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Study the Effect of Modified Newtonian Force on the Restricted 3-body Configuration in Non-linear Sense

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Abstract

This paper aims to investigate the non-linear stability of the triangular libration point in the restricted three-body problem (R3BP). The model, we use for our problem consists of a primary body as a heterogeneous spheroid with N-layers having different densities of each layer and a secondary body as a point mass that is producing the modified Newtonian Potential. We determine the equation of motion of the smallest body which is under the influence of the above-mentioned perturbations and also influenced by Coriolis as well as Centrifugal forces and then evaluated the Lagrangian for the evaluated system of equations. Afterwards, we write the first and second-order normalization of the Hamiltonian of the problem. By implying KAM theorem and the techniques used by Bhatnagar and Hallan, we discuss the non-linear stability analytically.

Keywords: Heterogeneous spheroid; Modified Newtonian force; Non-linear sense; Densities

MSC 2020 No.: 70F15, 70F07

1. Introduction

The three-body problem has one of the most promising topics of Celestial Mechanics and space dynamics for centuries for many mathematicians and astronomers. Due to no success in finding the general solutions to the three-body problem and also due to its broad range of applications, many astronomers, physicists, and mathematicians were inspired to study the three-body problem with some restrictions. Therefore, this problem with restrictions is known as restricted three-body problem (*R3BP)*. The history of *R3BP* begins with Euler, who found the three-collinear points $L_1,\ L_2,\ L_3$. Then, Lagrange discussed the problem further and discovered an additional two points, i.e., non-collinear $L_4,\ L_5$. The total of these five points is generally called Libration points or Lagrangian points. Along with them, many other scientists and astronomers who studied the *R3BP* with different aspects, and the results that came out, has given this problem to a different level.

From the last many decades, several papers have been published in the *R3BP* by taking into the consideration the effect of several force factors such as centrifugal and Coriolis force, Newtonian force, modified Newtonian force, Yukawa effects, viscous force, and the other kind of forces, on the motion of the smallest body such as Bhatnagar and Chawla (1977), Singh and Ishwar (1985), Kokubun (2004), Abdulraheem and Singh (2008), Lukyanov (2009), Ershkov and Shamin (2012), Abouelmagd and Shaboury (2012), Zoto (2016), Ansari et al. (2019).

There are many more, other aspects and configurations in *R3BP* where there were a large variety of scopes to conduct research, as many authors examined the stability in the linear sense of Libration points and the collinear points was found to be unstable whereas triangular points are stable in some range. Several mathematician have done their research on non-linear stability by taking different configuration in *R3BP*. Deprit and Bartholome (1967) discussed the non-linear stability of $L_4,\ L_5$ by applying Arnold's Theorem (Arnold (1961)). Bhatnagar and Hallan (1983) had examined the effect of perturbations in coriolis and centifugal forces on the non-linear stability of libration point in *R3BP*. Many other researchers, such as Gyorgyey (1985), Krzysztof et al. (1991), SubbaRao and Sharma (1997), Esteban and Vazquez (2001), Jain et al. (2001), Chandra and Kumar (2004), Kushvah et al. (2007), Singh (2011), Ishwar and Sharma (2012), Jain and Sinha (2014), Shalini et al. (2016), Ansari and Alam (2017), Ansari (2017), Ansari (2021a), have discussed the non-linear stability by taking different configuration in *R3BP*.

The R3BP has fascinated and astonished a very large section of researchers and mathematicians, since this problem is easier to analyze in comparison to the general three-body problem and we have gained a huge amount of knowledge about space and celestial bodies by working on this problem. In our problem, we have extended the work of Ansari (2021a) who has taken the primary as a heterogeneous spheroid and the secondary body that is producing Modified Newtonian Potential. He has studied the Stability of Libration points in a linear sense. We have taken the same combination and discussed the linear stability in a non-linear sense, by using the KAM theorem and following the methodology by Bhatnagar and Hallan (1983).

This paper has been outlined in seven sections. The first section is comprised of an introduction, where we have discussed the literature review of R3BP under various configurations. In the second section, we have shown the total gravitational potential exerted by both the bodies on the smallest body and the Equation of Motion along with the location of the two triangular libration points. In the third section, we have evaluated the Lagrangian for the system of the equation of motion. In the fourth and fifth sections, we have done First and Second-order Normalization, respectively. In the sixth section, we have calculated the non-linear stability of the triangular libration points. The seventh section contains the concluding remarks.

2. Equations of Motion

In our problem, we have considered three bodies: primary body having m_1 mass, secondary body with mass m_2, and the smallest body having negligible mass m_3. The primary body is taken as heterogeneous with N-layers having different densities of each layer. The secondary body is a point mass which is producing modified Newtonian potential with perturbing parameter ϵ. Both primary and secondary are revolving in circular orbits, about their common center of mass which is taken as origin O and the smallest body is moving in a plane of motion of mass m_1 and m_2 which is being influenced by their motion but not influencing them. In this system, we have also considered small perturbations in the Coriolis and centrifugal forces by taking parameters α and β, respectively, while the unperturbed value for each is unity. We have followed the synodic coordinate system, rotating with angular velocity n. The total gravitational potential employed by both the bodies (primary and secondary) on the smallest body will be (see Ansari (2021a)):

$$V = -\frac{Gm_1m_3}{r_1} - \frac{Gm_3}{2r_1^3}\left[h_{11} - \frac{3}{r_1^2}h_{12}y^2\right] - \frac{Gm_2m_3r_2}{(r_2^2+\epsilon)}. \tag{1}$$

Let's assume the non-dimensional units, as the sum of masses, be unity, i.e., $m_1 + m_2 = 1$. The distance R between the primary and secondary is also taken as one, i.e., $R = R_1 + R_2 = 1$ where R_1 and R_2 are the distances of primary and secondary from the origin O, respectively, and also $R_2 > R_1$ as well as $G = 1$. Let

$$\mu = \frac{m_2}{m_1+m_2} < \frac{1}{2},$$

then $m_2 = \mu$ and $m_1 = 1 - \mu$. Also, J_j is the dimensionless quantities of h_{1j} for $j = 1,\ 2$. Therefore, using dimensionless variables and synodic co-ordinate system, the equations of motion for the third body will be written as follows:

$$\begin{aligned} \ddot{x} - 2\,n\,\alpha\,\dot{y} &= U_x, \\ \ddot{y} + 2\,n\,\alpha\,\dot{x} &= U_y, \end{aligned} \tag{2}$$

with

$$
\begin{aligned}
n^2 &= 1 - 3\epsilon + \frac{3}{2}\left(\frac{J_1}{1-\mu}\right),\\
U &= \frac{n^2\,\beta}{2}\left(x^2+y^2\right) + \frac{1-\mu}{r_1} + \frac{1}{2\,r_1^3}\left(J_1 - \frac{3}{r_1^2}J_2\,y^2\right) + \frac{\mu\,r_2}{r_2^2+\epsilon}, \qquad (3)\\
U_x &= n^2\,\beta\,x - \frac{\mu(x-1+\mu)(r_2^2-\epsilon)}{r_2(r_2^2+\epsilon)^2} - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(x+\mu)}{2\,r_1^5}\left(J_1 - \frac{5}{r_1^2}J_2\,y^2\right),\\
U_y &= \left(n^2\,\beta - \frac{\mu(r_2^2-\epsilon)}{r_2(r_2^2+\epsilon)^2} - \frac{1-\mu}{r_1^3} - \frac{3}{2\,r_1^5}\left(J_1 + 2\,J_2 - \frac{5}{r_1^2}J_2\,y^2\right)\right)y, \qquad (4)\\
r_1^2 &= (x+\mu)^2 + y^2, \quad \text{and} \quad r_2^2 = (x-1+\mu)^2 + y^2. \qquad (5)
\end{aligned}
$$

Now, we can find the location of the libration points by equating all the derivatives with respect to time, given in system (2) to zero, i.e.,

$$
U_x = 0 \quad \text{and} \quad U_y = 0. \qquad (6)
$$

By solving Equation (6), we can obtain three collinear and two non-collinear libration points. The location of the non-collinear libration points $L_4(x,\ y)$ and $L_5(x,\ -y)$ are given as (see Ansari (2021a)),

$$
\begin{aligned}
x &= \frac{\gamma}{2} + \delta_1 - \delta_2,\\
y &= \pm\frac{\sqrt{3}}{2}\left(1 + \frac{2}{3}\left(\delta_1 + \delta_2\right)\right), \qquad (7)
\end{aligned}
$$

where δ_1 and δ_2 are the parameters due to perturbations and $\gamma = 1 - 2\,\mu$.

3. Lagrangian of the Equation of Motion

The Lagrangian of the equation of motion is given by

$$
L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + n^2\beta(x^2+y^2) + 2n(x\dot{y} - y\dot{x})\right) + \frac{1-\mu}{r_1} + \frac{J_1}{2\,r_1^3} - \frac{3\,J_2}{2\,r_1^5}\,y^2 + \frac{\mu\,r_2}{r_2^2+\epsilon}. \qquad (8)
$$

The origin of above equation is shifted to $L_4(x, y)$ and expand L in power series of x and y, and expressed as follows:

$$
L = L_0 + L_1 + L_2 + L_3 + L_4 \cdots,
$$

where

$$
\begin{aligned}
L_0 &= T_0,\\
L_1 &= -n\,\dot{x}\left(\frac{\delta_1}{\sqrt{3}} + \frac{\delta_2}{\sqrt{3}} + \frac{\sqrt{3}}{2}\right) + n\,\dot{y}\left(\frac{\gamma}{2} + \delta_1 - \delta_2\right) + T_1\,x + T_2\,y,\\
L_2 &= n\left(\dot{y}\,x - \dot{x}\,y\right) + \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + T_3\,x^2 + T_4\,y^2 + T_5\,xy,\\
L_3 &= T_6\,x^3 + T_7\,y^3 + T_8\,x^2y + T_9\,xy^2,\\
L_4 &= T_{10}\,x^4 + T_{11}\,y^4 + T_{12}\,x^3y + T_{13}\,xy^3 + T_{14}\,x^2y^2.
\end{aligned}
$$

The values of $T_i' s$ (where $i = 1, 2, 3, \ldots, 14$) are given in Appendix A.

The Lagrange equations of motion in the first order are

$$\begin{aligned} &\ddot{x} - 2n\dot{y} - 2T_3 x - T_5 y = 0, \\ &\ddot{y} + 2n\dot{x} - 2T_4 x - T_5 x = 0. \end{aligned} \tag{9}$$

The characteristic polynomial of Equation (9) is given in the following manner,

$$\lambda^4 + (4n^2 - 2T_4 - 2T_3)\lambda^2 + 4T_3T_4 - T_5^2 = 0. \tag{10}$$

Let's assume that $\pm\, i\, \omega_1'$ and $\pm\, i\, \omega_2'$ are the roots of Equation (10). Here, ω_1' and ω_2' are long term and short term perturbed frequencies, respectively, which is related to each other by the following relation,

$$\begin{aligned} &\omega_1'^2 + \omega_2'^2 = 4n^2 - 2T_4 - 2T_3, \\ &\omega_1'^2 * \omega_2'^2 = 4T_3T_4 - T_5^2, \qquad \left(0 < \omega_1' < \omega_2' < \frac{1}{2}\right). \end{aligned} \tag{11}$$

Let's consider ω_1 and ω_2 as unperturbed basic frequencies. Then we have,

$$\omega_1^2 + \omega_2^2 = 1, \quad \omega_1^2\omega_2^2 = \frac{27}{16}(1 - \gamma^2).$$

Now, there is a relation between perturbed frequencies $(\omega_1',\ \omega_2')$ and unperturbed frequencies $(\omega_1,\ \omega_2)$ which is given as,

$$\begin{aligned} &\omega_1' = \omega_1(1 + p_1J_1 + p_2J_2 + p_3\delta_1 + p_4\delta_2), \\ &\omega_2' = \omega_2(1 + q_1J_1 + q_2J_2 + q_3\delta_1 + q_4\delta_2), \end{aligned} \tag{12}$$

where

$$p_1 = \frac{3\left(45\gamma + 8\omega_1^2 - 69\right)}{32k^2\omega_1^2}, \qquad q_1 = \frac{3\left(45\gamma + 8\omega_2^2 - 69\right)}{32\left(-k^2\right)\omega_2^2},$$

$$p_2 = \frac{3\left(-585\gamma + 184\omega_1^2 + 273\right)}{128k^2\omega_1^2}, \qquad q_2 = \frac{3\left(184\omega_2^2 - 585\gamma + 273\right)}{128\left(-k^2\right)\omega_2^2},$$

$$p_3 = \frac{3\left(-33\gamma^2 + 8\gamma\omega_1^2 - 24\gamma + 8\omega_1^2 + 9\right)}{32k^2\omega_1^2}, \quad q_3 = \frac{3\left(+8\gamma\omega_2^2 - 24\gamma - 33\gamma^2 + 8\omega_2^2 + 9\right)}{32\left(-k^2\right)\omega_2^2},$$

$$p_4 = -\frac{3\left(33\gamma^2 + 8\gamma\omega_1^2 - 24\gamma - 8\omega_1^2 - 9\right)}{32k^2\omega_1^2}, \quad q_4 = -\frac{3\left(33\gamma^2 + 8\gamma\omega_2^2 - 24\gamma - 8\omega_2^2 - 9\right)}{32\left(-k^2\right)\omega_2^2},$$

and

$$k^2 = 2\,\omega_1^2 - 1 = 1 - 2\,\omega_2^2.$$

4. First Order Normalization

The Hamiltonian function, analogous to the Lagrangian L, is given as follows,

$$H = -\,L + p_x\dot{x} + p_y\dot{y}, \tag{13}$$

where p_x and p_y are momenta coordinates, which is given by

$$p_x = \dot{x} - ny, \quad p_y = \dot{y} + nx. \tag{14}$$

Using the values of $L,\ p_x$, and p_y into Equation (13), the Hamiltonian H transforms as follows,

$$\begin{aligned} H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}n^2(x^2 + y^2)(1-\beta) + n(yp_x - xp_y) \\ -\frac{1-\mu}{r_1} - \frac{J_1}{2r_1^3} + \frac{3J_2}{2r_1^5}y^2 - \frac{\mu r_2}{r_2^2 + \epsilon}. \end{aligned} \tag{15}$$

After using the following translation, and expanding in power series of x and y, the Hamiltonian in (15) transforms as

$$p_x \to p_x - n\frac{\sqrt{3}}{2}\left(\frac{2(\delta_1+\delta_2)}{3} + 1\right), \qquad p_y \to p_y + n\left(\frac{\gamma}{2} + \delta_1 - \delta_2\right),$$
$$x \to x + \frac{\gamma}{2} + \delta_1 - \delta_2, \qquad y \to y + \frac{\sqrt{3}}{2}\left(\frac{2(\delta_1+\delta_2)}{3} + 1\right).$$

$$H = \sum_{k=0}^{\infty} H_k, \tag{16}$$

where H_k is defined as the sum of the terms of k^{th} degree homogeneous in variable $x,\ y,\ p_x,\ p_y$. We have

$$H = H_0 + H_1 + H_2 + H_3 + H_4 \cdots,$$

where

$$\begin{aligned} &H_0 = -L_0, \qquad H_1 = E_1\, x + E_2\, y, \\ &H_2 = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \left(1 - \frac{3\epsilon}{2} + \frac{3J_1}{2}\right)(y\, p_x - x\, p_y) + E_3\, x^2 + E_4\, y^2 + E_5\, xy, \\ &H_3 = -L_3, \qquad H_4 = -L_4. \end{aligned}$$

The values of $E_i's$ (where $i = 1, 2, 3, 4, 5$) are given in Appendix A. The mechanism from Whittaker (1965) has been used to examine the stability of motion. For this we consider the set of linear equations in the variables x and y,

$$\begin{aligned} -\lambda p_x &= \frac{\partial H_2}{\partial x} = -np_y + 2E_3 x + E_5 y, \\ -\lambda p_y &= \frac{\partial H_2}{\partial y} = np_x + 2E_4 y + E_5 x, \\ \lambda x &= \frac{\partial H_2}{\partial P} = p_x + n\, y, \\ \lambda y &= \frac{\partial H_2}{\partial Q} = p_y - n\, x. \end{aligned} \tag{17}$$

The above equation can be written in matrix form as $A\,X = 0$, where

$$A = \begin{pmatrix} 2E_3 & E_5 & \lambda & -n \\ E_5 & 2E_4 & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}. \tag{18}$$

Equation (18) has non-zero solution if and only if $\det(A) = 0$ which implies

$$\lambda^4 + 2\lambda^2 \left(E_3 + E_4 + n^2\right) + n^4 - 2n^2(E_3 + E_4) + 4E_3E_4 - E_5^2 = 0. \tag{19}$$

Stability is insured only when the discriminant of the characteristic Equation (19) is greater than zero, and this implies that

$$\mu < \mu_c = \mu_0 + (1.13591)\epsilon + (-0.70627)J_1 + (-1.3162)J_2 + (-1.01035)\delta_1 + (0.100765)\delta_2,$$

where $\mu_0 = 0.0385208965...$

We have taken Equation (17) to express H_2 in its normal form and its solution is found in the following manner,

$$\frac{x}{-E_5 + 2n\lambda} = \frac{y}{2E_3 + \lambda^2 - n^2} = \frac{p_x}{n\lambda^2 - \lambda E_5 - 2nE_3 + n^3} = \frac{q_x}{\lambda^3 + n_2\lambda + 2E_3\lambda - nE_5}.$$

Using the methodology given by Whittaker (1965), we have implemented the canonical transformation from the phase space (x, y, p_x, p_y) into the phase space of the angles $(\phi_1,\ \phi_2)$ and the actions (I_1, I_2), to normalize the Hamiltonian H_2,

$$F = JM,$$

where,

$$F = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix}, \quad M = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix}, \quad J = (a_{ij})_{(1 \le i,j \le 4)}, \quad Q_i = \left(\frac{2I_i'}{\omega_i'}\right)^{\frac{1}{2}} \sin(\phi_i),$$

$$P_i = \left(2I_i'\,\omega_i'\right)^{\frac{1}{2}} \cos(\phi_i), \quad (i = 1,\ 2). \tag{20}$$

All the elements a_{ij} of J have been calculated and given below,

$$
\begin{aligned}
&a_{11}=0, \qquad a_{12}=0,\\
&a_{13}=\frac{l_1}{2k\omega_1}+\frac{3J_1}{64k^5l_1\omega_1^3}\Bigg[-8(45\gamma+29)\omega_1^4+4(659-405\gamma)\omega_1^2\\
&\quad+405\gamma+608\omega_1^6-621\Bigg]+\frac{3J_2}{256k^5l_1\omega_1^3}\Bigg[585\gamma\Bigg(8\omega_1^4+36\omega_1^2-9\Bigg)\\
&\quad-4\Bigg(376\omega_1^4+1182\omega_1^2+2551\Bigg)\omega_1^2+2457\Bigg]+\frac{\delta_1}{24k^5l_1\omega_1^3}\Bigg[4(127-99\gamma)\omega_1^6\\
&\quad+18(23\gamma-48)\omega_1^4+9(97\gamma+127)\omega_1^2-243(\gamma+1)+176\omega_1^8\Bigg]+\frac{\delta_2}{24k^5l_1\omega_1^3}\\
&\quad\Bigg[4(99\gamma+127)\omega_1^6-18(23\gamma+48)\omega_1^4+9(127-97\gamma)\omega_1^2+243(\gamma-1)+176\omega_1^8\Bigg]\\
&\quad+\frac{\epsilon}{4k^3l_1t_1\omega_1}\Bigg[-4(3\gamma-16)\omega_1^4-(239-123\gamma)\omega_1^2-27(15\gamma+7)-80\omega_1^6\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
a_{21}=&-\frac{4\omega_1}{kl_1}+\frac{3J_1}{8k^5l_1^3\omega_1}\Bigg[45\gamma\Bigg(8\omega_1^4+9\Bigg)-8\Bigg(32\omega_1^6-36\omega_1^4+95\omega_1^2-19\Bigg)\omega_1^2-621\Bigg]\\
&-\frac{3J_2}{32k^5l_1^3\omega_1}\Bigg[585\gamma\Bigg(8\omega_1^4+9\Bigg)+8\Bigg(188\omega_1^4-139\omega_1^2-367\Bigg)\omega_1^2-2457\Bigg]\\
&-\frac{\delta_1}{3k^5l_1^3\omega_1}\Bigg[4(99\gamma-17)\omega_1^6-54(\gamma-8)\omega_1^4-9(7\gamma+37)\omega_1^2+243(\gamma+1)+176\omega_1^8\Bigg]\\
&+\frac{\delta_2}{3k^5l_1^3\omega_1}\Bigg[4(99\gamma+17)\omega_1^6-54(\gamma+8)\omega_1^4+9(37-7\gamma)\omega_1^2+243(\gamma-1)-176\omega_1^8\Bigg]\\
&+\frac{\epsilon}{k^3l_1^3t_1}\Bigg[8(26-3\gamma)\omega_1^5+(750\gamma+638)\omega_1^3+54(6\gamma+5)\omega_1-288\omega_1^7\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
a_{23}=&-\frac{3\sqrt{3}\gamma}{2kl_1\omega_1}+\frac{3\sqrt{3}J_1}{64k^5l_1^3\omega_1^3}\Bigg[32(69\gamma-40)\omega_1^6+8(355-807\gamma)\omega_1^4+60(63-89\gamma)\omega_1^2+81(23\gamma\\
&-15)+640\omega_1^8\Bigg]+\frac{3\sqrt{3}J_2}{256k^5l_1^3\omega_1^3}\Bigg[160(27\gamma+104)\omega_1^6+8(3987\gamma-4615)\omega_1^4+12(1817\gamma\\
&-4095)\omega_1^2+1053(15-7\gamma)-8320\omega_1^8\Bigg]-\frac{\delta_1}{8\sqrt{3}k^5l_1^3\omega_1^3}\Bigg[16(33\gamma-32)\omega_1^8+12(141-97\gamma)\omega_1^6\\
&+2(456\gamma-499)\omega_1^4+81(29\gamma+35)\omega_1^2-729(\gamma+1)+448\omega_1^{10}\Bigg]+\frac{\delta_2}{8\sqrt{3}k^5l_1^3\omega_1^3}\Bigg[-16(33\gamma\\
&+32)\omega_1^8+12(97\gamma+141)\omega_1^6-2(456\gamma+499)\omega_1^4+81(35-29\gamma)\omega_1^2+729(\gamma-1)\\
&+448\omega_1^{10}\Bigg]+\frac{\epsilon}{4\sqrt{3}k^3l_1^3t_1\omega_1}\Bigg[48(13-15\gamma)\omega_1^6+4(594\gamma-1211)\omega_1^4+9(331\gamma+249)\omega_1^2\\
&+243(3\gamma+11)-64\omega_1^8\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
a_{31} = &-\frac{\omega_1}{2kl_1}\Big[4\omega_1^2+1\Big]+\frac{3J_1}{64k^5l_1^3\omega_1}\Big[-480(3\gamma+4)\omega_1^6+8(1219-585\gamma)\omega_1^4+4(1215\gamma\\
&-1897)\omega_1^2+405\gamma+1152\omega_1^8-621\Big]+\frac{3J_2}{256k^5l_1^3\omega_1}\Big[585\gamma\Big(4\omega_1^2\Big(8\omega_1^4+26\omega_1^2-27\Big)-9\Big)\\
&+39188\omega_1^2-8\Big(720\omega_1^4+2512\omega_1^2+3567\Big)\omega_1^4+2457\Big]+\frac{\delta_1}{24k^5l_1^3\omega_1}\Big[144(7\gamma+10)\omega_1^8\\
&+4(1125\gamma-977)\omega_1^6+18(246-131\gamma)\omega_1^4-9(119\gamma+233)\omega_1^2-243(\gamma+1)+704\omega_1^{10}\Big]\\
&+\frac{\delta_2}{24k^5l_1^3\omega_1}\Big[144(10-7\gamma)\omega_1^8-4(1125\gamma+977)\omega_1^6+18(131\gamma+246)\omega_1^4+9(119\gamma-233)\omega_1^2\\
&+243(\gamma-1)+704\omega_1^{10}\Big]+\frac{\omega_1\epsilon}{4k^3l_1^3t_1}\Big[\omega_1^2\Big(16(3\gamma+53)\omega_1^4-4(72\gamma+593)\omega_1^2-2487\gamma\\
&+320\omega_1^6+211\Big)+27(87\gamma+47)\Big],
\end{aligned}
$$

$$
\begin{aligned}
a_{33} = &\frac{3\sqrt{3}\gamma}{2kl_1\omega_1}+\frac{3\sqrt{3}J_1}{64k^5l_1^3\omega_1^3}\Big[128(6\gamma-5)\omega_1^8+32(40-39\gamma)\omega_1^6+40(123\gamma-71)\omega_1^4+12(481\gamma\\
&-315)\omega_1^2+81(15-23\gamma)\Big]+\frac{3\sqrt{3}J_2}{256k^5l_1^3\omega_1^3}\Big[-160(27\gamma+104)\omega_1^6+8(4615-3987\gamma)\omega_1^4\\
&+12(4095-1817\gamma)\omega_1^2+1053(7\gamma-15)+8320\omega_1^8\Big]+\frac{\delta_1}{8\sqrt{3}k^5l_1^3\omega_1^3}\Big[16(33\gamma-32)\omega_1^8\\
&+12(141-97\gamma)\omega_1^6+2(456\gamma-499)\omega_1^4+81(29\gamma+35)\omega_1^2-729(\gamma+1)+448\omega_1^{10}\Big]\\
&+\frac{\delta_2}{8\sqrt{3}k^5l_1^3\omega_1^3}\Big[16(33\gamma+32)\omega_1^8-12(97\gamma+141)\omega_1^6+2(456\gamma+499)\omega_1^4+81(29\gamma\\
&-35)\omega_1^2-729(\gamma-1)-448\omega_1^{10}\Big]+\frac{\epsilon}{4\sqrt{3}k^3l_1^3t_1\omega_1}\Big[-48(3\gamma+13)\omega_1^6+(4844\\
&-4536\gamma)\omega_1^4-9(349\gamma+249)\omega_1^2+64\omega_1^8-2673\Big],
\end{aligned}
$$

$$
\begin{aligned}
a_{41} = {} & \frac{3\sqrt{3}\gamma\omega_1}{2kl_1} + \frac{3\sqrt{3}J_1}{64k^5l_1^3\omega_1}\bigg[32(70-57\gamma)\omega_1^6 + 8(477\gamma-545)\omega_1^4 + 24(60-37\gamma)\omega_1^2 + 81(23\gamma \\
& - 15) + 640\omega_1^8\bigg] - \frac{3\sqrt{3}J_2}{256k^5l_1^3\omega_1}\bigg[32(910-141\gamma)\omega_1^6 + 8(417\gamma-7085)\omega_1^4 + 24(367\gamma+780)\omega_1^2 \\
& + 1053(7\gamma-15) + 8320\omega_1^8\bigg] + \frac{\delta_1}{8\sqrt{3}k^5l_1^3\omega_1}\bigg[-16(33\gamma+68)\omega_1^8 + 4(203-381\gamma)\omega_1^6 \\
& + 2(1704\gamma+479)\omega_1^4 - 27(59\gamma+29)\omega_1^2 + 729(\gamma+1) + 704\omega_1^{10}\bigg] - \frac{\delta_2}{8\sqrt{3}k^5l_1^3\omega_1}\bigg[16(33\gamma \\
& - 68)\omega_1^8 + 4(381\gamma+203)\omega_1^6 + 2(479-1704\gamma)\omega_1^4 + 27(59\gamma-29)\omega_1^2 - 729(\gamma-1) \\
& + 704\omega_1^{10}\bigg] + \frac{\omega_1\epsilon}{4\sqrt{3}k^3l_1^3t_1}\bigg[48(15\gamma-13)\omega_1^6 + 4(1211-594\gamma)\omega_1^4 - 9(331\gamma \\
& + 249)\omega_1^2 - 243(3\gamma+11) + 64\omega_1^8\bigg],
\end{aligned}
$$

$$
\begin{aligned}
a_{43} = {} & \frac{9-4\omega_1^2}{2kl_1\omega_1} + \frac{3J_1}{64k^5l_1^3\omega_1^3}\bigg[45\gamma\bigg(8\bigg(4\omega_1^4-27\omega_1^2-27\bigg)\omega_1^2+81\bigg) + 8\bigg(-128\omega_1^8+1040\omega_1^6 \\
& - 88\omega_1^4 + 705\omega_1^2 + 2196\bigg)\omega_1^2 - 5589\bigg] - \frac{3J_2}{256k^5l_1^3\omega_1^3}\bigg[585\gamma\bigg(8\bigg(4\omega_1^4-27\omega_1^2-27\bigg)\omega_1^2 \\
& + 81\bigg) + 8\bigg(2256\omega_1^6 + 2944\omega_1^4 + 7485\omega_1^2 + 7794\bigg)\omega_1^2 - 22113\bigg] - \frac{\delta_1}{24k^5l_1^3\omega_1^3}\bigg[16(297\gamma \\
& - 260)\omega_1^8 + 36(41\gamma+65)\omega_1^6 + 54(10-143\gamma)\omega_1^4 - 81(61\gamma+91)\omega_1^2 + 2187(\gamma+1) \\
& + 704\omega_1^{10}\bigg] + \frac{\delta_2}{24k^5l_1^3\omega_1^3}\bigg[16(297\gamma+260)\omega_1^8 + 36(41\gamma-65)\omega_1^6 - 54(143\gamma+10)\omega_1^4 \\
& + 81(91-61\gamma)\omega_1^2 + 2187(\gamma-1) - 704\omega_1^{10}\bigg] + \frac{\epsilon}{4k^3l_1^3t_1\omega_1}\bigg[-16(9\gamma+91)\omega_1^6 \\
& + 12(282\gamma-5)\omega_1^4 + 9(87\gamma-185)\omega_1^2 - 243(15\gamma+4) - 1856\omega_1^8\bigg],
\end{aligned}
$$

$$
\begin{aligned}
a_{44} = &-\frac{9-4\omega_2^2}{2kl_2\omega_2} + \frac{\epsilon}{4k^3l_2^3t_2\omega_2}\Bigg[1856\omega_2^8 + 16(9\gamma+91)\omega_2^6 + 12(5-282\gamma)\omega_2^4 + 9(185-87\gamma)\omega_2^2 \\
&+ 243(15\gamma+4)\Bigg] + \frac{3J_1}{64k^5l_2^3\omega_2^3}\Bigg[45\gamma\Bigg(8\Bigg(4\omega_2^4 - 27\omega_2^2 - 27\Bigg)\omega_2^2 + 81\Bigg) + 8\Bigg(-128\omega_2^8 \\
&+ 1040\omega_2^6 - 88\omega_2^4 + 705\omega_2^2 + 2196\Bigg)\omega_2^2 - 5589\Bigg] - \frac{3J_2}{256k^5l_2^3\omega_2^3}\Bigg[585\gamma\Bigg(8\Bigg(4\omega_2^4 - 27\omega_2^2 \\
&- 27\Bigg)\omega_2^2 + 81\Bigg) + 8\Bigg(2256\omega_2^6 + 2944\omega_2^4 + 7485\omega_2^2 + 7794\Bigg)\omega_2^2 - 22113\Bigg] \\
&- \frac{\delta_1}{24k^5l_2^3\omega_2^3}\Bigg[16(297\gamma-260)\omega_2^8 + 36(41\gamma+65)\omega_2^6 + 54(10-143\gamma)\omega_2^4 - 81(61\gamma \\
&+ 91)\omega_2^2 + 2187(\gamma+1) + 704\omega_2^{10}\Bigg] + \frac{\delta_2}{24k^5l_2^3\omega_2^3}\Bigg[16(297\gamma+260)\omega_2^8 + 36(41\gamma-65)\omega_2^6 \\
&- 54(143\gamma+10)\omega_2^4 + 81(91-61\gamma)\omega_2^2 + 2187(\gamma-1) - 704\omega_2^{10}\Bigg],
\end{aligned}
$$

$$
l_1 = \sqrt{9+4\omega_1^2}, \qquad t_1 = 3 + 4\,\omega_1^2.
$$

We can derive the values of $a_{ij}(j = 2,\ 4)$ from $a_{ij}(j = 1,\ 3)$ by replacing $\omega_1 \to \omega_2, l_1 \to l_2$, $t_1 \to t_2$ and without making any change in k whenever they arise.

After applying the above transformation, the Hamiltonian H_2 reduces to its normal form as

$$
H_2 = \omega_1' I_1 - \omega_2' I_2.
$$

The general solution of the corresponding equations of motion is

$$
I_i = \text{const.}\ (i = 1,\ 2),\ \phi_1 = \omega_1'\, t + \text{const.},\ \phi_2 = -\omega_2'\, t + \text{const.} \tag{21}
$$

5. Second Order Normalization

Hamiltonian has been transformed to Birkhoff's normal form using Moser's conditions. This transformation can be accomplished by Birkhoff's normalization in which the coordinates $(x,\ y)$ of the infinitesimal body are to be extended in double D'Alembert series,

$$
x = \sum_{n\geq 1} B_n^{1,0}, \quad y = \sum_{n\geq 1} B_n^{0,1}, \tag{22}
$$

where the homogeneous components $B_n^{1,0}$ and $B_n^{0,1}$ of degree n in $\sqrt{I_1},\ \sqrt{I_2}$, are of the form

$$
\sum_{0\leq m\leq n} I_1^{\frac{1}{2}(n-m)}\, I_2^{\frac{1}{2}m} \sum_{(i,\ j)} (C_{n-m,m,i,j}\ \cos(i\phi_1 + j\phi_2) + S_{n-m,m,i,j}\ \sin(i\phi_1 + j\phi_2)). \tag{23}
$$

The conditions on i, j are given as

- The component i works on those values of integers in the interval $0 \leq i \leq n-m$ whose association is the same as $n-m$.
- The component j works on those values of integers in the interval $-m \leq i \leq m$ whose association is the same as m.

I_1, I_2 are the action momenta coordinates which are considered as constant of integration and angle coordinates $\phi_1,\ \phi_2$ are taken as a linear functions of time such that

$$\dot{\phi_1} = \omega_1' + \sum_{n\geq 1} f_{2n}\ (I_1, I_2),\ \ \dot{\phi_2} = -\omega_2' + \sum_{n\geq 1} g_{2n}\ (I_1, I_2), \tag{24}$$

where f_{2n} and g_{2n} are of the form,

$$f_{2n} = \sum_{0\leq m\leq n} f'_{2(n-m),2m} I_1^{n-m}\ I_2^m,\ g_{2n} = \sum_{0\leq m\leq n} g'_{2(n-m),2m} I_1^{n-m}\ I_2^m. \tag{25}$$

The first order components $B_1^{1,0}$ and $B_1^{0,1}$ are the values of x and y given by (23). The second order components $B_2^{1,0}$ and $B_2^{0,1}$ can be attained as the solutions of the partial differential equations written as follows,

$$\Delta_1\ \Delta_2\ B_2^{1,0} = \Psi_1, \quad \Delta_1\ \Delta_2\ B_2^{0,1} = \Psi_2, \tag{26}$$

where

$$\begin{aligned}
\Delta_i &= (D^2 + \omega_i'^2), (i=1,2), \ \ D = \omega_1' \frac{\partial}{\partial \phi_1} - \omega_2' \frac{\partial}{\partial \phi_2}, \\
\Psi_1 &= (A_1 D^2 + A_2 D + A_3)(B_1^{1,0})^2 + (B_1 D^2 + B_2 D + B_3)(B_1^{0,1})^2 \\
&\quad + (C_1 D^2 + C_2 D + C_3)(B_1^{1,0})(B_1^{0,1}), \\
\Psi_2 &= (A_1' D^2 + A_2' D + A_3')(B_1^{1,0})^2 + (B_1' D^2 + B_2' D + B_3')(B_1^{0,1})^2 \\
&\quad + (C_1' D^2 + C_2' D + C_3')(B_1^{1,0})(B_1^{0,1}),
\end{aligned}$$

and the values of $A_i,\ B_i,\ C_i,\ A_i',\ B_i',\ C_i'\ (i=1,2,3)$, are given in Appendix A.

By following the Bhatnagar and Hallan (1983) method, the second-order components $B_2^{1,0}$ and $B_2^{0,1}$ are

$$\begin{aligned}
B_2^{1,0} &= r_1\ I_1 + r_2\ I_2 + r_3\ I_1\ \cos 2\phi_1 + r_4\ I_2\ \cos 2\phi_2 + r_5\ I_1\ \sin 2\phi_1 \\
&\quad + r_6\ I_2\ \sin 2\phi_2 + \{(r_7\ \cos(\phi_1+\phi_2) + r_8\ \cos(\phi_1-\phi_2) \\
&\quad + r_9\ sin(\phi_1+\phi_2) + r_{10}\ sin(\phi_1-\phi_2))\}\sqrt{I_1\ I_2}, \\
B_2^{0,1} &= s_1\ I_1 + s_2\ I_2 + s_3\ I_1\ \cos 2\phi_1 + s_4\ I_2\ \cos 2\phi_2 + s_5\ I_1\ \sin 2\phi_1 \\
&\quad + s_6\ I_2\ \sin 2\phi_2 + \{(s_7\ \cos(\phi_1+\phi_2) + s_8\ \cos(\phi_1-\phi_2) \\
&\quad + s_9\ \sin(\phi_1+\phi_2) + s_{10}\ \sin(\phi_1-\phi_2))\}\sqrt{I_1\ I_2},
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
r_i &= \alpha_{i,1} + \alpha_{i,2}\epsilon + \alpha_{i,3}J_1 + \alpha_{i,4}J_2 + \alpha_{i,5}\delta_1 + \alpha_{i,6}\delta_2, \\
s_i &= \beta_{i,1} + \beta_{i,2}\epsilon + \beta_{i,3}J_1 + \beta_{i,4}J_2 + \beta_{i,5}\delta_1 + \beta_{i,6}\delta_2 \qquad (i=1,2,...,10).
\end{aligned}$$

The values of $\alpha_{i,1}, \alpha_{i,2}, \alpha_{i,3}, \alpha_{i,4}, \alpha_{i,5}$ and $\beta_{i,1}, \beta_{i,2}, \beta_{i,3}, \beta_{i,4}, \beta_{i,5}$ are very lengthy and contained in a large number of pages so one can obtain these values from the author on request.

6. Non-Linear Stability

We have evaluated the non-linear stability of the libration points L_4 and L_5 numerically, using the KAM theorem. The first condition of given theorem has been flouted when, $\omega_1' = 2\omega_2'$ and $\omega_1' = 3\omega_2'$. So we have analyzed both the cases distinctly as follows.

Case I: $\omega_1' = 2\omega_2'$:

By the use of the above condition, we have cancelled out ω_1' and ω_2' from Equation (11) and then solving for $\mu = \mu_1$, we have obtained

$$\mu_1 = 0.0242939 + 0.960936\,\epsilon - 0.689806\,J_1 - 1.64008\,J_2 - 1.30833\,\delta_1 - 0.117218\,\delta_2. \tag{28}$$

Case II: $\omega_1' = 3\omega_2'$:

By the use of above condition, we have cancelled out ω_1' and ω_2' from Equation (11) and then solving for $\mu = \mu_2$, we have obtained

$$\mu_2 = 0.013516 + 1.1241\epsilon - 0.678806\,J_1 - 1.8776\,J_2 - 1.31961\,\delta_1 - 0.066281\,\delta_2. \tag{29}$$

We have deduced the first and second order normalization to get the Hamiltonian H in its normalised form, so now we can employ the Moser's modified version of Arnold (1961). Now, using Bhatnagar and Hallan (1983) method, we have chosen the relevant coefficients in the second order polynomials F_2, G_2 to annihilate the critical terms,

$$F_2 = F_{2,0}\,I_1 + F_{0,2}\,I_2,\; G_2 = G_{2,0}\,I_1 + G_{0,2}\,I_2, \tag{30}$$

where

$$F_{2,0} = \nu_1 + \nu_2 J_1 + \nu_3 J_2 + \nu_4 \delta_1 + \nu_5 \delta_2 = A, \tag{31}$$
$$F_{0,2} = G_{2,0} = \nu_6 + \nu_7 J_1 + \nu_8 J_2 + \nu_9 \delta_1 + \nu_{10} \delta_2 = B, \tag{32}$$
$$G_{0,2} = \nu_{11} + \nu_{12} J_1 + \nu_{13} J_2 + \nu_{14} \delta_1 + \nu_{15} \delta_2 = C. \tag{33}$$

The values of $\nu_i,\ (i = 1, 2, ..., 15)$ is given in Appendix B.

Now, the normalized Hamiltonian up to order 4 is given as:

$$H = \omega_1'\,I_1 - \omega_2'\,I_2 + \frac{1}{2}\left(A\,I_1^2 + 2B\,I_1\,I_2 + C\,I_2^2\right). \tag{34}$$

The Determinant D is:

$$D = -(A\omega_2'^2 + 2B\omega_1\omega_2 + C\omega_1^2). \tag{35}$$

Now, placing the values of A, B, C from Equation (31), Equation (32), Equation (33), using Equations (11) and (12) and the strategy given in Bhatnagar and Hallan (1983), we have obtained,

$$\begin{aligned}\mu_3 = 0.010936677... &+ (2.87552...)\,\epsilon + (-6.74161...)J_1 + (3.4985...)J_2\\ &+(-6.9897...)\delta_1 + (-6.8724...)\delta_2.\end{aligned} \tag{36}$$

7. Conclusion

We have investigated the non-linear stability of the triangular libration points by taking the combination of heterogeneous spheroid and point mass producing modified Newtonian potential. For investigating the non-linear stability of the triangular libration point, we discussed the Moser's modified version of Arnold's theorem and follow the procedure as adopted by Bhatnagar and Hallan. For this the Lagrangian function expanded in power series of the coordinates of the infinitesimal mass referred to the triangular libration point as origin.

We have computed the first order normalized Hamiltonian by following the method given in Whittaker. The canonical transformation from the phase space into the phase space of the angle coordinates and the action momenta, to the first order obtained so that the second order part of the Hamiltonian will be transformed to the normal form. Here co-ordinates have been expanded in double d'Alembert series, which are homogeneous components of order n. The fourth order part of the normalized Hamiltonian in Moser's theorem have been obtained. Also we have computed that the triangular points are stable for all mass ratios in the range of linear stability except for three mass ratios, which we have calculated with the help of Mathematica software where Moser's theorem has been failed to apply.

This analytical study is done under the effect of various perturbations, modified Newtonian potential produced by the point mass and other perturbations parameters. These perturbations makes this study different from the classical case. The values of these three mass ratios coincide with the classical case if we choose the perturbing parameters to be zero. The obtained results are highly valuable for examining the motion properties of an infinitesimal mass such as artificial satellites, minor planets, spacecraft, asteroid, etc., in our planetary system.

For our further study, we have taken the new model of Restricted three-body which consist of the heterogeneous primary having N-layers with distinct densities, the radiating finite straight segment as secondary body and the infinitesimal body. We will discuss the non-linear stability of the libration points analytically and numerically as well.

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Appendix A

$$
\begin{aligned}
T_0 &= \frac{3J_1\gamma^2}{8} - \frac{3\epsilon\gamma^2}{8} + \frac{\gamma^2}{8} + \frac{\epsilon\gamma}{2} + \frac{13J_1}{8} - \frac{9J_2}{8} - \frac{13\epsilon}{8} + \frac{11}{8},\\
T_1 &= \frac{3\gamma J_1}{2} - \frac{3J_1}{4} + \frac{45J_2}{16} + \frac{3\gamma\delta_1}{4} + \frac{3\delta_1}{4} + \frac{3\gamma\delta_2}{4} - \frac{3\delta_2}{4} - \frac{3\gamma\epsilon}{4} - \frac{3\epsilon}{4},\\
T_2 &= \frac{3J_1\sqrt{3}}{4} + \frac{21J_2\sqrt{3}}{16} + \frac{3}{4}\gamma\delta_1\sqrt{3} + \frac{3\delta_1\sqrt{3}}{4} - \frac{3}{4}\gamma\delta_2\sqrt{3} + \frac{3\delta_2\sqrt{3}}{4} - \frac{3}{4}\gamma\epsilon\sqrt{3} - \frac{3\epsilon\sqrt{3}}{4},\\
T_3 &= \frac{27J_1}{16} + \frac{21\gamma\delta_1}{16} + \frac{3\gamma\epsilon}{16} + \frac{3}{8} - \frac{3\delta_1}{16} - \frac{21\gamma\delta_2}{16} - \frac{3\delta_2}{16} - \frac{27\epsilon}{16} - \frac{135J_2}{64},\\
T_4 &= \frac{57J_1}{16} + \frac{33\gamma\delta_2}{16} + \frac{33\gamma\epsilon}{16} + \frac{9}{8} - \frac{33\gamma\delta_1}{16} - \frac{9\delta_1}{16} - \frac{9\delta_2}{16} - \frac{57\epsilon}{16} - \frac{141J_2}{64},\\
T_5 &= \frac{15J_1\sqrt{3}}{8} + \frac{3\gamma\sqrt{3}}{4} - \frac{11}{8}\gamma\delta_1\sqrt{3} - \frac{11}{8}\gamma\delta_2\sqrt{3} + \frac{3\delta_2\sqrt{3}}{8} - \frac{15}{8}\gamma\epsilon\sqrt{3} + \frac{15\epsilon\sqrt{3}}{8}\\
&\quad - \frac{3\delta_1\sqrt{3}}{8} - \frac{195J_2\sqrt{3}}{32},\\
T_6 &= \frac{25J_1}{32} + \frac{7\gamma}{16} + \frac{37\delta_2}{32} + \frac{25\epsilon}{32} - \frac{25\gamma\delta_1}{32} - \frac{37\delta_1}{32} - \frac{25\gamma\delta_2}{32} - \frac{25\gamma\epsilon}{32} - \frac{315J_2}{128},\\
T_7 &= \frac{15J_2\sqrt{3}}{128} + \frac{45}{32}\gamma\delta_1\sqrt{3} + \frac{\delta_1\sqrt{3}}{32} - \frac{45}{32}\gamma\delta_2\sqrt{3} + \frac{\delta_2\sqrt{3}}{32} - \frac{45}{32}\gamma\epsilon\sqrt{3} + \frac{45\epsilon\sqrt{3}}{32} - \frac{3\sqrt{3}}{16}\\
&\quad - \frac{45J_1\sqrt{3}}{32},\\
T_8 &= \frac{1215J_2\sqrt{3}}{128} - \frac{75}{32}\gamma\delta_1\sqrt{3} + \frac{41\delta_1\sqrt{3}}{32} + \frac{75}{32}\gamma\delta_2\sqrt{3} + \frac{41\delta_2\sqrt{3}}{32} - \frac{45}{32}\gamma\epsilon\sqrt{3} + \frac{45\epsilon\sqrt{3}}{32}\\
&\quad - \frac{3\sqrt{3}}{16} - \frac{45J_1\sqrt{3}}{32},\\
T_9 &= -\frac{255J_1}{32} + \frac{2685J_2}{128} + \frac{135\gamma\delta_1}{32} + \frac{123\delta_1}{32} + \frac{135\gamma\delta_2}{32} + \frac{255\gamma\epsilon}{32} - \frac{33\gamma}{16} - \frac{123\delta_2}{32} - \frac{255\epsilon}{32},\\
T_{10} &= -\frac{285J_1}{256} + \frac{7245J_2}{1024} + \frac{285\delta_1}{256} + \frac{115\gamma\delta_2}{256} + \frac{285\delta_2}{256} + \frac{285\epsilon}{256} - \frac{37}{128} - \frac{115\gamma\delta_1}{256} - \frac{285\gamma\epsilon}{256},\\
T_{11} &= \frac{555J_1}{256} + \frac{3525J_2}{1024} + \frac{165\delta_1}{256} + \frac{555\gamma\delta_2}{256} + \frac{165\delta_2}{256} + \frac{555\gamma\epsilon}{256} + \frac{3}{128} - \frac{555\gamma\delta_1}{256} - \frac{555\epsilon}{256},\\
T_{12} &= \frac{285\sqrt{3}\delta_1}{64} + \frac{215\gamma\delta_1}{64\sqrt{3}} + \frac{105J_2\sqrt{3}}{256} + \frac{105}{64}\gamma\epsilon\sqrt{3} - \frac{25\sqrt{3}\gamma}{32} - \frac{105\sqrt{3}J_1}{64} - \frac{285\sqrt{3}\delta_2}{64}\\
&\quad - \frac{105\sqrt{3}\epsilon}{64} + \frac{215\gamma\delta_2}{64\sqrt{3}},\\
T_{13} &= \frac{525J_1\sqrt{3}}{64} + \frac{45\gamma\sqrt{3}}{32} - \frac{185}{64}\gamma\delta_1\sqrt{3} - \frac{185}{64}\gamma\delta_2\sqrt{3} + \frac{345\delta_2\sqrt{3}}{64} - \frac{525}{64}\gamma\epsilon\sqrt{3} + \frac{525\epsilon\sqrt{3}}{64}\\
&\quad - \frac{345\delta_1\sqrt{3}}{64} - \frac{3885J_2\sqrt{3}}{256},\\
T_{14} &= \frac{1395J_1}{128} + \frac{645\gamma\delta_1}{128} + \frac{1395\gamma\epsilon}{128} + \frac{123}{64} - \frac{1035\delta_1}{128} - \frac{645\gamma\delta_2}{128} - \frac{1035\delta_2}{128} - \frac{1395\epsilon}{128} - \frac{29475J_2}{512},
\end{aligned}
$$

$$
\begin{aligned}
E_0 &= \frac{3\epsilon\gamma^2}{8} - \frac{3J_1\gamma^2}{8} - \frac{\gamma^2}{8} - \frac{\epsilon\gamma}{2} + \frac{9J_2}{8} + \frac{13\epsilon}{8} - \frac{13J_1}{8} - \frac{11}{8}, \\
E_1 &= -\frac{1}{2}3\gamma J_1 + \frac{3J_1}{4} + \frac{3\delta_2}{4} + \frac{3\gamma\epsilon}{4} + \frac{3\epsilon}{4} - \frac{3\gamma\delta_1}{4} - \frac{3\delta_1}{4} - \frac{3\gamma\delta_2}{4} - \frac{45J_2}{16}, \\
E_2 &= -\frac{1}{4}3\gamma\delta_1\sqrt{3} + \frac{3}{4}\gamma\delta_2\sqrt{3} + \frac{3}{4}\gamma\epsilon\sqrt{3} + \frac{3\epsilon\sqrt{3}}{4} - \frac{3J_1\sqrt{3}}{4} - \frac{3\delta_1\sqrt{3}}{4} - \frac{3\delta_2\sqrt{3}}{4} - \frac{21J_2\sqrt{3}}{16}, \\
E_3 &= -\frac{3J_1}{16} + \frac{135J_2}{64} + \frac{3\delta_1}{16} + \frac{21\gamma\delta_2}{16} + \frac{3\delta_2}{16} + \frac{3\epsilon}{16} + \frac{1}{8} - \frac{21\gamma\delta_1}{16} - \frac{3\gamma\epsilon}{16}, \\
E_4 &= -\frac{33J_1}{16} + \frac{141J_2}{64} + \frac{33\gamma\delta_1}{16} + \frac{9\delta_1}{16} + \frac{9\delta_2}{16} + \frac{33\epsilon}{16} - \frac{5}{8} - \frac{33\gamma\delta_2}{16} - \frac{33\gamma\epsilon}{16}, \\
E_5 &= \frac{195J_2\sqrt{3}}{32} + \frac{11}{8}\gamma\delta_1\sqrt{3} + \frac{3\delta_1\sqrt{3}}{8} + \frac{11}{8}\gamma\delta_2\sqrt{3} + \frac{15}{8}\gamma\epsilon\sqrt{3} - \frac{3\gamma\sqrt{3}}{4} - \frac{15J_1\sqrt{3}}{8} \\
&\quad - \frac{3\delta_2\sqrt{3}}{8} - \frac{15\epsilon\sqrt{3}}{8}, \\
E_6 &= -\frac{25J_1}{32} + \frac{315J_2}{128} + \frac{25\gamma\delta_1}{32} + \frac{37\delta_1}{32} + \frac{25\gamma\delta_2}{32} + \frac{25\gamma\epsilon}{32} - \frac{7\gamma}{16} - \frac{37\delta_2}{32} - \frac{25\epsilon}{32}, \\
E_7 &= \frac{45J_1\sqrt{3}}{32} - \frac{45}{32}\gamma\delta_1\sqrt{3} + \frac{45}{32}\gamma\delta_2\sqrt{3} + \frac{45}{32}\gamma\epsilon\sqrt{3} + \frac{3\sqrt{3}}{16} - \frac{\delta_1\sqrt{3}}{32} - \frac{\delta_2\sqrt{3}}{32} - \frac{45\epsilon\sqrt{3}}{32} \\
&\quad - \frac{15J_2\sqrt{3}}{128}, \\
E_8 &= \frac{45J_1\sqrt{3}}{32} + \frac{75}{32}\gamma\delta_1\sqrt{3} - \frac{75}{32}\gamma\delta_2\sqrt{3} + \frac{45}{32}\gamma\epsilon\sqrt{3} + \frac{3\sqrt{3}}{16} - \frac{41\delta_1\sqrt{3}}{32} - \frac{41\delta_2\sqrt{3}}{32} \\
&\quad - \frac{45\epsilon\sqrt{3}}{32} - \frac{1215J_2\sqrt{3}}{128}, \\
E_9 &= \frac{255J_1}{32} + \frac{33\gamma}{16} + \frac{123\delta_2}{32} + \frac{255\epsilon}{32} - \frac{135\gamma\delta_1}{32} - \frac{123\delta_1}{32} - \frac{135\gamma\delta_2}{32} - \frac{255\gamma\epsilon}{32} - \frac{2685J_2}{128}, \\
E_{10} &= \frac{285J_1}{256} + \frac{115\gamma\delta_1}{256} + \frac{285\gamma\epsilon}{256} + \frac{37}{128} - \frac{285\delta_1}{256} - \frac{115\gamma\delta_2}{256} - \frac{285\delta_2}{256} - \frac{285\epsilon}{256} - \frac{7245J_2}{1024}, \\
E_{11} &= -\frac{555J_1}{256} + \frac{555\gamma\delta_1}{256} + \frac{555\epsilon}{256} - \frac{3}{128} - \frac{165\delta_1}{256} - \frac{555\gamma\delta_2}{256} - \frac{165\delta_2}{256} - \frac{555\gamma\epsilon}{256} - \frac{3525J_2}{1024}, \\
E_{12} &= -\frac{1}{64}285\sqrt{3}\delta_1 - \frac{215\gamma\delta_1}{64\sqrt{3}} - \frac{105}{64}\sqrt{3}\gamma\epsilon + \frac{105J_1\sqrt{3}}{64} + \frac{25\gamma\sqrt{3}}{32} + \frac{285\delta_2\sqrt{3}}{64} + \frac{105\epsilon\sqrt{3}}{64} \\
&\quad - \frac{105\sqrt{3}J_2}{256} - \frac{215\gamma\delta_2}{64\sqrt{3}}, \\
E_{13} &= \frac{3885J_2\sqrt{3}}{256} + \frac{185}{64}\gamma\delta_1\sqrt{3} + \frac{345\delta_1\sqrt{3}}{64} + \frac{185}{64}\gamma\delta_2\sqrt{3} + \frac{525}{64}\gamma\epsilon\sqrt{3} - \frac{45\gamma\sqrt{3}}{32} - \frac{525J_1\sqrt{3}}{64} \\
&\quad - \frac{345\delta_2\sqrt{3}}{64} - \frac{525\epsilon\sqrt{3}}{64}, \\
E_{14} &= -\frac{1395J_1}{128} + \frac{29475J_2}{512} + \frac{1035\delta_1}{128} + \frac{645\gamma\delta_2}{128} + \frac{1035\delta_2}{128} + \frac{1395\epsilon}{128} - \frac{123}{64} - \frac{645\gamma\delta_1}{128} - \frac{1395\gamma\epsilon}{128}.
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= \frac{75J_1}{32} + \frac{21\gamma}{16} + \frac{111\delta_2}{32} + \frac{75\epsilon}{32} - \frac{75\gamma\delta_1}{32} - \frac{111\delta_1}{32} - \frac{75\gamma\delta_2}{32} - \frac{75\gamma\epsilon}{32} - \frac{945J_2}{128},\\
A_2 &= \frac{1215J_2\sqrt{3}}{64} - \frac{75}{16}\gamma\delta_1\sqrt{3} + \frac{41\delta_1\sqrt{3}}{16} + \frac{75}{16}\gamma\delta_2\sqrt{3} + \frac{41\delta_2\sqrt{3}}{16} - \frac{45}{16}\gamma\epsilon\sqrt{3} + \frac{27\epsilon\sqrt{3}}{8}\\
&\quad - \frac{27J_1\sqrt{3}}{8} - \frac{3\sqrt{3}}{8},\\
A_3 &= \frac{\delta_1\omega_1^4}{12} - \frac{\delta_2\omega_1^4}{12} - \frac{61\epsilon\omega_1^4}{12} + \frac{\delta_2\omega_1^2}{12} + \frac{61\epsilon\omega_1^2}{12} - \frac{\delta_1\omega_1^2}{12} + \frac{2565J_2}{128} + \frac{1737J_2\gamma}{64} + \frac{333\gamma\delta_1}{32}\\
&\quad + \frac{261\delta_1}{32} + \frac{333\gamma\delta_2}{32} + \frac{603\gamma\epsilon}{32} - \frac{27\gamma}{8} - \frac{261\delta_2}{32} - \frac{477\epsilon}{32} - \frac{405J_1}{64} - \frac{801J_1\gamma}{64},\\
B_1 &= -\frac{255J_1}{32} + \frac{2685J_2}{128} + \frac{135\gamma\delta_1}{32} + \frac{123\delta_1}{32} + \frac{135\gamma\delta_2}{32} + \frac{255\gamma\epsilon}{32} - \frac{33\gamma}{16} - \frac{123\delta_2}{32} - \frac{255\epsilon}{32},\\
B_2 &= \frac{45J_2\sqrt{3}}{64} + \frac{135}{16}\gamma\delta_1\sqrt{3} + \frac{3\delta_1\sqrt{3}}{16} - \frac{135}{16}\gamma\delta_2\sqrt{3} + \frac{3\delta_2\sqrt{3}}{16} - \frac{135}{16}\gamma\epsilon\sqrt{3} + \frac{81\epsilon\sqrt{3}}{8}\\
&\quad - \frac{81J_1\sqrt{3}}{8} - \frac{9\sqrt{3}}{8},\\
B_3 &= \frac{7\delta_1\omega_1^4}{12} - \frac{7\delta_2\omega_1^4}{12} - \frac{7\epsilon\omega_1^4}{12} + \frac{7\delta_2\omega_1^2}{12} + \frac{7\epsilon\omega_1^2}{12} - \frac{7\delta_1\omega_1^2}{12} + \frac{945J_1}{64} + \frac{333J_1\gamma}{64} + \frac{27\gamma}{8} + \frac{225\delta_2}{32}\\
&\quad + \frac{441\epsilon}{32} - \frac{297\gamma\delta_1}{32} - \frac{225\delta_1}{32} - \frac{297\gamma\delta_2}{32} - \frac{639\gamma\epsilon}{32} - \frac{531J_2\gamma}{64} - \frac{4725J_2}{128},\\
C_1 &= \frac{1215J_2\sqrt{3}}{64} - \frac{75}{16}\gamma\delta_1\sqrt{3} + \frac{41\delta_1\sqrt{3}}{16} + \frac{75}{16}\gamma\delta_2\sqrt{3} + \frac{41\delta_2\sqrt{3}}{16} - \frac{45}{16}\gamma\epsilon\sqrt{3} + \frac{45\epsilon\sqrt{3}}{16}\\
&\quad - \frac{3\sqrt{3}}{8} - \frac{45J_1\sqrt{3}}{16},\\
C_2 &= -\frac{1}{8}99\gamma J_1 - \frac{255J_1}{8} + \frac{2685J_2}{32} + \frac{135\gamma\delta_1}{8} + \frac{123\delta_1}{8} + \frac{135\gamma\delta_2}{8} + \frac{177\gamma\epsilon}{4} - \frac{33\gamma}{4} - \frac{123\delta_2}{8} - \frac{255\epsilon}{8},\\
C_3 &= 12\delta_1\sqrt{3}\gamma^2 + 12\delta_2\sqrt{3}\gamma^2 + \frac{315}{16}\epsilon\sqrt{3}\gamma^2 - \frac{99\sqrt{3}\gamma^2}{32} - \frac{315}{16}\sqrt{3}J_1\gamma - \frac{261}{16}\sqrt{3}\delta_2\gamma - \frac{189}{16}\sqrt{3}\epsilon\gamma\\
&\quad + \frac{7245}{128}J_2\sqrt{3}\gamma + \frac{261}{16}\delta_1\sqrt{3}\gamma - 9\sqrt{3}\epsilon + 9J_1\sqrt{3} + \frac{27\sqrt{3}}{32} - \frac{99\sqrt{3}\delta_1}{16} - \frac{99\sqrt{3}\delta_2}{16} - \frac{5679\sqrt{3}J_2}{128},
\end{aligned}
$$

$$
\begin{aligned}
A_1' &= \frac{1215J_2\sqrt{3}}{128} - \frac{75}{32}\gamma\delta_1\sqrt{3} + \frac{41\delta_1\sqrt{3}}{32} + \frac{75}{32}\gamma\delta_2\sqrt{3} + \frac{41\delta_2\sqrt{3}}{32} - \frac{45}{32}\gamma\epsilon\sqrt{3} + \frac{45\epsilon\sqrt{3}}{32} - \frac{3\sqrt{3}}{16}\\
&\quad - \frac{45J_1\sqrt{3}}{32},\\
A_2' &= -\frac{1}{16}63\gamma J_1 - \frac{75J_1}{16} + \frac{945J_2}{64} + \frac{75\gamma\delta_1}{16} + \frac{111\delta_1}{16} + \frac{75\gamma\delta_2}{16} + \frac{69\gamma\epsilon}{8} - \frac{21\gamma}{8} - \frac{111\delta_2}{16} - \frac{75\epsilon}{16},\\
A_3' &= -\frac{1}{16}57\sqrt{3}\delta_1\gamma^2 - \frac{57}{16}\sqrt{3}\delta_2\gamma^2 - \frac{135}{32}\sqrt{3}\epsilon\gamma^2 + \frac{63\sqrt{3}\gamma^2}{64} - \frac{3465}{256}\sqrt{3}J_2\gamma - \frac{27}{32}\sqrt{3}\delta_1\gamma\\
&\quad + \frac{135}{32}J_1\sqrt{3}\gamma + \frac{27}{32}\delta_2\sqrt{3}\gamma + \frac{171}{32}\epsilon\sqrt{3}\gamma + \frac{27J_1\sqrt{3}}{16} - \frac{27\sqrt{3}\epsilon}{16} - \frac{33\sqrt{3}\delta_1}{32} - \frac{33\sqrt{3}\delta_2}{32}\\
&\quad - \frac{2025\sqrt{3}J_2}{256},
\end{aligned}
$$

$$
\begin{aligned}
B_1' &= \frac{45J_2\sqrt{3}}{128} + \frac{135}{32}\gamma\delta_1\sqrt{3} + \frac{3\delta_1\sqrt{3}}{32} - \frac{135}{32}\gamma\delta_2\sqrt{3} + \frac{3\delta_2\sqrt{3}}{32} - \frac{135}{32}\gamma\epsilon\sqrt{3} + \frac{135\epsilon\sqrt{3}}{32} - \frac{9\sqrt{3}}{16} \\
&\quad - \frac{135J_1\sqrt{3}}{32}, \\
B_2' &= \frac{99\gamma J_1}{16} + \frac{255J_1}{16} + \frac{33\gamma}{8} + \frac{123\delta_2}{16} + \frac{255\epsilon}{16} - \frac{177\gamma\epsilon}{8} - \frac{135\gamma\delta_1}{16} - \frac{123\delta_1}{16} - \frac{135\gamma\delta_2}{16} - \frac{2685J_2}{64}, \\
B_3' &= \frac{35\epsilon\omega_1^4}{2\sqrt{3}} + \frac{32\delta_1\omega_1^4}{3\sqrt{3}} + \frac{32\delta_2\omega_1^4}{3\sqrt{3}} - \frac{11\omega_1^4}{4\sqrt{3}} - \frac{35\epsilon\omega_1^2}{2\sqrt{3}} - \frac{32\delta_1\omega_1^2}{3\sqrt{3}} - \frac{32\delta_2\omega_1^2}{3\sqrt{3}} + \frac{11\omega_1^2}{4\sqrt{3}} - \frac{315}{32}\sqrt{3}J_1\gamma \\
&\quad - \frac{63}{32}\sqrt{3}\gamma\delta_2 - \frac{207}{32}\sqrt{3}\gamma\epsilon + \frac{81J_1\sqrt{3}}{16} + \frac{7245}{256}J_2\gamma\sqrt{3} + \frac{63}{32}\gamma\delta_1\sqrt{3} + \frac{183\delta_1\sqrt{3}}{32} + \frac{183\delta_2\sqrt{3}}{32} \\
&\quad + \frac{153\epsilon\sqrt{3}}{32} - \frac{9\sqrt{3}}{8} - \frac{675\sqrt{3}J_2}{256}, \\
C_1' &= -\frac{255J_1}{16} + \frac{2685J_2}{64} + \frac{135\gamma\delta_1}{16} + \frac{123\delta_1}{16} + \frac{135\gamma\delta_2}{16} + \frac{255\gamma\epsilon}{16} - \frac{33\gamma}{8} - \frac{123\delta_2}{16} - \frac{255\epsilon}{16}, \\
C_2' &= \frac{27J_1\sqrt{3}}{4} + \frac{75}{8}\gamma\delta_1\sqrt{3} - \frac{75}{8}\gamma\delta_2\sqrt{3} + \frac{45}{8}\gamma\epsilon\sqrt{3} - \frac{27\epsilon\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} - \frac{41\delta_1\sqrt{3}}{8} - \frac{41\delta_2\sqrt{3}}{8} \\
&\quad - \frac{1215J_2\sqrt{3}}{32}, \\
C_3' &= \frac{\delta_1\omega_1^4}{6} - \frac{\delta_2\omega_1^4}{6} - \frac{17\epsilon\omega_1^4}{6} + \frac{\delta_2\omega_1^2}{6} + \frac{17\epsilon\omega_1^2}{6} - \frac{\delta_1\omega_1^2}{6} + \frac{315J_1}{32} + \frac{243J_1\gamma}{32} + \frac{405J_2\gamma}{16} + \frac{9\gamma}{4} \\
&\quad + \frac{81\delta_2}{16} + \frac{81\epsilon}{16} - \frac{9\gamma\delta_1}{16} - \frac{81\delta_1}{16} - \frac{9\gamma\delta_2}{16} - \frac{279\gamma\epsilon}{16} - \frac{1575J_2}{64},
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
\nu_1 &= \frac{\left(124\omega_1^4 - 696\omega_1^2 + 81\right)\omega_2^2}{72k^4\left(1 - 5\omega_1^2\right)} + \frac{\epsilon}{288k^6 11^2 t_1 \omega_1^4 \left(5\omega_1^2 - 1\right)\omega_2^2}\Bigg(- 153600\omega_1^{18} + 288(634\gamma \\
&\quad - 3029)\omega_1^{16} + 208(486\gamma - 407)\omega_1^{14} + 2(1740497 - 639378\gamma)\omega_1^{12} + (803430\gamma - 6716567)\omega_1^{10} \\
&\quad + 9(583481 - 446982\gamma)\omega_1^8 + 27(235778\gamma - 31583)\omega_1^6 - 81(63780\gamma + 36461)\omega_1^4 + 2916(616\gamma \\
&\quad + 577)\omega_1^2 - 196830(\gamma + 1)\Bigg), \\
\nu_2 &= -\frac{1}{192k^8\omega_1^2\left(5\omega_1^2 - 1\right)^2}\Bigg(385616\omega_1^{12} - 50400\gamma\omega_1^{10} - 729424\omega_1^{10} + 109260\gamma\omega_1^8 + 1101820\omega_1^8 \\
&\quad - 13410\gamma\omega_1^6 - 1051994\omega_1^6 + 39690\gamma\omega_1^4 + 367083\omega_1^4 - 11070\gamma\omega_1^2 - 56328\omega_1^2 + 4293\Bigg), \\
\nu_3 &= -\frac{1}{768k^8\omega_1^2\left(5\omega_1^2 - 1\right)^2}\Bigg(1863568\omega_1^{12} + 316800\gamma\omega_1^{10} - 5081552\omega_1^{10} + 2536200\gamma\omega_1^8 \\
&\quad + 12333344\omega_1^8 - 4748850\gamma\omega_1^6 - 11607874\omega_1^6 + 1711035\gamma\omega_1^4 + 5211966\omega_1^4 - 263250\gamma\omega_1^2 \\
&\quad - 968244\omega_1^2 + 25515\gamma + 55566\Bigg),
\end{aligned}
$$

$$\nu_4 = -\frac{1}{1728k^8\omega_1^2\left(5\omega_1^2-1\right)^2}\Big(2560\omega_1^{14} + 1345104\gamma\omega_1^{12} + 2141328\omega_1^{12} - 2925648\gamma\omega_1^{10} - 5019824\omega_1^{10} + 7846344\gamma\omega_1^{8} + 9525864\omega_1^{8} - 8492868\gamma\omega_1^{6} - 8914436\omega_1^{6} + 3473037\gamma\omega_1^{4} + 3513933\omega_1^{4} - 594270\gamma\omega_1^{2} - 597726\omega_1^{2} + 38637\gamma + 38637\Big),$$

$$\nu_5 = -\frac{1}{1728k^8\omega_1^2\left(5\omega_1^2-1\right)^2}\Big(2560\omega_1^{14} - 1345104\gamma\omega_1^{12} + 2141328\omega_1^{12} + 2925648\gamma\omega_1^{10} - 5019824\omega_1^{10} - 7846344\gamma\omega_1^{8} + 9525864\omega_1^{8} + 8492868\gamma\omega_1^{6} - 8914436\omega_1^{6} - 3473037\gamma\omega_1^{4} + 3513933\omega_1^{4} + 594270\gamma\omega_1^{2} - 597726\omega_1^{2} - 38637\gamma + 38637\Big),$$

$$\nu_6 = \frac{\epsilon}{48k^6 11^2 t_1 t_2\omega_1^3\omega_2^3(5\omega_1\omega_2-2)(5\omega_1\omega_2+2)}\Big(-972800\omega_1^{22} + 11848320\omega_1^{20} - 18319776\omega_1^{18} - 3163183\omega_1^{16} + 10176081\omega_1^{14} - 9264044\omega_1^{12} + 1534568\omega_1^{10} + 3830137\omega_1^{8} - 3765842\omega_1^{6} + 17063865\omega_1^{4} - 4893048\omega_1^{2} + 3\gamma\big(294400\omega_1^{20} + 58496\omega_1^{18} - 6020256\omega_1^{16} + 1364248\omega_1^{14} + 16770844\omega_1^{12} - 5193270\omega_1^{10} - 26882885\omega_1^{8} + 26891268\omega_1^{6} - 7578333\omega_1^{4} - 303264\omega_1^{2} + 244944\big) + 734832\Big) - \frac{\omega_1\omega_2\left(64\omega_1^2\omega_2^2+43\right)}{6\left(1-2\omega_1^2\right)\left(1-5\omega_1^2\right)\left(1-2\omega_2^2\right)\left(1-5\omega_2^2\right)},$$

$$\nu_7 = \frac{1}{256k^8\omega_1\omega_2^3 12^4(2-5\omega_1\omega_2)^2(5\omega_1\omega_2+2)^2}\Big(531609600\omega_1^{22} - 5757152000\omega_1^{20} + 27319498784\omega_1^{18} - 75579419888\omega_1^{16} + 142616722346\omega_1^{14} - 190768361115\omega_1^{12} + 178862243493\omega_1^{10} - 113994559537\omega_1^{8} + 46209858961\omega_1^{6} - 10220144968\omega_1^{4} + 798354676\omega_1^{2} - 45\gamma\big(24781600\omega_1^{18} - 249227200\omega_1^{16} + 1009541834\omega_1^{14} - 2095945887\omega_1^{12} + 2533819113\omega_1^{10} - 1880495925\omega_1^{8} + 845269901\omega_1^{6} - 208428608\omega_1^{4} + 21604132\omega_1^{2} - 499056\big) + 10323024\Big),$$

$$\nu_8 = \frac{1}{1024k^8 12^4\omega_1\omega_2^3(2-5\omega_1\omega_2)^2(5\omega_1\omega_2+2)^2}\Big(45\gamma\big(356554400\omega_1^{18} - 3601086400\omega_1^{16} + 1460222761\omega_1^{14} - 3031525960\omega_1^{12} + 3652242772\omega_1^{10} - 2688214613\omega_1^{8} + 1193397057\omega_1^{6} - 2901066872\omega_1^{4} + 296411700\omega_1^{2} - 6574256\big) + 408524800\omega_1^{22} - 5832070400\omega_1^{20} + 358376758\omega_1^{18} - 1341659966\omega_1^{16} + 3289126129\omega_1^{14} - 5224271704\omega_1^{12} + 51315283235\omega_1^{10} - 296974628731\omega_1^{8} + 98140850939\omega_1^{6} - 2075658744\omega_1^{4} + 3974309628\omega_1^{2} - 38498664\Big),$$

$$\nu_{13} = -\frac{1}{768k^8\omega_2^2\left(5\omega_2^2-1\right)^2}\Big(45\gamma(7040\omega_2^{10} + 56360\omega_2^{8} - 105530\omega_2^{6} + 38023\omega_2^{4} - 5850\omega_2^{2} + 567) - 968244\omega_2^{2} + 2\left(93178\omega_2^{8} - 254077\omega_2^{6} + 616667\omega_2^{4} - 580393\omega_2^{2} + 260598\right)\omega_2^{4} + 55566\Big),$$

$$\nu_9 = -\frac{1}{288k^8 l2^4\omega_1\omega_2^3(2-5\omega_1\omega_2)^2(5\omega_1\omega_2+2)^2}\Bigg(9\gamma\big(19180800\omega_1^{22} - 234020000\omega_1^{20} + 1214923472\omega_1^{18} - 3435237434\omega_1^{16} + 6050068715\omega_1^{14} - 7187656971\omega_1^{12} + 5860075650\omega_1^{10} - 319876577\omega_1^8 + 115237990\omega_1^6 - 29303525\omega_1^4 + 5619404\omega_1^2 - 479222\big) - 51812960\omega_1^{22} + 5753565600\omega_1^{20} - 27615046156\omega_1^{18} + 76856407420\omega_1^{16} - 143673957709\omega_1^{14} + 187940132913\omega_1^{12} - 16970262783\omega_1^{10} + 10220476053\omega_1^8 - 3887989859\omega_1^6 + 84330897\omega_1^4 - 83157323\omega_1^2 + 2193955\Bigg),$$

$$\nu_{10} = \frac{1}{288k^8 l2^4\omega_1\omega_2^3(2-5\omega_1\omega_2)^2(5\omega_1\omega_2+2)^2}\Bigg(9\gamma\big(1918080\omega_1^{22} - 2340200\omega_1^{20} + 12149234\omega_1^{18} - 3435237434\omega_1^{16} + 6050068715\omega_1^{14} - 7187656971\omega_1^{12} + 5860075650\omega_1^{10} - 3198765772\omega_1^8 + 115237990\omega_1^6 - 29303525\omega_1^4 + 5561940\omega_1^2 - 4792224\big) + 51812960\omega_1^{22} - 575356560\omega_1^{20} + 276150461\omega_1^{18} - 768564074\omega_1^{16} + 1436739570\omega_1^{14} - 1879401329\omega_1^{12} + 1697026278\omega_1^{10} - 102204760530\omega_1^8 + 38879898593\omega_1^6 - 8433089705\omega_1^4 + 831573236\omega_1^2 - 21939552\Bigg),$$

$$\nu_{11} = \frac{\left(124\omega_2^4 - 696\omega_2^2 + 81\right)\omega_1^2}{72k^4\left(1-5\omega_2^2\right)} + \frac{\epsilon}{288k^6 l2^2 t_2\omega_2^4\omega_1^2\left(5\omega_2^2-1\right)}\Bigg(- 153600\omega_2^{18} + 288(634\gamma - 3029)\omega_2^{16} + 208(486\gamma - 407)\omega_2^{14} + 2(1740497 - 639378\gamma)\omega_2^{12} + (803430\gamma - 6716567)\omega_2^{10} + 9(583481 - 446982\gamma)\omega_2^8 + 27(235778\gamma - 31583)\omega_2^6 - 81(63780\gamma + 36461)\omega_2^4 + 2916(616\gamma + 577)\omega_2^2 - 196830(\gamma + 1)\Bigg),$$

$$\nu_{12} = -\frac{1}{192k^8\omega_2^2\left(5\omega_2^2-1\right)^2}\Bigg(385616\omega_2^{12} - 16(3150\gamma + 45589)\omega_2^{10} + 20(5463\gamma + 55091)\omega_2^8 - 2(6705\gamma + 525997)\omega_2^6 + 9(4410\gamma + 40787)\omega_2^4 - 6(1845\gamma + 9388)\omega_2^2 + 4293\Bigg),$$

$$\nu_{14} = \frac{1}{1728k^8\omega_2^2\left(5\omega_2^2-1\right)^2}\Bigg(- 48(28023\gamma + 44611)\omega_2^{12} + 16(182853\gamma + 313739)\omega_2^{10} - 24(326931\gamma + 396911)\omega_2^8 + 4(2123217\gamma + 2228609)\omega_2^6 - 9(385893\gamma + 390437)\omega_2^4 + 54(11005\gamma + 11069)\omega_2^2 - 38637(\gamma + 1) - 2560\omega_2^{14}\Bigg),$$

$$\nu_{15} = \frac{1}{1728k^8\omega_2^2\left(5\omega_2^2-1\right)^2}\Bigg(48(28023\gamma - 44611)\omega_2^{12} + 16(313739 - 182853\gamma)\omega_2^{10} + 24(326931\gamma - 396911)\omega_2^8 + 4(2228609 - 2123217\gamma)\omega_2^6 + 9(385893\gamma - 390437)\omega_2^4 + 54(11069 - 11005\gamma)\omega_2^2 + 38637(\gamma - 1) - 2560\omega_2^{14}\Bigg).$$