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Numerical Computation of Travelling Wave Solutions for the Nonlinear Ito System

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Abstract: The Ito equation (a coupled nonlinear wave equation which generalizes the KdV equation) has previously been shown to admit a reduction to a single nonlinear Casimir equation governing the wave solutions. Some analytical properties of the solutions to this equation in certain parameter regimes have been studied recently. However, for general parameter regimes where the analytical approach is not so useful, a numerical method would be desirable. Therefore, in this paper, we proceed to show that the Casimir equation for the Ito system can be solved numerically by use of the shifted Jacobi-Gauss collocation (SJC) spectral method. First, we present the general solution method, which is follows by implementation of the method for specific parameter values. The presented results in this article demonstrate the accuracy and efficiency of the method. In particular, we demonstrate that relatively few notes permit very low residual errors in the approximate numerical solutions. We are also able to show that the coefficients of the higher order terms in the shifted Jacobi polynomials decrease exponentially, meaning that accurate solutions can be obtained after relatively few terms are used. With this, we have a numerical method which can accurately and efficiently capture the behavior of nonlinear waves in the Ito equation.

Keywords: Second-order initial value problem; Collocation method; Jacobi-Gauss quadrature; Shifted Jacobi polynomials; Ito equations

1 Introduction and formulation of the problem

The Ito system [1] reads

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial^3 U}{\partial x^3} + 3U\dfrac{\partial U}{\partial x} + V\dfrac{\partial U}{\partial x}, \\ \dfrac{\partial V}{\partial t} = \dfrac{\partial}{\partial x}(UV). \end{cases} \tag{1}$$

Ito showed that the system (1) is highly symmetric and possesses infinitely many conservation laws. It is an extension of the KdV equation, with an additional field variable V. Olver and Rosenau [2] introduced a dual bi-Hamiltonian system for the Ito system, which admits a Casimir functional and associated Casimir equations. Introducing a stream function for the Casimir equations, they then obtained the single partial differential equation

$$\frac{\partial^2 W}{\partial t^2} = \left(\frac{\partial W}{\partial t}\right)^2 \frac{\partial}{\partial x}\left\{\left(\frac{\partial W}{\partial t}\right)^2 \left(\frac{\partial W}{\partial x} \pm \frac{\partial^3 W}{\partial x^3}\right)\right\}. \tag{2}$$

While the Casimir equation (2) for the Ito system (1) has received relatively little attention in the recent literature, it is an interesting nonlinear partial differential equation related to an extension of the KdV equation. As such, it can be viewed physically as a generalization of KdV to account for more complicated nonlinear effects on the wave propagation in KdV. Despite being highly nonlinear, as shown in Van Gorder [3] it has been show to admit a variety of exact and analytical solutions. Van Gorder [3] obtained analytic solutions to this equation, which consist of explicit exact solutions in some cases and implicit integral relations in others. Then, Hausserman and Van Gorder [4] attempted to classify all possible series solutions to (i) travelling wave reductions of (2) and (ii) a class of self-similar reductions to (2). Some asymptotic solutions were also given.

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1.1 Travelling waves for the Casimir equation (2)

We will be interested in solving the travelling wave problem for the Casimir equation (2). To this end, let us assume a solution of the form $W(x,t) = \phi(z)$ where $z = x - \beta^{-1/2}t$ is the wave variable and $\beta > 0$ is a constant. Then (2) becomes

$$\phi'' = \frac{1}{\beta}\phi'^2\{\phi'^2(\phi' + \varepsilon\phi''')\}', \tag{3}$$

where prime denotes differentiation with respect to z and either $\varepsilon = 1$ or $\varepsilon = -1$ depending on the sign $(+$ or $-)$ in (2). Rearranging (3) assuming $\phi' = 0$ only if $\phi'' = 0$, and introducing a new function $f(z) = \phi'(z)$, we have

$$\frac{f'}{f^2} = \frac{1}{\beta}\{f^2(f + \varepsilon f'')\}', \tag{4}$$

and performing one integration with respect to z, we have

$$\alpha - \frac{\beta}{f} = f^2(f + \varepsilon f''), \tag{5}$$

where α is a constant of integration. If we rearrange (5) in terms of the second derivative, we find

$$f'' = \varepsilon\{\frac{\alpha}{f^2} - \frac{\beta}{f^3} - f\}, \tag{6}$$

which is a new transformation of (2). Since $\beta > 0$, we can choose the rescaling $g(z) = \beta^{1/4}f(z)$ so that (6) becomes

$$g'' = \varepsilon\{\frac{a}{g^2} - \frac{1}{g^3} - g\}, \tag{7}$$

where we define the constant $a = \alpha\beta^{-3/4}$. So, given either sign choice of ε there is a one parameter family of solutions depending on a.

We point out here that equation (7) is similar in form to the travelling wave version of a reaction-diffusion equation like the Nagumo equation. For that equation, one has a cubic reaction function such as $g'' = g(1-g)(\chi - g)$; see [5]-[7]. Here the nonlinear reaction function is much more complicated, involving negative powers of the unknown function. Still, this reduction is much simpler than (3). Therefore, obtaining solutions of equation (7) can be very challenging.

1.2 Restrictions on the existence regions for solutions

As addressed before [3], there are essentially two types of qualitative behavior associated with solutions of (7), depending on the sign of ε. When $\varepsilon = 1$, equation (7) is linearized like $g'' - g = 0$ so we expect solutions which may grow without bound in a hyperbolic manner. Meanwhile, when $\varepsilon = -1$, equation (7) is linearized like $g'' + g = 0$, so we expect solutions which oscillate when the amplitude is large. When the amplitude is small, the nonlinear terms dominate, and the behavior could be unpredictable by the linearization. In case of $\varepsilon = -1$, the solutions will be most interesting to study, so we consider the initial value problem

$$g'' + g - \frac{a}{g^2} + \frac{1}{g^3} = 0, \tag{8}$$

$$g(0) = A, \quad g'(0) = 0. \tag{9}$$

Then, (8)-(9) defines a two parameter nonlinear initial value problem with parameter space $a \in \mathbb{R}$ and $A > 0$. Note that a change in the length scales such as $z \Rightarrow \ell z$ manifests only as a change in the magnitude of ε. Therefore, we shall simulate solutions on the open interval $z \in (0,1)$ with the understanding that solutions can be found on any interval of the form $z \in (z_-, z_+)$ subject to an appropriate change in ε. Since the sign of ε is the only thing significant to the qualitative dynamics of the solutions, it is indeed sufficiently general to consider $z \in (0,1)$ and $\varepsilon = \pm 1$.

It is worthy to mention here that solutions might not exist for some parameter values. Hence, one must take care when solving this problem numerically. To demonstrate this point, consider the special case corresponds to $a = 0$. In such case, and if we multiply (8) with $2g'$ and integrating with respect to z, we have

$$g'^2 + g^2 - \frac{1}{g^2} = I(A) = A^2 - A^{-2}. \tag{10}$$

Separating variables in (10), we obtain

$$\begin{aligned} z &= \int_A^g \frac{d\xi}{\sqrt{I(A) - \xi^2 + \xi^{-2}}} \\ &= -\frac{1}{2}\tan^{-1}\left(\frac{I(A) - 2\xi^2}{2\sqrt{I(A)\xi^2 - \xi^4 + 1}}\right)\Big|_{\xi=A}^{\xi=g}, \end{aligned} \tag{11}$$

which can be inverted for g to obtain

$$g(z) = -\frac{1}{\sqrt{2}}\sqrt{I(A) - 2\sqrt{4 + I(A)^2}\sin(2[z - z_0(A)])}, \tag{12}$$

where $z_0(A)$ is a constant depending on A. To guarantee that the solution (12) is real valued, the condition $I(A) - 2\sqrt{4 + I(A)^2}\sin(2[z - z_0(A)]) \geq 0$ should be satisfied for all z. Yet, when $z = \frac{\pi}{2} + z_0(A)$, we have $I(A) - 2\sqrt{4 + I(A)^2} \geq 0$ and hence $I(A) \geq 2\sqrt{4 + I(A)^2} > 2 \mid I(A) \mid$, a contradiction. So, there is no such real valued solution when $a = 0$. This illustrates the fact that for a given value of the constant a, there may or may not exist solutions. Therefore, while we are interested in the numerical solution of (8)-(9), we must be careful in selecting the numerical method, due to the inherent challenges of solving the highly nonlinear equation (8).

1.3 Solution method and summary of results

A well-known advantage of a spectral method is that it achieves high accuracy with relatively fewer spatial grid points when compared with a finite-difference method (see, for instance [8]-[15]). On the other hand, spectral methods typically give rise to full matrices, partially negating the gain in efficiency due to the fewer number of grid points. In general, the use of Jacobi polynomials ($P_k^{(\theta,\vartheta)}$ with $\theta,\vartheta \in (-1,\infty)$ and n is the polynomial degree) has the advantage that the obtained solutions of differential equations are given in terms of the Jacobi parameters θ and ϑ (see, e.g. [16]-[19].

We mention that the spectral collocation method is very useful in providing highly accurate solutions to nonlinear differential equations (see, e.g. [20]-[28]). In the present paper we intend to extend the application of Jacobi-Gauss collocation method to solve the nonlinear initial value problem (8)-(9). This paper is organized as follows. In Section 2 we give an overview of shifted Jacobi polynomials and their relevant properties needed hereafter, and in Section 3, the way of constructing the collocation technique for equation (8)-(9) is described using the shifted Jacobi polynomials. In Section 4, we present some numerical results exhibiting the accuracy and efficiency of our numerical algorithms.

While the method outlined here has been applied to other nonlinear equations, such as the Lane-Emden problem [20], for the present problem there is the additional complication of the nonlinearity in (8). Indeed, such a nonlinearity involves inverse powers of the unknown function. Previous applications of such methods were only concerned with more standard power-law type nonlinearity. Therefore, the present paper demonstrates that the numerical method is rather versatile and can be used for a wide variety of problems.

2 Mathematical preliminaries

Let $\theta > -1$, $\vartheta > -1$ and $P_k^{(\theta,\vartheta)}(x)$ be the standard Jacobi polynomial of degree k, then we have the following special values

$$\begin{aligned} P_k^{(\theta,\vartheta)}(-x) &= (-1)^k P_k^{(\theta,\vartheta)}(x), \\ P_k^{(\theta,\vartheta)}(-1) &= \frac{(-1)^k \Gamma(k+\vartheta+1)}{k!\Gamma(\vartheta+1)}, \\ P_k^{(\theta,\vartheta)}(1) &= \frac{\Gamma(k+\theta+1)}{k!\Gamma(\theta+1)}. \end{aligned} \tag{13}$$

Besides,

$$D^m P_k^{(\theta,\vartheta)}(x) = 2^{-m} \frac{\Gamma(m+k+\theta+\vartheta+1)}{\Gamma(k+\theta+\vartheta+1)} P_{k-m}^{(\theta+m,\vartheta+m)}(x). \tag{14}$$

Let $w^{(\theta,\vartheta)}(x) = (1-x)^\theta (1+x)^\vartheta$, then we define the weighted space $L^2_{w^{(\theta,\vartheta)}}[-1,1]$ as usual, equipped with the following inner product and norm,

$$(u,v)_{w^{(\theta,\vartheta)}} = \int_{-1}^{1} u(x)\, v(x)\, w^{(\theta,\vartheta)}(x)\, dx,\ \|v\|_{w^{(\theta,\vartheta)}} = (v,v)^{\frac{1}{2}}_{w^{(\theta,\vartheta)}}.$$

The set of Jacobi polynomials forms a complete $L^2_{w^{\theta,\vartheta}}[-1,1]$-orthogonal system, and

$$\begin{aligned} \|P_k^{(\theta,\vartheta)}\|^2_{w^{(\theta,\vartheta)}} &= h_k^{(\theta,\vartheta)} \\ &= \frac{2^{\theta+\vartheta+1}\Gamma(k+\theta+1)\Gamma(k+\vartheta+1)}{(2k+\theta+\vartheta+1)\Gamma(k+1)\Gamma(k+\theta+\vartheta+1)}. \end{aligned} \tag{15}$$

If we define the shifted Jacobi polynomial of degree k by $J_k^{(\theta,\vartheta)}(x) = P_k^{(\theta,\vartheta)}(2x-1)$, and in virtue of (13) and (14), then it can be easily shown that

$$D^q J_k^{(\theta,\vartheta)}(0) = \frac{(-1)^{k-q}\Gamma(k+\vartheta+1)(k+\theta+\vartheta+1)_q}{\Gamma(k-q+1)\Gamma(q+\vartheta+1)}, \tag{16}$$

$$D^q J_k^{(\theta,\vartheta)}(1) = \frac{\Gamma(k+\theta+1)(k+\theta+\vartheta+1)_q}{\Gamma(k-q+1)\Gamma(q+\theta+1)}, \tag{17}$$

$$D^m J_k^{(\theta,\vartheta)}(x) = \frac{\Gamma(m+k+\theta+\vartheta+1)}{\Gamma(k+\theta+\vartheta+1)} J_{k-m}^{(\theta+m,\vartheta+m)}(x). \tag{18}$$

Next, let $\chi^{(\theta,\vartheta)}(x) = (1-x)^\theta x^\vartheta$, then we define the weighted space $L^2_{\chi^{(\theta,\vartheta)}}[0,1]$ in the usual way, with the following inner product and norm,

$$(u,v)_{\chi^{(\theta,\vartheta)}} = \int_{0}^{1} u(x)\, v(x)\, \chi^{(\theta,\vartheta)}(x)\, dx,\ \|v\|_{\chi^{(\theta,\vartheta)}} = (v,v)^{\frac{1}{2}}_{\chi^{(\theta,\vartheta)}}.$$

The set of shifted Jacobi polynomials forms a complete $L^2_{\chi^{(\theta,\vartheta)}}[0,1]$-orthogonal system. Moreover, and due to (15), we have

$$\|J_k^{(\theta,\vartheta)}\|^2_{\chi^{(\theta,\vartheta)}} = (\frac{1}{2})^{\theta+\vartheta+1} h_k^{(\theta,\vartheta)} = \eta_k^{(\theta,\vartheta)}. \tag{19}$$

For $\theta = \vartheta$ one recovers the shifted ultraspherical polynomials (symmetric Jacobi polynomials) and for $\theta = \vartheta = \mp\frac{1}{2}$, $\theta = \vartheta = 0$, the shifted Chebyshev of the first and second kinds and shifted Legendre polynomials respectively; and for the nonsymmetric shifted Jacobi polynomials, the two important special cases $\theta = -\vartheta = \pm\frac{1}{2}$ (shifted Chebyshev polynomials of the third and fourth kinds) are also recovered.

We denote by $x_{N,j}^{(\theta,\vartheta)}$, $0 \leqslant j \leqslant N$, the nodes of the standard Jacobi-Gauss interpolation on the interval $[-1,1]$. Their corresponding Christoffel numbers are $\varpi_{N,j}^{(\theta,\vartheta)}$, $0 \leqslant j \leqslant N$. The nodes of the shifted Jacobi-Gauss interpolation on the interval $[0,1]$ are the zeros of $J_{N+1}^{(\theta,\vartheta)}(x)$, which we denote by $\theta_{N,j}^{(\theta,\vartheta)}$, $0 \leqslant j \leqslant N$. Clearly $\theta_{N,j}^{(\theta,\vartheta)} = \frac{1}{2}(x_{N,j}^{(\theta,\vartheta)}+1)$, and their corresponding Christoffel numbers are $\vartheta_{N,j}^{(\theta,\vartheta)} = (\frac{1}{2})^{\theta+\vartheta+1}\varpi_{N,j}^{(\theta,\vartheta)}$, $0 \leqslant j \leqslant N$. Let $S_N[0,1]$ be the set of polynomials of degree at most N. Thanks to the property of the standard Jacobi-Gauss quadrature, it follows that for any $\phi \in S_{2N+1}[0,1]$,

$$
\begin{aligned}
&\int_0^1 (1-x)^{\theta}x^{\vartheta}\phi(x)dx \\
&= (\frac{1}{2})^{\theta+\vartheta+1}\int_{-1}^{1}(1-x)^{\theta}(1+x)^{\vartheta}\phi(\frac{1}{2}(x+1))dx \\
&= (\frac{1}{2})^{\theta+\vartheta+1}\sum_{j=0}^{N}\varpi_{N,j}^{(\theta,\vartheta)}\phi(\frac{1}{2}(x_{N,j}^{(\theta,\vartheta)}+1)) \\
&= \sum_{j=0}^{N}\vartheta_{N,j}^{(\theta,\vartheta)}\phi(\theta_{N,j}^{(\theta,\vartheta)}).
\end{aligned}
\tag{20}
$$

With the previous mathematical properties, we are now ready to obtain our desired collocation method.

3 Derivation of the shifted Jacobi-Gauss collocation method

In this section, we use the Jacobi-Gauss collocation method to obtain a general numerical method for the solutions of the following model problem:

$$
g^{''}(x) + g(x) - \frac{a}{g^2(x)} + \frac{1}{g^3(x)} = 0, \tag{21}
$$

$$
0 < x < 1, \quad a \in \mathbb{R},
$$

subject to

$$
g(0) = A, \quad g^{'}(0) = 0, \quad A > 0. \tag{22}
$$

The choice of collocation points is important for the convergence and efficiency of the collocation method. It should be noted that for a second-order differential equation with the singularity at $x = 0$ in the interval $(0,1)$, one is unable to apply the collocation method with Jacobi-Gauss-Radau points because the fixed node $x = 0$ is necessary to use as a one point from the collocation nodes. In fact, the collocation method with Jacobi-Gauss nodes are used to treat singular second-order differential equation; i.e., we collocate the singular nonlinear ODE only at the $(N-1)$ Jacobi-Gauss points that are the $(N-1)$ zeros of the shifted Jacobi polynomial on $(0,1)$. These equations together with two initial conditions generate $(N+1)$ nonlinear algebraic equations which can be solved. Let us first introduce some basic notation that will be used in the sequel. We set

$$
S_N(0,1) = \text{span}\{J_0^{(\theta,\vartheta)}(x), J_1^{(\theta,\vartheta)}(x), \ldots, J_N^{(\theta,\vartheta)}(x)\} \tag{23}
$$

and we define the discrete inner product and norm as follows:

$$
\begin{aligned}
&(u,v)_{\chi^{(\theta,\vartheta)},N} = \sum_{j=0}^{N} u(\zeta_{N,j}^{(\theta,\vartheta)})\, v(\zeta_{N,j}^{(\theta,\vartheta)})\, \eta_{N,j}^{(\theta,\vartheta)}, \\
&\| u \|_{\chi^{(\theta,\vartheta)},N} = \sqrt{(u,u)_{\chi^{(\theta,\vartheta)},N}}.
\end{aligned}
\tag{24}
$$

where $\zeta_{N,j}^{(\theta,\vartheta)}$ and $\eta_{N,j}^{(\theta,\vartheta)}$ are the nodes and the corresponding weights of the shifted Jacobi-Gauss quadrature formula on the interval $(0,1)$, respectively. Obviously,

$$
(u,v)_{\chi^{(\theta,\vartheta)},N} = (u,v)_{\chi^{(\theta,\vartheta)}}, \quad \forall u,v \in S_{2N-1}. \tag{25}
$$

Thus, for any $u \in S_N(0,1)$, the norms $\| u \|_{\chi^{(\theta,\vartheta)},N}$ and $\| u \|_{\chi^{(\theta,\vartheta)}}$ coincide. Associating with this quadrature rule, we denote by $I_N^{J^{(\theta,\vartheta)}}$ the shifted Jacobi-Gauss interpolation,

$$
I_N^{J^{(\theta,\vartheta)}} u(\zeta_{N,j}^{(\theta,\vartheta)}) = u(\zeta_{N,j}^{(\theta,\vartheta)}), \quad 0 \leq k \leq N.
$$

The shifted Jacobi-Gauss collocation method for solving (21) and (22) is to seek $g_N(x) \in S_N(0,1)$, such that

$$
\begin{aligned}
&g^{''}(\zeta_{N,k}^{(\theta,\vartheta)}) + g(\zeta_{N,k}^{(\theta,\vartheta)}) - \frac{a}{g^2(\zeta_{N,k}^{(\theta,\vartheta)})} + \frac{1}{g^3(\zeta_{N,k}^{(\theta,\vartheta)})} = 0, \\
&k = 0,1,\cdots,N-2, \\
&g(0) = A, \quad g^{'}(0) = 0.
\end{aligned}
\tag{26}
$$

We now derive the algorithm for solving (21) and (22). To do this, let

$$
g_N(x) = \sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x), \quad \mathbf{a} = (a_0, a_1, \ldots, a_N)^T. \tag{27}
$$

We first approximate $g(x)$ and $g^{''}(x)$, as Eq. (27). By substituting these approximation in Eq. (21), we get

$$
\begin{aligned}
&\sum_{j=0}^{N} a_j D^2 J_j^{(\theta,\vartheta)}(x) + \sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x) \\
&- \frac{a}{(\sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x))^2} + \frac{1}{(\sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x))^3} = 0.
\end{aligned}
\tag{28}
$$

Then, by virtue of (18), we deduce that

$$\sum_{j=0}^{N} a_j (j+\theta+\vartheta+1)_2 J_{j-2}^{(\theta+2,\vartheta+2)}(x) + \sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x) - \frac{a}{(\sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x))^2} + \frac{1}{(\sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x))^3} = 0. \tag{29}$$

Also, by substituting Eq. (27) in Eq. (22) we obtain

$$\sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(0) = A, \quad \sum_{j=0}^{N} a_j D J_j^{(\theta,\vartheta)}(0) = 0. \tag{30}$$

To find the solution $g_N(x)$, we first collocate Eq. (29) at the $(N-1)$ Jacobi rational roots, yields

$$\sum_{j=0}^{N} a_j (j+\theta+\vartheta+1)_2 J_{j-2}^{(\theta+2,\vartheta+2)}(x) + \sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x) - \frac{a}{(\sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x))^2} + \frac{1}{(\sum_{j=0}^{N} a_j J_j^{(\theta,\vartheta)}(x))^3} = 0. \tag{31}$$

Next, Eq. (30), After using (16), can be written as

$$\sum_{j=0}^{N} \frac{(-1)^j \Gamma(j+\vartheta+1)}{j!\,\Gamma(\vartheta+1)} a_j = A, \tag{32}$$

$$\sum_{j=1}^{N} (-1)^{j-1} \frac{(j+\theta+\vartheta+1)\Gamma(j+\vartheta+1)}{(j-1)!\,\Gamma(\vartheta+2)} a_j = 0. \tag{33}$$

Finally, from (31), (32) and (33), we get $(N+1)$ nonlinear algebraic equations which can be solved for the unknown coefficients a_j by using any standard iteration technique, like Newton's iteration method. Consequently, $g_N(x)$ given in Eq. (27) can be evaluated.

With this, we have developed the relevant collocation method. In the next section, we conduct numerical experiments in order to demonstrate the accuracy and efficiency of the method.

4 Numerical experiments

Having developed the collocation method in the previous section, we now apply the method by conducting numerical experiments for various values of the parameters A and α. In particular, in this section, we show the accuracy and fast convergence of the proposed algorithm.

Table 4.1 shows the approximations of $g_N(x)$ for $a=-2$, $A=6$, with $\theta=\vartheta=-\frac{1}{2}$ (first kind shifted Chebyshev collocation method), $\theta=\vartheta=0$ (shifted Legendre collocation method) and $\theta=\vartheta=\frac{1}{2}$ (second kind shifted Chebyshev collocation method) at $N=8$. Moreover, the last column of this table for $a=2$ with $A=6$, with $\theta=\vartheta=-\frac{1}{2}$. What we see here is that for each of the parameter choices taken (which determine the types of base functions obtained), the solutions are equivalent up to five decimal places. This is rather good agreement, seeing as only $N=8$ for these experiments.

Table 4.2 shows the approximations of $g_N(x)$ for $A=2,7$, $a=0$, with various choices of θ and ϑ at $N=8$. In this set of simulations, we see disagreements appearing at the last decimal place in the solutions. So, with $N=8$, the discrepancy in the solutions appears to be of order 10^{-5}. Increasing N shall help us increase the agreement in these approximations. Of course, 10^{-5} error is good for many applications, and it is certainly better than what is needed to accurately plot solutions. Still, for some applications, particularly nonlinear equations where precise knowledge of a function at a small value of z is indispensable in order to accurately obtain solutions at some larger value of z, improving the accuracy is more crucial.

Tables 4.3, 4.4, 4.5 and 4.6, show absolute residual errors of $g_N(x)$ for $A=6$, $a=-2$, $A=6$, $a=2$, $A=2$, $a=0$, and $A=7$, $a=0$, respectively, with various choices of θ and ϑ at $N=16$ and $N=24$. While there is no clear superior choice of θ and ϑ for all x in the domain, the error properties of each of the three choices are very similar. Increasing N results in a decrease in the residual errors, as anticipated. We gain roughly a power of 10 for ever two nodes we add. Note also that while the residual errors decrease with increasing N, they do not necessarily do so uniformly with respect to our choice of θ and ϑ. Indeed, fixing $x=x_0$, for $N=16$ once choice of θ and ϑ may be best, while for $N=24$, a different choice of θ and ϑ may be best. With these variations aside, for the three choices of θ and ϑ considered for each numerical simulation, the difference in residual errors is no more than a single order of magnitude.

The worst errors actually appear to correspond to small A and small a, as evidenced by Table 4.5. In the small a limit, we essentially have an ordinary differential equation of the form $g''+g+g^{-3}=0$, which becomes strongly singular for $g << 1$. So, if we pick initial data $g(0)=A$ so that A is small, we require more nodes in order to obtain an accurate numerical approximation.

The graph of the absolute value of the coefficients of shifted Jacobi polynomials of equation (21)-(22) are shown in Fig. 1 and demonstrate that the method has exponential convergence rate. Fig. 2 shows the absolute residual error functions for $A=10$, $a=2$, and $\theta=\vartheta=1$ at $N=24$. Note that solutions are demonstrating the most residual error near the boundary of the domain. In the case of $\theta=\vartheta=1$, the approximate solution is shown in Fig. 3, for $A=3$ and various values of a at $N=8$. Likewise, Fig. 4 shows the approximate solution for $a=-5$ and various A when $\theta=\vartheta=1$ at $N=8$.

Table 4.1
Approximate solutions for $A=6,\ N=8$.

x	$\theta=\vartheta=\frac{-1}{2}$	$\theta=\vartheta=0$	$\theta=\vartheta=\frac{1}{2}$	$\theta=\vartheta=\frac{-1}{2}$
	$(a=-2)$			$(a=2)$
0.0	6.00000	6.00000	6.00000	6.00000
0.1	5.96972	5.96972	5.96972	5.97028
0.2	5.87919	5.87919	5.87919	5.88142
0.3	5.72929	5.72929	5.72929	5.73433
0.4	5.52147	5.52147	5.52147	5.53049
0.5	5.25778	5.25778	5.25778	5.27200
0.6	4.94075	4.94075	4.94075	4.96147
0.7	4.57345	4.57345	4.57345	4.60211
0.8	4.15939	4.15939	4.15939	4.19761
0.9	3.70247	3.70247	3.70247	3.75218
1.0	3.20688	3.20688	3.20688	3.27049

Table 4.2
Approximate solutions for $a=0,\ N=8$.

x	$\theta=\vartheta=\frac{-1}{2}$	$\theta=\vartheta=0$	$\theta=\vartheta=\frac{-1}{2}$	$\theta=\vartheta=0$
	$(A=2)$		$(A=7)$	
0.0	2.00000	2.00000	7.00000	7.00000
0.1	1.98938	1.98938	6.96501	6.96501
0.2	1.95761	1.95761	6.86041	6.86041
0.3	1.90495	1.90495	6.68722	6.68722
0.4	1.83180	1.83180	6.44717	6.44717
0.5	1.73872	1.73872	6.14270	6.14270
0.6	1.62635	1.62635	5.77679	5.77679
0.7	1.49539	1.49539	5.35310	5.35310
0.8	1.34646	1.34646	4.87587	4.87587
0.9	1.17991	1.17991	4.34983	4.34983
1.0	0.995347	0.995362	3.78021	3.78021

Table 4.3
Absolute residual error functions for $A=6,\ a=-2$.

x	$\theta=\vartheta=0$	$\theta=\vartheta=\frac{1}{2}$	$\theta=\vartheta=0$	$\theta=\vartheta=\frac{1}{2}$
	$(N=16)$		$(N=24)$	
0.0	$1.02.10^{-9}$	$2.37.10^{-9}$	$4.16.10^{-14}$	$1.10.10^{-13}$
0.1	$2.85.10^{-10}$	$2.76.10^{-10}$	$8.28.10^{-15}$	$3.83.10^{-15}$
0.2	$1.18.10^{-10}$	$1.45.10^{-11}$	$1.19.10^{-14}$	$7.50.10^{-15}$
0.3	$3.30.10^{-10}$	$2.07.10^{-10}$	$1.05.10^{-14}$	$5.96.10^{-15}$
0.4	$4.60.10^{-10}$	$3.15.10^{-10}$	$3.44.10^{-15}$	$3.55.10^{-15}$
0.5	$6.14.10^{-10}$	$4.26.10^{-10}$	$2.03.10^{-14}$	$1.60.10^{-14}$
0.6	$8.53.10^{-10}$	$5.86.10^{-10}$	$6.21.10^{-15}$	$6.21.10^{-15}$
0.7	$1.18.10^{-9}$	$7.49.10^{-10}$	$3.19.10^{-14}$	$2.44.10^{-14}$
0.8	$9.20.10^{-10}$	$1.11.10^{-10}$	$5.72.10^{-14}$	$7.21.10^{-14}$
0.9	$6.51.10^{-9}$	$6.50.10^{-9}$	$1.80.10^{-14}$	$1.50.10^{-13}$
1.0	$6.04.10^{-7}$	$8.66.10^{-7}$	$5.22.10^{-11}$	$8.96.10^{-11}$

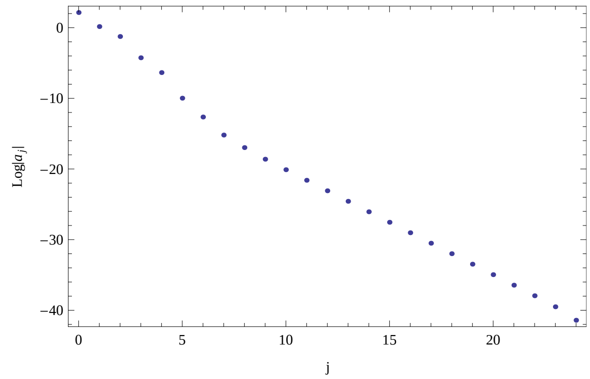

Fig. 1: Logarithmic graph of the absolute values of the coefficients, $|a_j|$, in the shifted Jacobi polynomial expansion for $A=10,\ a=2$ and $\theta=\vartheta=1$ at $N=24$.

Table 4.4
Absolute residual error functions for $A=6,\ a=2$.

x	$\theta=\vartheta=\frac{-1}{2}$	$\theta=\vartheta=\frac{1}{2}$	$\theta=\vartheta=\frac{-1}{2}$	$\theta=\vartheta=\frac{1}{2}$
	$(N=16)$		$(N=24)$	
0.0	$2.87.10^{-11}$	$2.36.10^{-10}$	$1.21.10^{-15}$	$5.62.10^{-16}$
0.1	$2.18.10^{-11}$	$2.73.10^{-11}$	$5.96.10^{-16}$	$2.49.10^{-16}$
0.2	$2.62.10^{-11}$	$1.38.10^{-12}$	$0.00.10^{00}$	$8.04.10^{-16}$
0.3	$4.76.10^{-11}$	$2.00.10^{-11}$	$3.33.10^{-16}$	$4.44.10^{-16}$
0.4	$6.25.10^{-11}$	$2.99.10^{-11}$	$6.66.10^{-16}$	$4.44.10^{-16}$
0.5	$8.15.10^{-11}$	$3.98.10^{-11}$	$2.10.10^{-15}$	$4.44.10^{-16}$
0.6	$1.10.10^{-10}$	$5.34.10^{-11}$	$4.44.10^{-16}$	$6.66.10^{-16}$
0.7	$1.55.10^{-10}$	$6.63.10^{-11}$	$2.22.10^{-16}$	$1.55.10^{-15}$
0.8	$1.75.10^{-10}$	$9.48.10^{-12}$	$3.10.10^{-15}$	$8.88.10^{-16}$
0.9	$3.95.10^{-10}$	$5.27.10^{-10}$	$5.77.10^{-15}$	$2.22.10^{-15}$
1.0	$2.81.10^{-8}$	$6.53.10^{-8}$	$6.69.10^{-13}$	$2.05.10^{-12}$

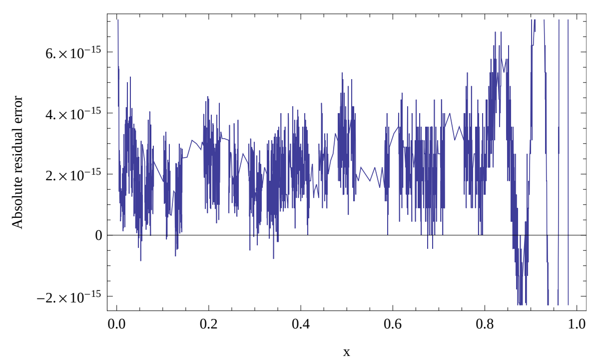

Fig. 2: Graph of absolute residual error function for $A=10,\ a=2$ and $\theta=\vartheta=1$ at $N=24$.

5 Conclusions

Here we have studied a class of travelling wave solutions to the Casimir equation for the nonlinear Ito system of PDEs. Since the Ito equation has been shown to admit a reduction to a single nonlinear Casimir equation, we assume a travelling wave solution and hence put this partial differential equation into the form of a nonlinear ordinary differential equation. The resulting ordinary differential equation takes the form of a second order highly nonlinear equation. This equation involves inverse powers of an unknown function, so, in addition to being nonlinear, it can become singular. As such, the equation governing travelling waves for the Ito system can be difficult for numerical and analytical solution methods.

We solve the ordinary differential equation and related initial value problem numerically on a finite interval by means of a shifted Jacobi-Gauss collocation (SJC)

Table 4.5

Absolute residual error functions for $A = 2,\ a = 0$.

x	$\theta = \vartheta = 0$	$\theta = \vartheta = \frac{1}{2}$	$\theta = \vartheta = 0$	$\theta = \vartheta = \frac{1}{2}$
	$(N = 16)$		$(N = 24)$	
0.0	$8.62.10^{-8}$	$1.81.10^{-7}$	$2.08.10^{-11}$	$5.66.10^{-11}$
0.1	$2.25.10^{-8}$	$2.14.10^{-8}$	$3.88.10^{-12}$	$1.68.10^{-12}$
0.2	$9.47.10^{-9}$	$1.10.10^{-9}$	$5.54.10^{-12}$	$3.88.10^{-12}$
0.3	$2.67.10^{-8}$	$1.64.10^{-8}$	$5.15.10^{-12}$	$2.98.10^{-12}$
0.4	$3.78.10^{-8}$	$2.53.10^{-8}$	$1.79.10^{-12}$	$1.75.10^{-12}$
0.5	$5.14.10^{-8}$	$3.50.10^{-8}$	$1.09.10^{-11}$	$7.35.10^{-12}$
0.6	$7.32.10^{-8}$	$4.92.10^{-8}$	$3.36.10^{-12}$	$3.39.10^{-12}$
0.7	$1.05.10^{-7}$	$6.50.10^{-8}$	$2.06.10^{-11}$	$1.15.10^{-11}$
0.8	$8.50.10^{-8}$	$1.00.10^{-8}$	$5.14.10^{-11}$	$3.44.10^{-11}$
0.9	$6.39.10^{-7}$	$6.25.10^{-7}$	$1.14.10^{-10}$	$4.30.10^{-11}$
1.0	$6.50.10^{-5}$	$9.13.10^{-5}$	$3.65.10^{-8}$	$6.03.10^{-8}$

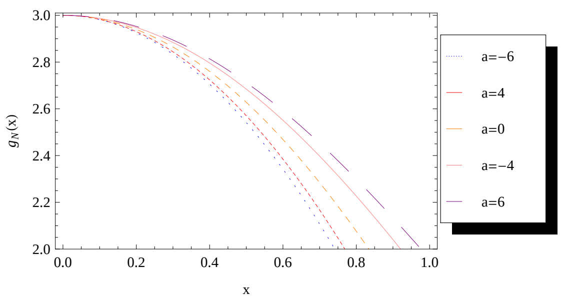

Fig. 3: Graphs of the travelling wave solution governed by (8)-(9) obtained by the present method for $a = 6,\ 4,\ 0,\ -4,\ -6$, fixed $A = 3$, and $\theta = \vartheta = 1$ at $N = 8$.

Table 4.6

Absolute residual error functions for $A = 7,\ a = 0$.

x	$\theta = \vartheta = \frac{-1}{2}$	$\theta = \vartheta = 0$	$\theta = \vartheta = \frac{-1}{2}$	$\theta = \vartheta = 0$
	$(N = 16)$		$(N = 24)$	
0.0	$5.85.10^{-11}$	$2.02.10^{-10}$	$1.87.10^{-15}$	$8.98.10^{-15}$
0.1	$4.49.10^{-11}$	$5.64.10^{-11}$	$1.45.10^{-15}$	$5.96.10^{-16}$
0.2	$5.46.10^{-11}$	$2.34.10^{-11}$	$4.02.10^{-15}$	$1.44.10^{-15}$
0.3	$1.00.10^{-10}$	$6.53.10^{-11}$	$3.99.10^{-15}$	$4.99.10^{-16}$
0.4	$1.34.10^{-10}$	$9.10.10^{-11}$	$4.44.10^{-16}$	$1.22.10^{-15}$
0.5	$1.78.10^{-10}$	$1.21.10^{-10}$	$3.77.10^{-15}$	$5.66.10^{-15}$
0.6	$2.48.10^{-10}$	$1.69.10^{-10}$	$4.44.10^{-16}$	$2.22.10^{-15}$
0.7	$3.60.10^{-10}$	$2.35.10^{-10}$	$1.19.10^{-14}$	$4.88.10^{-15}$
0.8	$4.21.10^{-10}$	$1.82.10^{-10}$	$2.04.10^{-14}$	$1.15.10^{-14}$
0.9	$1.00.10^{-9}$	$1.29.10^{-9}$	$5.63.10^{-14}$	$1.42.10^{-14}$
1.0	$7.68.10^{-8}$	$1.20.10^{-7}$	$5.57.10^{-12}$	$1.02.10^{-11}$

spectral method. The spatial approximation is based on shifted Jacobi polynomials $J_n^{(\theta,\vartheta)}(x)$ with $\theta, \vartheta \in (-1, \infty)$, $x \in (0,1)$ and n the polynomial degree. The shifted Jacobi-Gauss points are then used as collocation nodes. This method is developed for the general problem in Section 3, while a number of numerical experiments are highlighted in Section 4.

The numerical experiments presented here demonstrate both the reliability and the efficiency of the

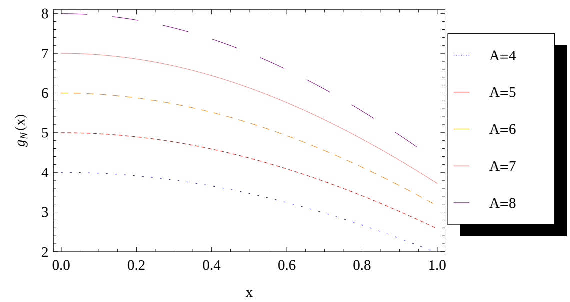

Fig. 4: Graphs of the travelling wave solution governed by (8)-(9) by the present method for $A = 8,\ 7,\ 6,\ 5,\ 4$, fixed $a = -5$ and $\theta = \vartheta = 1$ at $N = 8$.

method. Relatively few notes permit very low residual errors. The residuals increase at the initial data is small, owing to the fact that the governing ordinary differential equation involves inverse powers of the unknown function. Increasing the number of nodes is shown to improve the accuracy of the solutions (in terms of residual errors) by about an order of magnitude for ever two nodes added. We are also able to show that the coefficients of the higher order terms in the shifted Jacobi polynomials decrease exponentially; for one set of parameter values, this is demonstrated explicitly in Fig. 1. Hence, the proposed spectral method is rather accurate and efficient, granting us rather good residual errors upon employing relatively few nodes.

The results demonstrate the propagation of a wave solution to the Casimir equation for the Ito system. In order to recover the explicit solution $W(x,t)$ from the numerical solution for $g(z)$ as provided here, one can reverse the series of transformations given in Section 1, to obtain

$$W(x,t) = \beta^{-1/4} \int_0^{x-\beta^{-1/2}t} g(z)dz. \tag{34}$$

With this, we have given a numerical method by which one may obtain solutions $W(x,t)$ to the equation (2).

The equation we solve can be viewed as a type of reaction equation, with a rational response function. Clearly, the method here should certainly be applicable for reaction equations with simple polynomial reaction functions, such as the Fisher-Kolmogorov equation and the Nagumo equation [29]. In principle, the present method can be adapted to solve general problems of the form $g'' = R(g)$, where R is a reaction function, provided that $g(0) = A$ is selected so that $\lim_{g \to A} R(g)$ is finite.

Physically, the Ito system admits two types of solutions: hyperbolic solutions (which become large as $z \to \pm\infty$) and periodic solutions which remain bounded in space and time. The latter solutions are the most physically relevant. While some of these solutions were considered in [3], it is worth noting that such exact or analytical solutions are valid for restricted parameter regimes. What we have done in the present paper was to give a numerical algorithm by which we may approximate the periodic and bounded solutions to the Ito equation. Hence, we do not require restricted parameter regimes. Such a method is useful, as it is rather simple to apply (as opposed to analytical approximations, which may become rather involved).

The results suggest that physically interesting solutions are possible, even when analytical or exact solutions may not exist, or in cases where such analytical or exact solutions are not efficient to obtain. It is worth mentioning that the numerical approach has been shown to be both accurate and efficient for obtaining travelling wave solutions to the Ito equation.

The numerical solutions are given for a function $g(z)$. This solution may be mapped back into the solution $W(x,t)$ for the Casimir equation to the Ito equation by using the following transformation:

$$\begin{aligned} W(x,t) &= \phi(x-Ct) = \phi_0 + \int_0^{x-Ct} f(\sigma)d\sigma \\ &= \phi_0 + \sqrt{C}\int_0^{x-Ct} g(\sigma)d\sigma. \end{aligned} \tag{35}$$

Here, $C = \beta^{-1/2}$ is the wave speed. This, a travelling wave solution for $W(x,t)$ can be obtained numerically, upon integrating the function g obtained through the collocation method presented here. For this solution to be real-valued, one must have $C > 0$. Therefore, any such solution is propagating to the right in x.

In order to recover a solution pair $U(x,t)$ and $V(x,t)$ in the travelling wave case, we use the transforms given in [3]:

$$U(x,t) = \frac{\partial W}{\partial x} \quad \text{and} \quad V(x,t) = \frac{\partial W}{\partial t}. \tag{36}$$

We find that

$$U(x,t) = \sqrt{C}g(x-Ct) \quad \text{and} \quad V(x,t) = \frac{-1}{C^{3/2}g(x-Ct)}. \tag{37}$$

Interesting, this implies that UV is a conserved quantity in the travelling wave case, as

$$U(x,t)V(x,t) = \frac{-1}{C} \tag{38}$$

for travelling wave solutions.

The present problem has the additional complication of the nonlinearity in (8) which depends on inverse powers of g. Previous applications of such methods were only concerned with more standard power-law type nonlinearity. Therefore, the present paper demonstrates that the numerical method is rather versatile and can be used for a wide variety of problems. In particular, it is reasonable to apply the present method to problems of the form

$$g'' = \mathscr{F}(g) + \mathscr{G}\left(\frac{1}{g}\right), \tag{39}$$

where both $\mathscr{F}$ and $\mathscr{G}$ are polynomials.

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