given by

$$f(\varpi) = -\cos\varpi,\ \ g(\varpi) = \sin\varpi - 1,\ \ h(\varpi) = 0.$$

By taking into account the above equation, d.a.p. of closed ruled surfaces are

$$\begin{aligned} &\Lambda_{\widehat{T}} = \lambda_T - \varepsilon((\sin\varpi - 1)\lambda_B + \oint \cos\varpi),\ \ \Lambda_{\widehat{N}} = -\varepsilon(\cos\varpi\lambda_B + \oint \cos\varpi), \\ &\Lambda_{\widehat{B}} = \lambda_B - \varepsilon((1-\sin\varpi)\lambda_T + c), \end{aligned}$$

where c is an arbitrary constant known as the integration constant.

(i) (ii) (iii)

Figure 2. The figures (i), (ii) and (iii) show closed ruled surfaces corresponding to the $(\widehat{T})$, $(\widehat{N})$ and $(\widehat{B})$ dual spherical curves, respectively.

Example 2.3.

The Viviani's curve is formed by the intersection of a cylinder and a sphere. It is parametrized by

$$\alpha(\iota) = \left(a(1+\cos\iota), a\sin\iota, 2a\sin\frac{\iota}{2}\right).$$

Here, 2a is the radius of the sphere (Gray et al. (2017)). The expression for the Frenet invariants of Viviani's curve are given by

$$T(\iota)=\left(-\frac{\sqrt{2}\sin\iota}{\sqrt{3+\cos\iota}},\frac{\sqrt{2}\cos\iota}{\sqrt{3+\cos t}},\frac{\sqrt{2}\cos\frac{\iota}{2}}{\sqrt{3+\cos\iota}}\right),$$

$$N(\iota)=\left(\frac{-3-12\cos\iota-\cos 2\iota}{\sqrt{88\cos\iota+162+6\cos 2\iota}},\frac{-12\sin\iota-\sin 2\iota}{\sqrt{88\cos\iota+162+6\cos 2\iota}},\frac{2\sqrt{2}\sin\frac{\iota}{2}}{\sqrt{81+44\cos t+3\cos 2\iota}}\right),$$

$$B(\iota)=\left(\frac{3\sin\frac{\iota}{2}+\sin\frac{3\iota}{2}}{\sqrt{26+6\cos\iota}},\frac{-2\sqrt{2}\cos^3\frac{\iota}{2}}{\sqrt{13+3\cos\iota}},\frac{2\sqrt{2}}{\sqrt{13+3\cos\iota}}\right),$$

$$\kappa(\iota)=\frac{\sqrt{3\cos\iota+13}}{a(\cos\iota+3)^{\frac{3}{2}}},\quad \tau(\iota)=\frac{6\cos\frac{\iota}{2}}{3a\cos\iota+13a}.$$

Considering the dual expression of a ruled surface, we can write closed ruled surfaces as

$$\psi_{\widehat{T}}(\iota,v)=\left(2a\cos^2\frac{\iota}{2}-v\frac{\sqrt{2}\sin\iota}{\sqrt{3+\cos\iota}},a\sin\iota+v\frac{\sqrt{2}\cos\iota}{\sqrt{3+\cos\iota}},2a\sin\frac{\iota}{2}+v\frac{\sqrt{2}\cos\frac{\iota}{2}}{\sqrt{3+\cos\iota}}\right),$$

$$\begin{aligned}\psi_{\widehat{N}}(\iota,v)=\Bigg(&\frac{-2a\left(2\cos^8\frac{\iota}{2}+9\cos^6\frac{\iota}{2}+4\cos^4\frac{\iota}{2}-17\cos^2\frac{\iota}{2}+2\right)}{3\cos^4\frac{\iota}{2}+8\cos^2\frac{\iota}{2}+5}-v\frac{3+12\cos\iota+\cos 2\iota}{\sqrt{88\cos\iota+162+6\cos 2\iota}},\\ &\frac{-2a\sin\iota\left(\cos^6\frac{\iota}{2}+5\cos^4\frac{\iota}{2}+5\cos^2\frac{\iota}{2}-5\right)}{3\cos^4\frac{\iota}{2}+8\cos^2\frac{\iota}{2}+5}-v\frac{12\sin\iota+\sin 2\iota}{\sqrt{88\cos\iota+162+6\cos 2\iota}},\\ &\frac{32a\left(\cos^4\frac{\iota}{2}+3\cos^2\frac{\iota}{2}+1\right)\sin\frac{\iota}{2}}{22\cos\iota+39+3\cos^2\iota}+v\frac{2\sqrt{2}\sin\frac{\iota}{2}}{\sqrt{81+44\cos\iota+3\cos 2\iota}}\Bigg),\end{aligned}$$

$$\begin{aligned}\psi_{\widehat{B}}(\iota,v)=\Bigg(&\frac{4a\left(\cos^6\frac{\iota}{2}+3\cos^4\frac{\iota}{2}+\cos^2\frac{\iota}{2}-1\right)}{3\cos^2\frac{\iota}{2}+5}+v\frac{3\sin\frac{\iota}{2}+\sin\frac{3\iota}{2}}{\sqrt{26+6\cos\iota}},\\ &\frac{a(4\sin\frac{3\iota}{2}\cos^3\frac{\iota}{2}(\cos\iota+1)+3\sin 2\iota\cos^2\frac{\iota}{2}+13\sin\iota\cos^2\frac{\iota}{2}}{26+6\cos\iota}\\ &+\frac{6\sin\frac{3\iota}{2}\sin\iota\sin\frac{\iota}{2}-\cos^2\frac{3\iota}{2}\sin\iota+26\sin\iota}{26+6\cos\iota}-v\frac{2\sqrt{2}\cos^3\frac{\iota}{2}}{\sqrt{13+3\cos\iota}},\\ &\frac{2a\left(\cos^2\frac{\iota}{2}+1\right)\sin\frac{\iota}{2}}{3\cos^2\frac{\iota}{2}+5}+v\frac{2\sqrt{2}}{\sqrt{13+3\cos\iota}}\Bigg).\end{aligned}$$

These closed ruled surfaces are shown in Figure 3.

(i) (ii) (iii)

Figure 3. The figures (i), (ii) and (iii) show ruled surfaces corresponding to the $(\widehat{T})$, $(\widehat{N})$ and $(\widehat{B})$ dual spherical curves, respectively. Base curves of these surfaces are Vivian's curve.

3. Conclusion

In this study, the ruled surfaces corresponding to dual curves are written as a linear combination of Frenet vectors. According to the conditions, the dralls and the dual angles of the pitch of the surfaces are calculated. It is seen that the dual parts of the dual angles of pitch can be written in real terms. The fact that dual elements can be expressed in real terms will be a guide in the studies about dual space.

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