Applications and Applied Mathematics: An International Journal (AAM)

Volume 17 | Issue 2 Article 6

12-2022

# (R1894) Invariant Solution for Two-dimensional and Axisymmetric Jet of Power-Law Fluids

Bhavixa Bhagat
*Veer Narmad South Gujarat University*

M. G. Timol
*Veer Narmad South Gujarat University*

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 17, Issue 2 (December 2022), pp. 386 – 407

# Invariant Solution for Two-dimensional and Axisymmetric Jet of Power-Law Fluids

[1*]Bhavixa Bhagat and [2]M. G. Timol

[1]Department of Mathematics
Veer Narmad South Gujarat University
Surat-395007, Gujarat, INDIA
bhaviksha_bhagat@yahoo.com

[2]Department of Mathematics
Retired Professor
Veer Narmad South Gujarat University
Surat-395007, Gujarat, INDIA
mgtimol@gmail.com

*Corresponding author

Received: November 26, 2021; Accepted: October 21, 2022

## Abstract

An invariant solution is derived using the Lie symmetry technique for steady laminar two-dimensional and axisymmetric boundary layer jet flow of incompressible power-law fluids with appropriate boundary conditions. Using symmetry, the nonlinear partial differential equation of the jet flow problem is transformed into a nonlinear ordinary differential equation. The resultant nonlinear ordinary differential equation with boundary conditions is converted to an initial value problem using the Lie symmetry technique. A numerical solution for the resulting initial value problem is derived using Fehlberg's fourth-fifth order Runge-Kutta method through Maple software. The graphical representation of the characteristics of the velocity field for different physical parameters is also discussed.

**Keywords:** Non-Newtonian fluid; Boundary layer; Two-dimensional jet; Axisymmetric jet; Symmetries; Invariant solution; Runge-Kutta method

**MSC 2020 No.:** 76A05, 35B06, 65L06

## Nomenclature:

| | |
|---|---|
| $\dot{x}^*, \dot{y}^*, X, Y$ | axis of Cartesian co-ordinates |
| $\dot{u}^*, \dot{v}^*$ | Dimensional fluid velocity along $\dot{x}^*$- and $\dot{y}^*$- axis, respectively |

| | |
|---|---|
| $\bar{U}, \bar{V}$ | nondimensional fluid velocity along $X-$ and $Y-$ axis respectively |
| $n$ | Power-law index |
| $\bar{U}_0$ | characteristic velocity |
| $L$ | jet diameter |
| $\rho$ | Fluid density |
| $Re$ | Reynolds number |
| $\psi$ | stream function |
| $\xi$ | similarity independent variabl |
| $\nu$ | kinematic viscosity |
| $\xi^i, \eta, \zeta$ | Infinitesimals |
| $r$ | Index |

## 1. Introduction

A high-velocity stream of fluids that emanates under pressure from a small diameter opening or nozzle is defined as a jet. When the fluid is flowing in the absence of rigid boundaries is called free flow. This type of flow is important in many technical applications like fountains, fluid injection engines, aircraft propulsion, cooling systems, nasal spray device … .

The study of fluid mechanics was normally confined to the Newtonian fluids alone, but with the increasing importance of non-Newtonian fluids in fluid mechanics and chemical engineering, in particular, it has become necessary to analyze non-Newtonian fluids as well. The flow characteristics of non-Newtonian fluids under various geometries and physical condition is well discussed by Wilkinson (1960), Lee and Ames (1966), Kapur et al. (1982), Timol and Kalthia (1986), Oleinik et al. (1999), Chhabra and Richardson (2008), Patel et al. (2010, 2015), Shukla et al. (2019).

Many researchers have worked on laminar jet flow in the past. Laminar Newtonian jets have been studied extensively by Schlichting (1933). He was a pioneer in applying the boundary layer theory to such problems of the jet. He obtained similarity solutions of both axisymmetric and two-dimensional jets of Newtonian fluids separately and determined numerical solutions of the resulting ordinary differential equations. Later on, Bickley (1937) calculated analytical solutions to the differential equation.

Gutfinger et al. (1964) produced both theoretical and experimental results of a two-dimensional jet of power-law fluid. Lemieux-Unny (1968), Atkinson (1972) obtained a closed-form solution as well as numerical solutions for the two-dimensional jet of an incompressible pseudoplastic fluid. Kapur (1962) derived a similarity solution for the two-dimensional laminar jet of pseudoplastic fluid. He determined an analytical solution in terms of incomplete beta-gamma function. Rotem (1964) developed the similarity solution of the axisymmetrical laminar jet of a power-law fluid and explored the numerical results in detail. Pai-Hsieh (1972) investigated the general boundary layer equations for the two-dimensional, axisymmetric laminar jet of the incompressible Newtonian fluid. Kalthia (1974,1979) has discussed in detail the similarity analysis of different jets in his research outcome.

Mason (2002), Ruscic and Mason (2004) have derived a group invariant solution of two-dimensional as well as an axisymmetric laminar free jet of a Newtonian fluid through the Lie symmetry method. Pakdemirli et al. (2008) have found symmetries of modified power-law fluids through the Lie group method. Naz (2011) has derived group invariant solutions of several jets with fluid velocity finite at the orifice via Lie symmetry. Patel and Timol (2016) have derived similarity solutions for various jet flows of non-Newtonian fluid by the one-parameter group transformation technique of Morgan. Soid and Ishak (2017) have worked on boundary layer flows of a nanofluid using similarity transformation. Magan et al. (2016, 2017) have derived group invariant solution of free jet, liquid jet and solved the resultant equation analytically in parametric form.

The Lie symmetry technique is developed by Sophus Lie in the late nineteenth century, to determine the symmetries for differential equations, which leave a given family of equations invariant. It provides a systematic tool to generate the invariant solutions of the system of nonlinear partial differential equations with admissible initial or boundary conditions. The fundamental concept of the Lie symmetry approach can be found in the literature of Sheshadri and Na (1985), Bluman and Kumei (1989), Dresner (1999), Hydon (2000), Bluman and Anco (2002), Ibragimov and Kovalev (2009), Arrigo (2015). The Lie symmetry technique has recently been focused on by several researchers, including Bilige and Han (2018), Zeidan and Bira (2019), Halder et al. (2020) and Paliathanasis (2021), to solve differential equations. At the present, the fusion of symmetries and other solution methods, such as the numerical method, approximation method and analytical method, to solve boundary value problem (BVP) for differential equations is a promising concept for further investigation. The application of the Lie symmetry technique includes such variant fields as differential geometry, invariant theory, solid or fluid mechanics and across all areas of science and technology.

In the present paper, we have derived a general form of an invariant solution that applies to two-dimensional and axisymmetric jets of non-Newtonian power-law fluids using the Lie symmetry technique that was not investigated earlier. The invariant solution has been established by considering a linear combination of the point symmetries of the governing partial differential equation (PDE) in terms of stream function. This approach was used by Mason (2002) for the two-dimensional laminar jet of a Newtonian fluid. We have extended this approach to non-Newtonian power-law fluids. The fundamental boundary-layer equations for two-dimensional or axisymmetric laminar jet mixing for a Newtonian fluid were first presented by Pai and Hsieh (1972). We have extended those boundary layer equations for non-Newtonian power-law fluids in Section 2. Here, the Lie symmetry technique has been applied systematically without assuming a prior form for the stream function.

The (group) invariant solution is in the most general form as it supersedes the earlier derived solutions of Mason (2002) and Ruscic and Mason (2004). The governing PDE is converted into an ordinary differential equation (ODE) with appropriate boundary conditions using symmetry. Using Lie symmetry, a one-parameter group is derived that transforms BVP into an initial value problem (IVP). The numerical solution of the resulting IVP is obtained by the Runge-Kutta fourth-fifth order technique using the ODE solver of Maple programming. The present results are compared with those of Mason (2002) and Atkinson for the two-dimensional jet case and with Ruscic and Mason (2004) and Rotem (1964) for the axisymmetric jet case.

## 2. Mathematical formulation

The fundamental boundary-layer equations for a two-dimensional or an axisymmetric laminar steady jet mixing of an incompressible non-Newtonian power-law fluid can be written as:

$$\dot{u}^*\frac{\partial \dot{u}^*}{\partial \dot{x}^*}+\dot{v}^*\frac{\partial \dot{u}^*}{\partial \dot{y}^*}=\frac{\nu}{(\dot{y}^*)^r}\frac{\partial}{\partial \dot{y}^*}\left((\dot{y}^*)^r\left|\frac{\partial \dot{u}^*}{\partial \dot{y}^*}\right|^{n-1}\frac{\partial \dot{u}^*}{\partial \dot{y}^*}\right), \tag{1}$$

$$\frac{\partial\left((\dot{y}^*)^r\dot{u}^*\right)}{\partial \dot{x}^*}+\frac{\partial\left((\dot{y}^*)^r\dot{v}^*\right)}{\partial \dot{y}^*}=0, \tag{2}$$

with the boundary conditions

$$\dot{y}^*=0:\ \dot{v}^*=0,\quad \frac{\partial \dot{u}^*}{\partial \dot{y}^*}=0, \tag{3}$$

$$\dot{y}^*\to\infty:\ \dot{u}^*=0. \tag{4}$$

Since the pressure in the surrounding fluid is constant, the total momentum flux $J$ across any cross-section of the jet at any given value of $\dot{x}^*$ remains constant. It is given by

$$J=2\rho(\pi)^r\int_0^\infty(\dot{y}^*)^r(\dot{u}^*)^2\,dy=\text{constant}. \tag{5}$$

Here, $r=0$ is for the two-dimensional jet case and $r=1$ for the axisymmetric jet case.

Now apply the transformations (Acrivos et al. (1965)) listed below to convert the governing equations (1)-(5) into the dimensionless form equations (6)-(10).

$$X=\frac{\dot{x}^*}{L},\quad Y=(Re)^{1/n+1}\frac{\dot{y}^*}{L},\quad \bar{U}=\frac{\dot{u}^*}{\bar{U}_0},\quad \bar{V}=(Re)^{1/n+1}\frac{\dot{v}^*}{\bar{U}_0},\quad Re=\frac{\bar{U}_0^{\,2-n}L^n}{\nu}.$$

$$\bar{U}\frac{\partial \bar{U}}{\partial X}+\bar{V}\frac{\partial \bar{U}}{\partial Y}=\frac{1}{Y^r}\frac{\partial}{\partial Y}\left(Y^r\left|\frac{\partial \bar{U}}{\partial Y}\right|^{n-1}\frac{\partial \bar{U}}{\partial Y}\right). \tag{6}$$

$$\frac{\partial \bar{U}}{\partial X}+\frac{\partial \bar{V}}{\partial Y}=0. \tag{7}$$

Boundary conditions become

$$Y=0:\ \bar{V}=0,\ \frac{\partial \bar{U}}{\partial Y}=0, \tag{8}$$

$$Y\to\infty:\ \bar{U}=0, \tag{9}$$

and the momentum flux

$$J=\frac{2\rho(\pi)^r\bar{U}_0^2L^{r+1}}{(Re)^{r+1/n+1}}\int_0^\infty Y^r\bar{U}^2dY=\text{constant}. \tag{10}$$

Now, introduce $\psi(X,Y)$ as a stream function, is defined by

$$\bar{U}=\frac{1}{Y^r}\frac{\partial\psi}{\partial Y}\ \text{ and }\ \bar{V}=-\frac{1}{Y^r}\frac{\partial\psi}{\partial X}. \tag{11}$$

Then Equation (7) is identically satisfied by (11) and Equations (6) and (8) through (10) reduce to

$$\frac{1}{\left(Y^r\right)^2}\frac{\partial \psi}{\partial Y}\frac{\partial}{\partial X}\left(\frac{\partial \psi}{\partial Y}\right)-\frac{1}{Y^r}\frac{\partial \psi}{\partial X}\frac{\partial}{\partial Y}\left(\frac{1}{Y^r}\frac{\partial \psi}{\partial Y}\right)$$
$$=\frac{1}{Y^r}\frac{\partial}{\partial Y}\left\{Y^r\left|\frac{\partial}{\partial Y}\left(\frac{1}{Y^r}\frac{\partial \psi}{\partial Y}\right)\right|^{n-1}\frac{\partial}{\partial Y}\left(\frac{1}{Y^r}\frac{\partial \psi}{\partial Y}\right)\right\}, \tag{12}$$

$$Y=0 \ : \ \frac{1}{Y^r}\frac{\partial \psi}{\partial X}=0,\ \ \frac{\partial}{\partial Y}\left(\frac{1}{Y^r}\frac{\partial \psi}{\partial Y}\right)=0, \tag{13}$$

$$Y\to\infty: \ \frac{1}{Y^r}\frac{\partial \psi}{\partial Y}=0. \tag{14}$$

The equation of flux is transformed into

$$J=\frac{2\rho(\pi)^r\bar{U}_0^{\,2}L^{r+1}}{(Re)^{r+1/n+1}}\int_0^\infty\frac{1}{Y^r}\left(\frac{\partial \psi}{\partial Y}\right)^2 dY=\text{constant}. \tag{15}$$

## 3. Lie symmetry generators (Infinitesimal point symmetries)

Equation (12) can be written as follows:

$$\Delta\left(X,Y,\psi,\psi_X,\psi_Y,\psi_{XX},\psi_{XY},\psi_{YY},\psi_{XXX},\psi_{XXY},\psi_{XYY},\psi_{YYY}\right)=0, \tag{16}$$

where a subscript represents partial derivatives and

$$\Delta=\frac{1}{\left(Y^r\right)^2}\frac{\partial \psi}{\partial Y}\frac{\partial}{\partial X}\left(\frac{\partial \psi}{\partial Y}\right)-\frac{1}{Y^r}\frac{\partial \psi}{\partial X}\frac{\partial}{\partial Y}\left(\frac{1}{Y^r}\frac{\partial \psi}{\partial Y}\right)$$
$$-\frac{1}{Y^r}\frac{\partial}{\partial Y}\left\{Y^r\left|\frac{\partial}{\partial Y}\left(\frac{1}{Y^r}\frac{\partial \psi}{\partial Y}\right)\right|^{n-1}\frac{\partial}{\partial Y}\left(\frac{1}{Y^r}\frac{\partial \psi}{\partial Y}\right)\right\}=0. \tag{17}$$

The Lie symmetries generators

$$\hat{X}=\xi^1\left(X,Y,\psi\right)\frac{\partial}{\partial X}+\xi^2\left(X,Y,\psi\right)\frac{\partial}{\partial Y}+\eta\left(X,Y,\psi\right)\frac{\partial}{\partial \psi}, \tag{18}$$

are obtained by solving determining equations which will be generated by Lie's invariance condition

$$\hat{X}^{(3)}\left(\Delta\right)\big|_{\Delta=0}=0, \tag{19}$$

where

$$\hat{X}^{(3)} = \hat{X} + \zeta_1 \frac{\partial}{\partial \psi_X} + \zeta_2 \frac{\partial}{\partial \psi_Y} + \zeta_{11} \frac{\partial}{\partial \psi_{XX}} + \zeta_{12} \frac{\partial}{\partial \psi_{XY}} + \zeta_{22} \frac{\partial}{\partial \psi_{YY}} + \zeta_{111} \frac{\partial}{\partial \psi_{XXX}} + \zeta_{112} \frac{\partial}{\partial \psi_{XXY}} + \zeta_{122} \frac{\partial}{\partial \psi_{XYY}} + \zeta_{222} \frac{\partial}{\partial \psi_{YYY}}, \tag{20}$$

is the third extension of $\hat{X}$ and

$$\zeta_i = D_i(\eta) - \psi_m D_i(\xi^m), \tag{21}$$

$$\zeta_{ij} = D_j(\zeta_i) - \psi_{im} D_j(\xi^m), \tag{22}$$

$$\zeta_{ijk} = D_k(\zeta_{ij}) - \psi_{ijm} D_k(\xi^m), \tag{23}$$

with summation over repeated indices and total differential operators are

$$D_1 = D_X = \frac{\partial}{\partial X} + \psi_X \frac{\partial}{\partial \psi} + \psi_{XX} \frac{\partial}{\partial \psi_X} + \psi_{YX} \frac{\partial}{\partial \psi_Y} + ..., \tag{24}$$

$$D_2 = D_Y = \frac{\partial}{\partial Y} + \psi_Y \frac{\partial}{\partial \psi} + \psi_{XY} \frac{\partial}{\partial \psi_X} + \psi_{YY} \frac{\partial}{\partial \psi_Y} + ... . \tag{25}$$

Since $\Delta$ depends on derivatives $\psi_Y, \psi_{XY}, \psi_X, \psi_{YY}$ and $\psi_{YYY}$, the coefficients $\zeta_{122}$, $\zeta_{112}$, $\zeta_{111}$ and $\zeta_{11}$ need not be computed. The coefficient $\zeta_{222}$ depends on $\psi_{YYY}$ that is terminated from (19) using the PDE (16). Because $\xi^1, \xi^2$ and $\eta$ do not depend on the derivative of $\psi$. Equation (19) is separated according to the derivative of $\psi$. Hence, the infinitesimal symmetry of partial differential equation (12) is spanned by the following four linearly independent symmetries:

$$\hat{X} = C_1 \hat{X}_1 + C_2 \hat{X}_2 + C_3 \hat{X}_3 + C_4 \hat{X}_4, \tag{26}$$

where $C_1$, $C_2$, $C_3$ and $C_4$ are arbitrary constants and

$$\hat{X}_1 = X \frac{\partial}{\partial X} - \frac{\psi}{n-2} \frac{\partial}{\partial \psi}, \hat{X}_2 = \frac{\partial}{\partial X}, \hat{X}_3 = Y \frac{\partial}{\partial Y} + \frac{\left[r(n-2) + 2n - 1\right]}{(n-2)} \frac{\partial}{\partial \psi}, \hat{X}_4 = \frac{\partial}{\partial \psi}; \quad n > 0, n \neq 2, \tag{27}$$

and

$$\left.\begin{aligned}\hat{X}_1 &= X\frac{\partial}{\partial X}+\frac{Y}{\left[r(n-2)+2n-1\right]}\frac{\partial}{\partial Y},\hat{X}_2=\frac{\partial}{\partial X},\\ \hat{X}_3 &= \psi\frac{\partial}{\partial \psi}+\frac{(n-2)}{\left[r(n-2)+2n-1\right]}\frac{\partial}{\partial Y},\hat{X}_4=\frac{\partial}{\partial \psi}\end{aligned}\right\};n=2. \tag{27a}$$

## 4. Invariant solution

Now, $\psi=\Phi(X,Y)$ is an invariant solution of PDE (12) provided

$$\hat{X}\left(\psi-\Phi(X,Y)\right)\Big|_{\psi=\Phi}=0, \tag{28}$$

which may be written as

$$\left[(C_1X+C_2)\frac{\partial}{\partial X}+C_3Y\frac{\partial}{\partial Y}+\left\{\frac{\left[\left\{r(n-2)+2n-1\right\}C_3-C_1\right]\psi}{(n-2)}+C_4\right\}\frac{\partial}{\partial \psi}\right]\left(\psi-\Phi(X,Y)\right)\Big|_{\psi=\Phi}=0, \tag{29}$$

and therefore, in form of $\Phi(X,Y)$, Equation (28) is satisfying the quasi-linear first-order PDE:

$$(C_1X+C_2)\frac{\partial\Phi}{\partial X}+C_3Y\frac{\partial\Phi}{\partial Y}=\left\{\frac{\left[\left\{r(n-2)+2n-1\right\}C_3-C_1\right]\psi}{(n-2)}+C_4\right\}. \tag{30}$$

We will consider a solution for which $C_1\neq 0$ and $C_3\neq 0$. The characteristic equations of (30) are

$$\frac{dX}{C_1X+C_2}=\frac{dY}{C_3Y}=\frac{d\Phi}{\dfrac{\left[\left\{r(n-2)+2n-1\right\}C_3-C_1\right]\Phi}{(n-2)}+C_4}. \tag{31}$$

Take $C_3=1$ in (31). Now Equation (31) gives two independent solutions,

$$\frac{Y}{\left[X+\dfrac{C_2}{C_1}\right]^{\frac{1}{C_1}}}=\alpha_1,\quad \frac{\Phi+\dfrac{C_4(n-2)}{r(n-2)+2n-1-C_1}}{\left[X+\dfrac{C_2}{C_1}\right]^{\frac{r(n-2)+2n-1-C_1}{C_1(n-2)}}}=\alpha_2, \tag{32}$$

where $\alpha_1$ and $\alpha_2$ are constants. The general solution of (30) is

$$\alpha_2 = f(\alpha_1), \tag{33}$$

where $f$ is an arbitrary function. Thus, since $\psi = \Phi(X,Y)$, the invariant solution of (12) is

$$\psi = \left[X + \frac{C_2}{C_1}\right]^{\frac{r(n-2)+2n-1-C_1}{C_1(n-2)}} f(\xi) - \frac{C_4(n-2)}{r(n-2)+2n-1-C_1}, \tag{34}$$

where

$$\xi = \frac{Y}{\left[X + \frac{C_2}{C_1}\right]^{\frac{1}{C_1}}}. \tag{35}$$

Also, using Equations (27a), we obtain an invariant solution to Equation (12):

$$\psi = \left[X + \frac{C_2}{C_1}\right]^{\frac{1}{C_1}} f(\xi) - C_4; \quad \xi = \frac{Y}{\left[X + \frac{C_2}{C_1}\right]^{\frac{(n-2)+C_1}{C_1[r(n-2)+2n-1]}}}. \tag{35a}$$

Since the stream function is prescribed up to an arbitrary constant, we may choose $C_4 = 0$ without any loss of generality.

Now substituting (34) and (35) into (12), we obtain an ODE for $f(\xi)$:

$$\frac{1}{\xi^r}\left[\frac{n+1-C_1}{C_1(n-2)}\left(f'(\xi)\right)^2 - \frac{r(n-2)+2n-1-C_1}{C_1(n-2)} f(\xi) f''(\xi) + \frac{r(n-2)+2n-1-C_1}{C_1(n-2)}\frac{r}{\xi} f(\xi) f'(\xi)\right] = \frac{d}{d\xi}\left[\xi^r \left|\frac{d}{d\xi}\left(\frac{f'(\xi)}{\xi^r}\right)\right|^{n-1} \frac{d}{d\xi}\left(\frac{f'(\xi)}{\xi^r}\right)\right]. \tag{36}$$

Also, by substituting (35a) into (12), we get an ODE for $f(\xi)$:

$$\frac{1}{\xi^r}\left[\frac{n+1-C_1(r+1)}{C_1[r(n-2)+2n-1]}\left(f'(\xi)\right)^2 - \frac{1}{C_1} f(\xi) f''(\xi) + \frac{r}{C_1\xi} f(\xi) f'(\xi)\right] = \frac{d}{d\xi}\left[\xi^r \left|\frac{d}{d\xi}\left(\frac{f'(\xi)}{\xi^r}\right)\right|^{n-1} \frac{d}{d\xi}\left(\frac{f'(\xi)}{\xi^r}\right)\right]. \tag{36a}$$

To derive $C_1$, consider the flux condition $J$, defined by (15) is independent of $X$. Using (34) and (35) in (15) yields

$$J = \frac{2\rho(\pi)^r U_0{}^2 L^{r+1}}{(Re)^{r+1/n+1}}\left[X+\frac{C_2}{C_1}\right]^{\frac{r(n-2)+3n-2C_1}{C_1(n-2)}}\int_0^\infty \frac{1}{\xi^r}\left(f'(\xi)\right)^2 d\xi = \text{constant.} \tag{37}$$

Thus, the flux condition $J$ is independent of $X$ provided

$$C_1 = \frac{r(n-2)+3n}{2}. \tag{38}$$

Now, using (35a) in (15) gives

$$C_1 = \frac{r(n-2)+3n}{r+1}. \tag{38a}$$

Hence, the flux condition reduces to

$$J = \frac{2\rho(\pi)^r U_0{}^2 L^{r+1}}{(Re)^{r+1/n+1}}\int_0^\infty \frac{1}{\xi^r}\left(f'(\xi)\right)^2 d\xi = \text{constant.} \tag{39}$$

Then, using results (38) and (38a) in the differential equations (36) and (36a), respectively, can be written as follows:

$$\frac{r+1}{r(n-2)+3n}\cdot\frac{1}{\xi^r}\left[-\left(f'(\xi)\right)^2 - f(\xi)f''(\xi)+\frac{r}{\xi}f(\xi)f'(\xi)\right] = \frac{d}{d\xi}\left[\xi^r\left|\frac{d}{d\xi}\left(\frac{f'(\xi)}{\xi^r}\right)\right|^{n-1}\frac{d}{d\xi}\left(\frac{f'(\xi)}{\xi^r}\right)\right]. \tag{40}$$

Also, the boundary conditions (13) and (14) become

$$\xi = 0: \ \frac{f(\xi)}{\xi^r} - f'(\xi) = 0, \ \frac{d}{d\xi}\left(\frac{f'(\xi)}{\xi^r}\right) = 0. \tag{41}$$

$$\xi \to \infty: \ \frac{f'(\xi)}{\xi^r} = 0. \tag{42}$$

## 5. Two-dimensional jet flow

For $r = 0$ and $n > 0, n \neq 1$, the equation (40) with boundary conditions (41) and (42) will reduce into the non-Newtonian two-dimensional incompressible jet flow of power-law fluids. For $r = 0$ and $n = 1$, the equation (40) with boundary conditions (41) and (42) will reduce into Newtonian two-dimensional incompressible jet flow. For such flows, the boundary layer equations can be written as follows:

$$\frac{1}{3n}\frac{d}{d\xi}(f\,f') = \frac{d}{d\xi}(-f'')^{n}\ ;\ n>0,\quad \xi\geq 0, \tag{43a}$$

$$-\frac{1}{3n}\frac{d}{d\xi}(f\,f') = \frac{d}{d\xi}(f'')^{n}\ ;\ 0<n<1/2\,,\xi\geq 0,\xi\leq 0 \text{ and } n\geq 1/2\,,\xi\leq 0. \tag{43b}$$

The boundary conditions are

$$f(0)=0, f''(0)=0, f'(\infty)=0. \tag{44}$$

The flux condition (39) is transformed to

$$J = \frac{2\rho U_0^{\,2} L}{(Re)^{1/_{n+1}}}\int_0^{\infty}\left(f'(\xi)\right)^2 d\xi = \text{constant}, \tag{45}$$

where $\xi = \dfrac{Y}{\left[X+\dfrac{2C_2}{3n}\right]^{\frac{2}{3n}}}$ (Gutfinger and Shinnar (1964)).

Thus, Equation (43) is nonlinear BVP with boundary conditions (44).

## 6. Transformation of boundary value problem to initial value problem

The resultant BVP has been solved numerically by the Runge-Kutta fourth-fifth order method. Therefore, we have derived a one-parameter group transformation that transforms BVP into IVP. This extended method was introduced by Na (1979). But a parameter cannot be determined if the boundary condition is homogenous at the second point of infinity. To overcome this limitation, we will transform the homogeneous BVP (43a)-(44) into a non-homogenous BVP first. The procedure is as follows:

If we have a Lie symmetry generator, we can obtain one-parameter group transformations using Lie equations (Ibragimov (2009)). The Lie symmetry generators of Equations (43a) and (43b) are

$$X_1 = \frac{\partial}{\partial\xi}\quad,\quad X_2 = \xi\frac{\partial}{\partial\xi} + \frac{2n-1}{n-2}f\frac{\partial}{\partial f}\quad ;0<n<2, n>2, \tag{46}$$

$$X_1 = \frac{\partial}{\partial\xi}\quad,\quad X_2 = \frac{n-2}{2n-1}\xi\frac{\partial}{\partial\xi} + f\frac{\partial}{\partial f}\quad ;n=2. \tag{46a}$$

From $X_2$, we get a group transformation

$$\xi^{*} = e^{\beta}\xi \text{ and } f^{*} = e^{(2n-1/n-2)\beta} f\quad ;0<n<2, n>2, \tag{47}$$

$$\xi^{*} = e^{(n-2/2n-1)\beta} f \text{ and } f^{*} = e^{\beta} f\quad ;n=2, \tag{47a}$$

where " $\beta$ " is the parameter of group transformation.

Under the transformation (47), Equations (43a) and (43b) become invariant. Using this transformation, Equations (43a) and (43b) become

$$\frac{1}{3n}\frac{d}{d\xi^*}\left(f^* f^{*\prime}\right)=\frac{d}{d\xi^*}\left(-f^{*\prime\prime}\right)^n;\quad n>0,\xi^*\geq 0, \tag{48a}$$

$$-\frac{1}{3n}\frac{d}{d\xi^*}\left(f^* f^{*\prime}\right)=\frac{d}{d\xi^*}\left(f^{*\prime\prime}\right)^n; 0<n<1/2,\xi^*\geq 0,\xi^*\leq 0 \text{ and } n\geq 1/2,\xi^*\leq 0, \tag{48b}$$

and boundary conditions at zero are transformed to

$$\xi^*=0:\quad f^*=0,\quad f^{*\prime\prime}=0. \tag{49}$$

Boundary condition (44) shows that $f'(\xi)$ starts from the definite value at $\xi=0$ and decreases to zero as $\xi\to\infty$. Therefore,

$$\int_0^\infty f'(\xi)\,d\xi=\text{constant}. \tag{50}$$

Under the transformation (47), it will be

$$e^{-(2n-1/n-2)\beta}\left[f^*(\infty)-f^*(0)\right]=\text{constant}.$$

If we identify, constant = $e^{-(2n-1/n-2)\beta}$, we get

$$f^*(\infty)=1. \tag{51}$$

Thus, the boundary condition at the second point transforms into a non-homogeneous form and the flux condition transforms into

$$J=\frac{2\rho {U_0}^2 L}{(Re)^{1/n+1}}e^{-(3n/n-2)\beta}\int_0^\infty\left(f^{*\prime}\right)^2 d\xi^*=\text{constant}. \tag{52}$$

We now convert the BVP (48) with boundary conditions (49) and (51) into IVP using the group transformation

$$\bar{\xi}=e^{\beta}\xi^* \text{ and } \bar{f}=e^{(2n-1/n-2)\beta}f^*. \tag{53}$$

Under the transformation (53), Equations (48a), (48b) become

$$\frac{1}{3n}\frac{d}{d\bar{\xi}}\left(\bar{f}\,\bar{f}'\right)=\frac{d}{d\bar{\xi}}\left(-\bar{f}''\right)^n; n>0,\bar{\xi}\geq 0, \tag{54a}$$

$$-\frac{1}{3n}\frac{d}{d\bar{\xi}}\left(\bar{f}\,\bar{f}'\right)=\frac{d}{d\bar{\xi}}\left(\bar{f}''\right)^n; 0<n<1/2,\bar{\xi}\geq 0,\bar{\xi}\leq 0 \text{ and } n\geq 1/2,\bar{\xi}\leq 0, \tag{54b}$$

and boundary conditions in (49) become

$$\bar{f}(0)=0, \quad \bar{f}''(0)=0. \tag{55}$$

Now, set the missing initial condition is equal to the parameter of transformation,

$$f^{*\prime}(0)=\beta.$$

Under the transformation (53),

$$e^{-(n+1/n-2)\beta}\bar{f}'(0)=\beta, \tag{56}$$

which is independent of $\beta$ if $e^{-(n+1/n-2)\beta}=\beta$ and that implies

$$e^{-\beta}=(\beta)^{n-2/n+1}. \tag{57}$$

Hence, $\bar{f}'(0)=1$. (58)

To determine "$\beta$", from (51) and (53), we get

$$e^{-(2n-1/n-2)\beta}\bar{f}(\infty)=1,$$

that implies

$$\beta=\left[\frac{1}{\bar{f}(\infty)}\right]^{n+1/2n-1}. \tag{59}$$

Thus, we have the following initial value problem

$$\frac{1}{3n}\left(\bar{f}\,\bar{f}'\right)=\left(-\bar{f}''\right)^{n}; \quad n>0, \bar{\xi}\geq 0, \tag{60a}$$

$$-\frac{1}{3n}\left(\bar{f}\,\bar{f}'\right)=\left(\bar{f}''\right)^{n}; \; 0<n<1/2, \bar{\xi}\geq 0, \bar{\xi}\leq 0 \text{ and } n\geq 1/2, \bar{\xi}\leq 0, \tag{60b}$$

subject to initial conditions

$$\bar{f}(0)=0, \quad \bar{f}'(0)=1. \tag{61}$$

The flux condition transforms into

$$J=\frac{2\rho U_0{}^2 L}{(Re)^{1/n+1}}\int_0^\infty \left(\bar{f}'\right)^2 d\bar{\xi}=\text{constant}. \tag{62}$$

Similarly, we do proceed to calculate the $n=2$ case by using the group transformation (47a). If we put $n=1$ in (60a) or (60b), we get

$$\bar{f}'' + \frac{1}{3}\bar{f}\,\bar{f}' = 0, \tag{63}$$

with initial conditions

$$\bar{f}(0) = 0, \quad \bar{f}'(0) = 1. \tag{64}$$

Equation (63) is a standard second-order similarity equation for a free Laminar two-dimensional jet derived by Mason (2002). The exact solution of (63) is

$$\bar{f}(\bar{\xi}) = A\tanh\left[\frac{A\bar{\xi}}{6}\right], \tag{65}$$

where $A$ is integrating constant. Using flux condition $J$, we calculate an integrating constant $A$, by substituting (65) into (62):

$$A = \left(\frac{9J(Re)^{1/2}}{2\rho\bar{U}_0^{\,2}L}\right)^{1/3}. \tag{66}$$

The results (65) and (66) are similar to those derived by Mason (2002).

## 7. Axisymmetric jet flow

For $r=1$ and $n>0$, the equation (40) with boundary conditions (41) and (42) will reduce into the axisymmetric incompressible jet flow of power-law fluids. For such flows, the reduced boundary layer equations of (40) will be formed as:

$$\frac{1}{2n-1}\frac{d}{d\xi}\left(\pm\frac{f\,f'}{\xi}\right) = \frac{d}{d\xi}\left\{\xi^{1-n}\left(\frac{f'}{\xi} - f''\right)^n\right\};\ \xi > 0. \tag{67a}$$

Here, applies positive sign for $n > 1/2$ and negative sign for $0 < n < 1/2$,

$$\frac{1}{2n-1}\frac{d}{d\xi}\left(\mp\frac{f\,f'}{\xi}\right) = \frac{d}{d\xi}\left\{\xi^{1-n}\left(f'' - \frac{f'}{\xi}\right)^n\right\};\ \xi \le 0. \tag{67b}$$

Here, applies the negative sign for $n > 1/2$ and positive sign for $0 < n < 1/2$.

The boundary conditions:

$$f(0)=0, f''(0)\neq 0, f'(0)=0, \lim_{\xi\to\infty}\frac{f'}{\xi}=0. \tag{68}$$

The flux condition (39) is transformed to

$$J=\frac{2\rho\pi {U_0}^2L^2}{(Re)^{2/n+1}}\int_0^\infty \frac{1}{\xi}\left(f'\right)^2 d\xi=\text{constant}, \tag{69}$$

where $\xi=\dfrac{Y}{\left[X+\dfrac{C_2}{2n-1}\right]^{\frac{1}{2n-1}}}$; $n\neq\dfrac{1}{2}$ (Rotem (1964)). (70)

Thus, Equation (67) is a nonlinear boundary value problem with homogeneous boundary conditions (68). Now, we transform homogenous boundary conditions (68) into non-homogenous conditions.

The Lie symmetry generator of Equation (67) is

$$X_1=\frac{n-2}{3n-3}\xi\frac{\partial}{\partial\xi}+f\frac{\partial}{\partial f};\quad 0<n<1, n>1. \tag{71}$$

$$X_1=\xi\frac{\partial}{\partial\xi}+\frac{3n-3}{n-2}f\frac{\partial}{\partial f};\quad n=1. \tag{71a}$$

From $X_1$ in (71), we get a group transformation

$$\xi^*=e^{(n-2/3n-3)\beta}\xi \text{ and } f^*=e^{\beta}f;\ \ 0<n<1,\ \ n>1, \tag{72}$$

and from (71a):

$$\xi^*=e^{\beta}\xi \text{ and } f^*=e^{(3n-3/n-2)\beta}f;\ \ n=1, \tag{72a}$$

where "$\beta$" is the parameter of group transformation. Using the same procedure as discussed in the two-dimensional jet case, we get the following BVP with non-homogeneous boundary conditions at a second point:

$$\frac{1}{2n-1}\frac{d}{d\xi^*}\left(\pm\frac{f^*f^{*\prime}}{\xi^*}\right)=\frac{d}{d\xi^*}\left\{\xi^{*(1-n)}\left(\frac{f^{*\prime}}{\xi^*}-f^{*\prime\prime}\right)^n\right\};\xi^*>0, \tag{73a}$$

$$\frac{1}{2n-1}\frac{d}{d\xi^*}\left(\mp\frac{f^*f^{*\prime}}{\xi^*}\right)=\frac{d}{d\xi^*}\left\{\xi^{*(1-n)}\left(f^{*\prime\prime}-\frac{f^{*\prime}}{\xi^*}\right)^n\right\};\ \xi^*\leq 0. \tag{73b}$$

$$\xi^*=0:\ \ f^*=0,\ \ f^{*\prime}=0,\ \ f^{*\prime\prime}=1, \tag{74}$$

$$\xi^* \to \infty: \quad f^*(\infty)=1, \tag{75}$$

and the flux condition becomes

$$J=\frac{2\rho\pi U_0{}^2 L^2}{(Re)^{2/n+1}} e^{-2(2n-1/3n-3)\beta}\int_0^\infty \frac{1}{\xi^*}\left(f^{*\prime}\right)^2 d\xi^* = \text{constant}. \tag{76}$$

We now reduce the BVP (73)-(75) into IVP using the group transformation

$$\bar{\xi}=e^{(n-2/3n-3)\beta}\xi^* \text{ and } \bar{f}=e^{\beta} f^*. \tag{77}$$

Under the transformation (77), Equations (73) through (75) become

$$\frac{1}{2n-1}\frac{d}{d\bar{\xi}}\left(\pm\frac{\bar{f}\,\bar{f}'}{\bar{\xi}}\right)=\frac{d}{d\bar{\xi}}\left\{\bar{\xi}^{(1-n)}\left(\frac{\bar{f}'}{\bar{\xi}}-\bar{f}''\right)^n\right\};\bar{\xi}>0. \tag{78a}$$

Here, we apply positive sign for $n>1/2$ and negative sign for $0<n<1/2$,

$$\frac{1}{2n-1}\frac{d}{d\bar{\xi}}\left(\mp\frac{\bar{f}\,\bar{f}'}{\bar{\xi}}\right)=\frac{d}{d\bar{\xi}}\left\{\bar{\xi}^{(1-n)}\left(\bar{f}''-\frac{\bar{f}'}{\bar{\xi}}\right)^n\right\};\bar{\xi}\le 0, \tag{78b}$$

Apply negative sign for $n>1/2$ and positive sign for $0<n<1/2$.

The initial conditions:

$$\bar{f}(0)=0, \quad \bar{f}'(0)=0, \quad \bar{f}''(0)=1. \tag{79}$$

From the boundary condition at infinity, we have calculated the following parameter of the transformation

$$\beta=\left[\frac{1}{\bar{f}(\infty)}\right]^{n+1/3n-3}. \tag{80}$$

The flux condition is

$$J=\frac{2\rho\pi U_0{}^2 L^2}{(Re)^{2/n+1}}\int_0^\infty \frac{1}{\bar{\xi}}\left(\bar{f}'\right)^2 d\bar{\xi} = \text{constant}. \tag{81}$$

Similarly, we proceed for using transformations (72a) for the $n=1$ case. If we put $n=1$ in equation (78a) or (78b) and integrate it three times using (79), we get the exact solution (Ruscic and Mason (2004)) $f(\xi)=\dfrac{4k\xi^2}{1+k\xi^2}$, where $k$ is a constant.

# 8. Result and discussion

## 8.1 Two-dimensional Jet

The numerical solution of the IVP (60) with (61) is calculated by applying Fehlberg's fourth-fifth order Runge-Kutta method presented in Table 1. It shows that $\bar{f}$ tends to infinity, velocity $\bar{f}' = 0$ as $\bar{\xi}$ tends to infinity for $0 < n \le 1/2$ and $\bar{\xi}$ tends to infinity as $\bar{f}$ approaches some approximate value and $\bar{f}' = 0$ for $1/2 < n \le 1$, agreed with Atkinson (1972). For $n > 1$, $\bar{\xi}$ does not tend to infinity, it replaced by $\bar{\xi} = \bar{\xi}_n$, $\bar{f}$ approaches some approximate value and $\bar{f}' = 0$. From Table 1, we observed that $\bar{\xi}$ tends to infinity $\bar{f}$ approaches the approximate value of 2.4495 for the power index $n = 1$. Then, the parameter of transformation $\beta$ is obtained from (59) for $n = 1$, that is

$$\beta = 0.16667.$$

Therefore, (57) yields $e^{-\beta} = 2.4495$.

The solution to Equation (48) for $n = 1$ can be obtained by the transformation (53) as

$$\xi^* = 2.4495\bar{\xi}, \quad f^* = 0.4082\bar{f}.$$

By substituting the value of $\bar{f}(\infty)$ in the above relation, we obtain $f^*(\infty) = 1$ (Atkinson (1972)) that satisfies the boundary condition (51). Inspired by the research work of Atkinson (1972), we have obtained the following close-form solution of Equation (60a) with the condition (61):

$$\bar{f} = (2n-1)^{-n/n+1}(3n)^{1/n+1}(n+1)^{n/n+1}; \quad n > 1/2, \text{ and}$$

$$\bar{f}' = \exp\left(-4\bar{f}^3/27\right); 0 < n \le 1/2, \text{ therefore, } \bar{f} = \infty.$$

These results for different values of $n$ are well agreed with the numerical results presented in Table 1. The plot of $\bar{f}'$ verses $\bar{\xi}$ for different values of $n$ is shown in Figure 1.

**Table 1.** Numerical Solution of Equations (60a) - (61)

| $n$ | $\bar{\xi}$ | $\bar{f}$ | $\bar{f}'$ | $n$ | $\bar{\xi}$ | $\bar{f}$ | $\bar{f}'$ |
|---|---|---|---|---|---|---|---|
| 2 | 0 | 0 | 1 | 4/5 | 0 | 0 | 1 |
| | 2 | 1.4789 | 0.4134 | | 10 | 2.6130 | 0.010 |
| | 4 | 1.8156 | 0.0118 | | 30 | 2.6485 | 0.0002 |
| | 4.3 | 1.8171 | 0.0029 | | 89 | 2.6502 | 0 |
| | 4.3944841 | 1.8171 | 0 | | $\to\infty$ | 2.6502 | 0 |
| 1.5 | 0 | 0 | 1 | 1/2 | 0 | 0 | 1 |
| | 2 | 1.5652 | 0.4846 | | 10 | 2.7706 | 0.0428 |
| | 4 | 2.0497 | 0.0707 | | 30 | 3.1529 | 0.0096 |
| | 6 | 2.0866 | 0 | | 10000 | 4.2302 | 0 |
| | 6.0751819 | 2.0866 | 0 | | 20000000 | 5.0611 | 0 |
| 1 | 0 | 0 | 1 | 1/5 | 0 | 0 | 1 |
| | 10 | 2.3682 | 0.0652 | | 10 | 2.7706 | 0.0428 |
| | 15 | 2.4495 | 0.00001 | | 30 | 3.6201 | 0.0410 |
| | 20 | 2.4495 | 0 | | 100 | 5.4330 | 0.0182 |
| | $\to\infty$ | 2.4495 | 0 | | 1000 | 11.7261 | 0.0039 |

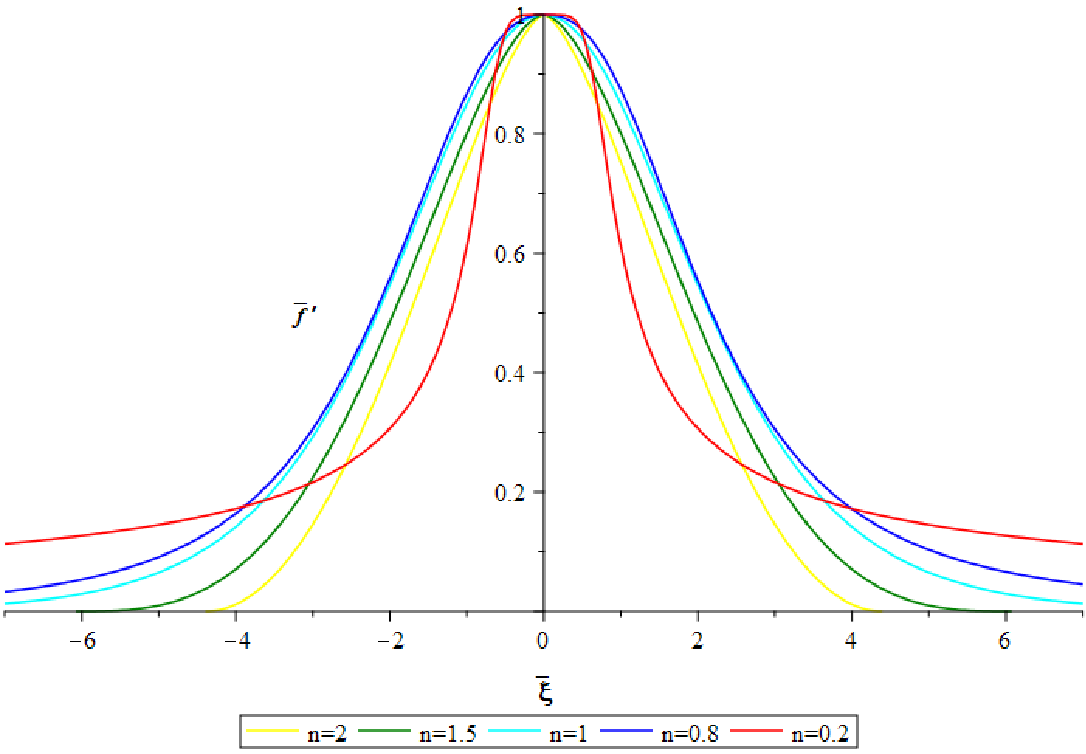


**Figure 1.** Velocity profile for Two-dimensional jet flow for different values of $n$

### 8.2 For Axisymmetric Jet

Numerical solution of the initial value problem (78) with (79) is calculated by applying Fehlberg's fourth-fifth Runge-Kutta Method using Maple software. It shows that $\bar{\xi}$ tends to infinity, $\bar{f}$ approaches the approximate value 4.0000 and velocity $\bar{f}'/\bar{\xi} = 0$ for power index $n = 1$ (agreed with Rotem (1964)) and $\bar{f}$ tends to infinity as $\bar{\xi}$ tends to infinity for $0 < n < 1$. For $n > 1$, $\bar{\xi}$ does not tend to infinity, it is replaced by $\bar{\xi} = \bar{\xi}_n$, $\bar{f}$ approaches some approximate value. Also, the velocity profile $\bar{f}'/\bar{\xi}$ is shown in Figure 2 satisfies the boundary condition (68) at $\bar{\xi} \to \infty$. The plot of $\bar{f}'/\bar{\xi}$ verses $\bar{\xi}$ for different values of $n$ is shown in Figure 2.

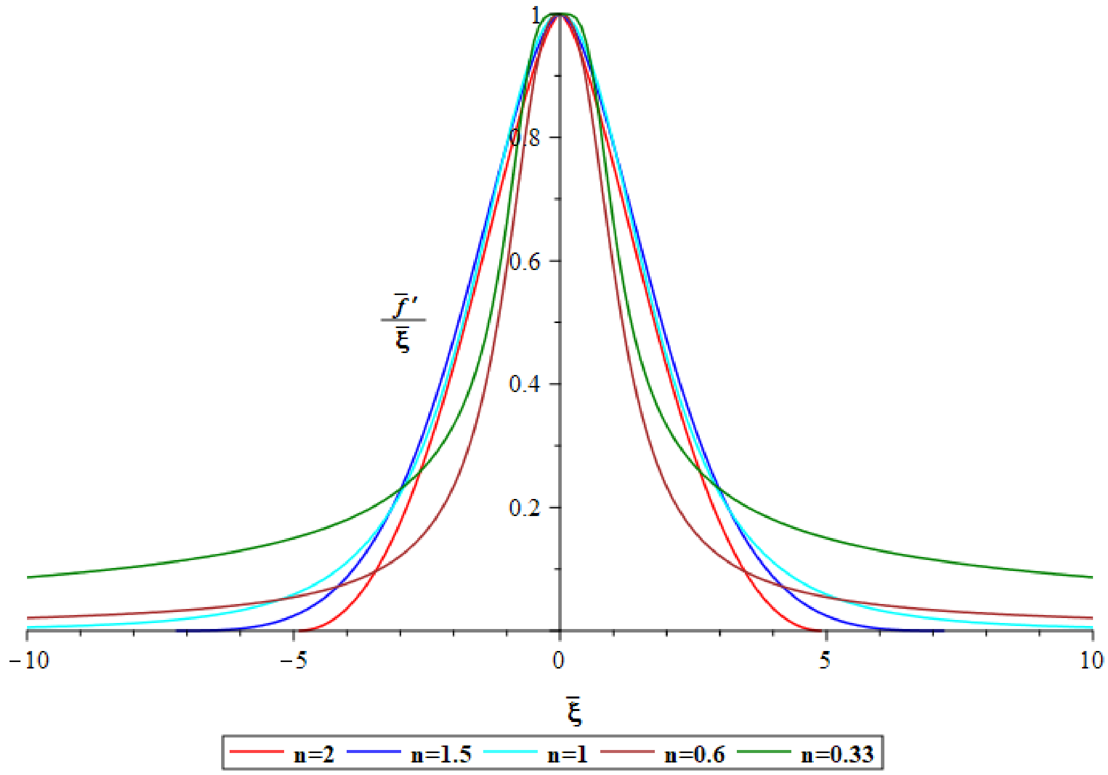


**Figure 2.** Velocity profile for Axisymmetric jet flow for different values of $n$

If we choose a similarity variable

$$\xi = \frac{Y}{(2n-1)^{\frac{1}{3(n-1)}}\left[X + \frac{C_2}{2n-1}\right]^{\frac{1}{2n-1}}}, \tag{82}$$

to remove singularity from the equations (67a) and (67b) for $n = 1/2$ case, then under this transformation equations (77a) and (77b) reduce to the following equations:

$$\frac{d}{d\bar{\xi}}\left(\frac{\bar{f}\,\bar{f}'}{\bar{\xi}}\right)=\frac{d}{d\bar{\xi}}\left\{\bar{\xi}^{(1-n)}\left(\frac{\bar{f}'}{\bar{\xi}}-\bar{f}''\right)^{n}\right\};\bar{\xi}>0,\ n>0, \tag{83a}$$

$$\frac{d}{d\bar{\xi}}\left(-\frac{\bar{f}\,\bar{f}'}{\bar{\xi}}\right)=\frac{d}{d\bar{\xi}}\left\{\bar{\xi}^{(1-n)}\left(\bar{f}''-\frac{\bar{f}'}{\bar{\xi}}\right)^{n}\right\};\bar{\xi}\le 0,\ n>0, \tag{83b}$$

with initial conditions $\bar{f}(0)=0,\ \ \bar{f}'(0)=0,\ \ \bar{f}''(0)=1.$ (84)

The equation (83a) with condition (84) is analogous to equations derived by Rotem (1964). Then, the plot of $\bar{f}'/\bar{\xi}$ verses $\bar{\xi}$ for different values of *n* corresponding Equations (83a)-(84) is shown in Figure 3.

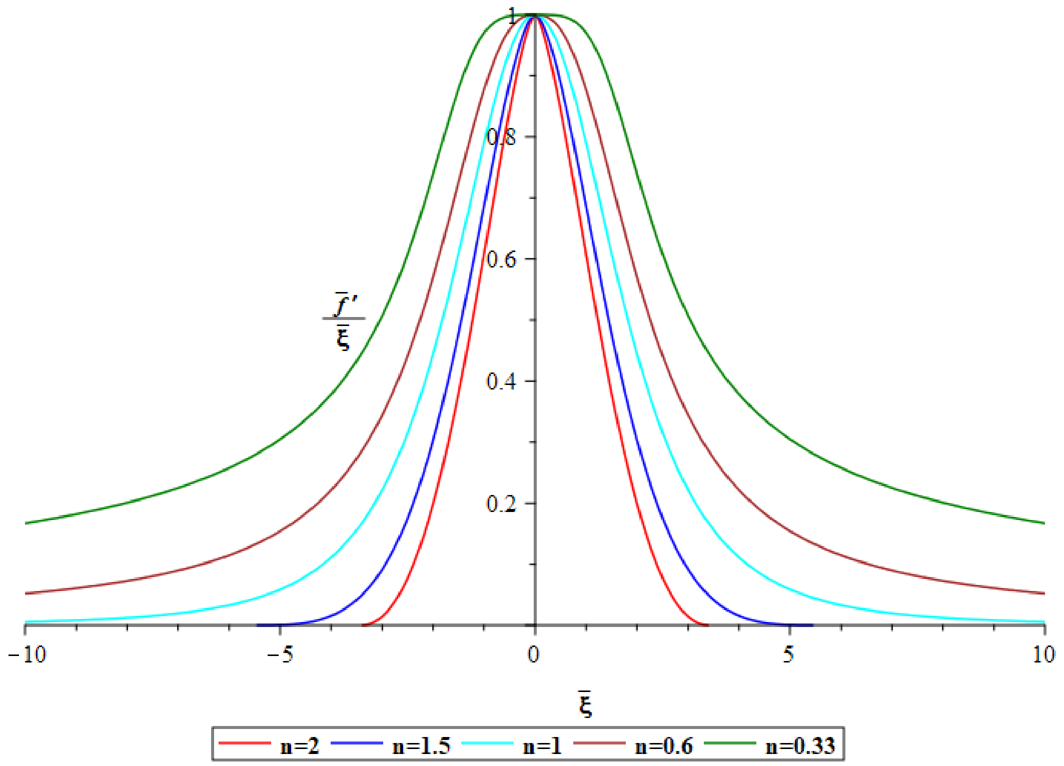


**Figure 3.** Velocity profile for Axisymmetric jet flow of equations (83a)-(84)

In the axisymmetric jet problem, the ODE system is singular at $\bar{\xi}=0$. Therefore, if initial conditions are started from $\bar{\xi}=0.00001$, the numerical results of the ODE system are executed for $n>0$.

## 9. Conclusion

Lie symmetric technique is applied to derive the most general form of an invariant solution that is exercisable to nonlinear PDEs governing the two-dimensional laminar jet as well as the axisymmetrical laminar jet of non-Newtonian power-law fluids. Using the symmetries, the non-linear PDE is transformed into a non-linear ODE with boundary conditions.

Here, the Lie symmetry generator is advantageously used to convert the entire homogeneous BVP into a non-homogeneous IVP. The converted IVP is solved numerically by the fourth-fifth order Runge-Kutta method. The velocity of two-dimensional and axisymmetric laminar, steady free jets increases as the power-law index decreases. The graphical representations are here demonstrating the importance of symmetries in applications. It is believed that the work contained in this research paper will be useful to other researchers working in technical applications like fluid injection engines, fluid flow in fountains and different types of jet flows.

## *Acknowledgment*

*The authors are very thankful to the honorable Professor Aliakbar M. Haghighi, the chief editor and the anonymous reviewers for their valuable comments and suggestions that helped to improve the present paper.*

## REFERENCES

Acrivos, A., Shah, M.J. and Petersen, E.E. (1965). On the two-dimensional boundary-layer flow equations for a non-Newtonian power-law fluid, Chemical Engineering Science, Vol. 20, pp. 101-105.

Arrigo, D.J. (2015). *Symmetry Analysis of Differential Equations*, John Wiley & Sons, Inc., New Jersey.

Atkinson, C. (1972). On the laminar two-dimensional free jet of an incompressible pseudoplastic fluid, Journal of Applied Mechanics, Vol. 39, Trans. ASME, Vol. 94, pp. 1162-1164.

Bickley, W.G. (1937). The plane jet, Phil. Mag., Vol. 23, pp. 727-731.

Bilige, S. and Han, Y. (2018). Symmetry reduction and numerical solution of a nonlinear boundary value problem in fluid mechanics, International Journal of Numerical Methods for Heat & Fluid Flow, Vol. 28, No. 3, pp. 518-531.

Bluman, G.W. and Anco, S.C. (2002). *Symmetry and Integration Methods for Differential Equations*. Springer-Verlag, New York.

Bluman, G.W. and Kumei, S. (1989). *Symmetries and Differential Equations, Applied Mathematical Science* 81**,** Springer-Verlag, New York.

Chhabra, R.P. and Richardson, J.F. (2008, Second Edition). *Non-Newtonian Flow and Applied Rheology: Engineering Applications*, Butterworth-Heinemann/IChemE.

Dresner, L. (1999). *Applications of Lie's Theory of Ordinary and Partial Differential Equations*, Institute of Physics Publishing, Bristol.

Gutfinger, C. and Shinnar, R. (1964). Velocity distribution in two-dimensional laminar liquid-into-liquid jets in power fluids, AIChE Journal, Vol. 10, pp. 631-639.

Halder, A.K., Leach, P.G.L and Paliathanasis, A. (2020). Similarity solutions and conservation laws for the Bogoyavlensky-Konopelchenko equation by Lie point symmetries, Quaestiones Mathematicae, Vol. 44, No. 6, pp. 815-827.

Hydon, P.E. (2000). *Symmetry Methods for Differential Equations: A Beginner's Guide*, Cambridge University Press, UK.

Ibragimov, N.H. and Kovalev, V.F. (2009). *Approximate and Renormgroup Symmetries*, Beijing and Springer-Verlag: Berlin, Germany.

Kalthia, N.L. (1979). Axisymmetric jet, Indian J. Pure Appl. Math., Vol. 10, No. 3, pp. 312-317.

Kalthia, N.L. and Jain, R.K. (1974). A weak jet of an incompressible pseudoplastic fluid, International Journal Pure and Applied Mathematics (IJPAM), Vol. 7, No. 1, pp. 110-15.

Kapur, J.N. (1962). On the two-dimensional jet of an incompressible pseudo-plastic fluid, Journal of the Physical Society of Japan, Vol. 17, pp. 1303-1309.

Kapur, J.N., Bhatt, B.S. and Sacheti, N.C. (1982). *Non–Newtonian Fluid Flows*, Pragati Prakashan, Meerut (India).

Lemieux, P.F. and Unny, T.E. (1968). The laminar two-dimensional free jet of an incompressible pseudoplastic fluid, Journal of Applied Mechanics, Vol. 35, Trans. ASME, Vol. 90, pp. 810-812.

Magan, A.B., Mason, D.P. and Mahomed, F.M. (2016). Analytical solution in parametric form for the two-dimensional free jet of a power-law fluid, International Journal of Non-Linear Mechanics International Journal of Non-Linear Mechanics, Vol. 85, pp. 94-108.

Magan, A.B., Mason, D.P., Mahomed, F.M. (2017). Analytical solution in parametric form for the two-dimensional liquid jet of a power-law fluid, International Journal of Non-Linear Mechanics, Vol. 93, pp. 53-64.

Mason, D.P. (2002). Group invariant solution and conservation law for a free laminar two-dimensional jet, Journal of Nonlinear Mathematical Physics, Vol. 9, pp. 92-101.

Na, T.Y. (1979). *Computational Method in Engineering Boundary Value Problems*, Academic Press, London.

Naz, R. (2011). Group-invariant solutions for two-dimensional free, wall, and liquid jets having finite fluid velocity at orifice, Mathematical Problems in Engineering Journal, Hindawi Publishing Corporation, Vol. 2011, Article ID 615612, pp. 1-12.

Oleinik, O.A. and Samokhin, V.M. (1999). *Mathematical Models in Boundary Layer Theory*, Applied Mathematics and Mathematical Computation 15, Taylor and Francis Group, CRC Press, Boca Raton, London, New York.

Pai, S.I. and Hsieh, T. (1972). Numerical solution of laminar jet mixing with and without free Stream, Appl. Sci. Res., Vol. 27, pp. 39-62.

Pakdemirli, M., Aksoy, Y. and Khalique, C.M. (2008). Symmetries of boundary layer equations of power-law fluids of second grade, Acta Mechanica Sinica, Vol. 24, No. 6, pp. 661-670.

Paliathanasis, A. (2021). Lie symmetry analysis for the Camassa-Choi equations, Analysis and Mathematical Physics, Vol. 11, No. 2, Article 57, pp. 1-13.

Patel, M., Patel, J. and Timol, M.G. (2015). Laminar boundary layer flow of Sisko fluid, Applications and Applied Mathematics: An International Journal (AAM), Vol. 10, No. 2, pp. 909-918.

Patel, M. and Timol, M.G. (2010). The general stress-strain relationship for some different visco-inelastic non-Newtonian fluids, International Journal of Applied Mechanics and Mathematics (IJAMM), Vol. 6, No. 12, pp. 79-93.

Patel, M. and Timol, M.G. (2016). Jet with variable fluid properties: Free jet and dissipative jet, International Journal of Non-Linear Mechanics, Vol. 85, pp. 54-61.

Rotem, Z. (1964). The axisymmetrical free laminar jet of an incompressible pseudoplastic fluid, Appl. Sci. Res., Section A, Vol.13, pp. 353-370.

Ruscic, I. and Mason, D.P. (2004). Group invariant solution and conservation law for a Steady Laminar Axisymmetric Free Jet, Quaestiones Mathematicae Vol. 27, No. 2, pp. 171-183.

Schlichting, H. (1933), Laminar Strahlauspreitung, ZAMM, Vol. 13, pp. 260- 263.

Sheshadri, R. and Na, T.Y. (1985). *Group Invariance in Engineering Boundary Value Problems*, Springer-Verlag, New York.

Shuka, H., Surati, H. and Timol M.G. (2019). Local non-similar solution of Powell-Eyring fluid flow over a vertical flat plate. Applications and Applied Mathematics: An International Journal (AAM), Vol. 14, No. 2, pp. 973-984.

Soid, S.K. and Ishak, A. (2017). Boundary layer flow and heat transfer of a nanofluid over a moving permeable surface, Journal of Computational and Theoretical Nanoscience, Vol. 23 (11), pp. 11153-11157.

Stephani, H. (1989). *Differential Equations: Their Solution using Symmetries*, Cambridge University Press, New York, USA.

Timol, M.G. and Kalthia, N.L. (1986). Similarity solutions of three-dimensional boundary layer equations of non-Newtonian fluids, Int. J. Non-Linear Mechanics, Vol. 21, No. 6, pp. 475-481.

Wilkinson, W.L. (1960). *Non-Newtonian Fluids*, Pergamon Press, London.

Zeidan, D. and Bira, B. (2019). Weak shock waves and its interaction with characteristic shocks in polyatomic gas, Mathematical Methods in the Applied Sciences, Vol. 42, No.14, pp. 4679-4687.