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Analytical and Numerical Solutions of a Fractional-Order Mathematical Model of Tumor Growth for Variable Killing Rate

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Abstract

This work intends to analyze the dynamics of the most aggressive form of brain tumor, glioblastomas, by following a fractional calculus approach. In describing memory preserving models, the non-local fractional derivatives not only deliver enhanced results but also acknowledge new avenues to be further explored. We suggest a mathematical model of fractional-order Burgess equation for new research perspectives of gliomas, which shall be interesting for biomedical and mathematical researchers. We replace the classical derivative with a non-integer derivative and attempt to retrieve the classical solution as a particular case. The prime motive is to acquire both analytical and numerical solutions to the posed problem. At first, we employ the transform method, and then the Adomian decomposition method to obtain the solutions that shall be useful to provide information about the effect of medical care in the annihilation of gliomas. Finally, we discuss the applicability of this model with numerical simulations and graphical representations.

Keywords: Fractional derivative; Modeling; Glioblastoma; Burgess equation; Laplace and Fourier transform; Adomian decomposition method

MSC 2010 No.: 26A33, 33E12, 35A22

1. Introduction

Bio-mathematical modeling is an interdisciplinary research area that acknowledges modernistic approaches to explain the behavioral dynamics of complex biological systems. It offers a broad analytical and quantitative understanding to both biologists and mathematicians. One of the extensively analyzed problems in this direction is understanding the pattern of tumor growth models described by integer-order differential equations. In the literature, some of the work in this direction include mathematical modeling of virtual and real brain tumors discussed by Swanson et al. (2003), diffusive tumors by Cruywagen et al. (2003), a spherically symmetric tumor growth model given by Ali et al. (2014), the response of cancer under immunological activity presented in Vladar and Gonjalez (2004) and Ghanbari et al. (2020), symmetry methods for a mathematical model of brain tumor by Moyo and Leach (2004), numerical analysis of reaction-diffusion epidemic model given by Ahmed et al. (2020), etc.

Established with the idea of arbitrary order derivatives, the research area of fractional calculus explored the possibility to explain the bio-mathematical systems depending upon historical data. The non-local property of fractional-order derivatives motivates researchers to analyze and enhance the existing tumor growth models in the sense of fractional calculus. For the existing literature, preliminaries and physical interpretations of fractional calculus, we advise the readers to the papers by Kilbas et al. (2006), Podlubny (1999), and Samko et al. (1993). Many of the fascinating problems related to this area include fractional optimal control problems discussed by Singha (2020) and Lotfi et al. (2011), a fuzzy Atangana-Baleanu fractional hybrid system given by Hasan et al. (2021), fractional variational problems presented in Singha and Nahak (2019) and Pooseh et al. (2013), fractional telegraph equation by Kumar (2014), numerical computations of coupled fractional resonant Schrödinger equations by Al-Smadi et al. (2020), fractionally convex functions introduced by Singha and Nahak (2020).

To construct a mathematical model of a biological system, fractional-order derivatives can be utilized efficiently which investigates the memory-preserving data more accurately. A detailed study of such models shall result in a better understanding of highly complex dynamics of tumor cells, and deliver analytical as well as approximate solutions. In this direction of research, one may comprehend the fractional-order model of malaria discussed by Pinto and Machado (2013), dengue fever given by Diethelm (2013), HIV model presented by Huo et al. (2015), the fractional epidemic model by Arqub and El-Ajou (2013), solving the population growth model by Hasan et al. (2020) and Al-Khaled (2005), fractional-order multiple chaotic FitzHugh-Nagumo neurons model by Momani et al. (2014), and evolution equation given by Doungmo et al. (2020), etc.

We investigate a mathematical model describing the growth of glioma cells at any time ‘t’, with a fractional diffusion equation. The underlying motive is to explore the fractional-order Burgess equation for analytical and numerical solutions. The introductory part deals with the existing results and motivation for the proposed work. Section 2 details the formulation of the fractional-order burgess equation, followed by an analytical solution scheme with the aid of transform technique, and an approximate solution scheme with the Adomian decomposition method in Section 3. In Section 4, numerical simulations are executed to interpret the efficiency of the proposed system.

2. The Fractional-Order Burgess Equation: A Mathematical Model of Brain Tumour Growth

In this section, we formulate a tumor growth model by introducing a fractional-order derivative in the characterization of Burgess equation. The Burgess equation illustrates the growth of tumors using the invasive diffusive properties of cancer cells discussed by Murray (2012), to be referred as the classical Burgess equation. Here, the medium of expansion of tumor cells is assumed to be isotropic and uniform, and these cells are supposed to possess spherical symmetry. Let N_0 denote the density of cancer cells at diagnostic time t_0 and at location r_0, that is, $\eta(r_0, t_0) = N_0$. With the initial condition $\eta(r_0, t_0) = N_0$, the classical Burgess equation represents a mathematical model for the growth of glioma cells.

The proliferation rate of tumor in the above mentioned problem has already been investigated by Tracqui et al. (1995) to study the effects of Chemotherapy, and Murray (2012) has obtained its solution in the absence of medical treatment. Using the solutions obtained, the authors have also discussed the expected survival time of a patient and the growth of untreated glioma.

Fractional-Order Burgess Equation The main aim is to analyze the Burgess equation with the assistance of Caputo non-integer order derivatives, denoted by ${}_0^cD_\tau^\alpha$. By assuming $\tau = 2Dt$, $U(r, \tau) = r\eta(r, t)$, $\omega = (p(t) - k(t))\,\eta(r, t)$, the classical Burgess equation can be rewritten as

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\frac{\partial^2 U}{\partial r^2} + \omega.$$

Thus, the fractional-order Burgess equation can be described as

$$(\mathbf{P}^*): \qquad {}_0^cD_\tau^\alpha U(r, \tau) = \frac{1}{2}\frac{\partial^2 U(r, \tau)}{\partial r^2} + \omega(r, \tau); \quad \alpha \in (0, 1),$$
$$U(r_0, t_0) = r_0\,\eta(r_0, t_0) = r_0 \cdot N_0.$$

Here,

- $\eta(r, t)$ $\rightarrow$ denotes the tumour cell concentration at a location r and time t.
- D $\rightarrow$ is the diffusion coefficient.
- p $\rightarrow$ represents the reproduction rate of glioblastoma cells.
- k $\rightarrow$ is the killing rate of glioblastoma cells varying with time t.

Note that the above problem **($\mathbf{P}^*$)** includes a time-fractional diffusion equation. The integer-order derivatives are local and thus don't possess memory and hereditary properties. However, the behavior of most of the biological models exhibits memory effects. Thus, practicing the fractional-order derivatives in place of ordinary derivatives is an effective way to incorporate memory effects in such models. Previously, significant consideration has already been given to the numerical solutions of fractional diffusion equations in Murray (2012); Swanson et al. (2000); Murray (2003). Next, we shall discuss the analytical and approximate solutions to the fractional-order tumor growth model **($\mathbf{P}^*$)**.

3. Solutions to the Fractional-Order Burgess Equation

In this section, we focus on finding the analytic and numerical solutions to the posed problem of cerebral tumor (glioblastoma) growth under medical care.

3.1. Closed form of the solution

Here, we apply the Laplace and Fourier transforms to obtain the solution to the problem **(P*)**. The advantage of using transform technique is that the analytical solution can be retrieved that shall be useful to explain any dynamical process. For various analytical and numerical solution techniques, we refer the reader to the work presented in the papersAl-Smadi and Arqub (2019), Momani et al. (2016), and Al-Smadi et al. (2016).

Let us consider the fractional-order Burgess equation

$$ {}_0^cD_\tau^\alpha U(r,\tau) = \frac{1}{2}\frac{\partial^2 U(r,\tau)}{\partial r^2} + \omega(r,\tau), \quad \alpha \in (0,1). \tag{1}$$

After applying the Laplace transform with respect to the variable τ, we get

$$ s^\alpha\, \bar{U}(r,s) - s^{\alpha-1}U(r,0) = \frac{1}{2}\frac{\partial^2}{\partial r^2}\bar{U}(r,s) + \bar{\omega}(r,s), \tag{2}$$

where s is the Laplace transform parameter. Note that, $\bar{U}(r,s)$ and $\bar{\omega}(r,s)$ denote the Laplace transform (with respect to τ) of $U(r,\tau)$ and $\omega(r,\tau)$, respectively. By assuming $U(r,0) = \phi(r)$, which can be accessed from the initial condition in **(P*)**, we rewrite the above equation as

$$ s^\alpha\, \bar{U}(r,s) - s^{\alpha-1}\phi(r) = \frac{1}{2}\frac{\partial^2}{\partial r^2}\bar{U}(r,s) + \bar{\omega}(r,s). $$

Next, apply the Fourier transform with respect to the variable r, that is

$$ s^\alpha\, \tilde{\bar{U}}(k,s) - s^{\alpha-1}\tilde{\phi}(k) = \frac{1}{2}(-ik)^2\tilde{\bar{U}}(k,s) + \tilde{\bar{\omega}}(k,s), \tag{3}$$

where k is the Fourier transform parameter. Also, $\tilde{\bar{U}}(k,s)$, $\tilde{\bar{\omega}}(k,s)$ and $\tilde{\phi}(k)$ denote the Fourier transform (with respect to r) of $\bar{U}(r,s)$, $\bar{\omega}(r,s)$ and $\phi(r)$, respectively. On simplifying the above equation for $\tilde{\bar{U}}(k,s)$, we arrive at

$$ \tilde{\bar{U}}(k,s) = \frac{s^{\alpha-1}\tilde{\phi}(k)}{s^\alpha + \frac{1}{2}k^2} + \frac{\tilde{\bar{\omega}}(k,s)}{s^\alpha + \frac{1}{2}k^2}. \tag{4}$$

After applying the inverse Fourier transform, we write

$$ \bar{U}(r,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{s^{\alpha-1}\,\tilde{\phi}(k)}{s^\alpha + \frac{1}{2}k^2}\, e^{-ikr}\, dk \; + \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\bar{\omega}}(k,s)\, e^{-ikr}\, dk. \tag{5}$$

On applying the inverse Laplace transform, we finally arrive at

$$U(r,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{\alpha,1}\left(-\frac{1}{2}k^2\tau^{\alpha}\right)\tilde{\phi}(k)e^{-ikr}dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikr}dk \times \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\tilde{\bar{\omega}}(k,s)}{s^{\alpha}+\frac{1}{2}k^2}\, e^{s\tau}\, ds, \tag{6}$$

where γ is a fixed real number. Finally, $U(r,\tau)$ in Equation (6) represents the required analytic solution for the problem **(P*)**.

3.2. Approximate Solutions to the Fractional-Order Burgess Equation

A fractional-order mathematical model of tumor cells exhibits a highly complex solution, one may look at Equation (6), which creates the necessity to obtain the approximate solutions. One may note that the transform method gives the analytical solution to the posed problem. To acquire the numerical solutions of **(P*)** in series form, we implement the Adomian decomposition method (ADM) introduced by Adomian (1990). ADM and its convergence have been previously discussed by many researchers like Adomian and Sarafyan (1981) and Singha and Nahak (2016), to obtain the numerical solutions of differential equations arising in various problems of science and engineering.

We denote the operators $L_1 \equiv {}_0^cD_{\tau}^{\alpha}$, $L_2 \equiv \frac{\partial^2}{\partial r^2}$, and N by the nonlinear part of U. Thus, the fractional-order Burgess equation in the suggested problem **(P*)** can be rewritten as

$$\begin{aligned} L_1U(r,\tau) &= \frac{1}{2}L_2U(r,\tau) + N(U(r,\tau)), \\ \text{or,}\quad U(r,\tau) &= U(r,0) + \frac{1}{2}L_1^{-1}L_2U(r,\tau) + L_1^{-1}N(U(r,\tau)), \end{aligned} \tag{7}$$

where $L_1^{-1} \equiv {}_0I_{\tau}^{\alpha}$ denotes the Riemann-Liouville fractional integration of order α.

By practicing the ADM, we write the series form of $U(r,\tau)$ as

$$U(r,\tau) = \sum_{n=0}^{\infty} U_n(r,\tau), \tag{8}$$

and the nonlinear term is given by

$$NU(r,\tau) = \sum_{n=0}^{\infty} A_n(U_0, U_1, ..., U_n), \tag{9}$$

where $\{A_n\}_{n=0}^{\infty}$ is the sequence of Adomian polynomials. For the insights of the ADM, we advise the reader to the work presented by Adomian and Sarafyan (1981) and Adomian (1994). Using Eqs. (8), (9) into the Equation (7), we get

$$\sum_{n=0}^{\infty} U_n(r,\tau) = U(r,0) + \frac{1}{2}L_1^{-1}L_2\sum_{n=0}^{\infty} U_n(r,\tau) + L_1^{-1}\sum_{n=0}^{\infty} A_n(U_0, U_1, ..., U_n),$$

where $U(r,0)$ can be obtained from the initial condition in problem **(P*)**.

Thus, we arrive at the following recurrence relation

$$U_0(r,\tau) = U(r,0), \tag{10}$$

$$U_{n+1}(r,\tau) = \frac{1}{2}L_1^{-1}\,L_2(U_n) + L_1^{-1}\,A_n,\ \ n = 0,1,2,... \tag{11}$$

Now we can establish a series of approximate solutions of the fractional-order Burgess equation as

$$U(r,\tau) \approx \sum_{n=0}^{k} U_n(r,\tau), \quad \text{where} \quad \lim_{k\to\infty}\sum_{n=0}^{k} U_n(r,\tau) = U(r,\tau).$$

The ADM provides the numerical solution, in the form of n^{th}-order approximation by series, of the problem with less computational work. To make the procedure evident, we shall now discuss some examples in the next section.

4. Numerical Simulation

Example 4.1.

Let us consider the fractional-order differential equation

$${}_0^c D_\tau^\alpha U(r,\tau) = \frac{1}{2}\frac{\partial^2 U}{\partial r^2} + \omega(r,\tau), \quad \alpha \in (0,1), \tag{12}$$

$$\text{with} \quad U(r,0) = \ln(r+2), \tag{13}$$

where the nonlinear source is given by $N(U) \equiv \omega(r,\tau) = e^{-U} + \frac{1}{2}e^{-2U}$. We may note that, for $\alpha = 1$, the above problem (12)-(13) reduces to a classical model of gliomas under some medical treatment discussed by González-Gaxiola and Bernal-Jaquez (2017).

By implementing the approximation scheme (10)-(11) explained in the last section, we write

$$U_0(r,\tau) = U(r,0), \tag{14}$$

$$U_{n+1}(r,\tau) = \frac{1}{2}L_1^{-1}\,L_2(U_n) + L_1^{-1}\,A_n,\ \ n = 0,1,2,... \tag{15}$$

where $L_1^{-1} \equiv {}_0I_\tau^\alpha$ and $L_2 \equiv \frac{\partial^2}{\partial r^2}$. Clearly,

$$U_0 = \ln(r+2), \tag{16}$$

$$A_0 = N(U_0) = e^{-U_0} + \frac{1}{2}e^{-2U}, \tag{17}$$

We first compute

$$\begin{aligned} U_1 &= \frac{1}{2}L_1^{-1}\,L_2(U_0) + L_1^{-1}\,A_0 \\ &= \frac{1}{2}{}_0I_\tau^\alpha\frac{\partial^2}{\partial r^2}(U_0) + {}_0I_\tau^\alpha A_0 \\ &= \frac{1}{(r+2)}\,\frac{\tau^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{18}$$

We first find the value of Adomian polynomial $A_1(U_0, U_1)$ as

$$\begin{aligned} A_1(U_0, U_1) &= -U_1\, e^{-U_0} - U_1\, e^{-2U_0} \\ &= -\left(\frac{1}{(r+2)^2} + \frac{1}{(r+2)^3}\right) \frac{\tau^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{19}$$

and thus the value of U_2 is then given by

$$\begin{aligned} U_2 &= \frac{1}{2} L_1^{-1}\, L_2(U_1) + L_1^{-1}\, A_1 \\ &= \frac{1}{2}\, {}_0I_\tau^\alpha \frac{\partial^2}{\partial r^2}(U_1) + {}_0I_\tau^\alpha A_1 \\ &= -\frac{1}{(r+2)^2}\, \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned} \tag{20}$$

Similarly, we obtain the values of A_2 and U_3 as

$$\begin{aligned} A_2(U_0, U_1, U_2) &= \frac{U_1^2}{2}\left(e^{-U_0} + 2e^{-2U_0}\right) + U_2\left(-e^{-U_0} - e^{-2U_0}\right) \\ &= \left(\frac{1}{(r+2)^3} + \frac{2}{(r+2)^4}\right) \frac{\tau^{2\alpha}}{2(\Gamma(\alpha+1))^2} \\ &\quad + \left(\frac{1}{(r+2)^3} + \frac{1}{(r+2)^4}\right) \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} U_3 &= \frac{1}{2} L_1^{-1}\, L_2(U_2) + L_1^{-1}\, A_2 \\ &= \frac{1}{2}\, {}_0I_\tau^\alpha \frac{\partial^2}{\partial r^2}(U_2) + {}_0I_\tau^\alpha A_2 \\ &= \left[\frac{1}{(r+2)^3} + \frac{\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^2(r+2)^3}\right. \\ &\quad \left. + \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2(r+2)^4} - \frac{2}{(r+2)^4}\right] \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \tag{22}$$

Lastly, the third order approximation of U is given by

$$\begin{aligned} U(r,\tau) &\approx \sum_{n=0}^{3} U_n(r,\tau) \\ &= U_0(r,\tau) + U_1(r,\tau) + U_2(r,\tau) + U_3(r,\tau) \\ &= \ln(r+2) + \frac{1}{(r+2)}\, \frac{\tau^\alpha}{\Gamma(\alpha+1)} - \frac{1}{(r+2)^2}\, \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad + \left[\frac{1}{(r+2)^3} + \frac{\Gamma(2\alpha+1)}{2(\Gamma(\alpha+1))^2(r+2)^3}\right. \\ &\quad \left. + \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2(r+2)^4} - \frac{2}{(r+2)^4}\right] \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \tag{23}$$

In order to reduce complexities and to make the present mechanism clear, we have obtained the third-order approximation to the solutions of the present problem (12)-(13). One may also use any

mathematical software to deduce higher-order approximations. At this point, we would also like to mention that

$$U_1\big|_{\alpha=1} = \frac{\tau}{(r+2)}, \qquad U_2\big|_{\alpha=1} = -\frac{\tau^2}{2(r+2)^2},$$
$$U_3\big|_{\alpha=1} = \frac{\tau^3}{3(r+2)^3}, \qquad U_4\big|_{\alpha=1} = -\frac{\tau^4}{4(r+2)^4}, \ldots$$
$$\text{and} \qquad U_n\big|_{\alpha=1} = (-1)^{n+1}\frac{\tau^n}{n(r+2)^n}.$$

Finally, for $\alpha = 1$, the n^{th}-order approximation of U can also be deduced as

$$\begin{aligned} U(r,\tau)\big|_{\alpha=1} &\approx \sum_{n=0}^{\infty} U_n(r,\tau)\big|_{\alpha=1} \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\tau^n}{n(r+2)^n} \\ &= \ln(r+\tau+2), \end{aligned} \tag{24}$$

which meets the exact solution to the classical Burgess equation given by González-Gaxiola and Bernal-Jaquez (2017). We have also provided the 3D graphical representations to the profile of the concentration of glioblastoma cells in Figures 1-4, for various values of $\alpha \in (0,1)$.

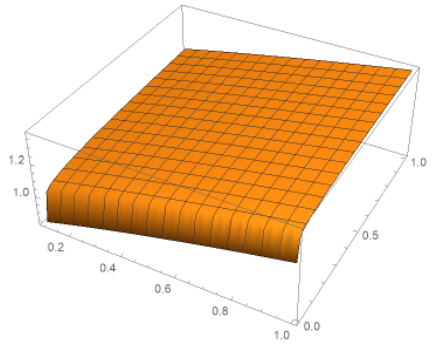

Figure 1. $U|_{\alpha=0.1}$, $(r,\tau) \in (0,1] \times (0,1]$

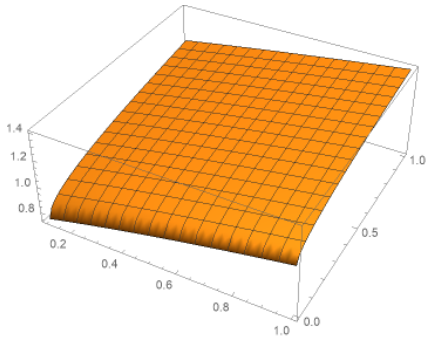

Figure 2. $U|_{\alpha=0.5}$, $(r,\tau) \in (0,1] \times (0,1]$

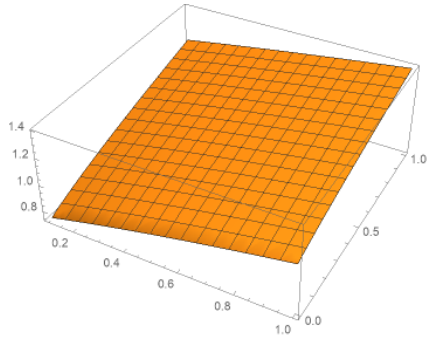

Figure 3. $U|_{\alpha=0.9}$, $(r,\tau) \in (0,1] \times (0,1]$

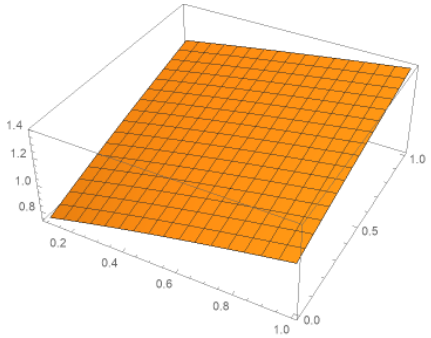

Figure 4. $U|_{\alpha=1}$, $(r,\tau) \in (0,1] \times (0,1]$

Example 4.2.

For $\alpha \in (0,1)$, let us consider the α^{th}-order model for the growth of glioblastoma cells described as

$$ {}_0^c D_\tau^\alpha U(r,\tau) = \frac{1}{2}\frac{\partial^2 U}{\partial r^2} + \frac{1}{2}U(r,\tau), \tag{25} $$

$$ \text{with} \quad U(r,0) = e^r. \tag{26} $$

Equation (26) yields the initial profile of tumor growth as a priori assumption $u(r,0) = e^r$. That is, at the time of diagnosis, the glioma cells are expanding exponentially. We shall now extract the solution to the problem (25)-(26) by the approximation scheme (10)-(11), and in similar steps, as explained in the previous example. Here, $N(U) = \frac{1}{2}U$ and $A_0 = N(U_0) = \frac{e^r}{2}$.

We first compute

$$ \begin{aligned} U_1 &= \frac{1}{2}L_1^{-1}\, L_2(U_0) + L_1^{-1}\, A_0 \\ &= \frac{1}{2}{}_0I_\tau^\alpha \frac{\partial^2}{\partial r^2}(e^r) + {}_0I_\tau^\alpha \frac{e^r}{2} \\ &= e^r\, \frac{\tau^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{27} $$

Again, $A_1 = N'(U_0)\, U_1 = \frac{e^r\, \tau^\alpha}{2\,\Gamma(\alpha+1)}$ and

$$ \begin{aligned} U_2 &= \frac{1}{2}L_1^{-1}\, L_2(U_1) + L_1^{-1}\, A_1 \\ &= \frac{1}{2}{}_0I_\tau^\alpha \frac{\partial^2}{\partial r^2}\left[e^r\, \frac{\tau^\alpha}{\Gamma(\alpha+1)}\right] + {}_0I_\tau^\alpha \left[\frac{e^r\, \tau^\alpha}{2\,\Gamma(\alpha+1)}\right] \\ &= e^r\, \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned} \tag{28} $$

Also, $A_2 = \frac{e^r\, \tau^{2\alpha}}{2\Gamma(2\alpha+1)}$ and

$$ \begin{aligned} U_3 &= \frac{1}{2}L_1^{-1}\, L_2(U_2) + L_1^{-1}\, A_2 \\ &= \frac{1}{2}{}_0I_\tau^\alpha \frac{\partial^2}{\partial r^2}\left[e^r\, \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\right] + {}_0I_\tau^\alpha \left[\frac{e^r\, \tau^{2\alpha}}{2\Gamma(2\alpha+1)}\right] \\ &= e^r\, \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \tag{29} $$

Repeating the same process, we arrive at

$$ U_n(r,\tau) = e^r\, \frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}. $$

At last, the n^{th}-order approximation of $U(r,\tau)$ is given by

$$\begin{aligned} U(r,\tau) &\approx \sum_{n=0}^{\infty} U_n(r,\tau) \\ &= \sum_{n=0}^{\infty} e^r \, \frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)} \\ &= e^r \, E_\alpha(t^\alpha). \end{aligned} \tag{30}$$

The solution $U(r,\tau)$ to the problem (25)-(26) delivers the density of cancer cells at every point (r,τ). For $\alpha = 1$, $U(r,\tau) = e^{r+t}$ represents the solution of the corresponding classical model of tumor growth. In the Figures 5-8, we have provided the 3D plot of $U(r,\tau)$ for distinct values of α that signifies the exponential growth of tumor cells.

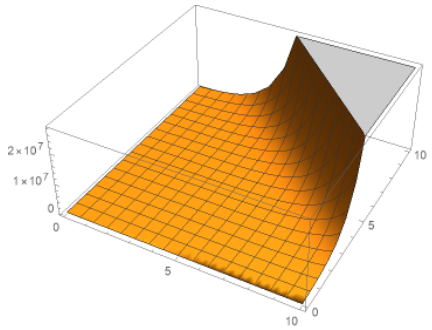

Figure 5. $U|_{\alpha=0.1} = e^r \, E_{0.1}(t^{0.1})$

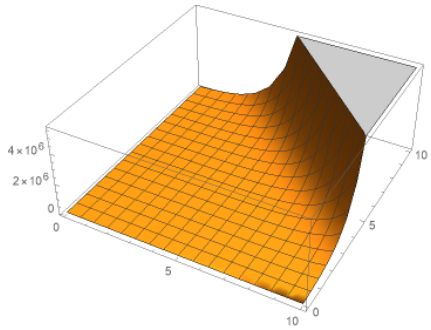

Figure 6. $U|_{\alpha=0.5} = e^r \, E_{0.5}(t^{0.5})$

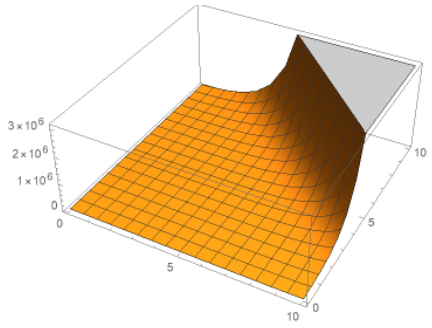

Figure 7. $U|_{\alpha=0.9} = e^r \, E_{0.9}(t^{0.9})$

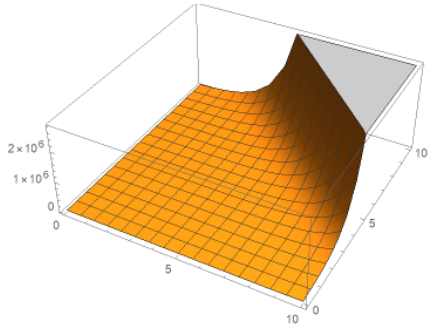

Figure 8. $U(r,\tau)|_{\alpha=1} = e^{r+t}$

5. Conclusions

We have discussed a fractional-order model of Burgess equation outlining the growth of brain tumor, glioblastoma, with a non-linear source depicting the effect of medical treatment. The obtained results present the consequence of using fractional derivatives on the amount of growth of glioma cells. In addition, we have deduced the analytic and approximation scheme to find the numerical solutions of the suggested model. Firstly, the closed-form of the solution has been obtained by employing Laplace and Fourier Transforms. Next, the numerical technique is executed in an organized manner to explain the concentration of glioma cells and visualizes the growth of tumor cells

with the assistance of 3D plots. The suggested model could also be used for comparing the growth in tumor cell radius with the effect of medical treatment (chemotherapy or radiotherapy), that is a prospective research work of the author.

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