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Permanent of Toeplitz-Hessenberg Matrices with Generalized Fibonacci and Lucas entries

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Abstract

In the present paper, we evaluate the permanent and determinant of some Toeplitz-Hessenberg matrices with generalized Fibonacci and generalized Lucas numbers as entries. We develop identities involving sums of products of generalized Fibonacci numbers and generalized Lucas numbers with multinomial coefficients using the matrix structure, and then we present an application of the determinant of such matrices.

Keywords: Generalized Fibonacci numbers; Generalized Lucas numbers; Toeplitz-Hessenberg matrix; Permanent; Determinant; Recurrent sequences; Fibonacci sums

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1. Introduction

Generalized Fibonacci numbers have garnered a lot of attention in the recent two decades. Siar and Keskin (2013) used generalized Fibonacci numbers to establish several identities. Kilic and Tasci (2010) discovered some connections between generalized Fibonacci numbers and Hessenberg matrices permanent and determinant. The sums of products of generalized Fibonacci and Lucas numbers were examined by Belbachir and Bencherif (To appear). In addition, generalized Lucas numbers are used in primality testing; see the work of Di Porto and Filipponi (1988).

In the paper of Cerda-Morales (2013), matrix methods were used to represent generalized Fibonacci and generalized Lucas numbers and to establish Binet-like formulas. Also, Goy and Shattuck (2019) investigated the determinant of Toeplitz-Hessenberg matrices with various subsequences of the Fibonacci numbers as entries. A similar work of Goy and Shattuck (2021) was established with Jacobsthal numbers as entries. The generating functions approach were used by Goy and Shattuck (2020) to provide formulas for the determinant of Toeplitz-Hessenberg matrices with generalized Fibonacci entries.

In the present paper, we establish formulas for the permanent of Toeplitz-Hessenberg matrices with generalized Fibonacci and Lucas entries. Our work follows the results of Goy and Shattuck (2019-2021) for determinant, extended to case of permanent.

The paper is structured as follows. In Section 2, we determine explicit formulas for the permanent and determinant of Toeplitz-Hessenberg matrices when generalized Fibonacci and Lucas numbers are used as entries. In Section 3, we use a formula for the permanent of the Toeplitz-Hessenberg Matrix to establish new combinatorial identities for the sums of products of generalized Fibonacci numbers and generalized Lucas numbers. Finally, an application of the determinant of those matrices is given.

2. Permanent identities

The present paper deals with the matrix polynomial **permanent**. Let $A = (a_{ij})$ be an $n \times n$ matrix. The permanent of A, written $\operatorname{per}(A)$, is defined by

$$\operatorname{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all elements σ of the symmetric group S_n.

A lower Toeplitz-Hessenberg matrix is a square matrix of the form

$$M_n(a_0, a_1, \ldots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix}, \tag{1}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

Expanding along the first row, the determinant formula of (1) is as follows,

$$\det(M_n) = \sum_{k=1}^{n} (-a_0)^{k-1} a_k \det(M_{n-k}),$$

where $\det(M_0) = 1$.

We are interested in computing the permanent of (1) in the current paper, and we obtain the following recurrence by expanding along the first row continuously,

$$\operatorname{per}(M_n) = \sum_{k=1}^{n} a_0^{k-1} a_k \operatorname{per}(M_{n-k}), \tag{2}$$

where $\operatorname{per}(M_0) = 1$.

In the present paper, we derive explicit formulas for the permanent and determinant of (1) with generalized Fibonacci entries.

For any $p, q \in \mathbb{Z}$, the generalized Fibonacci sequence, denoted $\{U_n\}_{n\geq 0}$, is defined by $U_0 = 0$, $U_1 = 1$, and the following recurrence relation

$$U_n = pU_{n-1} + qU_{n-2}.$$

The generalized Lucas sequence, denoted $\{V_n\}_{n\geq 0}$, is defined by $V_0 = 2$, $V_1 = p$, and the recurrence relation

$$V_n = pV_{n-1} + qV_{n-2}.$$

For simplicity, we use the notation $\operatorname{per}(a_1, a_2, \ldots, a_n)$ and $\det(a_1, a_2, \ldots, a_n)$ instead of $\operatorname{per}(M_n(1, a_1, a_2, \ldots, a_n))$ and $\det(M_n(1, a_1, a_2, \ldots, a_n))$, respectively.

We evaluate the expressions

$$\operatorname{per}(U_{as+b}, U_{a(s+1)+b}, \ldots, U_{a(s+n-1)+b}), \quad \text{and} \quad \operatorname{per}(V_{as+b}, V_{a(s+1)+b}, \ldots, V_{a(s+n-1)+b}).$$

Before stating the main results, we present recurrences related to $(U_{as+b})_s$ and $(V_{as+b})_s$; see the paper of Shannon and Horadam (1979). They established for $a, b, s \in \mathbb{N}, b < a$ that the generalized Fibonacci sequence and the generalized Lucas sequence satisfy the following recurrences,

$$U_{as+b} = V_a U_{a(s-1)+b} - (-q)^a U_{a(s-2)+b},$$

and

$$V_{as+b} = V_a V_{a(s-1)+b} - (-q)^a V_{a(s-2)+b}.$$

Now, we give the main results.

Theorem 2.1.

For integers $n, s, a \geq 1$ and $0 \leq b < a$. Let M_n be a Toeplitz-Hessenberg matrix with generalized Fibonacci entries, then the permanent of M_n is

$$\begin{aligned}
&\operatorname{per}(U_{as+b}, U_{a(s+1)+b}, \ldots, U_{a(s+n-1)+b}) \\
&\quad = \frac{U_{as+b}(U_{as+b} - V_a + \sqrt{A}) + 2U_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{U_{as+b} + V_a + \sqrt{A}}{2} \right)^{n-1} \\
&\quad + \frac{U_{as+b}(-U_{as+b} + V_a + \sqrt{A}) - 2U_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{U_{as+b} + V_a - \sqrt{A}}{2} \right)^{n-1},
\end{aligned}$$

where $A = (U_{as+b} + V_a)^2 - 4(-q)^a(U_{a(s-1)+b} + 1)$.

Proof:

Let us denote

$$\alpha = \tfrac{1}{2\sqrt{A}} \left[U_{as+b}(U_{as+b} - V_a + \sqrt{A}) + 2U_{a(s+1)+b} \right],$$

and

$$\beta = \tfrac{1}{2\sqrt{A}} \left[U_{as+b}(-U_{as+b} + V_a + \sqrt{A}) - 2U_{a(s+1)+b} \right].$$

We will prove the theorem by induction on n.

Case $n = 1$:

$$\operatorname{per}(U_{as+b}) = \alpha + \beta = U_{as+b}.$$

Case $n = 2$:

$$\begin{aligned}
\operatorname{per}(U_{as+b}, U_{a(s+1)+b}) &= \alpha \left(\tfrac{U_{as+b} + V_a + \sqrt{A}}{2} \right) + \beta \left(\tfrac{U_{as+b} + V_a - \sqrt{A}}{2} \right) \\
&= U_{as+b}^2 + U_{a(s+1)+b}.
\end{aligned}$$

Suppose now Formula (2.1) is true, for $3 \leq k \leq n-1$, and let us prove it for n.

Let $\operatorname{per}_n := per(U_{as+b}, U_{a(s+1)+b}, \ldots, U_{a(s+n-1)+b})$. Using recurrence (2), we have,

$$\begin{aligned}
\operatorname{per}_n &= \sum_{j=1}^{n} U_{a(s+j-1)+b} \operatorname{per}_{n-j} \\
&= U_{as+b} \operatorname{per}_{n-1} + \sum_{j=2}^{n} U_{a(s+j-1)+b} \operatorname{per}_{n-j}
\end{aligned}$$

$$
\begin{aligned}
&= U_{as+b}\mathrm{per}_{n-1} + \sum_{j=2}^{n}(V_aU_{a(s+j-2)+b} - (-q)^aU_{a(s+j-3)+b})\mathrm{per}_{n-j}\\
&= U_{as+b}\mathrm{per}_{n-1} + V_a\sum_{j=2}^{n}U_{a(s+j-2)+b}\mathrm{per}_{n-j} - (-q)^a\sum_{j=2}^{n}U_{a(s+j-3)+b}\mathrm{per}_{n-j}\\
&= U_{as+b}\mathrm{per}_{n-1} + V_a\sum_{j=1}^{n-1}U_{a(s+j-1)+b}\mathrm{per}_{n-j-1}\\
&-(-q)^aU_{a(s-1)+b}\mathrm{per}_{n-2} - (-q)^a\sum_{j=3}^{n}U_{a(s+j-3)+b}\mathrm{per}_{n-j}\\
&= (U_{as+b}+V_a)\mathrm{per}_{n-1} - (-q)^aU_{a(s-1)+b}\mathrm{per}_{n-2} - (-q)^a\sum_{j=1}^{n-2}U_{a(s+j-1)+b}\mathrm{per}_{n-j-2}\\
&= (U_{as+b}+V_a)\mathrm{per}_{n-1} + (-(-q)^aU_{a(s-1)+b} - (-q)^a)\mathrm{per}_{n-2}\\
&= (U_{as+b}+V_a)\mathrm{per}_{n-1} - (-q)^a(U_{a(s-1)+b}+1)\mathrm{per}_{n-2},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{per}_n &= (U_{as+b}+V_a)\left[\alpha\left(\frac{U_{as+b}+V_a+\sqrt{A}}{2}\right)^{n-2} + \beta\left(\frac{U_{as+b}+V_a-\sqrt{A}}{2}\right)^{n-2}\right]\\
&-(-q)^a(U_{a(s-1)+b}+1)\left[\alpha\left(\frac{U_{as+b}+V_a+\sqrt{A}}{2}\right)^{n-3} + \beta\left(\frac{U_{as+b}+V_a-\sqrt{A}}{2}\right)^{n-3}\right]\\
&= \alpha\left(\frac{U_{as+b}+V_a+\sqrt{A}}{2}\right)^{n-3}\left[(U_{as+b}+V_a)\left(\frac{U_{as+b}+V_a+\sqrt{A}}{2}\right) - (-q)^a(U_{a(s-1)+b}+1)\right]\\
&+\beta\left(\frac{U_{as+b}+V_a-\sqrt{A}}{2}\right)^{n-3}\left[(U_{as+b}+V_a)\left(\frac{U_{as+b}+V_a-\sqrt{A}}{2}\right) - (-q)^a(U_{a(s-1)+b}+1)\right]\\
&= \alpha\left(\frac{U_{as+b}+V_a+\sqrt{A}}{2}\right)^{n-3}\left(\frac{U_{as+b}+V_a+\sqrt{A}}{2}\right)^{2}\\
&+\beta\left(\frac{U_{as+b}+V_a-\sqrt{A}}{2}\right)^{n-3}\left(\frac{U_{as+b}+V_a-\sqrt{A}}{2}\right)^{2}\\
&= \alpha\left(\frac{U_{as+b}+V_a+\sqrt{A}}{2}\right)^{n-1} + \beta\left(\frac{U_{as+b}+V_a-\sqrt{A}}{2}\right)^{n-1}.
\end{aligned}
$$

Thus, we conclude the proof. ■

Theorem 2.2.

For integers $n, s, a \geq 1$ and $0 \leq b < a$. Let M_n be a Toeplitz-Hessenberg matrix with generalized

Fibonacci numbers entries. Then the determinant of M_n is

$$\det(U_{as+b}, U_{a(s+1)+b}, \dots, U_{a(s+n-1)+b})$$
$$= \frac{(U_{as+b} + V_a + \sqrt{A})U_{as+b} - 2U_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{U_{as+b} - V_a + \sqrt{A}}{2} \right)^{n-1}$$
$$- \frac{(U_{as+b} + V_a - \sqrt{A})U_{as+b} - 2U_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{U_{as+b} - V_a - \sqrt{A}}{2} \right)^{n-1},$$

where $A = (U_{as+b} - V_a)^2 + 4(-q)^a(U_{a(s-1)+b} - 1)$.

Proof:

By induction on n, we use the general recurrence of $\det_n$, and we obtain

$$\begin{aligned}\det{}_n &= \sum_{j=1}^{n}(-1)^j U_{a(s+j-1)+b}\det{}_{n-j} \\ &= -U_{as+b}\det{}_{n-1} + \sum_{j=2}^{n}(-1)^j U_{a(s+j-1)+b}\det{}_{n-j} \\ &= -U_{as+b}\det{}_{n-1} + \sum_{j=2}^{n}(-1)^j (V_a U_{a(s+j-2)+b} - (-q)^a U_{a(s+j-3)+b})\det{}_{n-j},\end{aligned}$$

$$\begin{aligned}\det{}_n = -U_{as+b}\det{}_{n-1} &+ V_a \sum_{j=1}^{n-1}(-1)^j U_{a(s+j-1)+b}\det{}_{n-j-1} \\ &-(-q)^a U_{a(s-1)+b}\det{}_{n-2} - (-q)^a \sum_{j=3}^{n}(-1)^j U_{a(s+j-3)+b}\det{}_{n-j} \\ &= -(U_{as+b} + V_a)\det{}_{n-1} + (-(-q)^a U_{a(s-1)+b} - (-q)^a)\det{}_{n-2}.\end{aligned}$$

By induction hypothesis we get the desired result. ■

Next, in Theorem 2.3 we give the expression of the permanent of Toeplitz-Hessenberg matrices with generalized Lucas entries.

Theorem 2.3.

For integers $n, s, a \geq 1$ and $0 \leq b < a$. Let M_n be a Toeplitz-Hessenberg matrix with generalized Lucas numbers entries, then the permanent of M_n is

$$\operatorname{per}(V_{as+b}, V_{a(s+1)+b}, \dots, V_{a(s+n-1)+b})$$
$$= \frac{V_{as+b}(V_{as+b} - V_a + \sqrt{A}) + 2V_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{V_{as+b} + V_a + \sqrt{A}}{2} \right)^{n-1}$$
$$+ \frac{V_{as+b}(-V_{as+b} + V_a + \sqrt{A}) - 2V_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{V_{as+b} + V_a - \sqrt{A}}{2} \right)^{n-1},$$

where $A = (V_{as+b} + V_a)^2 - 4(-q)^a(V_{a(s-1)+b} + 1)$.

Proof:

By induction on n, using the recurrence (2) we have

$$\begin{aligned}
\mathrm{per}_n &= \sum_{j=1}^{n} V_{a(s+j-1)+b}\mathrm{per}_{n-j} \\
&= V_{as+b}\mathrm{per}_{n-1} + \sum_{j=2}^{n} V_{a(s+j-1)+b}\mathrm{per}_{n-j} \\
&= V_{as+b}\mathrm{per}_{n-1} + \sum_{j=2}^{n}(V_a V_{a(s+j-2)+b} - (-q)^a V_{a(s+j-3)+b})\mathrm{per}_{n-j} \\
&= V_{as+b}\mathrm{per}_{n-1} + V_a \sum_{j=1}^{n-1} V_{a(s+j-1)+b}\mathrm{per}_{n-j-1} \\
&-(-q)^a V_{a(s-1)+b}\mathrm{per}_{n-2} - (-q)^a \sum_{j=3}^{n} V_{a(s+j-3)+b}\mathrm{per}_{n-j} \\
&= (V_{as+b} + V_a)\mathrm{per}_{n-1} - (-q)^a V_{a(s-1)+b}\mathrm{per}_{n-2} - (-q)^a \sum_{j=1}^{n-2} V_{a(s+j-1)+b}\mathrm{per}_{n-j-2} \\
&= (V_{as+b} + V_a)\mathrm{per}_{n-1} - (-q)^a(V_{a(s-1)+b} + 1)\mathrm{per}_{n-2}.
\end{aligned}$$

We get the desired result using the induction hypothesis. ■

Theorem 2.4.

For integers $n, s, a \geq 1$ and $0 \leq b < a$. Let M_n be a Toeplitz-Hessenberg matrix with generalized Lucas numbers entries. Then the determinant of M_n is

$$\begin{aligned}
&\det(V_{as+b}, V_{a(s+1)+b}, \ldots, V_{a(s+n-1)+b}) \\
&\quad = \frac{V_{as+b}(V_{as+b} + V_a + \sqrt{A}) - 2V_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{V_{as+b} - V_a + \sqrt{A}}{2}\right)^{n-1} \\
&\qquad + \frac{V_{as+b}(-V_{as+b} - V_a + \sqrt{A}) + 2V_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{V_{as+b} - V_a - \sqrt{A}}{2}\right)^{n-1},
\end{aligned}$$

where $A = (V_{as+b} - V_a)^2 + 4(-q)^a(V_{a(s-1)+b} - 1)$.

Proof:

By induction on n, we use the general recurrence of $\det_n$, and we get

$$\begin{aligned}\det_n &= \sum_{j=1}^{n}(-1)^j V_{a(s+j-1)+b}\det_{n-j}\\ &= -V_{as+b}\det_{n-1} + \sum_{j=2}^{n}(-1)^j V_{a(s+j-1)+b}\det_{n-j}\\ &= -V_{as+b}\det_{n-1} + \sum_{j=2}^{n}(-1)^j (V_a V_{a(s+j-2)+b} - (-q)^a V_{a(s+j-3)+b})\det_{n-j}\\ &= -V_{as+b}\det_{n-1} + V_a\sum_{j=1}^{n-1}(-1)^j V_{a(s+j-1)+b}\det_{n-j-1}\\ &\quad -(-q)^a V_{a(s-1)+b}\det_{n-2} - (-q)^a\sum_{j=3}^{n}(-1)^j V_{a(s+j-3)+b}\det_{n-j}\\ &= -(V_{as+b}+V_a)\det_{n-1} + (-(-q)^a V_{a(s-1)+b} - (-q)^a)\det_{n-2}.\end{aligned}$$

By induction hypothesis and after simplification, we get the desired result. ■

Remark 2.1.

Our results generalize some previous results, by setting $p = q = 1$ in Theorems 2.2 and 2.4 we find the equations of Goy and Shattuck (2019).

Similar results for the permanent with special sequences as entries are derived from Theorems 2.1 and 2.3.

For $p = q = 1$ in Theorem 2.1, the permanent of Toeplitz-Hessenberg matrix with Fibonacci F_n numbers entries is as follows:

$$\begin{aligned}&\operatorname{per}(F_{as+b}, F_{a(s+1)+b}, \ldots, F_{a(s+n-1)+b})\\ &\quad= \frac{F_{as+b}(F_{as+b} - L_a + \sqrt{A}) + 2F_{a(s+1)+b}}{2\sqrt{A}}\left(\frac{F_{as+b}+L_a+\sqrt{A}}{2}\right)^{n-1}\\ &\quad\quad+ \frac{F_{as+b}(-F_{as+b} + L_a + \sqrt{A}) - 2F_{a(s+1)+b}}{2\sqrt{A}}\left(\frac{F_{as+b}+L_a-\sqrt{A}}{2}\right)^{n-1},\end{aligned}$$

where $A = (F_{as+b} + L_a)^2 - 4(-1)^a(F_{a(s-1)+b} + 1)$.

The Pell numbers ($A000129$ in the OEIS) are defined by, $P_n = 2P_{n-1} + P_{n-2}$, with $P_0 = 0$ and $P_1 = 1$. The Pell-Lucas numbers ($A002203$ in the OEIS) are defined by, $Q_n = 2Q_{n-1} + Q_{n-2}$, with $Q_0 = 2$ and $Q_1 = 2$.

For $p = 2$, $q = 1$ in Theorem 2.1, the permanent of Toeplitz-Hessenberg matrix with Pell P_n

numbers entries is given by

$$\begin{aligned}&\operatorname{per}(P_{as+b},P_{a(s+1)+b},\ldots,P_{a(s+n-1)+b})\\&=\frac{P_{as+b}(P_{as+b}-Q_a+\sqrt{A})+2P_{a(s+1)+b}}{2\sqrt{A}}\left(\frac{P_{as+b}+Q_a+\sqrt{A}}{2}\right)^{n-1}\\&+\frac{P_{as+b}(-P_{as+b}+Q_a+\sqrt{A})-2P_{a(s+1)+b}}{2\sqrt{A}}\left(\frac{P_{as+b}+Q_a-\sqrt{A}}{2}\right)^{n-1},\end{aligned}$$

where $A=(P_{as+b}+Q_a)^2-4(-1)^a(P_{a(s-1)+b}+1)$.

The Jacobsthal numbers ($A001045$ in the OEIS) are defined by, $J_n=J_{n-1}+2J_{n-2}$, with $J_0=0$ and $J_1=1$. And the Jacobsthal-Lucas numbers ($A014551$ in the OEIS) are defined by, $j_n=j_{n-1}+2j_{n-2}$, with $j_0=2$ and $j_1=1$.

For $p=1$, $q=2$ in Theorem 2.1, the permanent of Toeplitz-Hessenberg matrix with Jacobsthal J_n numbers entries is as follows

$$\begin{aligned}&\operatorname{per}(J_{as+b},J_{a(s+1)+b},\ldots,J_{a(s+n-1)+b})\\&=\frac{J_{as+b}(J_{as+b}-j_a+\sqrt{A})+2J_{a(s+1)+b}}{2\sqrt{A}}\left(\frac{J_{as+b}+j_a+\sqrt{A}}{2}\right)^{n-1}\\&+\frac{J_{as+b}(-J_{as+b}+j_a+\sqrt{A})-2J_{a(s+1)+b}}{2\sqrt{A}}\left(\frac{J_{as+b}+j_a-\sqrt{A}}{2}\right)^{n-1},\end{aligned}$$

where $A=(J_{as+b}+j_a)^2-2^{a+2}(-1)^a(J_{a(s-1)+b}+1)$.

3. New identities for generalized Fibonacci and Lucas numbers

Trudi's formula for $\det(M_n)$, where M_n is Toeplitz-Hessenberg matrix (see the book of Muir (1960)), is given by

$$\det(M_n)=\sum_{\substack{t_1,t_2,\ldots,t_n\geq 0\\t_1+2t_2+\cdots+nt_n=n}}(-a_0)^{n-\sum_{i=1}^n t_i}\binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n}a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n}.$$

We are interested to an analogues of Trudi's formula in the case of permanent. Using generating functions approach, we obtain the following.

Lemma 3.1.

Let M_n be the matrix defined in (1). Then

$$\operatorname{per}(M_n)=\sum_{\substack{t_1,t_2,\ldots,t_n\geq 0\\t_1+2t_2+\cdots+nt_n=n}}a_0^{n-\sum_{i=1}^n t_i}\binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n}a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n}. \quad (3)$$

We use Formula (3) of the permanent of Toeplitz-Hessenberg matrices with generalized Fibonacci and Lucas entries to derive the following combinatorial identities.

Theorem 3.1.

For integers $n, s, a \geq 1$ and $0 \leq b < a$, we have

$$\sum_{\substack{t_1,t_2,\ldots,t_n \geq 0 \\ t_1+2t_2+\cdots+nt_n=n}} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} U_{as+b}^{t_1} U_{a(s+1)+b}^{t_2} \cdots U_{a(s+n-1)+b}^{t_n}$$

$$= \frac{U_{as+b}(U_{as+b}-V_a+\sqrt{A})+2U_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{U_{as+b}+V_a+\sqrt{A}}{2}\right)^{n-1} \tag{4}$$

$$+\frac{U_{as+b}(-U_{as+b}+V_a+\sqrt{A})-2U_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{U_{as+b}+V_a-\sqrt{A}}{2}\right)^{n-1},$$

and,

$$\sum_{\substack{t_1,t_2,\ldots,t_n \geq 0 \\ t_1+2t_2+\cdots+nt_n=n}} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} V_{as+b}^{t_1} V_{a(s+1)+b}^{t_2} \cdots V_{a(s+n-1)+b}^{t_n}$$

$$= \frac{V_{as+b}(V_{as+b}-V_a+\sqrt{B})+2V_{a(s+1)+b}}{2\sqrt{B}} \left(\frac{V_{as+b}+V_a+\sqrt{B}}{2}\right)^{n-1} \tag{5}$$

$$+\frac{V_{as+b}(-V_{as+b}+V_a+\sqrt{B})-2V_{a(s+1)+b}}{2\sqrt{B}} \left(\frac{V_{as+b}+V_a-\sqrt{B}}{2}\right)^{n-1},$$

where $A = (U_{as+b}-V_a)^2+4(-q)^a(U_{a(s-1)+b}-1)$ and $B = (V_{as+b}-V_a)^2+4(-q)^a(V_{a(s-1)+b}-1)$.

Proof:

We prove identity (4). From Equation (3), for $\operatorname{per}_n := \operatorname{per}(U_{as+b}, U_{a(s+1)+b}, \ldots, U_{a(s+n-1)+b})$, we get

$$\operatorname{per}_n = \sum_{\substack{t_1,t_2,\ldots,t_n \geq 0 \\ t_1+2t_2+\cdots+nt_n=n}} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} U_{as+b}^{t_1} U_{a(s+1)+b}^{t_2} \cdots U_{a(s+n-1)+b}^{t_n}.$$

By Theorem 2.1, we get the identity. ∎

As an application, we use Identities (4), (5) and the determinant of Toeplitz-Hessenberg matrix with generalized Fibonacci and Lucas entries to give the following theorem.

Theorem 3.2.

Let M_n be a Toeplitz-Hessenberg matrix with generalized Fibonacci numbers entries and let

$(T_m)_{m>-n}$ the recurrent sequence defined by

$$\begin{cases} T_{-j} = 0 & \text{for } 1 \leq j \leq n-1, \\ T_0 = 1, \\ T_m = -U_{as+b}T_{m-1} - U_{a(s+1)+b}T_{m-2} - \cdots - U_{a(s+n-1)+b}T_{m-n}. \end{cases}$$

Then, $T_n = (-1)^n \det(M_n)$.

Proof:

For the proof we use Theorem 2 of Belbachir and Bencherif (2006).

We set $a_1 = -U_{as+b}$, $a_2 = -U_{a(s+1)+b}$, $\ldots$, $a_n = -U_{a(s+n-1)+b}$. Then, we have

$$\begin{aligned} T_m &= \sum_{\substack{t_1,t_2,\ldots,t_n \geq 0 \\ t_1+2t_2+\cdots+nt_n=m}} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} (-U_{as+b})^{t_1}(-U_{a(s+1)+b})^{t_2}\cdots(-U_{a(s+n-1)+b})^{t_n} \\ &= \sum_{\substack{t_1,t_2,\ldots,t_n \geq 0 \\ t_1+2t_2+\cdots+nt_n=m}} (-1)^{-\sum_{i=1}^n t_i} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} U_{as+b}^{t_1}U_{a(s+1)+b}^{t_2}\cdots U_{a(s+n-1)+b}^{t_n} \\ &= (-1)^n \sum_{\substack{t_1,t_2,\ldots,t_n \geq 0 \\ t_1+2t_2+\cdots+nt_n=m}} (-1)^{n-\sum_{i=1}^n t_i} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} U_{as+b}^{t_1}U_{a(s+1)+b}^{t_2}\cdots U_{a(s+n-1)+b}^{t_n}. \end{aligned}$$

For $m = n$, we ge,

$$T_n = (-1)^n \sum_{\substack{t_1,t_2,\ldots,t_n \geq 0 \\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{n-\sum_{i=1}^n t_i} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} U_{as+b}^{t_1}U_{a(s+1)+b}^{t_2}\cdots U_{a(s+n-1)+b}^{t_n}.$$

By (4), $T_n = (-1)^n \det(M_n)$. ■

Theorem 3.3.

Let M_n be a Toeplitz-Hessenberg matrix with generalized Lucas numbers entries and let $(T_m)_{m>-n}$ the recurrent sequence defined by

$$\begin{cases} T_{-j} = 0 & \text{for } 1 \leq j \leq n-1, \\ T_0 = 1, \\ T_m = -V_{as+b}T_{m-1} - V_{a(s+1)+b}T_{m-2} - \cdots - V_{a(s+n-1)+b}T_{m-n}. \end{cases}$$

Then, $T_n = (-1)^n \det(M_n)$.

Proof:

We set $a_1 = -V_{as+b}$, $a_2 = -V_{a(s+1)+b}$, $\ldots$, $a_n = -V_{a(s+n-1)+b}$ in Theorem 2 of Belbachir and Bencherif (2006). We get $T_n = (-1)^n \det(M_n)$. ■

For fixed $p = q = 1$ in Theorem 3.2 and hence classical Fibonacci numbers, let $(T_m)_{m>-n}$ the recurrent sequence defined by

$$\begin{cases} T_{-j} = 0 \qquad \text{for } 1 \le j \le n-1, \\ T_0 = 1, \\ T_m = -F_{as+b}T_{m-1} - F_{a(s+1)+b}T_{m-2} - \cdots - F_{a(s+n-1)+b}T_{m-n}. \end{cases}$$

Then,

$$T_n = (-1)^n \left[\frac{F_{as+b}(F_{as+b} + L_a + \sqrt{A}) - 2F_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{F_{as+b} - L_a + \sqrt{A}}{2} \right)^{n-1} + \frac{F_{as+b}(-F_{as+b} - L_a + \sqrt{A}) + 2F_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{F_{as+b} - L_a - \sqrt{A}}{2} \right)^{n-1} \right],$$

where $A = (F_{as+b} - L_a)^2 + 4(-1)^a(F_{a(s-1)+b} - 1)$.

Also, for $p = q = 1$ in Theorem 3.3 and hence classical Lucas numbers, let $(T_m)_{m>-n}$ the recurrent sequence defined by

$$\begin{cases} T_{-j} = 0 \qquad \text{for } 1 \le j \le n-1, \\ T_0 = 1, \\ T_m = -L_{as+b}T_{m-1} - L_{a(s+1)+b}T_{m-2} - \cdots - L_{a(s+n-1)+b}T_{m-n}. \end{cases}$$

Then,

$$T_n = (-1)^n \left[\frac{L_{as+b}(L_{as+b} + L_a + \sqrt{A}) - 2L_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{L_{as+b} - L_a + \sqrt{A}}{2} \right)^{n-1} + \frac{L_{as+b}(-L_{as+b} - L_a + \sqrt{A}) + 2L_{a(s+1)+b}}{2\sqrt{A}} \left(\frac{L_{as+b} - L_a - \sqrt{A}}{2} \right)^{n-1} \right],$$

where $A = (L_{as+b} - L_a)^2 + 4(-1)^a(L_{a(s-1)+b} - 1)$.

4. Conclusion

Fibonacci and Lucas sequences, as well as their generalizations, are well-known. A number of approaches may be used to obtain combinatorial identities for these sequences. We established closed formulas for the permanent and determinant using Toeplitz-Hessenberg matrices with generalized Fibonacci and Lucas as entries. For sums of products of generalized Fibonacci and Lucas numbers, we derive identities. Finally, we show how to find the general term of some sequences using the determinant of these matrices. The benefit here is in terms of complexity theory, as we can use fast algorithms for Toeplitz-Hessenberg matrices to establish terms of special sequences without having to do recurrent computation.

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