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On the Approximation of Eventual Periodicity Of Linearized KdV Type Equations using RBF-PS Method

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Abstract

Water wave propagation phenomena still attract the interest of researchers from many areas and with various objectives. The dispersive equations, including a large body of classes, are widely used models for a great number of problems in the fields of physics, chemistry and biology. For instance, the Korteweg-de Vries (KdV) equation is one of the famous dispersive wave equation appeared in the theories of shallow water waves with the assumption of small wave-amplitude and large wave length, also its various modifications serve as the modeling equations in several physical problems. Another interesting qualitative characteristic of solutions of some dispersive wave equations indicated through experiments that are connected with their large-time behavior termed as Eventual Time Periodicity which is exhibited by solutions of initial-boundary-value problems (IBVPs henceforth). Laboratory experiments in a channel with a flap-type or piston-type wave maker mounted at one end of a channel exposed this interesting phenomena. Here in this study we numerically investigate the solutions periodicity for linearized KdV type equations on a finite (bounded) domain with periodic boundary conditions using meshfree technique known as Radial basis function pseudo spectral (RBF-PS) method.

Keywords: RBF Meshless Method; RBF-PS Method; KdV type equations; Eventual Periodicity

MSC 2010 No.: 34K28, 35G61, 35Q53, 34A08, 34K13

1. Introduction

The Korteweg-de Vries (KdV) equation was first derived by Boussinesq in 1870, and then rediscovered by Korteweg and de Vries in 1895 (Korteweg (1895)) with the assumption of small wave-amplitude and large wave length. In numerous dispersive and dissipative nonlinear physical systems, the evolution of long wave approximations (one-dimensional) in many physical setting described by KdV type equation. The KdV equation and its various modifications serve as the modeling equations in several physical problems; see, for example, Pava (2009), Sulem (1999), Newell (1992), Ablowitz (1979), Brenner (1981), Tao (2006), Uddin (2016), Biazar (2021) and Haghighi (2013). The analytical and numerical study on classical and fractional KdV equations has resulted in a rich area of research in applied mathematics, physics and other related disciplines throughout the last century (look at the references, for example, Bona (1989), Fornberg (1978), Al (2001), Kumar (2015), Alquran (2011), Momani (2005), Wang (2006), Wang (2007), El (2015) and Khattak (2008)).

The eventual periodicity of initial-boundary-value problem (IBVP) solutions is another specific qualitative characteristic of some dispersive wave equations solutions that has been experimentally demonstrated and is related to their large-time behaviour. A piston-type or paddle-type wave maker fitted at one end of a channel in laboratory experiments show this attractive event. When the wavemaker periodically oscillates with a period $\mathrm{T}_0 > 0$, it is observed that the amplitude of the wave becomes periodic of the same period at each point along the channel after some time (Bona (1981), Bona (1989)). This interesting phenomena of eventual periodicity investigated by Bona and Wu (Bona (2009)) and has been elaborated previously in separate studies for the generalized Benjamin-Bona-Mahony equation denoted by BBM (Benjamin et al. (1972)) and KdV equations and also for their dissipative counterparts respectively which include Burger-type term (for more details look at the references Shen (2007), Usman (2007), Usman (2009), Uddin (2020), Uddin (2021), Uddin (2022a), Uddin (2022b), Al (2018), Jan (2021), Hussain (2021) and Jan (2022)).

The radial basis function (RBF) meshfree approach is the most utilized tool in the field of multivariate approximation theory with no meshing or minimum of meshing for which the traditionally used mesh-based methods are not suited like Finite volumes, Finite differences, Finite elements, Moving least square, Element free galerkin, Point interpolation method, Reproducing kernel particle method and Boundary element free method. RBF approximation method is a generalized refinement of Multiquadric (MQ) method. The MQ RBF has a rich history of theoretical development and application first studied and developed by Rolland L. Hardy (Hardy (1971), Hardy (1990)). Researchers of several distinct areas initiated performing the MQ method after Hardy announcement. The MQ method is very favorable in geology, geophysics, geodesy and other fields (Hardy (1990)). A valuable advancement in the field of MQ was done in 1979 by Richard Franke (Franke (1979)), when he studied and compared various methods to interpolate a test surface. He declared, "One of the most exciting method in these tests is the Hardy MQ method." The next momentous time in RBF history was in 1986 when Charles Micchelli (Micchelli (1986)) reintroduced the MQ method theory and added enough criteria that guarantee the nonsingularity of the system matrix.

Consequences which create invertibility of the system matrix are cerdited to Schoenberg in 1938

(Schoenberg (1938)). Micchelli later deduced that Schoenberg conditions could be eased to include many more functions and turned over enough conditions for function to make secure that the system matrix would be nonsingular. Edward Kansa, a physicist, was the first to admit that MQ may be used to solve differential equations in 1990 (Sarra (2009)). This discovery sparked a surge in RBF research, and RBF are now used to solve numerically partial differential equations and meshless methods in a systematic approach (Sarra (2008), Martinez (2008)). In 1992, results from Madych and Nelson (Madych (1992)) showed the spectral convergence rate of MQ interpolation. This finding speedily developed research in RBF and RBF are now thought-out an efficient way to solve numerically partial differential equations and meshless methods on irregular domain in comparison to other state-of-the-arts methods (Belytschko (1996), Buhmann (2003)). Many branches of applied sciences have a large list of mathematical applications of RBF utilized in numerical techniques for solving PDEs with high accuracy in multi-dimensions (see, for example, Buhmann (2000), Buhmann (2003), Fasshauer (2007) and Fasshauer (2015)).

In the present work we investigate the periodic behavior for linearized KdV type (IBVPs) equations on bounded domain with periodic boundary conditions using RBF-PS method.

2. Model of KdV type IBVPs equations

Consider the following model representing linearized KdV type (IBVPs) equations

$$\begin{cases} w_t(x,t) + \eta w(x,t) + \xi w_x(x,t) - \delta w_{xx}(x,t) + \zeta w_{xxx}(x,t) = f(x,t), \text{ for} \\ \qquad x \in \Omega \subset \mathbf{R}^d, \ d \geq 1, \text{ and } t > 0, \end{cases} \tag{1}$$

the initial and boundary conditions with boundary operator $\mathcal{B}$ are given by

$$w(x,t) = w_0(x), \text{ for } x \in \Omega, \text{ and } t = 0, \tag{2}$$

$$\mathcal{B}w(x,t) = g(x,t), \text{ for } x \in \partial\Omega, \text{ and } t \geq 0, \tag{3}$$

where η, ξ, δ, ζ are some parameters and $f(x,t)$ is the source function.

For eventual periodicity the conditions listed below will be added to the above model equation,

$$w(x,0) = 0, \text{ for } x \geq 0, \text{ and } w(x,t) = g(t), \text{ for } t \geq 0, \tag{4}$$

condition on function g being periodic with T_0 as a period alternatively $g(t+T_0) = g(t)$, where $t \geq 0$.

3. RBF Pseudo-spectral approximation scheme

Fasshauer connected RBF collocation approach to Pseudo spectral (PS) scheme, known as RBF-PS method (Fasshauer (2005)), and used this approach to approximate two-dimensional Helmholtz and Laplace models, as well as the Allen-Cahn model with piecewise boundary conditions (Fasshauer (2007)). This approach was exploited and implemented by several authors to evaluate and solve various model PDEs (see the references Ferreira (2006), Ferreira (2007), Roque (2011), Uddin (2016), Uddin (2013) and Nikan (2019)). We also use this approach here for investigating the eventual periodicity of model equations (1)-(4).

For the set of N scattered nodes $x_j \in \Omega \subset \mathbf{R}^d, \quad d \geq 1$, the RBF interpolant is defined as a linear combination of radial basis function and is given in the following equation,

$$w(x,t) = \sum_{x_j \in \Omega} \lambda \kappa_j(\|x - x_j\|), x \in \Omega, \tag{5}$$

where λ denote the unknown expansion coefficients at any time t and κ_j denote an RBF centered at $x_j \in \Omega$ and $\|.\|$ is any norm, usually the distance norm (Euclidean norm) in $\mathbf{R}^d, d \geq 1$. Now collocating Equation (5) on the grid points x_i, one obtains,

$$w(x_i,t) = \sum_{j=1}^{N} \lambda \kappa_j(x_i, x_j), 1 \leq i \leq N. \tag{6}$$

The matrix form of Equation (6) can be given by

$$w = E\lambda, \tag{7}$$

where the matrix E is usually a square matrix, called a system matrix, whose entries are given by $E_{ij} = \kappa_j(\|x_i - x_j\|)$. Now differentiation of w, that is, w_x using Equation (7), is obtained by differentiating and re-evaluating the RBF function for each position x_i, where $1 \leq i \leq N$. We arrived at the matrix-vector representation

$$w_x = E_x\lambda, \tag{8}$$

where matrix E_x entries are stated as $\dfrac{d\kappa(x,x_j)_{x=x_i}}{dx}$ for $1 \leq i,j \leq N$. Upon solution of Equations (7) and (8) in terms of unknown values λ, the differentiation matrix is obtained in the subsequent format,

$$w_x = E_x E^{-1} w = H_x w, \tag{9}$$

where $H_x = E_x E^{-1}$ referred to as the differentiation matrix. It should be noted that the differentiation matrix depends on the invertibility of the matrix E. It is well known that the matrix E is always invertible for distinct set of collocation points. Thus, we are able to write in a similar manner

$$w_{xx} = E_{xx} E^{-1} w = H_{xx} w, \tag{10}$$

where $H_{xx} = E_{xx}E^{-1}$ containing entries of the form $\dfrac{d^2\kappa(x,x_j)_{x=x_i}}{dx^2}$ for $1 \leq i,j \leq N$. In a similar manner higher-order differentiation matrices are possible to build. The numerical approach for solving Equations (1) through (4), utilizing differentiation matrices described above, is shown below,

$$w' + \eta w + \xi H_x w - \delta H_{xx} w + \zeta H_{xxx} w = f(x,t). \tag{11}$$

Here, w' signify derivative with respect to time. The above Equation (11) can be written in the following as

$$w' = f(x,t) - \eta w - \xi H_x w + \delta H_{xx} w - \zeta H_{xxx} w. \tag{12}$$

Equation (11) is also represented by

$$w' = F(w). \tag{13}$$

ODE solvers like ode45, ode113, ode23 can now be utilized to solve the discretize ODE system Equation (13) in time. w_0 is the initial solution. To address ODE system stiffness, each efficacious ODE solver will choose an appropriate period of time δt to fix the stiffness of the ODE system.

4. Error analysis and stability of the presented numerical scheme

The time-dependent partial differential equation transformed into an ODE system in time based on RBF-PS method has been proposed. The method of lines is the name for this type of procedure. Hence, using the finite difference approach, we can solve this system of coupled ODEs in time, for example, Runge-Kutta technique and so on. The well-known rule of thumb can be used to address the method of lines stability. It is shown in the work (Trefethen (2000)) that the method of lines will be stable, when the eigenvalues of spatial discretization operator, linearized and scaled by step size δt, lie in region of stability of the corresponding time-discretization operator. The stability area is one facet of a complex plane comprising the eigenvalues for which the scheme produce a bounded solution.

5. Application of proposed scheme for eventual periodicity

Now we will show the results of our strategy for determining eventual periodicity of linearized KdV type Equations (1) through (4) in graphical form along with appropriate periodic boundary data $g(t) = \sin(20\pi t)\tanh(5t)$. The initial data w_0 is not necessarily necessary in examining eventual periodicity so we simply take it zero. The wave amplitudes $w(x, t)$ at time $t \in [0, 1.8]$ using $\delta t = 0.001$ produced in six graphs at these particular points $x = -0.950670$, -0.808460, -0.587280, -0.308720, 0.0 and 0.999650 in domain $[-1, 1]$ considering $N = 200$ total points in this domain and utilizing parameters η, ξ, δ and ζ for different values (see Shen (2007)). The plots below clearly confirm the subsequent periodic activity of the solutions in the specified domain at these particular positions. The X and Y axes are representative in these graphs of time t and amplitude w, respectively. The last graph shows the amplitude remains zero in every problem.

6. Conclusion

The Radial basis function pseudo-spectral (RBF-PS) approach is implemented to investigate the eventual periodicity to the initial and boundary value problems (IBVPs) for linearized KdV type equations on bounded domain with periodic boundary condition in this study. For spatial derivative approximation, we used RBF-PS method while for time integration we employed Runge-Kutta of order four (RK-4) approach. Hence, the applied numerical scheme has a great capability to approximate numerically many complicated problems with ease and accuracy.

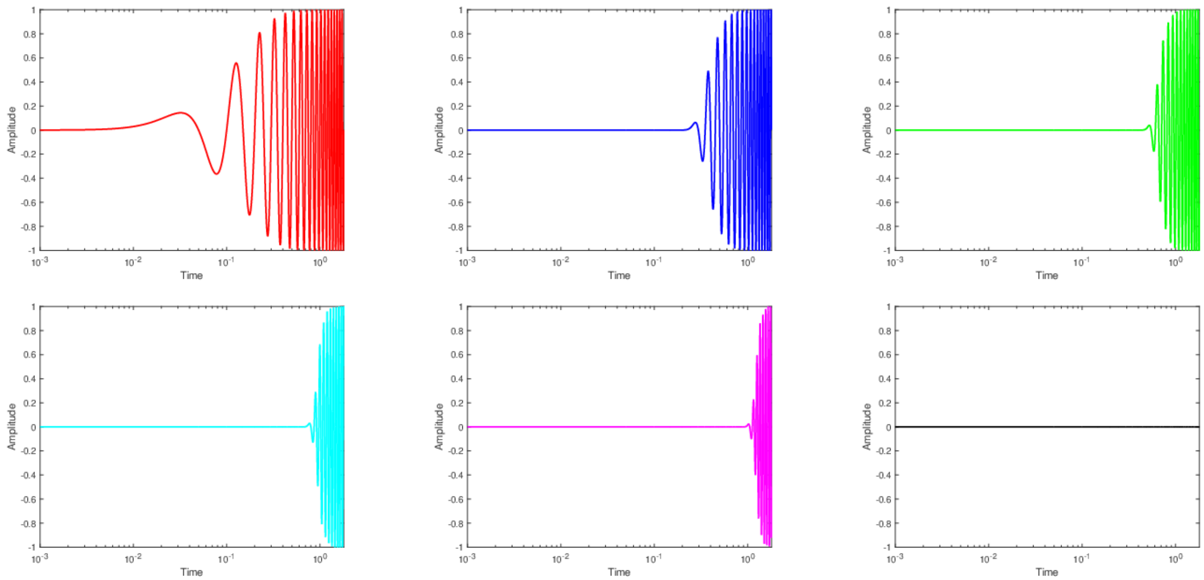

Figure 1. Linear KdV equation eventual periodicity for $x = -0.950670$ (red), -0.808460 (blue), -0.587280 (green), -0.308720 (cyan), 0 (magenta) and 0.999650 (black) in domain $[-1, 1]$, for $N = 200$, $\eta = 0$, $\xi = 1$, $\delta = 0$, $\zeta = 10^{-5}$, $\delta t = 0.001$, at time $t \in [0, 1.8]$ and $g(t) = \sin(20\pi t)\tanh(5t)$ with MQ-RBF, $\varepsilon = 0.1$, corresponding to model problem (1)-(4).

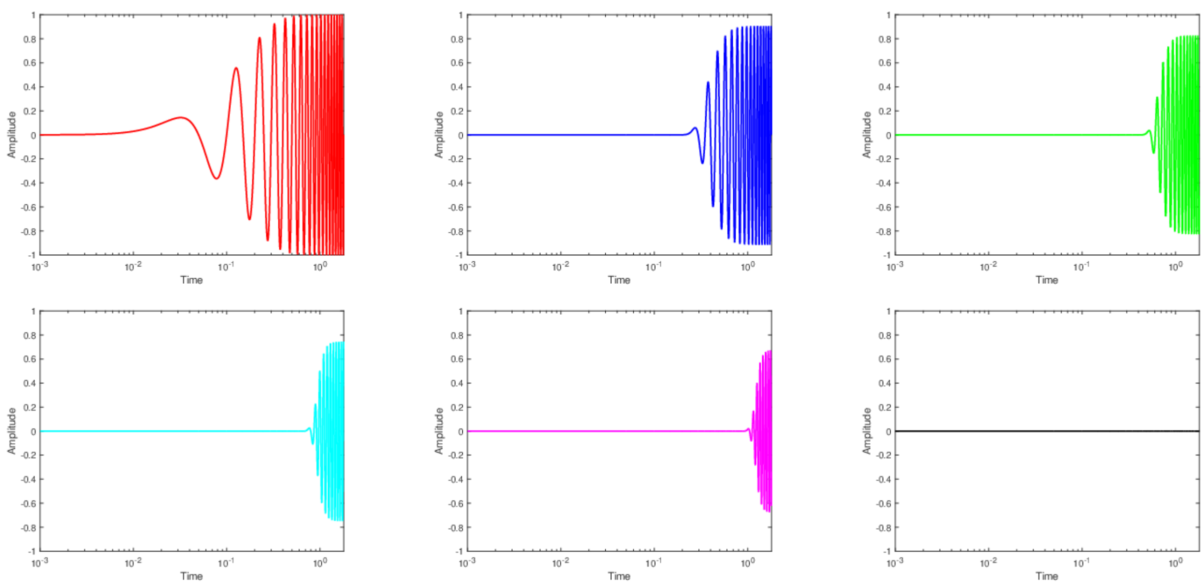

Figure 2. Linear KdV-Burger equation eventual periodicity for $x = -0.950670$ (red), -0.808460 (blue), -0.587280 (green), -0.308720 (cyan), 0 (magenta) and 0.999650 (black) in domain $[-1, 1]$, for $N = 200$, $\eta = 0$, $\xi = 1$, $\delta = 10^{-4}$, $\zeta = 10^{-5}$, $\delta t = 0.001$, at time $t \in [0, 1.8]$ and $g(t) = \sin(20\pi t)\tanh(5t)$ with MQ-RBF, $\varepsilon = 0.1$, corresponding to model problem (1)-(4).

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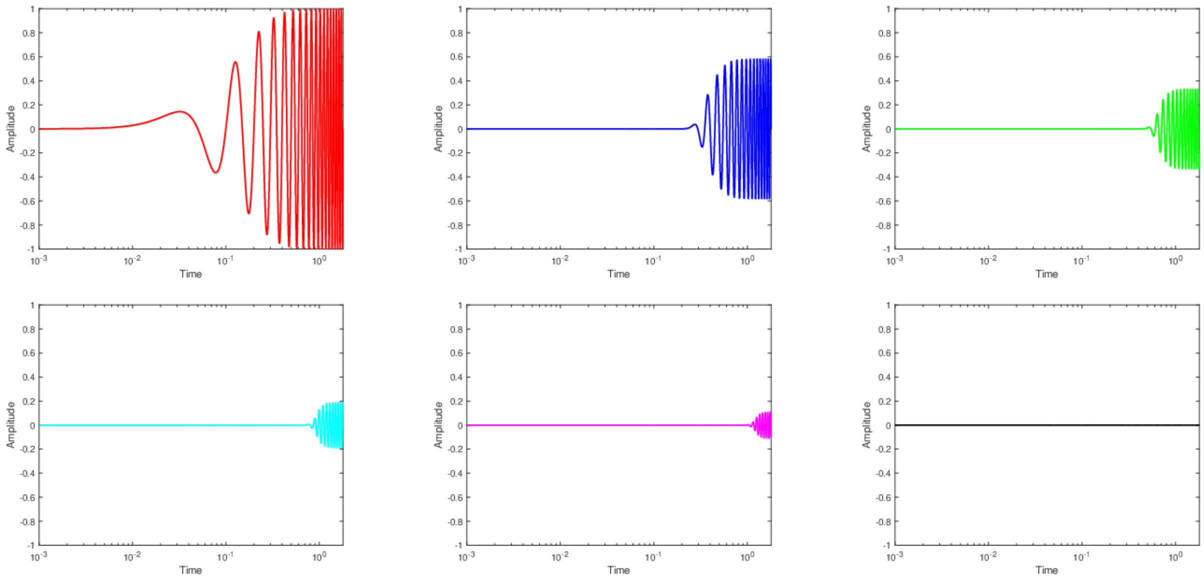

Figure 3. Linear Damped KdV equation eventual periodicity for $x = -0.950670$ (red), -0.808460 (blue), -0.587280 (green), -0.308720 (cyan), 0 (magenta) and 0.999650 (black) in domain $[-1, 1]$, for $N = 200$, $\eta = 2$, $\xi = 1$, $\delta = 0$, $\zeta = 10^{-5}$, $\delta t = 0.001$, at time $t \in [0, 1.8]$ and $g(t) = \sin(20\pi t)\tanh(5t)$ with MQ-RBF, $\varepsilon = 0.1$, corresponding to model problem (1)-(4).

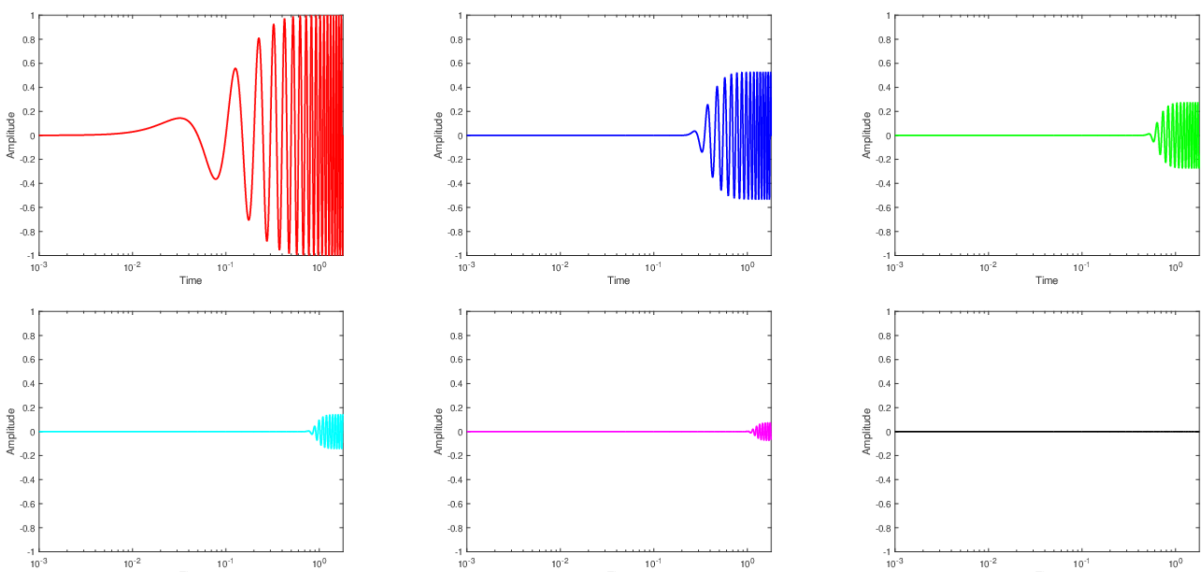

Figure 4. Linear Damped KdV-Burger equation eventual periodicity for $x = -0.950670$ (red), -0.808460 (blue), -0.587280 (green), -0.308720 (cyan), 0 (magenta) and 0.999650 (black) in domain $[-1, 1]$, for $N = 200$, $\eta = 2$, $\xi = 1$, $\delta = 10^{-4}$, $\zeta = 10^{-5}$, $\delta t = 0.001$, at time $t \in [0, 1.8]$ and $g(t) = \sin(20\pi t)\tanh(5t)$ with MQ-RBF, $\varepsilon = 0.1$, corresponding to model problem (1)-(4).

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